Ahmad A. Kamal

1000 Solved Problems in Classical Physics

An Exercise Book

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Dedicated to my Parents

Preface

This book complements the book *1000 Solved Problems in Modern Physics* by the same author and published by Springer-Verlag so that bulk of the courses for undergraduate curriculum are covered. It is targeted mainly at the undergraduate students of USA, UK and other European countries and the M.Sc. students of Asian countries, but will be found useful for the graduate students, students preparing for graduate record examination (GRE), teachers and tutors. This is a by-product of lectures given at the Osmania University, University of Ottawa and University of Tebriz over several years and is intended to assist the students in their assignments and examinations. The book covers a wide spectrum of disciplines in classical physics and is mainly based on the actual examination papers of UK and the Indian universities. The selected problems display a large variety and conform to syllabi which are currently being used in various countries.

The book is divided into 15 chapters. Each chapter begins with basic concepts and a set of formulae used for solving problems for quick reference, followed by a number of problems and their solutions.

The problems are judiciously selected and are arranged section-wise. The solutions are neither pedantic nor terse. The approach is straightforward and step-by-step solutions are elaborately provided. There are approximately 450 line diagrams, one-fourth of them in colour for illustration. A subject index and a problem index are provided at the end of the book.

Elementary calculus, vector calculus and algebra are the prerequisites. The areas of mechanics and electromagnetism are emphasized. No book on problems can claim to exhaust the variety in the limited space. An attempt is made to include the important types of problems at the undergraduate level.

It is a pleasure to thank Javid, Suraiya and Techastra Solutions (P) Ltd. for typesetting and Maryam for her patience. I am grateful to the universities of UK and India for permitting me to use their question papers; to R.W. Norris and W. Seymour, *Mechanics via Calculus*, Longmans, Green and Co., 1923; to Robert A. Becker, *Introduction to Theoretical Mechanics*, McGraw-Hill Book Co. Inc, 1954, for one problem; and Google Images for the cover page. My thanks are to Springer-Verlag,

in particular Claus Ascheron, Adelheid Duhm and Elke Sauer, for constant encouragement.

Murphy, Texas *Ahmad A. Kamal*
November 2010

Contents

Chapter 1
Kinematics and Statics

Abstract *Chapter 1* is devoted to problems based on one and two dimensions. The use of various kinematical formulae and the sign convention are pointed out. Problems in statics involve force and torque, centre of mass of various systems and equilibrium.

1.1 Basic Concepts and Formulae

Motion in One Dimension

The notation used is as follows: u = initial velocity, v = final velocity, a = acceleration, s = displacement, t = time (Table 1.1).

Table 1.1 Kinematical equations

	U	V	A	S	t
(i) $v = u + at$	✓	✓	✓	X	✓
(ii) $s = ut + 1/2at^2$	✓	X	✓	✓	✓
(iii) $v^2 = u^2 + 2as$	✓	✓	✓	✓	X
(iv) $s = \frac{1}{2}(u + v)t$	✓	✓	X	✓	✓

In each of the equations u is present. Out of the remaining four quantities only three are required. The initial direction of motion is taken as positive. Along this direction u and s and a are taken as positive, t is always positive, v can be positive or negative. As an example, an object is dropped from a rising balloon. Here, the parameters for the object will be as follows:

u = initial velocity of the balloon (as seen from the ground)

$u = +$ve, $a = -g$. $t = +$ve, $v = +$ve or $-$ve depending on the value of t, $s = +$ve or $-$ve, if $s = -$ve, then the object is found below the point it was released.

Note that (ii) and (iii) are quadratic. Depending on the value of u, both the roots may be real or only one may be real or both may be imaginary and therefore unphysical.

v–t and *a–t* Graphs

The area under the v–t graph gives the displacement (see prob. 1.11) and the area under the a–t graph gives the velocity.

Motion in Two Dimensions – Projectile Motion

$$\text{Equation: } y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha} \tag{1.1}$$

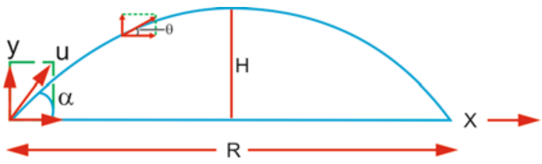


Fig. 1.1 Projectile Motion

$$\text{Time of flight: } T = \frac{2u\sin\alpha}{g} \tag{1.2}$$

$$\text{Range: } R = \frac{u^2\sin 2\alpha}{g} \tag{1.3}$$

$$\text{Maximum height: } H = \frac{u^2\sin^2\alpha}{2g} \tag{1.4}$$

$$\text{Velocity: } v = \sqrt{g^2t^2 - 2ug\sin\alpha.t + u^2} \tag{1.5}$$

$$\text{Angle: } \tan\theta = \frac{u\sin\alpha - gt}{u\ \cos\alpha} \tag{1.6}$$

Relative Velocity

If v_{A} is the velocity of A and v_{B} that of B, then the relative velocity of A with respect to B will be

$$v_{\mathrm{AB}} = v_{\mathrm{A}} - v_{\mathrm{B}} \tag{1.7}$$

Motion in Resisting Medium

In the absence of air the initial speed of a particle thrown upward is equal to that of final speed, and the time of ascent is equal to that of descent. However, in the presence of air resistance the final speed is less than the initial speed and the time of descent is greater than that of ascent (see prob. 1.21).

Equation of motion of a body in air whose resistance varies as the velocity of the body (see prob. 1.22).

Centre of mass is defined as

$$\boldsymbol{r}_{\text{cm}} = \frac{\Sigma m_i r_i}{\Sigma m_i} = \frac{1}{M}\Sigma m_i \boldsymbol{r}_i \tag{1.8}$$

Centre of mass velocity is defined as

$$\boldsymbol{V}_{\text{c}} = \frac{1}{M}\Sigma m_i \dot{r}_i \tag{1.9}$$

The centre of mass moves as if the mass of various particles is concentrated at the location of the centre of mass.

Equilibrium

A system will be in translational equilibrium if $\Sigma \boldsymbol{F} = 0$. In terms of potential $\dfrac{\partial V}{\partial x} = 0$, where V is the potential. The equilibrium will be stable if $\dfrac{\partial^2 V}{\partial x^2} < 0$. A system will be in rotational equilibrium if the sum of the external torques is zero, i.e. $\Sigma \tau_i = 0$

1.2 Problems

1.2.1 Motion in One Dimension

1.1 A car starts from rest at constant acceleration of $2.0\,\text{m/s}^2$. At the same instant a truck travelling with a constant speed of 10 m/s overtakes and passes the car.

(a) How far beyond the starting point will the car overtake the truck?
(b) After what time will this happen?
(c) At that instant what will be the speed of the car?

1.2 From an elevated point A, a stone is projected vertically upward. When the stone reaches a distance h below A, its velocity is double of what it was at a height h above A. Show that the greatest height obtained by the stone above A is $5h/3$.

[Adelaide University]

1.3 A stone is dropped from a height of 19.6 m, above the ground while a second stone is simultaneously projected from the ground with sufficient velocity to enable it to ascend 19.6 m. When and where the stones would meet.

1.4 A particle moves according to the law $x = A \sin \pi t$, where x is the displacement and t is time. Find the distance traversed by the particle in 3.0 s.

1.5 A man of height 1.8 m walks away from a lamp at a height of 6 m. If the man's speed is 7 m/s, find the speed in m/s at which the tip of the shadow moves.

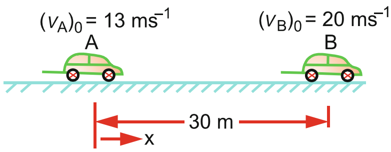
1.6 The relation $3t = \sqrt{3x} + 6$ describes the displacement of a particle in one direction, where x is in metres and t in seconds. Find the displacement when the velocity is zero.

1.7 A particle projected up passes the same height h at 2 and 10 s. Find h if $g = 9.8\,\mathrm{m/s^2}$.

1.8 Cars A and B are travelling in adjacent lanes along a straight road (Fig. 1.2). At time, $t = 0$ their positions and speeds are as shown in the diagram. If car A has a constant acceleration of $0.6\,\mathrm{m/s^2}$ and car B has a constant deceleration of $0.46\,\mathrm{m/s^2}$, determine when A will overtake B.

[University of Manchester 2007]

Fig. 1.2



1.9 A boy stands at A in a field at a distance 600 m from the road BC. In the field he can walk at 1 m/s while on the road at 2 m/s. He can walk in the field along AD and on the road along DC so as to reach the destination C (Fig. 1.3). What should be his route so that he can reach the destination in the least time and determine the time.

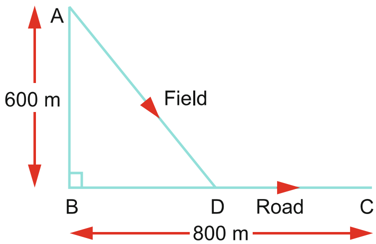


Fig. 1.3

1.10 Water drips from the nozzle of a shower onto the floor 2.45 m below. The drops fall at regular interval of time, the first drop striking the floor at the instant the third drop begins to fall. Locate the second drop when the first drop strikes the floor.

1.11 The velocity–time graph for the vertical component of the velocity of an object thrown upward from the ground which reaches the roof of a building and returns to the ground is shown in Fig. 1.4. Calculate the height of the building.

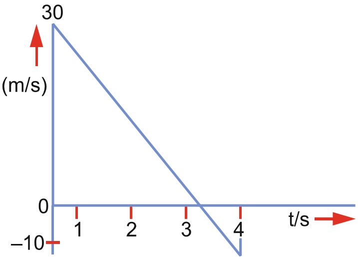


Fig. 1.4

1.12 A ball is dropped into a lake from a diving board 4.9 m above the water. It hits the water with velocity v and then sinks to the bottom with the constant velocity v. It reaches the bottom of the lake 5.0 s after it is dropped. Find

(a) the average velocity of the ball and
(b) the depth of the lake.

1.13 A stone is dropped into the water from a tower 44.1 m above the ground. Another stone is thrown vertically down 1.0 s after the first one is dropped. Both the stones strike the ground at the same time. What was the initial velocity of the second stone?

1.14 A boy observes a cricket ball move up and down past a window 2 m high. If the total time the ball is in sight is 1.0 s, find the height above the window that the ball rises.

1.15 In the last second of a free fall, a body covered three-fourth of its total path:

(a) For what time did the body fall?
(b) From what height did the body fall?

1.16 A man travelling west at 4 km/h finds that the wind appears to blow from the south. On doubling his speed he finds that it appears to blow from the southwest. Find the magnitude and direction of the wind's velocity.

1.17 An elevator of height h ascends with constant acceleration a. When it crosses a platform, it has acquired a velocity u. At this instant a bolt drops from the top of the elevator. Find the time for the bolt to hit the floor of the elevator.

1.18 A car and a truck are both travelling with a constant speed of 20 m/s. The car is 10 m behind the truck. The truck driver suddenly applies his brakes, causing the truck to decelerate at the constant rate of 2 m/s^2. Two seconds later the driver of the car applies his brakes and just manages to avoid a rear-end collision. Determine the constant rate at which the car decelerated.

1.19 Ship A is 10 km due west of ship B. Ship A is heading directly north at a speed of 30 km/h, while ship B is heading in a direction 60° west of north at a speed of 20 km/h.

(i) Determine the magnitude and direction of the velocity of ship B relative to ship A.
(ii) What will be their distance of closest approach?

[University of Manchester 2008]

1.20 A balloon is ascending at the rate of 9.8 m/s at a height of 98 m above the ground when a packet is dropped. How long does it take the packet to reach the ground?

1.2.2 Motion in Resisting Medium

1.21 An object of mass m is thrown vertically up. In the presence of heavy air resistance the time of ascent (t_1) is no longer equal to the time of descent (t_2). Similarly the initial speed (u) with which the body is thrown is not equal to the final speed (v) with which the object returns. Assuming that the air resistance F is constant show that

$$\frac{t_2}{t_1} = \sqrt{\frac{g + F/m}{g - F/m}}; \quad \frac{v}{u} = \sqrt{\frac{g - F/m}{g + F/m}}$$

1.22 Determine the motion of a body falling under gravity, the resistance of air being assumed proportional to the velocity.

1.23 Determine the motion of a body falling under gravity, the resistance of air being assumed proportional to the square of the velocity.

1.24 A body is projected upward with initial velocity u against air resistance which is assumed to be proportional to the square of velocity. Determine the height to which the body will rise.

1.25 Under the assumption of the air resistance being proportional to the square of velocity, find the loss in kinetic energy when the body has been projected upward with velocity u and return to the point of projection.

1.2.3 Motion in Two Dimensions

1.26 A particle moving in the xy-plane has velocity components $\mathrm{d}x/\mathrm{d}t = 6 + 2t$ and $\mathrm{d}y/\mathrm{d}t = 4 + t$
where x and y are measured in metres and t in seconds.

(i) Integrate the above equation to obtain x and y as functions of time, given that the particle was initially at the origin.
(ii) Write the velocity $\boldsymbol{v}$ of the particle in terms of the unit vectors $\hat{i}$ and $\hat{j}$.

(iii) Show that the acceleration of the particle may be written as $a = 2\hat{i} + \hat{j}$.
(iv) Find the magnitude of the acceleration and its direction with respect to the x-axis.

[University of Aberystwyth Wales 2000]

1.27 Two objects are projected horizontally in opposite directions from the top of a tower with velocities u_1 and u_2. Find the time when the velocity vectors are perpendicular to each other and the distance of separation at that instant.

1.28 From the ground an object is projected upward with sufficient velocity so that it crosses the top of a tower in time t_1 and reaches the maximum height. It then comes down and recrosses the top of the tower in time t_2, time being measured from the instant the object was projected up. A second object released from the top of the tower reaches the ground in time t_3. Show that $t_3 = \sqrt{t_1 t_2}$.

1.29 A shell is fired at an angle θ with the horizontal up a plane inclined at an angle α. Show that for maximum range, $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$.

1.30 A stone is thrown from ground level over horizontal ground. It just clears three walls, the successive distances between them being r and $2r$. The inner wall is 15/7 times as high as the outer walls which are equal in height. The total horizontal range is nr, where n is an integer. Find n.

[University of Dublin]

1.31 A boy wishes to throw a ball through a house via two small openings, one in the front and the other in the back window, the second window being directly behind the first. If the boy stands at a distance of 5 m in front of the house and the house is 6 m deep and if the opening in the front window is 5 m above him and that in the back window 2 m higher, calculate the velocity and the angle of projection of the ball that will enable him to accomplish his desire.

[University of Dublin]

1.32 A hunter directs his uncalibrated rifle toward a monkey sitting on a tree, at a height h above the ground and at distance d. The instant the monkey observes the flash of the fire of the rifle, it drops from the tree. Will the bullet hit the monkey?

1.33 If α is the angle of projection, R the range, h the maximum height, T the time of flight then show that
(a) $\tan\alpha = 4h/R$ and **(b)** $h = gT^2/8$

1.34 A projectile is fired at an angle of 60° to the horizontal with an initial velocity of 800 m/s:

(i) Find the time of flight of the projectile before it hits the ground
(ii) Find the distance it travels before it hits the ground (range)
(iii) Find the time of flight for the projectile to reach its maximum height

(iv) Show that the shape of its flight is in the form of a parabola $y = bx + cx^2$, where b and c are constants [acceleration due to gravity $g = 9.8\,\mathrm{m/s^2}$].

[University of Aberystwyth, Wales 2004]

1.35 A projectile of mass 20.0 kg is fired at an angle of 55.0° to the horizontal with an initial velocity of 350 m/s. At the highest point of the trajectory the projectile explodes into two equal fragments, one of which falls vertically downwards with no initial velocity immediately after the explosion. Neglect the effect of air resistance:

(i) How long after firing does the explosion occur?
(ii) Relative to the firing point, where do the two fragments hit the ground?
(iii) How much energy is released in the explosion?

[University of Manchester 2008]

1.36 An object is projected horizontally with velocity 10 m/s. Find the radius of curvature of its trajectory in 3 s after the motion has begun.

1.37 A and B are points on opposite banks of a river of breadth a and AB is at right angles to the flow of the river (Fig. 1.4). A boat leaves B and is rowed with constant velocity with the bow always directed toward A. If the velocity of the river is equal to this velocity, find the path of the boat (Fig. 1.5).

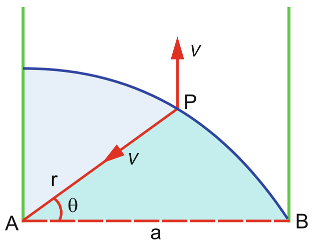


Fig. 1.5

1.38 A ball is thrown from a height h above the ground. The ball leaves the point located at distance d from the wall, at 45° to the horizontal with velocity u. How far from the wall does the ball hit the ground (Fig. 1.6)?

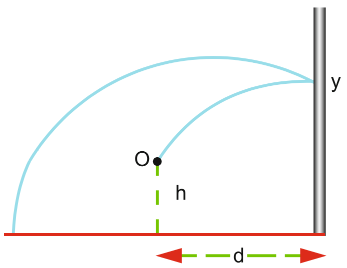


Fig. 1.6

1.2.4 Force and Torque

1.39 Three vector forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ act on a particle of mass $m = 3.80\,\text{kg}$ as shown in Fig. 1.7:

(i) Calculate the magnitude and direction of the net force acting on the particle.
(ii) Calculate the particle's acceleration.
(iii) If an additional stabilizing force $\boldsymbol{F}_4$ is applied to create an equilibrium condition with a resultant net force of zero, what would be the magnitude and direction of $\boldsymbol{F}_4$?

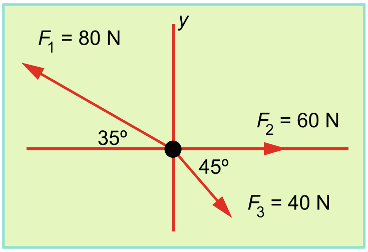


Fig. 1.7

1.40 **(a)** A thin cylindrical wheel of radius $r = 40\,\text{cm}$ is allowed to spin on a frictionless axle. The wheel, which is initially at rest, has a tangential force applied at right angles to its radius of magnitude 50 N as shown in Fig. 1.8a. The wheel has a moment of inertia equal to $20\,\text{kg}\,\text{m}^2$.

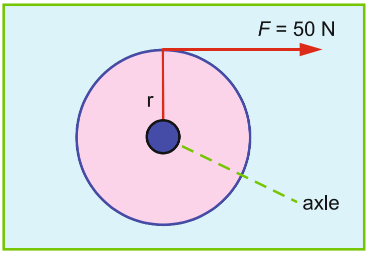


Fig. 1.8a

Calculate

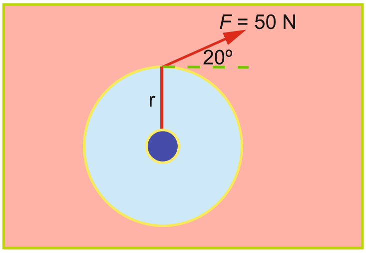
(i) The torque applied to the wheel
(ii) The angular acceleration of the wheel
(iii) The angular velocity of the wheel after 3 s
(iv) The total angle swept out in this time

(b) The same wheel now has the same force applied but inclined at an angle of 20° to the tangent as shown in Fig. 1.8b. Calculate

(i) The torque applied to the wheel
(ii) The angular acceleration of the wheel

[University of Aberystwyth, Wales 2005]

Fig. 1.8b



1.41 A container of mass 200 kg rests on the back of an open truck. If the truck accelerates at 1.5 m/s^2, what is the minimum coefficient of static friction between the container and the bed of the truck required to prevent the container from sliding off the back of the truck?

[University of Manchester 2007]

1.42 A wheel of radius r and weight W is to be raised over an obstacle of height h by a horizontal force F applied to the centre. Find the minimum value of F (Fig. 1.9).

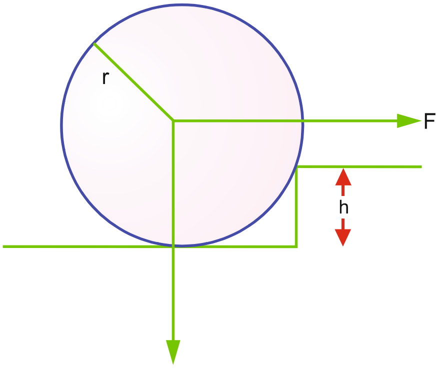


Fig. 1.9

1.2.5 Centre of Mass

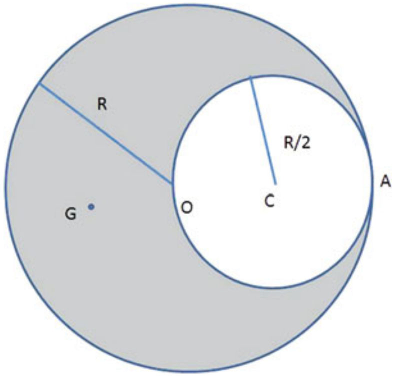
1.43 A thin uniform wire is bent into a semicircle of radius R. Locate the centre of mass from the diameter of the semicircle.

1.44 Find the centre of mass of a semicircular disc of radius R and of uniform density.

1.45 Locate the centre of mass of a uniform solid hemisphere of radius R from the centre of the base of the hemisphere along the axis of symmetry.

1.46 A thin circular disc of uniform density is of radius R. A circular hole of radius ½R is cut from the disc and touching the disc's circumference as in Fig. 1.10. Find the centre of mass.

Fig. 1.10



1.47 The mass of the earth is 81% the mass of the moon. The distance between the centres of the earth and the moon is 60 times the radius of earth $R = 6400\,\text{km}$. Find the centre of mass of the earth–moon system.

1.48 The distance between the centre of carbon and oxygen atoms in CO molecule is 1.13 Å. Locate the centre of mass of the molecule relative to the carbon atom.

1.49 The ammonia molecule NH_3 is in the form of a pyramid with the three H atoms at the corners of an equilateral triangle base and the N atom at the apex of the pyramid. The H–H distance = 1.014 Å and N–H distance = 1.628 Å. Locate the centre of mass of the NH_3 molecule relative to the N atom.

1.50 A boat of mass 100 kg and length 3 m is at rest in still water. A boy of mass 50 kg walks from the bow to the stern. Find the distance through which the boat moves.

1.51 At one end of the rod of length L, a body whose mass is twice that of the rod is attached. If the rod is to move with pure translation, at what fractional length from the loaded end should it be struck?

1.52 Find the centre of mass of a solid cone of height h.

1.53 Find the centre of mass of a wire in the form of an arc of a circle of radius R which subtends an angle 2α symmetrically at the centre of curvature.

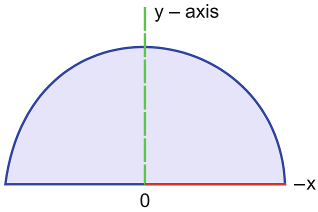
1.54 Five identical pigeons are flying together northward with speed v_0. One of the pigeons is shot dead by a hunter and the other four continue to fly with the same speed. Find the centre of mass speed of the rest of the pigeons which continue to fly with the same speed after the dead pigeon has hit the ground.

1.55 The linear density of a rod of length L is directly proportional to the distance from one end. Locate the centre of mass from the same end.

1.56 Particles of masses m, $2m$, $3m \ldots nm$ are collinear at distances L, $2L$, $3L \ldots nL$, respectively, from a fixed point. Locate the centre of mass from the fixed point.

1.57 A semicircular disc of radius R has density ρ which varies as $\rho = cr^2$, where r is the distance from the centre of the base and $c$is a constant. The centre of mass will lie along the y-axis for reasons of symmetry (Fig. 1.11). Locate the centre of mass from O, the centre of the base.

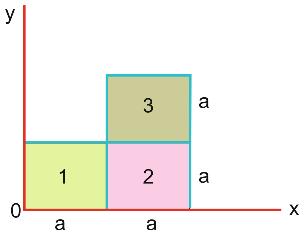
Fig. 1.11



1.58 Locate the centre of mass of a water molecule, given that the OH bond has length 1.77 Å and angle HOH is 105°.

1.59 Three uniform square laminas are placed as in Fig. 1.12. Each lamina measures 'a' on side and has mass m. Locate the CM of the combined structure.

Fig. 1.12



1.2.6 Equilibrium

1.60 Consider a particle of mass m moving in one dimension under a force with the potential $U(x) = k(2x^3 - 5x^2 + 4x)$, where the constant $k > 0$. Show that the point $x = 1$ corresponds to a stable equilibrium position of the particle.
[University of Manchester 2007]

1.61 Consider a particle of mass m moving in one dimension under a force with the potential $U(x) = k(x^2 - 4xl)$, where the constant $k > 0$. Show that the point $x = 2l$ corresponds to a stable equilibrium position of the particle.
Find the frequency of a small amplitude oscillation of the particle about the equilibrium position.
[University of Manchester 2006]

1.62 A cube rests on a rough horizontal plane. A tension parallel to the plane is applied by a thread attached to the upper surface. Show that the cube will slide or topple according to the coefficient of friction is less or greater than 0.5.

1.63 A ladder leaning against a smooth wall makes an angle α with the horizontal when in a position of limiting equilibrium. Show that the coefficient of friction between the ladder and the ground is $\frac{1}{2}\cot\alpha$.

1.3 Solutions

1.3.1 Motion in One Dimension

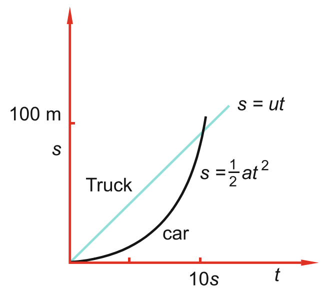
1.1 **(a)** Equation of motion for the truck: $s = ut$ (1)

Equation of motion for the car: $s = \dfrac{1}{2}at^2$ (2)

The graphs for (1) and (2) are shown in Fig. 1.13. Eliminating t between the two equations

$$s\left(1 - \frac{1}{2}\frac{as}{u^2}\right) = 0 \tag{3}$$

Fig. 1.13



Either $s = 0$ or $1 - \dfrac{1}{2}\dfrac{as}{u^2} = 0$. The first solution corresponds to the result that the truck overtakes the car at $s = 0$ and therefore at $t = 0$.

The second solution gives $s = \dfrac{2u^2}{a} = \dfrac{2 \times 10^2}{2} = 100\,\text{m}$

(b) $t = \dfrac{s}{u} = \dfrac{100}{10} = 10\,\text{s}$

(c) $v = at = 2 \times 10 = 20\,\text{m/s}$

1.2 When the stone reaches a height h above A

$$v_1^2 = u^2 - 2gh \tag{1}$$

and when it reaches a distance h below A

$$v_2^2 = u^2 + 2gh \tag{2}$$

since the velocity of the stone while crossing A on its return journey is again u vertically down.

Also, $v_2 = 2v_1$ (by problem) (3)

Combining (1), (2) and (3) $u^2 = \frac{10}{3}gh$ (4)

Maximum height

$$H = \frac{u^2}{2g} = \frac{10}{3}\frac{gh}{2g} = \frac{5h}{3}$$

1.3 Let the stones meet at a height s m from the earth after t s. Distance covered by the first stone

$$h - s = \frac{1}{2}gt^2 \tag{1}$$

where $h = 19.6\,\text{m}$. For the second stone

$$s = ut = \frac{1}{2}gt^2 \tag{2}$$

$$v^2 = 0 = u^2 - 2gh$$

$$u = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 19.6} = 19.6\,\text{m/s} \tag{3}$$

Adding (1) and (2)

$$h = ut,\ t = \frac{h}{u} = \frac{19.6}{19.6} = 1\,\text{s}$$

From (2),

$$s = 19.6 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 14.7\,\text{m}$$

1.4 $x = A \sin \pi t = A \sin \omega t$

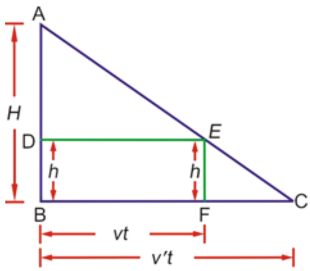
where ω is the angular velocity, $\omega = \pi$

Time period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi} = 2\,\text{s}$

In $\frac{1}{2}$s (a quarter of the cycle) the distance covered is A. Therefore in 3 s the distance covered will be $6A$.

1.5 Let the lamp be at A at height H from the ground, that is AB $= H$, Fig. 1.14. Let the man be initially at B, below the lamp, his height being equal to BD $= h$, so that the tip of his shadow is at B. Let the man walk from B to F in time t with speed v, the shadow will go up to C in the same time t with speed v':

Fig. 1.14



BF $= vt$; BC $= v't$

From similar triangles EFC and ABC

$$\frac{\text{FC}}{\text{BC}} = \frac{\text{EF}}{\text{AB}} = \frac{h}{H}$$

$$\frac{\text{FC}}{\text{BC}} = \frac{\text{EF}}{\text{AB}} = \frac{h}{H} \rightarrow \frac{v't - vt}{v't} = \frac{h}{H}$$

or

$$v' = \frac{Hv}{H - h} = \frac{6 \times 7}{(6 - 1.8)} = 10\,\text{m/s}$$

1.6 $\sqrt{3x} = 3t - 6$ (1)

Squaring and simplifying $x = 3t^2 - 12t + 12$ (2)

$$v = \frac{dx}{dt} = 6t - 12$$

$$v = 0 \text{ gives } t = 2\,\text{s} \tag{3}$$

Using (3) in (2) gives displacement $x = 0$

1.7 $$s = ut + \frac{1}{2}at^2 \tag{1}$$

$$\therefore \quad h = u \times 2 - \frac{1}{2}g \times 2^2 \tag{2}$$

$$h = u \times 10 - \frac{1}{2}g \times 10^2 \tag{3}$$

Solving (2) and (3) $h = 10g = 10 \times 9.8 = 98\,\text{m}$.

1.8 Take the origin at the position of A at $t = 0$. Let the car A overtake B in time t after travelling a distance s. In the same time t, B travels a distance $(s - 30)$ m:

$$s = ut + \frac{1}{2}at^2 \tag{1}$$

$$s = 13t + \frac{1}{2} \times 0.6\,t^2 \quad \text{(Car A)} \tag{2}$$

$$s - 30 = 20t - \frac{1}{2} \times 0.46\,t^2 \quad \text{(Car B)} \tag{3}$$

Eliminating s between (2) and (3), we find $t = 0.9$ s.

1.9 Let BD $= x$. Time t_1 for crossing the field along AD is

$$t_1 = \frac{\text{AD}}{v_1} = \frac{\sqrt{x^2 + (600)^2}}{1.0} \tag{1}$$

Time t_2 for walking on the road, a distance DC, is

$$t_2 = \frac{\text{DC}}{v_2} = \frac{800 - x}{2.0} \tag{2}$$

$$\text{Total time } t = t_1 + t_2 = \sqrt{x^2 + (600)^2} + \frac{800 - x}{2} \tag{3}$$

Minimum time is obtained by setting $dt/dx = 0$. This gives us $x = 346.4$ m. Thus the boy must head toward D on the round, which is 800–346.4 or 453.6 m away from the destination on the road.
The total time t is obtained by using $x = 346.4$ in (3). We find $t = 920$ s.

1.10 Time taken for the first drop to reach the floor is

$$t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 2.45}{9.8}} = \frac{1}{\sqrt{2}}\,\text{s}$$

As the time interval between the first and second drop is equal to that of the second and the third drop (drops dripping at regular intervals), time taken by the second drop is $t_2 = \frac{1}{2\sqrt{2}}$ s; therefore, distance travelled by the second drop is

$$S = \frac{1}{2}gt_2^2 = \frac{1}{2} \times 9.8 \times \left(\frac{1}{2\sqrt{2}}\right)^2 = 0.6125\,\text{m}$$

1.11 Height h = area under the $\upsilon - t$ graph. Area above the t-axis is taken positive and below the t-axis is taken negative. h = area of bigger triangle minus area of smaller triangle.
Now the area of a triangle = base × altitude

$$h = \frac{1}{2} \times 3 \times 30 - \frac{1}{2} \times 1 \times 10 = 40\,\text{m}$$

1.12 **(a)** Time for the ball to reach water $t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 4.9}{9.8}} = 1.0\,\text{s}$
Velocity of the ball acquired at that instant $\upsilon = gt_1 = 9.8 \times 1.0 = 9.8\,\text{m/s}$.
Time taken to reach the bottom of the lake from the water surface

$$t_2 = 5.0 - 1.0 = 4.0\,\text{s}.$$

As the velocity of the ball in water is constant, depth of the lake,

$$d = \upsilon t_2 = 9.8 \times 4 = 39.2\,\text{m}.$$

(b) $< \upsilon >= \frac{\text{total displacement}}{\text{total time}} = \frac{4.9 + 39.2}{5.0} = 8.82\,\text{m/s}$

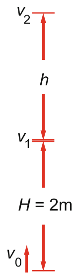
1.13 For the first stone time $t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 44.1}{9.8}} = 3.0\,\text{s}$.
Second stone takes $t_2 = 3.0 - 1.0 = 2.0$ s to strike the water

$$h = ut_2 + \frac{1}{2}gt_2^2$$

Using $h = 44.1$ m, $t_2 = 2.0$ s and $g = 9.8\,\text{m/s}^2$, we find $u = 12.25$ m/s

1.14 Transit time for the single journey = 0.5 s.
When the ball moves up, let υ_0 be its velocity at the bottom of the window, υ_1 at the top of the window and $\upsilon_2 = 0$ at height h above the top of the window (Fig. 1.15)

Fig. 1.15



$$v_1 = v_0 - gt = v_0 - 9.8 \times 0.5 = v_0 - 4.9 \quad (1)$$

$$v_1^2 = v_0^2 - 2gh = v_0^2 - 2 \times 9.8 \times 2 = v_0^2 - 39.2 \quad (2)$$

Eliminating v_1 between (1) and (2)

$$v_0 = 6.45\,\text{m/s} \quad (3)$$

$$v_2^2 = 0 = v_0^2 - 2g\,(H + h)$$

$$H + h = \frac{v_0^2}{2g} = \frac{(6.45)^2}{2 \times 9.8} = 2.1225\,\text{m}$$

$$h = 2.1225 - 2.0 = 0.1225\,\text{m}$$

Thus the ball rises 12.25 cm above the top of the window.

1.15 **(a)** $S_n = g\left(n - \frac{1}{2}\right)$ $\qquad S = \frac{1}{2}gn^2$

By problem $S_n = \frac{3s}{4}$

$$g\left(n - \frac{1}{2}\right) = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)gn^2$$

Simplifying $3n^2 - 8n + 4 = 0$, $n = 2$ or $\frac{2}{3}$

The second solution, $n = \frac{2}{3}$, is ruled out as $n < 1$.

(b) $s = \frac{1}{2}gn^2 = \frac{1}{2} \times 9.8 \times 2^2 = 19.6\,\text{m}$

1.16 In the triangle ACD, CA represents magnitude and apparent direction of wind's velocity w_1, when the man walks with velocity DC $= v = 4\,\text{km/h}$ toward west, Fig. 1.16. The side DA must represent actual wind's velocity because

$$\boldsymbol{W}_1 = \boldsymbol{W} - \boldsymbol{v}$$

When the speed is doubled, DB represents the velocity $2\boldsymbol{v}$ and BA represents the apparent wind's velocity $\boldsymbol{W}_2$. From the triangle ABD,

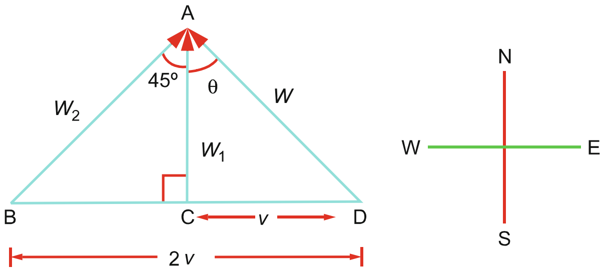


Fig. 1.16

$$\boldsymbol{W}_2 = \boldsymbol{W} - 2\boldsymbol{v}$$

By problem angle CAD $= \theta = 45°$. The triangle ACD is therefore an isosceles right angle triangle:

$$\mathrm{AD} = \sqrt{2}\mathrm{CD} = 4\sqrt{2}\,\mathrm{km/h}$$

Therefore the actual speed of the wind is $4\sqrt{2}$ km/h from southeast direction.

1.17 Choose the floor of the elevator as the reference frame. The observer is inside the elevator. Take the downward direction as positive.
Acceleration of the bolt relative to the elevator is

$$a' = g - (-a) = g + a$$

$$h = \frac{1}{2}a't^2 = \frac{1}{2}(g + a)t^2 \qquad t = \sqrt{\frac{2h}{g + a}}$$

1.18 In 2 s after the truck driver applies the brakes, the distance of separation between the truck and the car becomes

$$d_{\mathrm{rel}} = d - \frac{1}{2}\mathrm{at}^2 = 10 - \frac{1}{2} \times 2 \times 2^2 = 6\,\mathrm{m}$$

The velocity of the truck 2 becomes $20 - 2 \times 2 = 16$ m/s.
Thus, at this moment the relative velocity between the car and the truck will be

$$u_{\mathrm{rel}} = 20 - 16 = 4\,\mathrm{m/s}$$

Let the car decelerate at a constant rate of a_2. Then the relative deceleration will be

$$a_{\mathrm{rel}} = a_2 - a_1$$

If the rear-end collision is to be avoided the car and the truck must have the same final velocity that is

$$v_{\text{rel}} = 0$$

$$\text{Now } v_{\text{rel}}^2 = u_{\text{rel}}^2 - 2\, a_{\text{rel}}\, d_{\text{rel}}$$

$$a_{\text{rel}} = \frac{v_{\text{rel}}^2}{2\, d_{\text{rel}}} = \frac{4^2}{2 \times 6} = \frac{4}{3}\,\text{m/s}^2$$

$$\therefore \quad a_2 = a_1 + a_{\text{rel}} = 2 + \frac{4}{3} = 3.33\,\text{m/s}^2$$

1.19 $\boldsymbol{v}_{\text{BA}} = \boldsymbol{v}_{\text{B}} - \boldsymbol{v}_{\text{A}}$

From Fig. 1.17a

$$\boldsymbol{v}_{\text{BA}} = \sqrt{v_{\text{B}}^2 + v_{\text{A}}^2 - 2 v_{\text{B}} v_{\text{A}} \cos 60^\circ}$$

$$= \sqrt{20^2 + 30^2 - 2 \times 20 \times 30 \times 0.5} = 10\sqrt{7}\,\text{km/h}$$

The direction of $\boldsymbol{v}_{\text{BA}}$ can be found from the law of sines for ΔABC, Fig. 1.17a:

(i) $$\frac{\text{AC}}{\sin\theta} = \frac{\text{BC}}{\sin 60}$$

$$\text{or } \sin\theta = \frac{\text{AC}}{\text{BC}} \sin 60 = \frac{v_{\text{B}}}{v_{\text{BA}}} \sin 60^\circ = \frac{20 \times 0.866}{10\sqrt{7}} = 0.6546$$

$$\theta = 40.9^\circ$$

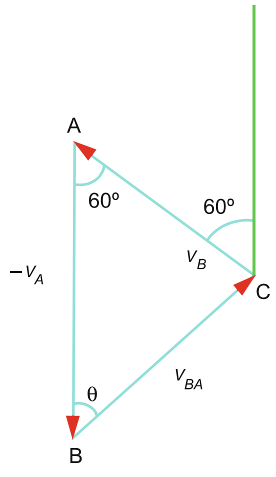


Fig. 1.17a

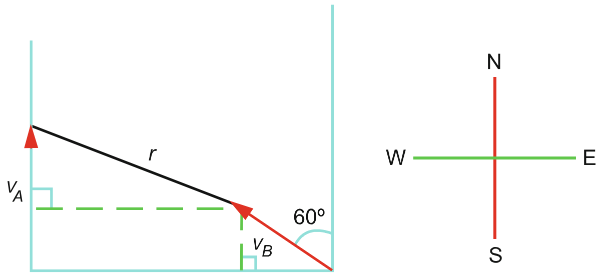


Fig. 1.17b

Thus $\boldsymbol{v}_{\mathrm{BA}}$ makes an angle 40.9° east of north.

(ii) Let the distance between the two ships be r at time t. Then from the construction of Fig. 1.17b

$$r = [(v_{\mathrm{A}}t - v_{\mathrm{B}}t \cos 60°)^2 + (10 - v_{\mathrm{B}}t \sin 60°)^2]^{1/2} \tag{1}$$

Distance of closest approach can be found by setting $\mathrm{d}r/\mathrm{d}t = 0$. This gives $t = \frac{\sqrt{3}}{7}h$. When $t = \frac{\sqrt{3}}{7}$ is inserted in (1) we get $r_{\min} = 20/\sqrt{7}$ or 7.56 km.

1.20 The initial velocity of the packet is the same as that of the balloon and is pointing upwards, which is taken as the positive direction. The acceleration due to gravity being in the opposite direction is taken negative. The displacement is also negative since it is vertically down:

$$u = 9.8\,\mathrm{m/s},\ a = -g = -9.8\,\mathrm{m/s^2};\ S = -98\,\mathrm{m}$$

$$s = ut + \frac{1}{2}at^2;\ -98 = 9.8t - \frac{1}{2} \times 9.8\,t^2 \quad \text{or} \quad t^2 - 2t - 20 = 0,$$

$$t = 1 \pm \sqrt{21}$$

The acceptable solution is $1 + \sqrt{21}$ or 5.58 s. The second solution being negative is ignored. Thus the packet takes 5.58 s to reach the ground.

1.3.2 Motion in Resisting Medium

1.21 Physically the difference between t_1 and t_2 on the one hand and v and u on other hand arises due to the fact that during ascent both gravity and air resistance act downward (friction acts opposite to motion) but during descent gravity and air resistance are oppositely directed. Air resistance F actually increases with the velocity of the object ($F \propto v$ or v^2 or v^3). Here for simplicity we assume it to be constant.

For upward motion, the equation of motion is

$$ma_1 = -(F + mg)$$

or

$$a_1 = -\left(\frac{F}{m} + g\right) \tag{1}$$

For downward motion, the equation of motion is

$$ma_2 = mg - F$$

or

$$a_2 = g - \frac{F}{m} \tag{2}$$

For ascent

$$v_1 = 0 = u + a_1 t = u - \left(\frac{F}{m} + g\right) t_1$$

$$t_1 = \frac{u}{g + \frac{F}{m}} \tag{3}$$

$$v_1{}^2 = 0 = u^2 + 2a_1 h$$

$$u = \sqrt{\frac{2h}{\left(g + \frac{F}{m}\right)}} \tag{4}$$

where we have used (1). Using (4) in (3)

$$t_1 = \sqrt{\frac{2h}{g + \frac{F}{m}}} \tag{5}$$

For descent $v^2 = 2a_2 h$

$$v = \sqrt{2h\left(g - \frac{F}{m}\right)} \tag{6}$$

where we have used (2)

$$t_2 = \frac{v}{a_2} = \sqrt{\frac{2h}{g - \frac{F}{m}}}, \tag{7}$$

where we have used (2) and (6)
From (5) and (7)

$$\frac{t_2}{t_1} = \sqrt{\frac{g + \frac{F}{m}}{g - \frac{F}{m}}} \tag{8}$$

It follows that $t_2 > t_1$, that is, time of descent is greater than the time of ascent. Further, from (4) and (6)

$$\frac{v}{u} = \sqrt{\frac{g - \frac{F}{m}}{g + \frac{F}{m}}} \tag{9}$$

It follows that $v < u$, that is, the final speed is smaller than the initial speed.

1.22 Taking the downward direction as positive, the equation of motion will be

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv \tag{1}$$

where k is a constant. Integrating

$$\int \frac{\mathrm{d}v}{g - kv} = \int \mathrm{d}t$$
$$\therefore \quad -\frac{1}{k}\ln\left(\frac{g - kv}{c}\right) = t$$

where c is a constant:

$$g - kv = ce^{-kt} \tag{2}$$

This gives the velocity at any instant.
As t increases e^{-kt} decreases and if t increases indefinitely $g - kv = 0$, i.e.

$$v = \frac{g}{k} \tag{3}$$

This limiting velocity is called the terminal velocity. We can obtain an expression for the distance x traversed in time t. First, we identify the constant c in (2). Since it is assumed that $v = 0$ at $t = 0$, it follows that $c = g$.
Writing $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ in (2) and putting $c = g$, and integrating

$$g - k\frac{\mathrm{d}x}{\mathrm{d}t} = g\mathrm{e}^{-kt}$$
$$\int g\mathrm{d}t - k\int \mathrm{d}x = g\int \mathrm{e}^{-kt}\mathrm{d}t + D$$
$$gt - kx = -\frac{g}{k}\mathrm{e}^{-kt} + D$$

At $x = 0$, $t = 0$; therefore, $D = \frac{g}{k}$

$$x = \frac{gt}{k} - \frac{g}{k^2}\left(1 - e^{-kt}\right) \tag{4}$$

1.23 The equation of motion is

$$\frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2 \tag{1}$$

$$\frac{dv}{dt} = g - kv^2 \tag{2}$$

$$\therefore \quad \frac{1}{k}\int \frac{dv}{\frac{g}{k} - v^2} = t + c \tag{3}$$

writing $V^2 = \frac{g}{k}$ and integrating

$$\ln \frac{V+v}{V-v} = 2kV(t+c) \tag{4}$$

If the body starts from rest, then $c = 0$ and

$$\ln \frac{V+v}{V-v} = 2kVt = \frac{2gt}{V}$$

$$\therefore \quad t = \frac{V}{2g} \ln \frac{V+v}{V-v} \tag{5}$$

which gives the time required for the particle to attain a velocity $\upsilon = 0$. Now

$$\frac{V+v}{V-v} = e^{2kVt}$$

$$\therefore \quad \frac{v}{V} = \frac{e^{2kVt} - 1}{e^{2kVt} + 1} = \tanh kVt \tag{6}$$

i.e.

$$v = V \tanh \frac{gt}{V} \tag{7}$$

The last equation gives the velocity υ after time t. From (7)

$$\frac{dx}{dt} = V \tanh \frac{gt}{V}$$

$$x = \frac{V^2}{g} \ln \cosh \frac{gt}{V} \tag{8}$$

$$x = \frac{V^2}{g} \ln \frac{e^{gt/v} + e^{-gt/v}}{2} \tag{9}$$

no additive constant being necessary since $x = 0$ when $t = 0$. From (6) it is obvious that as t increases indefinitely υ approaches the value V. Hence V is the terminal velocity, and is equal to $\sqrt{g/k}$.
The velocity υ in terms of x can be obtained by eliminating t between (5) and (9).

From (9),

$$\mathrm{e}^{kx} = \frac{\mathrm{e}^{kVt} + \mathrm{e}^{-kVt}}{2}$$

Squaring $4\mathrm{e}^{2kx} = \mathrm{e}^{2kVt} + \mathrm{e}^{-2kVt} + 2$

$$= \frac{V+\upsilon}{V-\upsilon} + \frac{V-\upsilon}{V+\upsilon} + 2 \quad \text{from (5)}$$

$$= \frac{4V^2}{V^2 - \upsilon^2}$$

$$\therefore \quad \upsilon^2 = V^2(1 - \mathrm{e}^{-2kx})$$

$$= V^2\left(1 - \mathrm{e}^{-\frac{2gx}{V^2}}\right) \tag{10}$$

1.24 Measuring x upward, the equation of motion will be

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -g - k\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \tag{1}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \frac{\mathrm{d}\upsilon}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \upsilon\frac{\mathrm{d}\upsilon}{\mathrm{d}x}$$

$$\therefore \quad \upsilon\frac{\mathrm{d}\upsilon}{\mathrm{d}x} = -g - k\upsilon^2 \tag{2}$$

$$\therefore \quad \frac{1}{2k}\int \frac{\mathrm{d}\left(\upsilon^2\right)}{(g/k) + \upsilon^2} = -\int \mathrm{d}x$$

Integrating, $\ln\left(\frac{(g/k) + \upsilon^2}{c}\right) = -2kx$

$$\text{or} \qquad \frac{g}{k} + \upsilon^2 = c\mathrm{e}^{-2kx} \tag{3}$$

When $x = 0$, $\upsilon = u$; $\therefore c = \frac{g}{k} + u^2$ and writing $\frac{g}{k} = V^2$, we have

$$\frac{V^2 + \upsilon^2}{V^2 + u^2} = \mathrm{e}^{-\frac{2gx}{V^2}} \tag{4}$$

$$\therefore \quad \upsilon^2 = (V^2 + u^2)\mathrm{e}^{-\frac{2gx}{V^2}} - V^2 \tag{5}$$

The height h to which the particle rises is found by putting $\upsilon = 0$ at $x = h$ in (5)

$$\frac{V^2+u^2}{V^2} = \mathrm{e}^{\frac{2gh}{V^2}}$$

$$h = \frac{V^2}{2g}\ln\left(1+\frac{u^2}{V^2}\right) \tag{6}$$

1.25 The particle reaches the height h given by

$$h = \frac{V^2}{2g}\ln\left(1+\frac{u^2}{V^2}\right) \quad \text{(by prob. 1.24)}$$

The velocity at any point during the descent is given by

$$v^2 = V^2\left(1-\mathrm{e}^{-\frac{2gx}{V^2}}\right) \quad \text{(by prob. 1.23)}$$

The velocity of the body when it reaches the point of projection is found by substituting h for x:

$$\therefore \quad v^2 = V^2\left\{1-\frac{V^2}{V^2+u^2}\right\} = \frac{u^2V^2}{V^2+u^2}$$

$$\text{Loss of kinetic energy} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}mu^2\left\{1-\frac{V^2}{V^2+u^2}\right\} = \frac{1}{2}mu^2\left(\frac{u^2}{V^2+u^2}\right)$$

1.3.3 Motion in Two Dimensions

1.26 **(i)**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6+2t$$

$$\int \mathrm{d}x = 6\int \mathrm{d}t + 2\int t\,\mathrm{d}t$$

$$x = 6t + t^2 + C$$

$$x = 0, t = 0; C = 0$$

$$x = 6t + t^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4+t$$

$$\int \mathrm{d}y = 4\int \mathrm{d}t + \int t\,\mathrm{d}t$$

$$y = 4t + \frac{t^2}{2} + D$$

$$y = 0, t = u; D = u$$

$$y = u + 4t + \frac{t^2}{2}$$

(ii) $\vec{v} = (6 + 2t)\hat{i} + (4 + t)\hat{j}$

(iii) $\vec{a} = \frac{dv}{dt} = 2\hat{i} + \hat{j}$

(iv) $a = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$\tan\theta = \frac{1}{2};\ \theta = 26.565°$$

Acceleration is directed at an angle of $26°34'$ with the x-axis.

1.27 Take upward direction as positive, Fig. 1.18. At time t the velocities of the objects will be

$$\boldsymbol{v}_1 = u_1\hat{i} - gt\,\hat{j} \tag{1}$$

$$\boldsymbol{v}_2 = -u_2\hat{i} - gt\,\hat{j} \tag{2}$$

If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are to be perpendicular to each other, then $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$, that is

$$\left(u_1\hat{i} - gt\,\hat{j}\right) \cdot \left(-u_2\hat{i} - gt\,\hat{j}\right) = 0$$

$$\therefore \quad -u_1u_2 + g^2t^2 = 0$$

$$\text{or} \quad t = \frac{1}{g}\sqrt{u_1u_2} \tag{3}$$

The position vectors are $\boldsymbol{r}_1 = u_1t\,\hat{i} - \frac{1}{2}gt^2\hat{j}, \quad \boldsymbol{r}_2 = -u_2t\,\hat{i} - \frac{1}{2}gt^2\hat{j}$.

The distance of separation of the objects will be

$$r_{12} = |\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2| = (u_1 + u_2)t$$

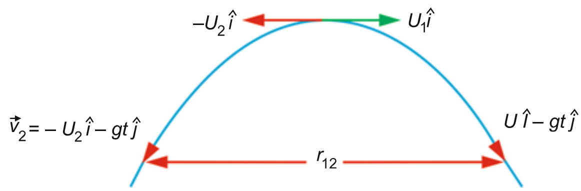


Fig. 1.18

or

$$r_{12} = \frac{(u_1 + u_2)}{g}\sqrt{u_1 u_2} \tag{4}$$

where we have used (2).

1.28 Consider the equation

$$s = ut + \frac{1}{2}at^2 \tag{1}$$

Taking upward direction as positive, $a = -g$ and let $s = h$, the height of the tower, (1) becomes

$$h = ut - \frac{1}{2}gt^2$$

or

$$\frac{1}{2}gt^2 - ut + h = 0 \tag{2}$$

Let the two roots be t_1 and t_2. Compare (2) with the quadratic equation

$$ax^2 + bx + c = 0 \tag{3}$$

The product of the two roots is equal to c/a. It follows that

$$t_1 t_2 = \frac{2h}{g} \text{ or } \sqrt{t_1 t_2} = \sqrt{\frac{2h}{g}} = t_3$$

which is the time taken for a free fall of an object from the height h.

1.29 Let the shell hit the plane at $p(x, y)$, the range being AP $= R$, Fig. 1.19. The equation for the projectile's motion is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \tag{1}$$

Now $\quad y = R\sin\alpha \quad (2)$

$$x = R\cos\alpha \tag{3}$$

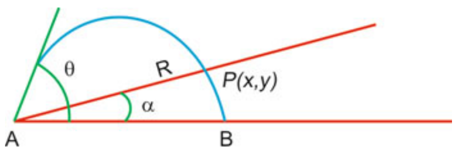


Fig. 1.19

Using (2) and (3) in (1) and simplifying

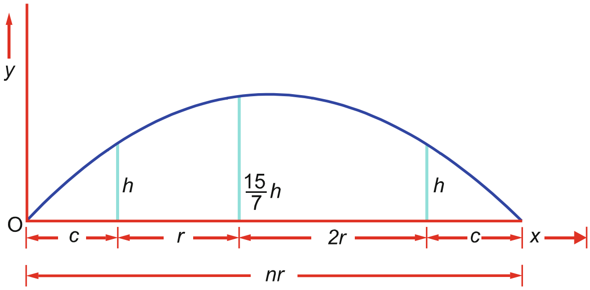
$$R = \frac{2u^2 \cos\theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

The maximum range is obtained by setting $\dfrac{\mathrm{d}R}{\mathrm{d}\theta} = 0$, holding u, α and g constant. This gives $\cos(2\theta - \alpha) = 0$ or $2\theta - \alpha = \dfrac{\pi}{2}$

$$\therefore\ \alpha = \frac{\theta}{2} + \frac{\pi}{4}$$

1.30 As the outer walls are equal in height (h) they are equally distant (c) from the extremities of the parabolic trajectory whose general form may be written as (Fig. 1.20)

Fig. 1.20



$$y = ax - bx^2 \tag{1}$$

$y = 0$ at $x = R = nr$, when R is the range

This gives $a = bnr$ (2)

The range $R = c + r + 2r + c = nr$, by problem

$$\therefore\quad c = (n-3)\frac{r}{2} \tag{3}$$

The trajectory passes through the top of the three walls whose coordinates are (c, h), $\left(c + r, \frac{15}{7}h\right)$, $(c + 3r, h)$, respectively. Using these coordinates in (1), we get three equations

$$h = ac - bc^2 \tag{4}$$

$$\frac{15h}{7} = a(c+r) - b(c+r)^2 \tag{5}$$

$$h = a(c+3r) - b(c+3r)^2 \tag{6}$$

Combining (2), (3), (4), (5) and (6) and solving we get $n = 4$.

1.31 The equation to the parabolic path can be written as

$$y = ax - bx^2 \tag{1}$$

$$\text{with } a = \tan\theta;\ b = \frac{g}{2u^2\cos^2\theta} \tag{2}$$

Taking the point of projection as the origin, the coordinates of the two openings in the windows are (5, 5) and (11, 7), respectively. Using these coordinates in (1) we get the equations

$$5 = 5a - 25b \tag{3}$$

$$7 = 11a - 121b \tag{4}$$

with the solutions, $a = 1.303$ and $b = 0.0606$. Using these values in (2), we find $\theta = 52.5°$ and $u = 14.8\,\text{m/s}$.

1.32 Let the rifle be fixed at A and point in the direction AB at an angle α with the horizontal, the monkey sitting on the tree top at B at height h, Fig. 1.21. The bullet follows the parabolic path and reaches point D, at height H, in time t.

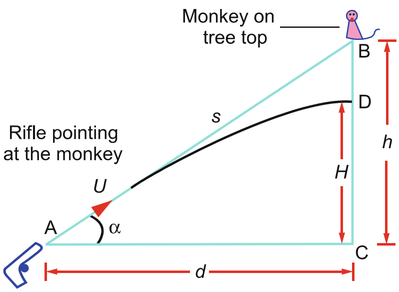


Fig. 1.21

The horizontal and initial vertical components of velocity of bullet are

$$u_x = u\cos\alpha;\ u_y = u\sin\alpha$$

Let the bullet reach the point D, vertically below B in time t, the coordinates of D being (d, H). As the horizontal component of velocity is constant

$$d = u_x t = (u\cos\alpha)t = \frac{udt}{s}$$

where $s = \text{AB}$:

$$t = \frac{s}{u}$$

The vertical component of velocity is reduced due to gravity.

In the same time, the y-coordinate at D is given by

$$y = H = u_y t - \frac{1}{2}gt^2 = u(\sin\alpha)t - \frac{1}{2}gt^2$$
$$H = u\left(\frac{h}{s}\right)\left(\frac{s}{u}\right) - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2$$
$$\text{or} \quad h - H = \frac{1}{2}gt^2$$
$$\therefore \quad t = \sqrt{\frac{2(h-H)}{g}}$$

But the quantity $(h–H)$ represents the height through which the monkey drops from the tree and the right-hand side of the last equation gives the time for a free fall. Therefore, the bullet would hit the monkey independent of the bullet's initial velocity.

1.33 $R = \dfrac{u^2 \sin 2\alpha}{g}, \quad h = \dfrac{u^2 \sin^2\alpha}{g}, \quad T = \dfrac{2u\sin\alpha}{g}$

(a) $\dfrac{h}{R} = \dfrac{1}{4}\tan\alpha \rightarrow \tan\alpha = \dfrac{4h}{R}$

(b) $\dfrac{h}{T^2} = \dfrac{g}{8} \rightarrow h = \dfrac{gT^2}{8}$

1.34 **(i)** $T = \dfrac{2u\sin\alpha}{g} = \dfrac{2 \times 800 \sin 60°}{9.8} = 141.4\,\text{s}$

(ii) $R = \dfrac{u^2\sin 2\alpha}{g} = \dfrac{(800)^2\sin(2\times 60)}{9.8} = 5.6568 \times 10^4\text{m} = 56.57\,\text{km}$

(iii) Time to reach maximum height $= \frac{1}{2}T = \frac{1}{2} \times 141.4 = 707\,\text{s}$

(iv) $$x = (u\cos\alpha)t \tag{1}$$
$$y = (u\sin\alpha)t - \frac{1}{2}gt^2 \tag{2}$$

Eliminating t between (1) and (2) and simplifying

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha} \tag{3}$$

which is of the form $y = bx + cx^2$, with $b = \tan\alpha$ and $c = -\dfrac{1}{2}\dfrac{g}{u^2\cos^2\alpha}$.

1.35 **(i)** $T = \dfrac{u\sin\alpha}{g} = \dfrac{350\,\sin 55°}{9.8} = 29.25\,\text{s}$

(ii) At the highest point of the trajectory, the velocity of the particle is entirely horizontal, being equal to $u\cos\alpha$. The momentum of this particle at the highest point is $p = mu\cos\alpha$, when m is its mass. After the explosion, one fragment starts falling vertically and so does not carry any momentum initially. It would fall at half of the range, that is

$$\frac{R}{2} = \frac{1}{2}\frac{u^2 \sin 2\alpha}{g} = \frac{(350)^2 \sin(2 \times 55°)}{2 \times 9.8} = 5873\,\text{m, from the firing point.}$$

The second part of mass $\frac{1}{2}m$ proceeds horizontally from the highest point with initial momentum p in order to conserve momentum. If its velocity is v then

$$p = \frac{m}{2}v = mu\cos\alpha$$
$$v = 2u\cos\alpha = 2 \times 350\cos 55° = 401.5\,\text{m/s}$$

Then its range will be

$$R' = v\sqrt{\frac{2h}{g}} \tag{1}$$

But the maximum height

$$h = \frac{u^2 \sin^2\alpha}{2g} \tag{2}$$

Using (2) in (1)

$$R' = \frac{vu\sin\alpha}{g} = \frac{(401.5)(350)(\sin 55°)}{9.8} = 11746\,\text{m}$$

The distance form the firing point at which the second fragment hits the ground is

$$\frac{R}{2} + R' = 5873 + 11746 = 17619\,\text{m}$$

(iii) Energy released = (kinetic energy of the fragments) − (kinetic energy of the particle) at the time of explosion

$$= \frac{1}{2}\frac{m}{2}v^2 - \frac{1}{2}m(u\cos\alpha)^2$$
$$= \frac{20}{4} \times (401.5)^2 - \frac{20}{2}(350\cos\ 55°)^2 = 4.03 \times 10^5\,\text{J}$$

1.36 The radius of curvature

$$\rho = \frac{\left[1 + (\mathrm{d}y/\mathrm{d}x)^2\right]^{3/2}}{\mathrm{d}^2 y/\mathrm{d}x^2} \tag{1}$$

$$x = v_o t = 10 \times 3 = 30\,\mathrm{m}$$

$$y = \frac{1}{2} g t^2 = \frac{1}{2} \times 9.8 \times 3^2 = 44.1\,\mathrm{m}.$$

$$\therefore y = \frac{1}{2} g \frac{x^2}{v_0^2}$$

$$v_0^2 = \frac{9.8 \times 30}{10^2} = 2.94 \tag{2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{g}{v_0^2} = \frac{9.8}{10^2} = 0.098 \tag{3}$$

Using (2) and (3) in (1) we find $\rho = 305\,\mathrm{m}$.

1.37 Let P be the position of the boat at any time, Let AP $= r$, angle $B\hat{A}P = \theta$, and let v be the magnitude of each velocity, Fig. 1.5:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -v + v \sin\theta$$

$$\text{and} \quad \frac{r\mathrm{d}\theta}{\mathrm{d}t} = v \cos\,\theta$$

$$\therefore \quad \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{-1 + \sin\theta}{\cos\theta}$$

$$\therefore \quad \int \frac{\mathrm{d}r}{r} = \int [-\sec\theta + \tan\theta]\,\mathrm{d}\theta$$

$$\therefore \quad \ln r = -\ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \ln\cos\theta + \ln C \text{ (a constant)}$$

When $\theta = 0, r = a,$ so that $C = a$

$$\therefore \quad r = \frac{a}{\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\cos\theta}$$

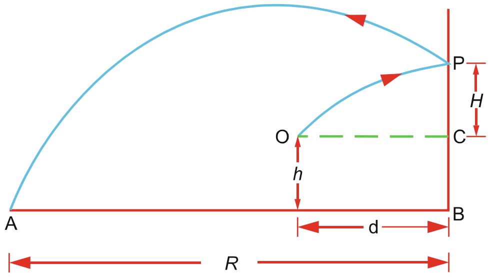
The denominator can be shown to be equal to $1 + \sin\theta$:

$$\therefore \quad r = \frac{a}{1 + \sin\theta}$$

This is the equation of a parabola with AB as semi-latus rectum.

1.38 Take the origin at O, Fig. 1.22. Draw the reference line OC parallel to AB, the ground level. Let the ball hit the wall at a height H above C. Initially at O,

Fig. 1.22



$$u_x = u\cos\alpha = u\cos 45^\circ = \frac{u}{\sqrt{2}}$$
$$u_y = u\sin\alpha = u\sin 45^\circ = \frac{u}{\sqrt{2}}$$

When the ball hits the wall, $y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$
Using $y = H$, $x = d$ and $\alpha = 45^\circ$

$$H = d\left(1 - \frac{gd}{u^2}\right) \tag{1}$$

If the collision of the ball with the wall is perfectly elastic then at P, the horizontal component of the velocity (u'_x) will be reversed, the magnitude remaining constant, while both the direction and magnitude of the vertical component v'_y are unaltered. If the time taken for the ball to bounce back from P to A is t and the range BA = R

$$y = v'_y t - \frac{1}{2}gt^2 \tag{2}$$

Using $$t = \frac{R}{u\cos 45^\circ} = \sqrt{2}\frac{R}{u} \tag{3}$$

$$y = -(H + h) \tag{4}$$

$$v'_y t = u\sin 45^\circ - g\frac{d}{u\cos 45^\circ} = \frac{u}{\sqrt{2}} - \sqrt{2}\frac{gd}{u} \tag{5}$$

Using (3), (4) and (5) in (2), we get a quadratic equation in R which has the acceptable solution

$$R = \frac{u^2}{2g} + \sqrt{\frac{u^2}{4g^2} + H + h}$$

1.3.4 Force and Torque

1.39 Resolve the force into x- and y-components:

$$F_x = -80\cos 35^\circ + 60 + 40\cos 45^\circ = 22.75\,\text{N}$$
$$F_y = 80\sin 35^\circ + 0 - 40\sin 45^\circ = 17.6\,\text{N}$$

(i) $\boldsymbol{F}_{\text{net}} = \sqrt{F_x^2 + F_y^2} = \sqrt{(22.75)^2 + (17.6)^2} = 28.76\,\text{N}$

$$\tan\theta = \frac{F_y}{F_x} = \frac{17.6}{22.75} = 0.7736 \rightarrow \theta = 37.7^\circ$$

The vector $\boldsymbol{F}_{\text{net}}$ makes an angle of 37.7° with the x-axis.

(ii) $a = \dfrac{\boldsymbol{F}_{\text{net}}}{m} = \dfrac{28.76\,N}{3.8\,kg} = 7.568\,\text{m/s}^2$

(iii) $\boldsymbol{F}_4$ of magnitude 28.76 N must be applied in the opposite direction to $\boldsymbol{F}_{\text{net}}$

1.40 **(a)** **(i)** $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$

$$\tau = rF\sin\theta = (0.4\,\text{m})(50\,\text{N})\sin 90^\circ = 20\,\text{N} - \text{m}$$

(ii) $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{20}{20} = 1.0\,\text{rad/s}^2$$

(iii) $\omega = \omega_0 + \alpha t = 0 + 1 \times 3 = 3\,\text{rad/s}$

(iv) $\omega^2 = \omega_0^2 + 2\alpha\theta,\ \theta = \frac{3^2-0}{2\times 1} = 4.5\,\text{rad}$

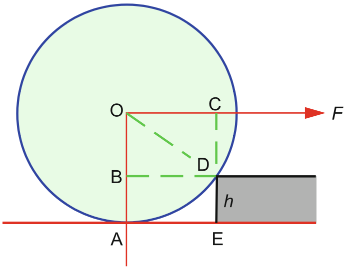
(b) **(i)** $\tau = 0.4 \times 50 \times \sin(90 + 20) = 18.794\,\text{N m}$

(ii) $\alpha = \dfrac{\tau}{I} = \dfrac{18.794}{20} = 0.9397\,\text{rad/s}^2$

1.41 Force applied to the container $F = ma$
Frictional force = $F_r = \mu\, mg$

$$F_r = F$$
$$\mu\, mg = ma$$
$$\mu = \frac{a}{g} = \frac{1.5}{9.8} = 0.153$$

Fig. 1.23



1.42 Taking torque about D, the corner of the obstacle, $(F)\text{CD} = (W)\text{BD}$ (Fig. 1.23)

$$F = W\frac{\text{BD}}{\text{CD}} = \sqrt{\frac{\text{OD}^2 - \text{OB}^2}{\text{CE} - \text{DE}}}$$
$$= \sqrt{\frac{r^2 - (r-h)^2}{r-h}} = \frac{\sqrt{h(2r-h)}}{r-h}$$

1.3.5 Centre of Mass

1.43 Let λ be the linear mass density (mass per unit length) of the wire. Consider an infinitesimal line element $\mathrm{d}s = R\,\mathrm{d}\theta$ on the wire, Fig. 1.24. The corresponding mass element will be $\mathrm{d}m = \lambda \mathrm{d}s = \lambda R\,\mathrm{d}\theta$. Then

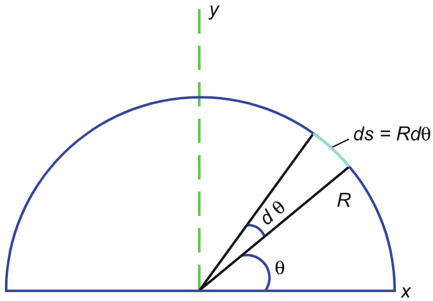
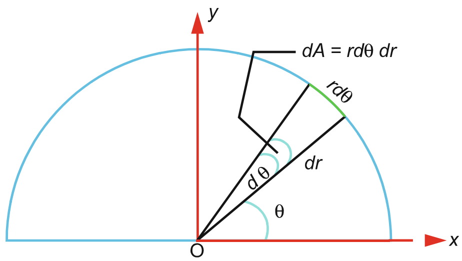


Fig. 1.24

$$y_{\text{CM}} = \frac{\int y\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^\pi (R\sin\theta)(\lambda R\,\mathrm{d}\theta)}{\int_0^\pi \lambda R\,\mathrm{d}\theta}$$
$$= \frac{\lambda R^2 \int_0^\pi \sin\theta\,\mathrm{d}\theta}{\lambda R\int_0^\pi \mathrm{d}\theta} = \frac{2R}{\pi}$$

1.44 Let the x-axis lie along the diameter of the semicircle. The centre of mass must lie on y-axis perpendicular to the flat base of the semicircle and through O, the centre of the base, Fig. 1.25.

Fig. 1.25



For continuous mass distribution

$$y_{CM} = \frac{1}{M}\int y\,\mathrm{d}m$$

Let σ be the surface density (mass per unit area), so that

$$M = \frac{1}{2}\pi R^2 \sigma$$

In polar coordinates $\mathrm{d}m = \sigma\,\mathrm{d}A = \sigma r\,\mathrm{d}\theta\,\mathrm{d}r$
where $\mathrm{d}A$ is the element of area. Let the centre of mass be located at a distance y_{CM} from O along y-axis for reasons of symmetry:

$$y_{CM} = \frac{1}{\frac{1}{2}\pi R^2 \sigma}\int_0^R \int_0^\pi (r\sin\theta)(\sigma r\,\mathrm{d}\theta\,\mathrm{d}r) = \frac{2}{\pi R^2}\int_0^R r^2 \int_0^\pi \sin\theta\,\mathrm{d}\theta = \frac{4R}{3\pi}$$

1.45 Let O be the origin, the centre of the base of the hemisphere, the z-axis being perpendicular to the base. From symmetry the CM must lie on the z-axis, Fig. 1.26. If ρ is the density, the mass element, $\mathrm{d}m = \rho\,\mathrm{d}V$, where $\mathrm{d}V$ is the volume element:

$$Z_{CM} = \frac{1}{M}\int Z\,\mathrm{d}m = \frac{1}{M}\int Z\rho\,\mathrm{d}V \tag{1}$$

In polar coordinates, $Z = r\cos\theta$ (2)

$$\mathrm{d}V = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}r \tag{3}$$

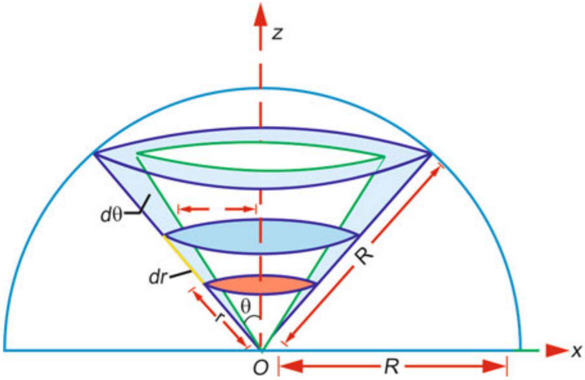
$$0 < r < R;\, 0 < \theta < \frac{\pi}{2}; \qquad 0 < \phi < 2\pi$$

The mass of the hemisphere

$$M = \rho\frac{2}{3}\pi R^3 \tag{4}$$

Using (2), (3) and (4) in (1)

Fig. 1.26



$$Z_{CM}\frac{\int_0^R r^3 \mathrm{d}r \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi}{\frac{2\pi R^3}{3}} = \frac{3R}{8}$$

1.46 The mass of any portion of the disc will be proportional to its surface area. The area of the original disc is πR^2, that corresponding to the hole is $\frac{1}{4}\pi R^2$ and that of the remaining portion is $\pi R^2 - \frac{\pi R^2}{4} = \frac{3}{4}\pi R^2$.

Let the centre of the original disc be at O, Fig. 1.10. The hole touches the circumference of the disc at A, the centre of the hole being at C. When this hole is cut, let the centre of mass of the remaining part be at G, such that

$$\mathrm{OG} = x \text{ or } \mathrm{AG} = \mathrm{AO} + \mathrm{OG} = R + x$$

If we put back the cut portion of the hole and fill it up then the centre of the mass of this small disc (C) and that of the remaining portion (G) must be located at the centre of the original disc at O

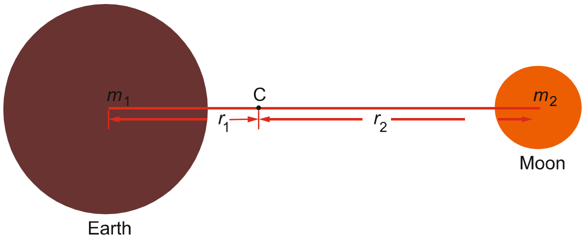
$$\mathrm{AO} = R = \frac{\mathrm{AC}\pi(R^2/4) + \mathrm{AG}\frac{3\pi}{4}R^2}{\pi R^2/4 + 3\pi R^2/4} = \frac{R}{8} + \frac{3}{4}(R + x)$$

$$\therefore \quad x = \frac{R}{6}$$

Thus the C:M of the remaining portion of the disc is located at distance $R/6$ from O on the left side.

1.47 Let m_1 be the mass of the earth and m_2 that of the moon. Let the centre of mass of the earth–moon system be located at distance r_1 from the centre of the earth and at distance r_2 from the centre of the moon, so that $r = r_1 + r_2$ is the distance between the centres of earth and moon, Fig. 1.27. Taking the origin at the centre of mass

Fig. 1.27



$$\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = 0$$
$$m_1 r_1 - m_2 r_2 = 0$$
$$r_1 = \frac{m_2 r_2}{m_1} = \frac{m_2(r - r_1)}{81 m_2} = \frac{60R - r_1}{81}$$
$$r_1 = 0.7317R = 0.7317 \times 6400 = 4683\,\text{km}$$

along the line joining the earth and moon; thus, the centre of mass of the earth–moon system lies within the earth.

1.48 Let the centre of mass be located at a distance r_c from the carbon atom and at r_0 from the oxygen atom along the line joining carbon and oxygen atoms. If r is the distance between the two atoms, m_c and m_o the mass of carbon and oxygen atoms, respectively

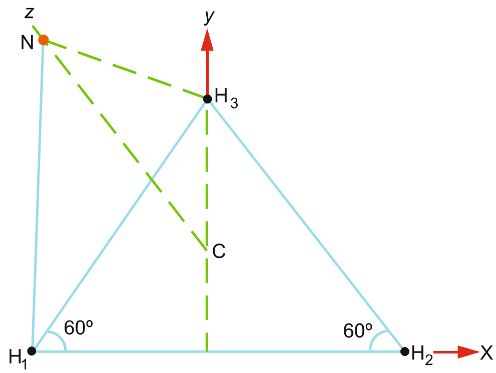
$$m_c r_c = m_o r_o = m_o(r - r_c)$$
$$r_c = \frac{m_o r}{m_o + m_o} = \frac{16 \times 1.13}{12 + 16} = 0.646\,\text{Å}$$

1.49 Let C be the centroid of the equilateral triangle formed by the three H atoms in the xy-plane, Fig. 1.28. The N–atom lies vertically above C, along the z-axis. The distance r_{CN} between C and N is

$$r_{CN} - \sqrt{r_{NH_3}^2 - r_{CH_3}^2}$$
$$r_{CN} = \frac{r_{H_1H_2}^2}{\sqrt{3}} = \frac{1.628}{1.732} = 0.94\,\text{Å}$$
$$r_{CN} = \sqrt{(1.014)^2 - (0.94)^2} = 0.38\,\text{Å}$$

Now, the centre of mass of the three H atoms $3m_H$ lies at C. The centre of mass of the NH_3 molecule must lie along the line of symmetry joining N and C and is located below N atom at a distance

Fig. 1.28 Centre of mass of NH_3 molecule



$$Z_{CM} = \frac{3m_H}{3m_H + m_N} \times r_{CN} = \frac{3m_H}{3m_H + 14m_H} \times 0.38 = 0.067\,\text{Å}$$

1.50 Take the origin at A at the left end of the boat, Fig. 1.29. Let the boy of mass m be initially at B, the other end of the boat. The boat of mass M and length L has its centre of mass at C. Let the centre of mass of the boat + boy system be located at G, at a distance x from the origin. Obviously $AC = 1.5\,m$:

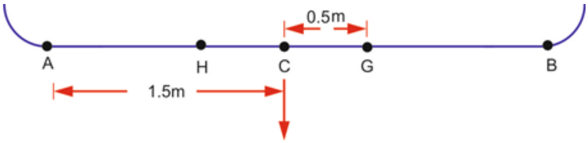


Fig. 1.29

$$AG = x = \frac{M\,AC + m\,AB}{M + m}$$
$$= \frac{100 \times 1.5 + 50 \times 3}{100 + 50} = 2\,\text{m}$$

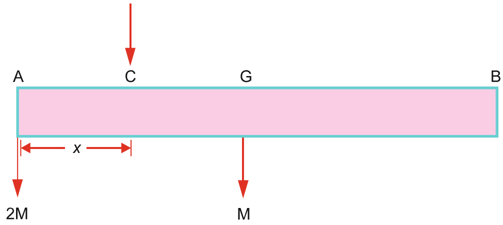
Thus $CG = AG - AC$

$= 2.0 - 1.5 = 0.5\,\text{m}$

When the boy reaches A, from symmetry the CM of boat + boy system would have moved to H by a distance of 0.5 m on the left side of C. Now, in the absence of external forces, the centre of mass should not move, and so to restore the original position of the CM the boat moves towards right so that the point H is brought back to the original mark G. Since $HG = 0.5 + 0.5 = 1.0$, the boat in the mean time moves through 1.0 m toward right.

1.51 If the rod is to move with pure translation without rotation, then it should be struck at C, the centre of mass of the loaded rod. Let C be located at distance x from A so that
GC $= \frac{1}{2}L - x$, Fig. 1.30. Let M be the mass of the rod and $2M$ be attached at A. Take torques about C

Fig. 1.30



$$2Mx = M\left(\frac{L}{2} - x\right) \quad \therefore x = \frac{L}{6}$$

Thus the rod should be struck at a distance $\frac{L}{6}$ from the loaded end.

1.52 Volume of the cone, $V = \frac{1}{3}\pi R^2 h$ where R is the radius of the base and h is its height, Fig. 1.31. The volume element at a depth z below the apex is $\mathrm{d}V = \pi r^2 \mathrm{d}z$, the mass element $\mathrm{d}m = \rho \mathrm{d}V = \pi r^2 \mathrm{d}z f$

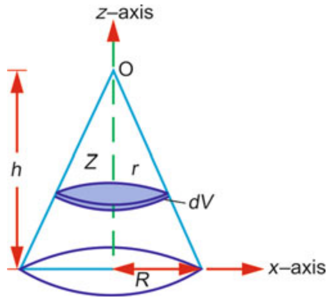


Fig. 1.31

$$\mathrm{d}m = \rho \mathrm{d}v = \rho \pi r^2 \mathrm{d}z$$

$$\frac{z}{r} = \frac{h}{R} \quad \therefore \mathrm{d}z = \frac{h}{R}\mathrm{d}r$$

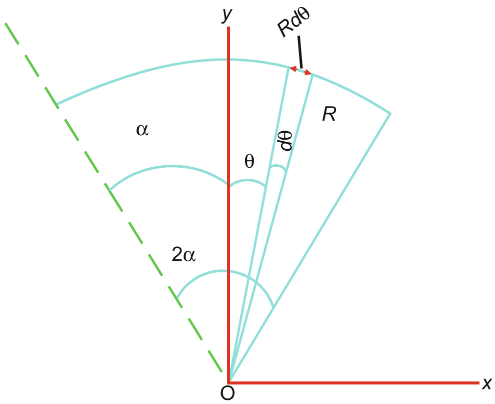
For reasons of symmetry, the centre of mass must lie on the axis of the cone. Take the origin at O, the apex of the cone:

$$Z_{\mathrm{CM}} = \frac{\int Z\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^R \left(\frac{hr}{R}\right)\rho\pi r^2 \left(\frac{h}{r}\,\mathrm{d}r\right)}{\frac{1}{3}\pi R^2 h\rho} = \frac{3h}{4}$$

Thus the CM is located at a height $h - \frac{3}{4}h = \frac{1}{4}h$ above the centre of the base of the cone.

1.53 Take the origin at O, Fig. 1.32. Let the mass of the wire be M. Consider mass element dm at angles θ and $\theta + \mathrm{d}\theta$

Fig. 1.32



$$\mathrm{d}m = \frac{MR\,\mathrm{d}\theta}{2\alpha R} = \frac{M\,\mathrm{d}\theta}{2\alpha} \tag{1}$$

From symmetry the CM of the wire must be on the y-axis.
The y-coordinate of dm is $y = R\sin\theta$

$$y_{\mathrm{CM}} = \frac{1}{M}\int y_{dm} = \int\limits_{90-\alpha}^{90+\alpha} \frac{R\sin\theta\,\mathrm{d}\theta}{2\alpha} = \frac{R\sin\alpha}{\alpha}$$

Note that the results of prob. (1.43) follow for $\alpha = \frac{1}{2}\pi$.

1.54 $V_{\mathrm{CM}} = \dfrac{\Sigma m_i v_i}{\Sigma m_i} = \dfrac{4mv_0 + (m)(0)}{5m} = \dfrac{4v_0}{5}$

1.55 $\rho = cx(c = \text{constant});\ \ \mathrm{d}m = \rho\,\mathrm{d}x = cx\,\mathrm{d}x$

$$x_{\mathrm{CM}} = \frac{\int x\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^L xcx\,\mathrm{d}x}{\int_0^L cx\,\mathrm{d}x} = \frac{2}{3}L$$

1.56 $$x_{\mathrm{CM}} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{mL + (2m)(2L) + (3m)(3L) + \cdots + (nm)(nL)}{m + 2m + 3m + \cdots + nm}$$
$$= \frac{(1+4+9+\cdots+n^2)L}{1+2+3+\cdots+n} = \frac{(\text{sum of squares of natural numbers})L}{\text{sum of natural numbers}}$$
$$= \frac{n(n+1)(2n+1)L/6}{n(n+1)/2} = (2n+1)\frac{L}{3}$$

1.57 The diagram is the same as for prob. (1.44)

$$
\begin{aligned}
y &= r\sin\theta \\
\mathrm{d}A &= r\ \mathrm{d}\theta\mathrm{d}r \\
\mathrm{d}m &= r\ \mathrm{d}\theta\mathrm{d}r\rho = r\mathrm{d}\theta\mathrm{d}r\,cr^2 = cr^3\ \mathrm{d}r\ \mathrm{d}\theta
\end{aligned}
$$

$$
\text{Total mass}\quad M = \int \mathrm{d}m = c\int_0^R r^3\mathrm{d}r\int_0^\pi \mathrm{d}\theta = \frac{\pi c R^4}{4} \tag{1}
$$

$$
\begin{aligned}
y_{\mathrm{CM}} &= \frac{1}{M}\int y\,\mathrm{d}m = \frac{1}{M}\int\int (r\sin\theta)cr^3\ \mathrm{d}r\mathrm{d}\theta \\
&= \frac{C}{M}\int_0^R r^4\mathrm{d}r\int_0^\pi \sin\theta\ \mathrm{d}\theta \\
&= \frac{C}{M}\frac{2}{5}R^5 = \frac{8a}{5\pi}
\end{aligned} \tag{2}
$$

where we have used (1).

1.58 The CM of the two H atoms will be at G the midpoint joining the atoms, Fig. 1.33. The bisector of $\widehat{\mathrm{HOH}}$

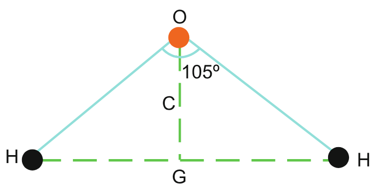


Fig. 1.33

$$
\mathrm{OG} = (\mathrm{OH})\cos\left(\frac{105^\circ}{2}\right) = 1.77\times 0.06088 = 1.0775\ \text{Å}
$$

Let the CM of the O atom and the two H atoms be located at C at distance y_{CM} from O on the bisector of angle $\mathrm{H\hat{O}H}$

$$
y_{\mathrm{CM}} = \frac{2M_{\mathrm{H}}}{M_0}\times OG = \frac{2\times 1}{16}\times 1.0775 = 0.1349\ \text{Å}
$$

1.59 The CM coordinates of three individual laminas are

$$
\mathrm{CM}(1) = \left(\frac{a}{2},\frac{a}{2}\right), \mathrm{CM}(2) = \left(\frac{3a}{2},\frac{a}{2}\right), \mathrm{CM}(3) = \left(\frac{3a}{2},\frac{3a}{2}\right)
$$

The CM coordinates of the system of these three laminas will be

$$
x_{\mathrm{CM}} = \frac{m\frac{a}{2}+m\frac{3a}{2}+m\frac{3a}{2}}{m+m+m} = \frac{7a}{6}\qquad y_{\mathrm{CM}} = \frac{m\frac{a}{2}+m\frac{a}{2}+m\frac{3a}{2}}{m+m+m} = \frac{5a}{6}
$$

1.3.6 Equilibrium

1.60 $U(x) = k(2x^3 - 5x^2 + 4x)$ (1)

$$\frac{dU(x)}{dx} = k(6x^2 - 10x + 4) \quad (2)$$

$$\therefore \quad \frac{dU(x)}{dx}\bigg|_{x=1} = k(6x^2 - 10x + 4)\big|_{x=1} = 0$$

which is the condition for maximum or minimum. For stable equilibrium position of the particle it should be a minimum. To this end we differentiate (2) again:

$$\frac{d^2U(x)}{dx^2} = k(12x - 10)$$

$$\therefore \quad \frac{d^2U(x)}{dx^2}\bigg|_{x=1} = +2k$$

This is positive because k is positive, and so it is minimum corresponding to a stable equilibrium.

1.61 $U(x) = k(x^2 - 4xl)$ (1)

$$\frac{dU(x)}{dx} = 2k(x - 2l) \quad (2)$$

$$\text{At } x = 2l, \ \frac{dU(x)}{dx} = 0 \quad (3)$$

Differentiating (2) again

$$\frac{d^2U}{dx^2} = 2k$$

which is positive. Hence it is a minimum corresponding to a stable equilibrium. Force

$$F = -\frac{dU}{dx} = -2k(x - 2l)$$

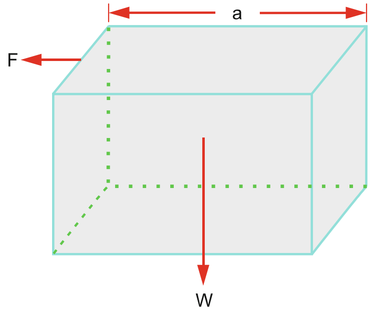
Put $X = x - 2l$, $\ddot{X} = \ddot{x}$

acceleration $\ddot{X} = \frac{F}{m} = -\frac{2k}{m}X = -\omega^2 X$

$$\therefore f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$$

1.62 Let 'a' be the side of the cube and a force F be applied on the top surface of the cube, Fig. 1.34. Take torques about the left-hand side of the edge. The condition that the cube would topple is

Fig. 1.34



Counterclockwise torque > clockwise torque

$$Fa > W\frac{a}{2}$$

or

$$F > 0.5W \tag{1}$$

Condition for sliding is

$$F > \mu W \tag{2}$$

Comparing (1) and (2), we conclude that the cube will topple if $\mu > 0.5$ and will slide if $\mu < 0.5$.

1.63 In Fig. 1.35 let the ladder AB have length L, its weight mg acting at G, the CM of the ladder (middle point). The weight mg produces a clockwise torque τ_1 about B:

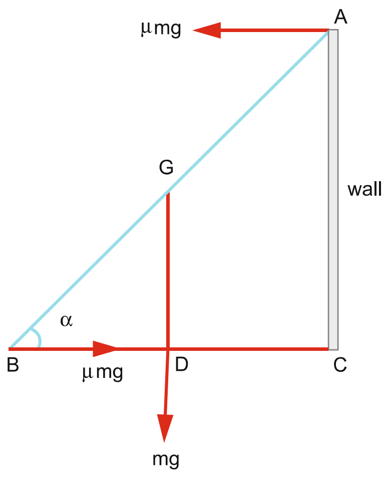


Fig. 1.35

$$\tau_1 = (mg)(\mathrm{BD}) = (mg)\left(\frac{\mathrm{BD}}{\mathrm{BG}}\mathrm{BG}\right) = mg\frac{L}{2}\cos\alpha \tag{1}$$

The friction with the ground, which acts toward right produces a counterclockwise torque τ_2:

$$\tau_2 = (\mu\, mg)\mathrm{AC} = \mu mg\frac{\mathrm{AC}}{\mathrm{AB}}\mathrm{AB} = \mu mgL\sin\alpha \tag{2}$$

For limiting equilibrium $\tau_1 = \tau_2$

$$\therefore \quad mg\frac{L}{2}\cos\alpha = \mu mgL\sin\alpha$$

$$\therefore \quad \mu = \frac{1}{2}\cot\alpha$$

Chapter 2
Particle Dynamics

Abstract *Chapter 2* is concerned with motion of blocks on horizontal and inclined planes with and without friction, work, power and energy. Elastic, inelastic and partially elastic collisions in both one dimension and two dimensions are treated. Problems on variable mass cover rocket motion, falling of rain drops, etc.

2.1 Basic Concepts and Formulae

Internal and External Forces

Forces acting upon a system due to external agencies are called external forces. As an example a body placed on a surface is acted by earth's gravitation which is an external force.

Forces that act between pairs of particles which constitute the body or a system are all internal to the system and are called internal forces. The size of a system is entirely arbitrary and is defined by the convenience of the situation. If a system is made sufficiently extensive then all forces become internal forces. By Newton's third law of motion internal forces between pairs of particles get cancelled. Hence net internal force is zero. Internal forces cannot cause motion.

Inertial and Gravitational Mass

If mass is determined by Newton's second law, that is, $m = F/a$, then it is called inertial mass.

If the mass is determined by the gravitational force exerted on it by another body, say the earth of mass M, that is, $m' = Fr^2/\mathrm{GM}$, then it is called the gravitational mass. It turns out that $m = m'$.

Frames of Reference

A frame of reference (coordinate system) is necessary in order to measure the motion of particles. A reference frame is called an inertial frame if Newton's laws are found to be valid in that frame. It is found that inertial frames move with constant velocity with respect to one another.

Conservation Laws

(i) If the total force **F** is zero, then linear momentum **p** is conserved.
(ii) If the total external torque is zero, then the angular momentum **J** is conserved.
(iii) If the forces acting on a particle are conservative, then the total mechanical energy (kinetic + potential) of the particle is conserved.

Conservative Force

If the force field is such that the work done around a closed orbit is zero, i.e.

$$\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = 0 \tag{2.1}$$

then the force and the system are said to be conservative. A system cannot be conservative if a dissipative force like friction is present. Since the quantity $F\mathrm{d}s$ due to friction will always be negative and the integrand cannot vanish, by Stokes theorem the condition for conservative forces given by (2.1) becomes

$$\nabla \times \boldsymbol{F} = 0 \tag{2.2}$$

Since the curl of a gradient always vanishes, it follows that F must be the gradient of the scalar quantity V, i.e.

$$F = -\nabla V \tag{2.3}$$

V is called the potential energy.

Centre of Mass

$$\boldsymbol{r}_{\mathrm{c}} = \frac{1}{M} \sum\nolimits_{i=1}^{n} m_i \boldsymbol{r}_i \tag{2.4}$$

where $\boldsymbol{r}_{\mathrm{c}}$ is the position vector of the centre of mass from the origin and $M = \Sigma m_i$ is the total mass. The centre of mass moves as if it were a single particle of mass equal to the total mass of the system, acted upon by the total external force and independent of the nature of the internal forces.

The reduced mass (μ) of two bodies of mass m_1 and m_2 is given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.5}$$

A two-body problem is reduced to a one-body problem through the introduction of the reduced mass μ.

Motion of a Body of a Variable Mass

It is well known that in relativistic mechanics the mass of a particle increases with increasing velocity. However, in Newtonian mechanics too one can give meaning to variable mass as in the following example. Consider an open wagon moving on rails on a horizontal plane under steady heavy shower. As rain is collected the mass of the wagon increases at constant rate. Other examples are rocket, motion of jet propelled vehicles, an engine taking water on the run.

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(mv) = m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} \tag{2.6}$$

Motion of a Rocket

If m is the mass of the rocket plus fuel at any time t and v_r the velocity of the ejected gases relative to the rocket then

Resultant force on rocket = (upward thrust on rocket) – (weight of the rocket)

$$m\frac{\mathrm{d}v}{\mathrm{d}r} = v_\mathrm{r}\frac{\mathrm{d}m}{\mathrm{d}t} - mg \tag{2.7}$$

Therefore, acceleration of the rocket

$$a = \frac{dv}{dt} = \frac{v_\mathrm{r}}{m}\frac{dm}{dt} - g \tag{2.8}$$

Assuming that v_r and g remain constant and at $t = 0$, $v = 0$ and $m = m_0$,

$$v_\mathrm{B} = v_\mathrm{r} \ln\left(\frac{m_0}{m_\mathrm{B}}\right) - gt \tag{2.9}$$

where m_0 is the initial mass of the system and m_B the mass at burn-out velocity v_B (the velocity at which all the fuel is burnt out is called the burn-out velocity).

Now

$$m = m_0 e^{-v/v_\mathrm{r}} \tag{2.10}$$

Time taken for the rocket to reach the burn-out velocity is given by

$$t = t_0 = \frac{m_0 - m}{\alpha} \tag{2.11}$$

where $\alpha = -\mathrm{d}m/\mathrm{d}t$ is a positive constant.

Elastic Collisions (One-Dimensional Head-On)

By definition total kinetic energy is conserved. If u_1 and u_2 be the respective initial velocities of m_1 and m_2, v_1 and v_2 being the corresponding final velocities, then

$$u_1 - u_2 = v_2 - v_1 \tag{2.12}$$

Thus, the relative velocity of approach before the collision is equal to the relative velocity of separation after the collision:

$$v_1 = \left[\frac{m_1 - m_2}{m_1 + m_2}\right] u_1 + \frac{2m_2 u_2}{m_1 + m_2} \tag{2.13}$$

$$v_2 = \frac{2m_1 u_1}{m_1 + m_2} + \left[\frac{m_2 - m_1}{m_1 + m_2}\right] u_2 \tag{2.14}$$

Inelastic Collisions, Direct Impact

The bodies stick together in the course of collision and are unable to separate out. After the collision they travel as one body with common velocity v given by

$$v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \tag{2.15}$$

$$\text{Energy wasted} = \frac{1}{2}\mu(u_1 - u_2)^2 \tag{2.16}$$

where μ is the reduced mass.

Ballistic pendulum is a device for measuring the velocity of a bullet. The pendulum consists of a large wooden block of mass M which is supported vertically by two cords. A bullet of mass m hits the block horizontally with velocity v and is lodged within it. As a result of collision the block is raised through maximum height h (see prob. 2.44). Applying momentum conservation for the initial collision process, and energy conservation for the subsequent motion, it can be shown that

$$v = \left(1 + \frac{M}{m}\right)\sqrt{2gh} \tag{2.17}$$

Partially elastic collisions are collisions which fall in between perfectly elastic collisions and totally inelastic collisions. The coefficient of restitution e which defines the degree of inelasticity is given by

$$e = -\frac{\text{relative velocity of separation}}{\text{relative velocity of approach}} = \frac{v_1 - v_2}{u_2 - u_1} \tag{2.18}$$

For perfectly elastic collisions $e = 1$, for totally inelastic collisions $e = 0$ and for partially elastic collisions $0 < e < 1$.

Relations of Quantities in the Lab System (LS) and Centre of Mass System (CMS)

Quantities that are unprimed refer to LS and primed refer to CMS.

Lab system CM system

$$m_1 : u_1; \qquad u_1^* = \frac{m_2 u_1}{m_1 + m_2} \tag{2.19}$$

$$m_2 : u_2 = 0; \qquad u_2^* = \frac{m_1 u_1}{m_1 + m_2} \tag{2.20}$$

Scattering Angle

Because of elastic scattering

$$v_1^* = u_1^*; \; v_2^* = u_2^*$$

The centre of mass velocity

$$v_c = -\frac{m_1 u_1}{m_1 + m_2} = -u_2^* \tag{2.21}$$

$$\tan\theta = \frac{\sin\theta^*}{\cos\theta^* + m_1/m_2} \tag{2.22}$$

$$\tan\theta^* = \frac{\sin\theta}{\cos\theta - m_1/m_2} \tag{2.23}$$

If $m_1 < m_2$, all scattering angles for m_1 in LS are possible.
If $m_1 = m_2$, scattering only in the forward hemisphere ($\theta \le 90°$) is possible.
If $m_1 > m_2$, the maximum scattering angle is possible, $\theta_{\max}$, being given by

$$\theta_{\max} = \sin^{-1}(m_2/m_1) \tag{2.24}$$

Recoil Angle

$$\tan\varphi = \frac{\sin\varphi^*}{\cos\varphi^* + 1} = \tan\frac{\varphi^*}{2} \tag{2.25}$$

$$\text{Or} \quad \phi = \frac{1}{2}\varphi^* \tag{2.26}$$

Recoiling angle is limited to $\varphi \le 90°$.

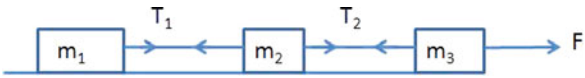
2.2 Problems

2.2.1 Motion of Blocks on a Plane

2.1 Three blocks of mass m_1, m_2 and m_3 interconnected by cords are pulled by a constant force F on a frictionless horizontal table, Fig. 2.1. Find

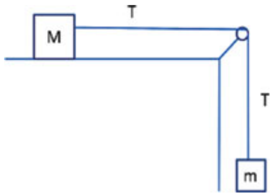
(a) Common acceleration 'a'
(b) Tensions T_1 and T_2

Fig. 2.1



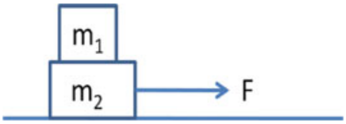
2.2 A block of mass M on a rough horizontal table is driven by another block of mass m connected by a thread passing over a frictionless pulley. Assuming that the coefficient of friction between the mass M and the table is μ, find (a) acceleration of the masses (b) tension in the thread (Fig. 2.2).

Fig. 2.2



2.3 A block of mass m_1 sits on a block of mass m_2, which rests on a smooth table, Fig. 2.3. If the coefficient of friction between the blocks is μ, find the maximum force that can be applied to m_2 so that m_1 may not slide.

Fig. 2.3



2.4 Two blocks m_1 and m_2 are in contact on a frictionless table. A horizontal force F is applied to the block m_1, Fig. 2.4. (a) Find the force of contact between the blocks. (b) Find the force of contact between the blocks if the same force is applied to m_2 rather than to m_1, Fig. 2.5.

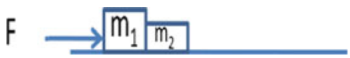


Fig. 2.4

Fig. 2.5

2.5 A box of weight mg is dragged with force F at an angle θ above the horizontal. (a) Find the force exerted by the floor on the box. (b) Find the acceleration of the box if the coefficient of friction with the floor is μ. (c) How would the results be altered if the box is pushed with the same force?

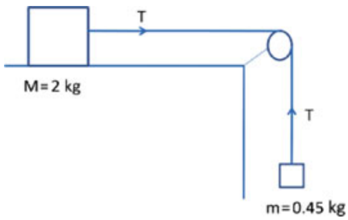
2.6 A uniform chain of length L lies on a table. If the coefficient of friction is μ, what is the maximum length of the part of the chain hanging over the table such that the chain does not slide?

2.7 A uniform chain of length L and mass M is lying on a smooth table and one-third of its length is hanging vertically down over the edge of the table. Find the work required to pull the hanging part on the table.

2.8 A block of metal of mass 2 kg on a horizontal table is attached to a mass of 0.45 kg by a light string passing over a frictionless pulley at the edge of the table. The block is subjected to a horizontal force by allowing the 0.45 kg mass to fall. The coefficient of sliding friction between the block and table is 0.2. Calculate (a) the initial acceleration, (b) the tension in the string, (c) the distance the block would continue to move if, after 2 s of motion, the string should break (Fig. 2.6).

[University of New Castle]

Fig. 2.6



2.2.2 *Motion on Incline*

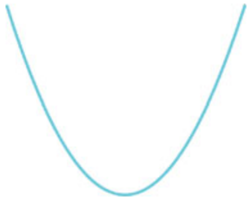
2.9 A block of mass of 2 kg slides on an inclined plane that makes an angle of 30° with the horizontal. The coefficient of friction between the block and the surface is $\sqrt{3}/2$.

(a) What force should be applied to the block so that it moves down without any acceleration?
(b) What force should be applied to the block so that it moves up without any acceleration?

[Indian Institute of Technology 1976]

2.10 A block is placed on a ramp of parabolic shape given by the equation $y = x^2/20$, Fig. 2.7. If $\mu_s = 0.5$, what is the maximum height above the ground at which the block can be placed without slipping?

Fig. 2.7



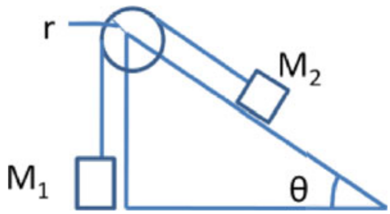
2.11 A block slides with constant velocity down an inclined plane that has slope angle $\theta = 30°$.

(a) Find the coefficient of kinetic friction between the block and the plane.
(b) If the block is projected up the same plane with initial speed $v_0 = 2.5\,\text{m/s}$, how far up the plane will it move before coming to rest? What fraction of the initial kinetic energy is transformed into potential energy? What happens to the remaining energy?
(c) After the block comes to rest, will it slide down the plane again? Justify your answer.

2.12 Consider a fixed inclined plane at angle θ. Two blocks of mass M_1 and M_2 are attached by a string passing over a pulley of radius r and moment of inertia I_1 as in Fig. 2.8:

(a) Find the net torque acting on the system comprising the two masses, pulley and the string.
(b) Find the total angular momentum of the system about the centre of the pulley when the blocks are moving with speed v.
(c) Calculate the acceleration of the blocks.

Fig. 2.8



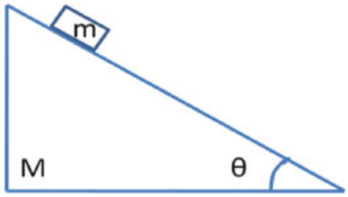
2.13 A box of mass 1 kg rests on a frictionless inclined plane which is at an angle of 30° to the horizontal plane. Find the constant force that needs to be applied parallel to the incline to move the box

(a) up the incline with an acceleration of 1 m/s^2
(b) down the incline with an acceleration of 1 m/s^2

[University of Aberystwyth, Wales 2008]

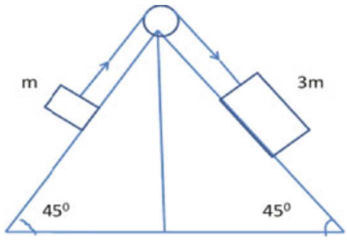
2.14 A wedge of mass M is placed on a horizontal floor. Another mass m is placed on the incline of the wedge. Assume that all surfaces are frictionless, and the incline makes an angle θ with the horizontal. The mass m is released from rest on mass M, which is also initially at rest. Find the accelerations of M and m (Fig. 2.9).

Fig. 2.9



2.15 Two smooth inclined planes of angles 45° and hinged together back to back. Two masses m and $3m$ connected by a fine string passing over a light pulley move on the planes. Show that the acceleration of their centre of mass is $\sqrt{5}/8\, g$ at an angle $\tan^{-1}$ ½ to the horizon (Fig. 2.10).

Fig. 2.10



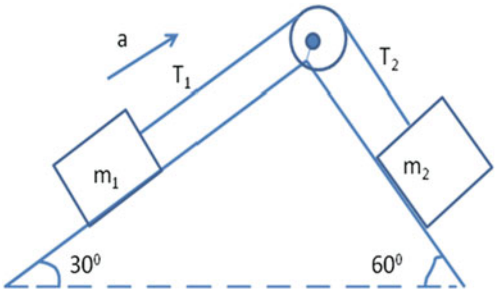
2.16 Two blocks of masses m_1 and m_2 are connected by a string of negligible mass which passes over a pulley of mass M and radius r mounted on a frictionless axle. The blocks move with an acceleration of magnitude a and direction as shown in the diagram. The string does not slip on the pulley, so the tensions T_1 and T_2 are different. You can assume that the surfaces of the inclines are frictionless. The moment of inertia of the pulley is given by $I = ½Mr^2$:

(a) Draw free body diagrams for the two blocks and the pulley.
(b) Write down the equations for the translational motion of the two blocks and the rotational motion of the pulley.
(c) Show that the magnitude of the acceleration of the blocks is given by

$$a = \frac{g(\sqrt{3}m_2 - m_1)}{M + 2(m_2 + m_1)}$$

2.17 Two masses in an Atwood machine are 1.9 and 2.1 kg, the vertical distance of the heavier body being 20 cm above the lighter one. After what time would the lighter body be above the heavier one by the same vertical distance? Neglect the mass of the pulley and the cord (Fig. 2.11).

Fig. 2.11



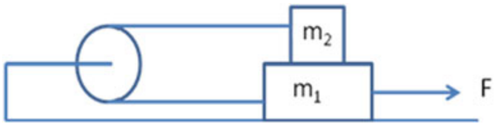
2.18 A body takes 4/3 times as much time to slide down a rough inclined plane as it takes to slide down an identical but smooth inclined plane. Find the coefficient of friction if the angle of incline is 45°.

2.19 A body slides down an incline which has coefficient of friction $\mu = 0.5$. Find the angle θ if the incline of the normal reaction is twice the resultant downward force along the incline.

2.20 Two masses m_1 and m_2 are connected by a light inextensible string which passes over a smooth massless pulley. Find the acceleration of the centre of mass of the system.

2.21 Two blocks with masses m_1 and m_2 are attached by an unstretchable string around a frictionless pulley of radius r and moment of inertia I. Assume that there is no slipping of the string over the pulley and that the coefficient of kinetic friction between the two blocks and between the lower one and the floor is identical. If a horizontal force F is applied to m_1, calculate the acceleration of m_1 (Fig. 2.12).

Fig. 2.12



2.2.3 Work, Power, Energy

2.22 The constant forces $\mathbf{F}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ N and $\mathbf{F}_2 = 4\hat{i} - 5\hat{j} - 2\hat{k}$ N act together on a particle during a displacement from position $\mathbf{r}_2 = 7\hat{k}$ cm to position $\mathbf{r}_1 = 20\hat{i} + 15\hat{j}$ cm. Determine the total work done on the particle.

[University of Manchester 2008]

2.23 The potential energy of an object is given by

$$U(x) = 5x^2 - 4x^3$$

where U is in joules and x is in metres.

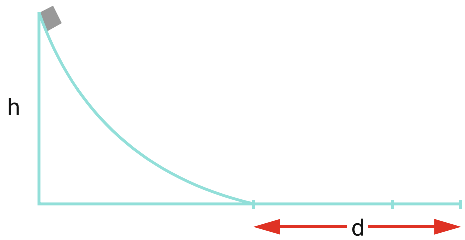
(i) What is the force, $F(x)$, acting on the object?
(ii) Determine the positions where the object is in equilibrium and state whether they are stable or unstable.

2.24 A body slides down a rough plane inclined to the horizontal at 30°. If 70% of the initial potential energy is dissipated during the descent, find the coefficient of sliding friction.

[University of Bristol]

2.25 A ramp in an amusement park is frictionless. A smooth object slides down the ramp and comes down through a height h, Fig. 2.13. What distance d is necessary to stop the object on the flat track if the coefficient of friction is μ.

Fig. 2.13



2.26 A spring is used to stop a crate of mass 50 kg which is sliding on a horizontal surface. The spring has a spring constant $k = 20\,\text{kN/m}$ and is initially in its equilibrium state. In position A shown in the top diagram the crate has a velocity of 3.0 m/s. The compression of the spring when the crate is instantaneously at rest (position B in the bottom diagram) is 120 mm.

(i) What is the work done by the spring as the crate is brought to a stop?
(ii) Write an expression for the work done by friction during the stopping of the crate (in terms of the coefficient of kinetic friction).
(iii) Determine the coefficient of friction between the crate and the surface.
(iv) What will be the velocity of the crate as it passes again through position A after rebounding off the spring (Fig. 2.14a, b)?

[University of Manchester 2007]

Fig. 2.14a

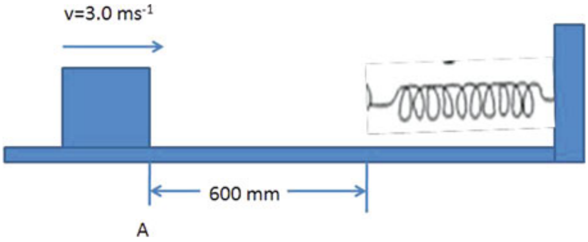
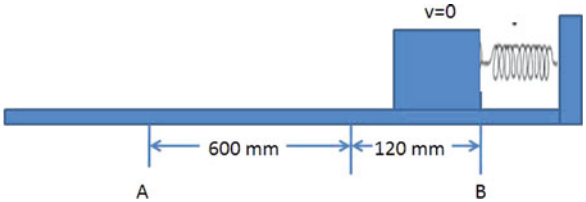


Fig. 2.14b



2.2.4 Collisions

2.27 Observed in the laboratory frame, a body of mass m_1 moving at speed v collides elastically with a stationary mass m_2. After the collision, the bodies move at angles θ_1 and θ_2 relative to the original direction of motion of m_1. Find the velocity of the centre of mass (CM) frame of m_1 and m_2.
Hence show that before the collision in the CM frame m_1 and m_2 are approaching each other, m_1 with speed $m_2 v/(m_1 + m_2)$ and m_2 with speed $m_1 v/(m_1 + m_2)$.
In the CM frame after the collision m_1 moves off with speed $m_2 v/(m_1 + m_2)$ at an angle θ to its original direction. Draw a diagram showing the direction and speed of m_2 in the CM frame after the collision.
Find an expression for the speed m_1 after the collision in the laboratory frame in terms of m_1, m_2, v and the angle θ.

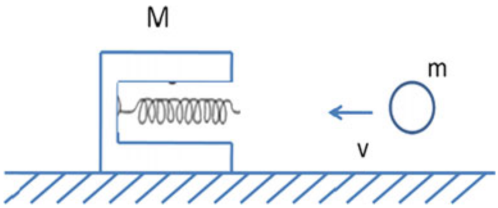
[University of Durham 2002]

2.28 Consider an off-centre elastic scattering of two objects of equal mass when one is initially at rest.

(a) Show that the final velocity vectors of the two objects are orthogonal.
(b) Show that neither ball can be scattered in the backward direction.

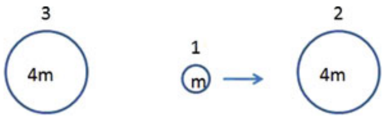
2.29 A small ball of mass m is projected horizontally with velocity v. It hits a spring of spring constant k attached inside an opening of a block resting on a frictionless horizontal surface. Find the compression of the spring noting that the block will slide due to the impact (Fig. 2.15).

Fig. 2.15



2.30 Two equal spheres of mass 4 m are at rest and another sphere of mass m is moving along their lines of centres between them. How many collisions will there be if the spheres are perfectly elastic (Fig. 2.16)?

Fig. 2.16



2.31 Two particles of mass m_1 and m_2 and velocities u_1 and $\alpha u_2 (\alpha > 0)$ make an elastic collision. If the initial kinetic energies of the two particles are equal, what should be the ratios u_1/u_2 and m_1/m_2 so that m_1 will be at rest after the collision?

2.32 Two bodies A and B, having masses m_A and m_B, respectively, collide in a totally inelastic collision.

(i) If body A has initial velocity v_A and B has initial velocity v_B, write down an expression for the common velocity of the merged bodies after the collision, assuming there are no external forces.

(ii) If $\boldsymbol{v}_\mathrm{A} = 5\hat{i} + 3\hat{j}$ m/s and $\boldsymbol{v}_\mathrm{B} = -\hat{i} + 4\hat{j}$ m/s and $m_\mathrm{A} = 3m_\mathrm{B}/2$, show that the common velocity after the collision is

$$v = 2.6\hat{i} + 3.4\hat{j}\ \mathrm{m/s}$$

(iii) Given that the mass of body A is 1200 kg and that the collision lasts for 0.2 s, determine the average force vectors acting on each body during the collision.

(iv) Determine the total kinetic energy after the collision.

2.33 A particle has an initial speed v_0. It makes a glancing collision with a second particle of equal mass that is stationary. After the collision the speed of the first particle is v and it has been deflected through an angle θ. The velocity of the second particle makes an angle β with the initial direction of the first particle.
Using the conservation of linear momentum principle in the x- and y-directions, respectively, show that $\tan\beta = v\sin\theta/(v_0 - v\cos\theta)$ and show that if the collision is elastic, $v = v_0\cos\theta$ (Fig. 2.17a,b).

Fig. 2.17a

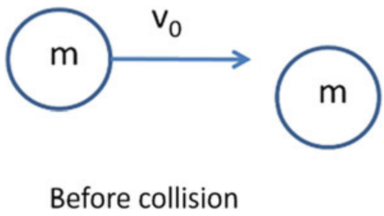
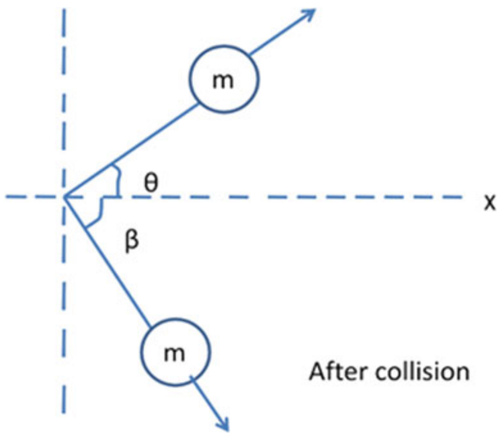


Fig. 2.17b



2.34 A carbon-14 nucleus which is radioactive decays into a beta particle, a neutrino and N-14 nucleus. In a particular decay, the beta particle has momentum p and the nitrogen nucleus has momentum of magnitude $4p/3$ at an angle of $90°$ to $\boldsymbol{p}$. In what direction do you expect the neutrino to be emitted and what would be its momentum?

2.35 If a particle of mass m collides elastically with one of mass m at rest, and if the former is scattered at an angle θ and the latter recoils at an angle φ with respect to the line of motion of the incident particle, then show that $\dfrac{m}{M} = \dfrac{\sin(2\varphi + \theta)}{\sin\theta}$.

2.36 A body of mass M rests on a smooth table and another of mass m moving with a velocity u collides with it. Both are perfectly elastic and smooth and no rotations are set up by this collision. The body M is driven in a direction at angle φ to the initial line of motion of the body m. Show that the velocity of M is $\dfrac{2m}{M+m} u \cos\varphi$.

2.37 A nucleus A of mass 2 m moving with velocity u collides inelastically with a stationary nucleus B of mass 10 m. After collision the nucleus A travels at 90°

with the incident direction while B proceeds at an angle 37° with the incident direction.

(a) Find the speeds of A and B after the collision.
(b) What fraction of the initial kinetic energy is gained or lost due to the collision.

2.38 A neutron moving with velocity v_0 collides head-on with carbon nucleus of mass number 12. Assuming that the collision is elastic

(a) calculate the fraction of neutron's kinetic energy transferred to the carbon nucleus and
(b) calculate the velocities of the neutron and the carbon nucleus after the collision.

2.39 Show that in an elastic collision between a very light body and a heavy body proceeds with twice the initial velocity of the heavy body.

2.40 A moving body makes a completely inelastic collision with a stationary body of equal mass at rest. Show that half of the original kinetic energy is lost.

2.41 A bullet weighing 5 g is fired horizontally into a 2 kg wooden block resting on a horizontal table. The bullet is arrested within the block which moves 2 m. If the coefficient of kinetic friction between the block and surface of the table is 0.2, find the speed of the bullet.

2.42 A particle of mass m with initial velocity u makes an elastic collision with a particle of mass M initially at rest. After the collision the particles have equal and opposite velocities. Find (a) the ratio M/m; (b) the velocity of centre of mass; (c) the total kinetic energy of the two particles in the centre of mass; and (d) the final kinetic energy of m in the laboratory system.

2.43 Consider an elastic collision between an incident particle of mass m with M initially at rest ($m > M$). Show that the largest possible scattering angle $\theta_{\text{max}} = \sin^{-1}(M/m)$.

2.44 The ballistic pendulum is a device for measuring the velocity v of a bullet of mass m. It consists of a large wooden block of mass M which is supported by two vertical cords. When the bullet is fired at the block, it is dislodged and the block is set in motion reaching maximum height h. Show that $v = (1 + M/m)\sqrt{2gh}$

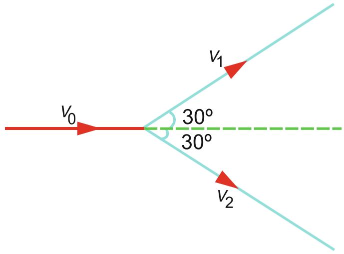
2.45 A fire engine directs a water jet onto a wall at an angle θ with the wall. Calculate the pressure exerted by the jet on the wall assuming that the collision with the wall is elastic, in terms of ρ, the density of water, A the area of the nozzle, and v the jet velocity.

2.46 Repeat the calculation of (2.45) assuming normal incidence and completely inelastic collision.

2.47 A ball moving with a speed of 9 m/s strikes an identical stationary ball such that after collision, the direction of each ball makes an angle 30° with the original line of motion (see Fig. 2.18). Find the speeds of the two balls after the collision. Is the kinetic energy conserved in the collision process?

[Indian Institute of Technology 1975]

Fig. 2.18



2.48 A ball is dropped from a height h onto a fixed horizontal plane. If the coefficient of restitution is e, calculate the total time before the ball comes to rest.

2.49 In prob. (2.48), calculate the total distance travelled.

2.50 In prob. (2.48), calculate the height to which the ball goes up after it rebounds for the nth time.

2.51 In the case of completely inelastic collision of two bodies of mass m_1 and m_2 travelling with velocities u_1 and u_2 show that the energy that is imparted is proportional to the square of the relative velocity of approach.

2.52 A projectile is fired with momentum p at an angle θ with the horizontal on a plain ground at the point A. It reaches the point B. Calculate the magnitude of change in momentum at A and B.

2.53 A shell is fired from a cannon with a velocity v at angle θ with the horizontal. At the highest point in its path, it explodes into two pieces of equal masses. One of the pieces retraces its path towards the cannon. Find the speed of the other fragment immediately after the explosion.

2.54 A helicopter of mass 500 kg hovers when its rotating blades move through an area of 45 m^2. Find the average speed imparted to air (density of air = 1.3 kg/m^3 and $g = 9.8\,\mathrm{m/s^2}$)

2.55 A machine gun fires 100 g bullets at a speed of 1000 m/s. The gunman holding the machine gun in his hands can exert an average force of 150 N against the gun. Find the maximum number of bullets that can be fired per minute.

2.56 The scale of balance pan is adjusted to read zero. Particles fall from a height of 1.6 m before colliding with the balance. If each particle has a mass of 0.1 kg and collisions occur at 441 particles/min, what would be the scale reading in kilogram weight if the collisions of the particles are perfectly elastic?

2.57 In prob. (2.56), assume that the collisions are completely inelastic. In this case, what would be the scale reading after time t?

2.58 A smooth sphere of mass m moving with speed v on a smooth horizontal surface collides directly with a second sphere of the same size but of half the mass that is initially at rest. The coefficient of restitution is e.

(i) Show that the total kinetic energy after collision is $\frac{mv^2}{6}\left(2+e^2\right)$.
(ii) Find the kinetic energy lost during the collision.

[University of Aberystwyth, Wales 2008]

2.59 A car of mass $m = 1200\,\text{kg}$ and length $l = 4\,\text{m}$ is positioned such that its rear end is at the end of a flat-top boat of mass $M = 8000\,\text{kg}$ and length $L = 18\,\text{m}$. Both the car and the boat are initially at rest and can be approximated as uniform in their mass distributions and the boat can slide through the water without significant resistance.

(a) Assuming the car accelerates with a constant acceleration $a = 4\,\text{m/s}^2$ relative to the boat, how long does it take before the centre of mass of the car reaches the other end of the boat (and therefore falls off)?
(b) What distance has the boat travelled relative to the water during this time?
(c) Use momentum conservation to find a relation between the velocity of the car relative to the boat and the velocity of the boat relative to the water. Hence show that the distance travelled by the boat, until the car falls off, is independent of the acceleration of the car.

[University of Durham 2005]

2.2.5 Variable Mass

2.60 A rocket has an initial mass of m and a burn rate of

$$a = -\mathrm{d}m/\mathrm{d}t$$

(a) What is the minimum exhaust velocity that will allow the rocket to lift off immediately after firing? Obtain an expression for (b) the burn-out velocity; (c) the time the rocket takes to attain the burn-out velocity ignoring g; and (d) the mass of the rocket as a function of rocket velocity.

2.61 A rocket of mass 1000 t has an upward acceleration equal to $0.5\,g$. How many kilograms of fuel must be ejected per second at a relative speed of 2000 m/s to produce the desired acceleration.

2.62 For the Centaur rocket use the data given below:
Initial mass $m_0 = 2.72 \times 10^6\,\text{kg}$
Mass at burn-out velocity, $m_\text{B} = 2.52 \times 10^6\,\text{kg}$
Relative velocity of exhaust gases $v_\text{r} = 55\,\text{km/s}$
Rate of change of mass, $\mathrm{d}m/\mathrm{d}t = 1290\,\text{kg/s}$.

Find

(a) the rocket thrust,
(b) net acceleration at the beginning,
(c) time to reach the burn-out velocity,
(d) the burn-out velocity.

2.63 A 5000 kg rocket is to be fired vertically. Calculate the rate of ejection of gas at exhaust speed 100 m/s in order to provide necessary thrust to

(a) support the weight of the rocket and
(b) impart an initial upward acceleration of $2\,g$.

2.64 A flexible rope of length L and mass per unit length μ slides over the edge of a frictionless table. Initially let a length y_0 of it be hanging at rest over the edge and at time t let a length y moving with a velocity $\mathrm{d}y/\mathrm{d}t$ be over the edge. Obtain the equation of motion and discuss its solution.

2.65 An open railway car of mass W is running on smooth horizontal rails under rain falling vertically down which it catches and retains in the car. If v_0 is the initial velocity of the car and k the mass of rain falling into the car per unit time, show that the distance travelled in time t is $(Wv_0/k)\ln(1+kt/W)$.

[with courtesy from R.W. Norris and W. Seymour, Mechanics via Calculus, Longmans, Green and Co., 1923]

2.66 A heavy uniform chain of length L and mass M hangs vertically above a horizontal table, its lower end just touching the table. When it falls freely, show that the pressure on the table at any instant during the fall is three times the weight of the portion on the table.

[with courtesy from R.W. Norris and W. Seymour, Mechanics via Calculus, Longmans, Green and Co., 1923]

2.67 A spherical rain drop of radius R cm falls freely from rest. As it falls it accumulates condensed vapour proportional to its surface. Find its velocity when it has fallen for t s.

[with courtesy from R.W. Norris and W. Seymour, Mechanics via Calculus, Longmans, Green and Co., 1923]

2.3 Solutions

2.3.1 Motion of Blocks on a Plane

2.1 **(a)** $\text{Acceleration} = \dfrac{\text{Force}}{\text{Total mass}}$

$$a = \frac{F}{(m_1 + m_2 + m_3)} \tag{1}$$

(b) Tension T_1 = Force acting on m_1

$$T_2 = m_1 a = \frac{m_1 F}{(m_1 + m_2 + m_3)} \tag{2}$$

where we have used (1).

Applying Newton's second law to m_2

$$m_2 a = T_2 - T_1$$

or $\quad T_2 = m_2 a + T_1 = (m_1 + m_2)a$

$$T_2 = \frac{(m_1 + m_2)F}{(m_1 + m_2 + m_3)} \tag{3}$$

where we have used (1) and (2).

2.2 **(a)** The equations of motion are

$$ma = mg - T \tag{1}$$
$$Ma = T - \mu Mg \tag{2}$$

Solving (1) and (2)

$$a = \frac{(m - \mu M)g}{m + M} \tag{3}$$
$$T = \frac{Mm}{M + m}(1 + \mu)g \tag{4}$$

Thus with the introduction of friction, the acceleration is reduced and tension is increased compared to the motion on a smooth surface ($\mu = 0$).

2.3 $F_{\max} = (m_1 + m_2)a \quad (1)$

The condition that m_1 may not slide is

$$a = \mu g \tag{2}$$

Using (2) in (1)

$$F_{\max} = (m_1 + m_2)\mu g$$

2.4 **(a)** The force of contact F_c between the blocks is equal to the force exerted on m_2:

$$F_c = m_2 a \tag{1}$$

where the acceleration of the whole system is

$$a = \frac{F}{m_1 + m_2} \tag{2}$$

$$\therefore \quad F_c = \frac{m_2 F}{m_1 + m_2}$$

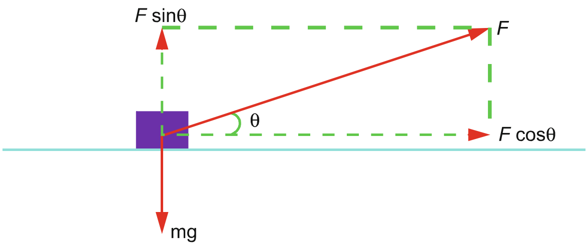
(b) Here the contact force F_c' is given by

$$F_c' = m_1 a = \frac{m_1 F}{m_1 + m_2}$$

Notice that $F_c' \neq F_c$ simply because $m_1 \neq m_2$.

2.5 **(a)** When the box is dragged, the horizontal component of F is $F\cos\theta$ and the vertical component (upward) is $F \sin\theta$ as in Fig. 2.19. The reaction force N on the box by the floor will be

Fig. 2.19



$$N = mg - F\sin\theta \tag{1}$$

(b) The equation of motion will be

$$ma = F\cos\theta - \mu N = F\cos\theta - \mu(mg - F\sin\theta)$$

$$\therefore \quad a = \frac{F}{m}(\cos\theta + \mu\sin\theta) - \mu g \tag{2}$$

(c) When the box is pushed the horizontal component of F will be $F\cos\theta$ and the vertical component $F\sin\theta$ (downwards), Fig. 2.20. The reaction force exerted by the floor on the box will be

$$N' = mg + F\sin\theta \tag{3}$$

which is seen to be greater than N (the previous case).

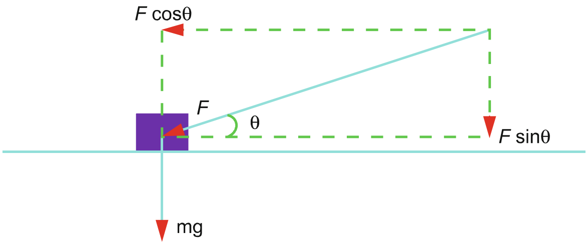
The equation of motion will be

$$ma' = F\cos\theta - \mu N' = F\cos\theta - \mu(mg + F\sin\theta)$$

$$\therefore \quad a' = \frac{F}{m}(\cos\theta - \mu\sin\theta) - \mu g \tag{4}$$

a value which is less than a (the previous case). It therefore pays to pull rather than push at an angle with the horizontal. The difference arises due to the smaller value of the reaction in pulling than in pushing. This fact is exploited in handling a manual road roller or mopping a floor, which is pulled rather than pushed.

Fig. 2.20



2.6 Let x be the length of the chain hanging over the table. The length of the chain resting on the table will be $L - x$. For equilibrium, gravitational force on the hanging part of the chain = frictional force on the part of the chain resting on the table. If M is the mass of the entire chain then

$$\frac{Mg\,x}{L} = \frac{M(L-x)}{L}\mu g$$

$$\therefore \quad x = \frac{\mu L}{\mu + 1}$$

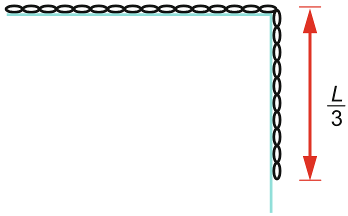
2.7 *First method:* The centre of mass of the hanging part of the chain is located at a distance $L/6$ below the edge of the table, Fig. 2.21. The mass of the hanging part of the chain is $M/3$. The work done to pull the hanging part on the table

$$W = \frac{Mg}{3}\frac{L}{6} = \frac{MgL}{18}$$

Second method: We can obtain the same result by calculus. Consider an element of length $\mathrm{d}x$ of the hanging part at a distance x below the edge. The mass of the length $\mathrm{d}x$ is $\frac{M\,\mathrm{d}x}{L}$. The work required to lift the element of length $\mathrm{d}x$ through a distance x is

$$\mathrm{d}W = \frac{M\,\mathrm{d}x}{L}g\,x$$

Fig. 2.21



Work required to lift the entire hanging part is

$$W = \int \mathrm{d}W = \int\limits_{o}^{L/3} \frac{Mg}{L} x \,\mathrm{d}x = \frac{MgL}{18}$$

2.8 **(a)** The equations of motion are

$$Ma = mg - T \quad (1)$$

$$ma = T - Mg\mu \quad (2)$$

Solving (1) and (2)

$$a = \frac{(m - \mu M)g}{M + m} = \frac{(0.45 - 0.2 \ \times 2)9.8}{2 + 0.45} = 0.2\,\mathrm{m/s^2}$$

(b) $T = m(g - a) = 0.45(9.8 - 0.2) = 4.32\,\mathrm{N}$

(c) After 2 s, the velocity will be

$$\upsilon_1 = 0 + at = 0.2 \times 2 = 0.4\,\mathrm{m/s^1}$$

When the string breaks, the acceleration will be $a_1 = -\mu g = -0.2 \times 9.8 = -1.96\,\mathrm{m/s^2}$ and final velocity $\upsilon_2 = 0$:

$$S = \frac{\upsilon_2^2 - \upsilon_1^2}{2a_1} = \frac{0 - (0.4)^2}{(2)(-1.96)} = 0.0408\,\mathrm{m} = 4.1\,\mathrm{cm}$$

2.3.2 Motion on Incline

2.9 **(a)** Gravitational force down the incline is $Mg \sin\theta$. Frictional force up the incline is $\mu mg \cos\theta$. Net force

$$F = \mu Mg\cos\theta - Mg\sin\theta = Mg(\mu\cos\theta - \sin\theta)$$
$$= 2 \times 9.8\left(\frac{\sqrt{3}}{2}\cos 30^\circ - \sin 30^\circ\right) = 4.9\,\text{N}$$

(b) $$F' = Mg\sin\theta + \mu Mg\cos\theta = Mg(\sin\theta + \mu\cos\theta)$$
$$= 2 \times 9.8\left(\sin 30^\circ + \frac{\sqrt{3}}{2}\cos 30^\circ\right) = 24.5\,\text{N}$$

2.10 $y = \dfrac{x^2}{20} \qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = \dfrac{x}{10}$

For equilibrium, $mg\sin\theta - \mu mg\cos\theta = 0$

$$\therefore \quad \tan\theta = \mu = 0.5$$
$$x = 10\tan\theta = 10 \times 0.5 = 5$$
$$y = \frac{x^2}{20} = \frac{5^2}{20} = 1.25\,\text{m}$$

2.11 **(a)** $\mu = \tan\theta = \tan 30^\circ = 0.577$
(b) $ma = -(mg\sin\theta + \mu\, mg\cos\theta)$

$$\therefore \quad a = -g(\sin\theta + \mu\cos\theta) = -g(\sin\theta + \tan\theta\cos\theta)$$
$$= -9.8(2\sin 30^\circ) = -9.8$$
$$s = \frac{v_0^2}{-2a} = \frac{(2.5)^2}{2 \times 9.8} = 0.319\,\text{m}$$

Initial kinetic energy

$$K = \frac{1}{2}mv_0^2$$

Potential energy $\quad U = mgh = mgs\sin\theta$

$$\therefore \quad \frac{U}{K} = \frac{2\,mgs\sin\theta}{mv_0^2} = \frac{2 \times 9.8 \times 0.319 \times \sin 30^\circ}{(2.5)^2} = 0.5$$

The remaining energy goes into heat due to friction.

(c) It will not slide down as the coefficient of static friction is larger than the coefficient of kinetic friction.

2.12 (**a**) The torque due to the external gravitational force on M_1 will be $M_1 gr$, and the torque due to the external gravitational force on M_2 will be the component of $M_2 g$ along the string times r, i.e. $(M_2 g \sin\theta)gr$. Now, these two torques act in opposite directions. Taking the counterclockwise rotation of the pulley as positive and assuming that the mass M_1 is falling down, the net torque is

$$\tau = M_1 gr - (M_2 g \sin\theta)r = (M_1 - M_2 \sin\theta)gr \tag{1}$$

and pointing out of the page.

(**b**) When the string is moving with speed υ, the pulley will be rotating with angular velocity $\omega = \upsilon/r$, so that its angular momentum is

$$L_{\text{pulley}} = I\omega = \frac{I\upsilon}{r}$$

and that of the two blocks will be

$$L_{M_1} = rM_1\upsilon \qquad L_{M_2} = rM_2\upsilon$$

All the angular momenta point in the same direction, positive if M_1 is assumed to fall. The total angular momentum is then given by

$$L_{\text{total}} = \upsilon\left[(M_1 + M_2)r + \frac{I}{r}\right] \tag{2}$$

(**c**) Using (1) and (2)

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}\upsilon}{\mathrm{d}t}\left[(M_1 + M_2)r + \frac{1}{r}\right] = [M_1 - M_2 \sin\theta]\, gr$$

The acceleration $a = \dfrac{\mathrm{d}\upsilon}{\mathrm{d}t} = \dfrac{[M_1 - M_2 \sin\theta]\, g}{(M_1 + M_2) + \frac{1}{r^2}}$

2.13 (**i**) $ma = F - mg\sin\theta$ (equation of motion, up the incline)

$$F = ma + mg\sin\theta = m(a + g\sin\theta)$$
$$= (1.0)(1 + 9.8 \times 0.5) = 5.9\,\mathrm{N}\ (\theta = 30°)$$

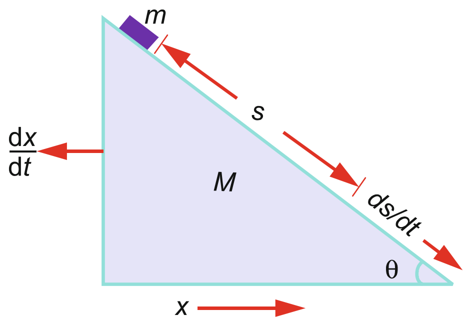
(**ii**) $ma = F + mg\sin\theta$ (equation of motion, down the incline)

$$\therefore \quad F = ma - mg\sin\theta = m(a - g\sin\theta)$$
$$= (1.0)(1 - 9.8 \times 0.5) = -3.9\,\mathrm{N}$$

The negative sign implies that the force F is to be applied up the incline.

2.14 The displacement on the edge is measured by s while that on the floor by x. As the mass m goes down the wedge the wedge itself would start moving towards left, Fig. 2.22. Since the external force in the horizontal direction is zero, the component of momentum along the x-direction must be conserved:

Fig. 2.22



$$(M+m)\frac{\mathrm{d}x}{\mathrm{d}t} - m\frac{\mathrm{d}s}{\mathrm{d}t}\cos\alpha = 0 \tag{1}$$

Since the wedge is smooth, the only force acting down the plane is $mg \sin\alpha$

$$m\left(\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} - \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\cos\alpha\right) = mg\sin\alpha \tag{2}$$

Differentiating (1)

$$(M+m)\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - m\cos\alpha\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 0 \tag{3}$$

Solving (2) and (3)

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{(M+m)g\sin\alpha}{M+m\sin^2\alpha} \quad \text{(acceleration of } m\text{)}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha} \quad \text{(acceleration of } M\text{)}$$

2.15 The lighter body of mass $m_1 = m$ moves up the plane with acceleration a_1 and the heavier one of mass $m_2 = 3\,\mathrm{m}$ moves down the plane with acceleration a_2. Assuming that the string is taut, the acceleration of the two masses must be numerically equal, i.e.
$a_2 = a_1 = a$. Let the tension in the string be T.
The equations of motion of the two masses are

$$F_1 = m_1 a_1 = ma = T - mg\sin\theta \tag{1}$$

$$F_2 = m_2 a_2 = 3ma = 3mg\sin\theta - T \tag{2}$$

Adding (1) and (2)

$$a = \frac{g}{2\sqrt{2}} \tag{3}$$

$$\boldsymbol{a}_{\text{CM}} = \frac{m_1\,\boldsymbol{a}_1 + m_2\,\boldsymbol{a}_2}{m_1 + m_2} = \frac{\boldsymbol{a}_1 + 3\boldsymbol{a}_2}{4} \tag{4}$$

In Fig. 2.23 BA represents $\boldsymbol{a}_1$ and AC represents $3\boldsymbol{a}_2$. Therefore, BC the third side of the ΔABC represents $|\boldsymbol{a}_1 + 3\boldsymbol{a}_2|$. Obviously $\mathrm{B\hat{A}C}$ is a right angle so that

$$|\boldsymbol{a}_1 + 3\boldsymbol{a}_2| = \text{BC} = \sqrt{a_1^2 + (3a_2)^2} = \sqrt{10}a$$

$$\therefore \quad a_{\text{CM}} = \frac{1}{4}\sqrt{10}a = \frac{\sqrt{10}}{4}\frac{g}{2\sqrt{2}} = \frac{\sqrt{5}}{8}g \tag{5}$$

In Fig. 2.23, BD is parallel to the base so that $\mathrm{A\hat{B}D} = 45^\circ$. Let $\mathrm{C\hat{B}D} = \alpha$.

$$\text{Now}\tan(\alpha + 45^\circ) = \frac{\tan\alpha + \tan 45^\circ}{1 - \tan\alpha\tan 45^\circ} = \frac{\tan\alpha + 1}{1 - \tan\alpha} \tag{6}$$

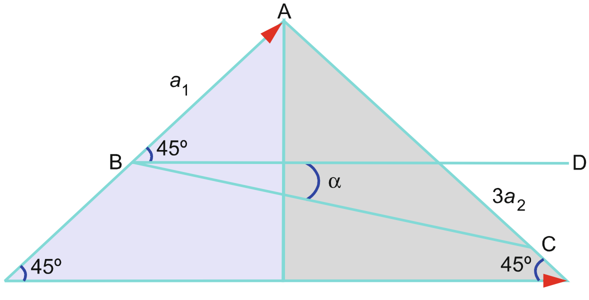
Further, in the right angle triangle ABC,

$$\tan\widehat{\text{ABC}} = \tan(\alpha + 45^\circ) = \frac{\text{AC}}{\text{AB}} = 3 \tag{7}$$

Combining (6) and (7) $\quad \tan\alpha = \frac{1}{2} \quad \text{or} \quad \alpha = \tan^{-1}\left(\frac{1}{2}\right)$.

Thus a_{CM} is at an angle $\tan^{-1}\left(\frac{1}{2}\right)$ to the horizon.

Fig. 2.23



2.16 (**a**) Free body diagram (Fig. 2.24)

(**b**) $m_1 a = T_1 - m_1 g \sin 30^\circ$ (1)

$$m_2 a = m_2 \sin 60^\circ - T_2 \tag{2}$$

$$(T_2 - T_1)r = I\alpha = \left(\frac{1}{2}Mr^2\right)\left(\frac{a}{r}\right) \tag{3}$$

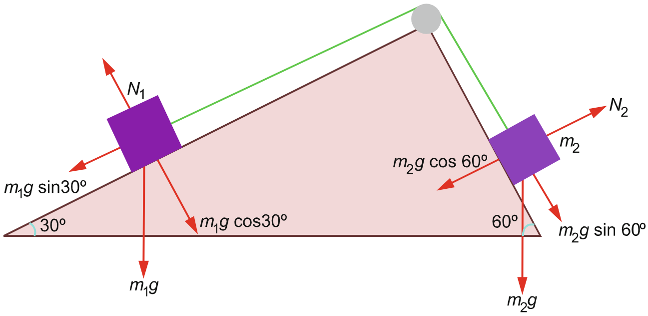


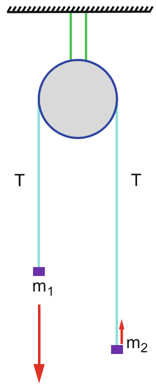
Fig. 2.24

(**c**) Combining (1), (2) and (3) and simplifying

$$a = \frac{g\left(\sqrt{3}m_2 - m_1\right)}{M + 2(m_2 + m_1)}$$

2.17 Let $m_1 = 2.1$ kg and $m_2 = 1.9$ kg, Fig. 2.25. As the pulley is weightless the tension is the same on either side of the pulley. Equations of motion are as follows:

Fig. 2.25



$$m_1 a = m_1 g - T \tag{1}$$
$$m_2 a = T - m_2 g \tag{2}$$

Adding (1) and (2)

$$(m_1 + m_2)a = (m_1 - m_2)g$$
$$\therefore \quad a = \frac{(m_1 - m_2)\,g}{m_1 + m_2} = \frac{(2.1 - 1.9)9.8}{2.1 + 1.9} = 0.49\,\text{m/s}^2$$

Distance travelled by either mass, $s = 40\,\text{cm}$. Time taken

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 0.4}{0.49}} = 1.28\,\text{s}.$$

2.18 Equations of motion are

$$ma_1 = mg\sin\theta - \mu mg\cos\theta \qquad \text{(rough incline)}$$
$$ma_2 = mg\sin\theta \qquad \text{(smooth incline)}$$
$$\therefore \quad a_1 = (\sin\theta - \mu\cos\theta)g$$
$$a_2 = g\sin\theta$$
$$t_1 = \sqrt{\frac{2s}{a_1}} \quad t_2 = \sqrt{\frac{2s}{a_2}} \quad \therefore \quad \frac{t_1}{t_2} = \frac{4}{3}$$
$$\sqrt{\frac{\sin\theta}{\sin\theta - \mu\cos\theta}} = \sqrt{\frac{\sin 45^\circ}{\sin 45^\circ - \mu\cos 45^\circ}} = \frac{1}{\sqrt{1-\mu}}$$
$$\therefore \quad \mu = \frac{7}{16}$$

2.19 The normal reaction $N = mg\cos\theta$
Resultant downward force $F = mg\sin\theta - \mu mg\cos\theta$
Given that $N = 2F$

$$mg\ \cos\theta = 2\,mg(\sin\theta - 0.5\cos\theta)$$
$$\therefore \quad \tan\theta = 1 \rightarrow \theta = 45^\circ$$

2.20 By prob. (2.17), each mass will have acceleration

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2}$$

The heaver mass m_1 will have acceleration a_1 vertically down while the lighter mass m_2 will have acceleration a_2 vertically up:

$$\boldsymbol{a}_2 = -\boldsymbol{a}_1$$

The acceleration of the centre of mass of the system will be

$$\boldsymbol{a}_{\text{CM}} = \frac{m_1\,\boldsymbol{a}_1 + m_2\,\boldsymbol{a}_2}{m_1 + m_2} = \frac{(m_1 - m_2)\boldsymbol{a}_1}{m_1 + m_2}$$
$$\therefore \quad a_{\text{CM}} = \frac{(m_1 - m_2)^2 g}{(m_1 + m_2)^2}$$

2.21 Free body diagrams for the two blocks and the pulley are shown in Fig. 2.26. The forces acting on m_2 are tension T_2 due to the string, gravity, frictional force f_2 due to the movement of m_1 and the normal force which m_1 exerts on it to prevent if from moving vertically. The forces on m_1 due to m_2 are equal and opposite to those of m_1 on m_2. By Newton's third law the tensions T_1 and T_2 in the thread are not equal as the pulley has mass. The equations of motion for m_1, m_2 and the pulley are

$$m_1 a = F - f_1 - f_2 - T_1 \tag{1}$$

$$m_2 a = T_2 - f_2 \tag{2}$$

$$\alpha I = I\frac{a}{r} = r(T_1 - T_2) \tag{3}$$

Balancing the vertical forces

$$N_2 = m_2 g$$

$$N_1 = N_2 + m_1 g = (m_1 + m_2)\,g$$

Frictional forces are

$$f_2 = \mu N_2 = \mu m_2 g \tag{4}$$

$$f_1 = \mu N_1 = \mu(m_1 + m_2)g \tag{5}$$

Combining (1), (2), (3), (4) and (5), eliminating f_1, f_2 and T

$$a = \frac{F - \mu(m_1 + 3m_2)g}{m_1 + m_2 + \frac{I}{r^2}}$$

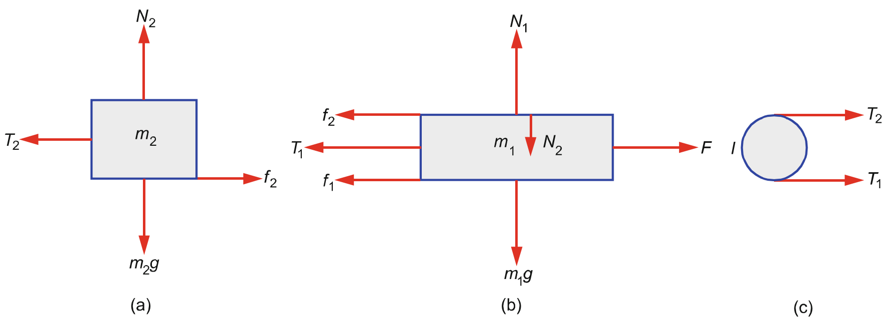


Fig. 2.26

2.3.3 *Work, Power, Energy*

2.22 Net force $\quad \boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = (\hat{i} + 2\hat{j} + 3\hat{k}) + (4\hat{i} - 5\hat{j} - 2\hat{k})$

$$= 5\hat{i} - 3\hat{j} + \hat{k}$$

Displacement $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2 = 7\hat{k} - (20\hat{\imath} + 15\hat{\jmath}) = (-20\hat{\imath} - 15\hat{\jmath} - 7\hat{k})$ cm

Work done $W = \boldsymbol{F} \cdot \boldsymbol{r}_{12} = (5\hat{\imath} - 3\hat{\jmath} + \hat{k}) \cdot (-0.20\hat{\imath} - 0.15\hat{\jmath} + 0.07\hat{k})$

$= -0.48\,\text{J}$.

2.23 **(i)** $U(x) = 5x^2 - 4x^3$

$$F(x) = -\frac{\mathrm{d}U}{\mathrm{d}x} = -(10x - 12x^2) = 12x^2 - 10x$$

(ii) For equilibrium $F(x) = 0$

$$x(12x - 10) = 0 \quad \text{or} \quad x = 5/6\,\text{m or } 0$$

$$\frac{\mathrm{d}F}{\mathrm{d}x} = 24x - 10$$

$$\frac{\mathrm{d}F}{\mathrm{d}x}|_{x=0} = (24x - 10)|_{x=0} = -10$$

The position $x = 0$ is stable:

$$\frac{\mathrm{d}F}{\mathrm{d}x}\Big|_{x=\frac{5}{6}} = (24x - 10)\Big|_{x=\frac{5}{6}} = +10$$

The position $x = 5/6$ is unstable.

2.24 Let the body travel a distance s on the incline and come down through a height h.
Potential energy lost $= mgh = mgs \sin\theta$.
Work down against friction $W = fs = \mu mg \cos\theta \cdot s$.

By problem $\mu\, mg \cos\theta s = \dfrac{70}{100} mgs\ \sin\theta$

$\therefore \quad \mu = 0.7 \tan\theta = 0.7 \tan 30^\circ = 0.404$

2.25 At the bottom of the ramp the kinetic energy K available is equal to the loss of potential energy, mgh:

$$K = mgh$$

On the flat track the entire kinetic energy is used up in the work done against friction

$$W = fd = \mu mgd$$

$$\therefore \quad \mu\, mgd = mgh$$

$$\mu = \frac{h}{d}$$

2.26 **(i)** Work done by the spring $W_s = \frac{1}{2}kx^2 = \frac{1}{2} \times 20 \times 10^3 \times (0.12)^2 = 144\,\text{J}$

(ii) Work done by friction $W_f = \frac{1}{2}mv^2 - W_s = \frac{1}{2} \times 50 \times 3^2 - 144 = 81\,\text{J}$

(iii) $W_f = \mu mgs$

$$\therefore \quad \mu = \frac{W_f}{mgs} = \frac{81}{50 \times 9.8 \times (0.60 + 0.12)} = 0.2296$$

(iv) If v_1 is the velocity of the crate as it passes position A after rebonding

$$\frac{1}{2}mv_1^2 = W_s - \mu\, mgs$$

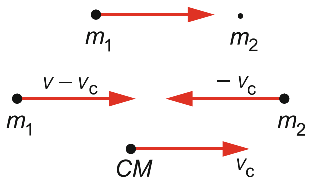
$$\frac{1}{2} \times 50v_1^2 = 144 - 0.2296 \times 50 \times 9.8 \times (0.60 + .012) = 63$$

$$\therefore \quad v_1 = 1.587\,\text{m/s}$$

2.3.4 Collisions

2.27 In the CMS the velocity of m_1 will be $v_1^* = v - v_c$ and that of m_2 will be $v_2^* = -v_c$, Fig. 2.27. By definition in the CMS total momentum is zero:

Fig. 2.27



$$m_1\vec{v}_1^* + m_2\vec{v}_2^* = 0$$

$$\therefore \quad m_1(v - v_c) - m_2 v_c = 0$$

$$\therefore \quad v_c = v_2^* = \frac{m_1 v}{m_1 + m_2} \tag{1}$$

$$\therefore \quad v_1^* = v - v_c = \frac{m_2 v}{m_1 + m_2} \tag{2}$$

Note that as the collision is elastic, the velocities of m_1 and m_2 after the collision in the CMS remain unchanged. The lab velocity v_1 of m_1 is obtained by the vectorial addition of v_1^* and v_c^*. From the triangle ABC, Fig. 2.28. After collision

$$v_1^2 = v_1^{*2} + v_c^2 - 2v_1^* v_c \cos(180° - \theta) \tag{3}$$

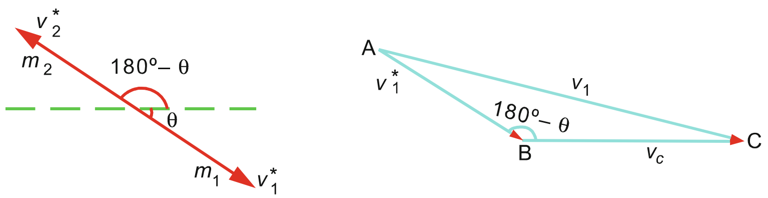


Fig. 2.28

Using (1) and (2) and simplifying

$$v_1 = \frac{v}{m_1 + m_2}\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos\theta} \tag{4}$$

2.28 (a) Let the initial momentum of one object be $\boldsymbol{p}$. After scattering let the momenta be $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, with the angle θ between them, Fig. 2.29.

$\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ (momentum conservation)

$$\therefore \quad (\boldsymbol{p} \cdot \boldsymbol{p}) = \boldsymbol{p}^2 = (\boldsymbol{p}_1 + \boldsymbol{p}_2) \cdot (\boldsymbol{p}_1 + \boldsymbol{p}_2)$$

$$= \boldsymbol{p}_1 \cdot \boldsymbol{p}_1 + \boldsymbol{p}_2 \cdot \boldsymbol{p}_2 + \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 + \boldsymbol{p}_2 \cdot \boldsymbol{p}_1 = \boldsymbol{p}_1^2 + \boldsymbol{p}_2^2 + 2\boldsymbol{p}_1 + \boldsymbol{p}_2 \tag{1}$$

$$\frac{\boldsymbol{p}^2}{2m} = \frac{\boldsymbol{p}_1^2}{2m} + \frac{\boldsymbol{p}_2^2}{2m} \text{ (energy conservation) or } \boldsymbol{p}^2 = \boldsymbol{p}_1^2 + \boldsymbol{p}_2^2 \tag{2}$$

Combining (1) and (2), $2\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = 0$

$\therefore \quad \boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are orthogonal

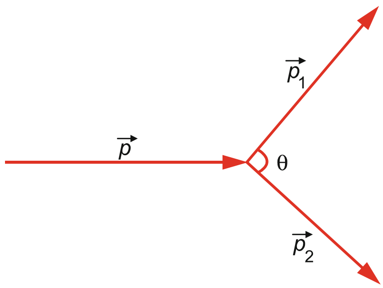


Fig. 2.29a

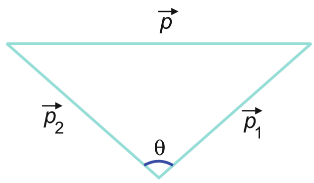
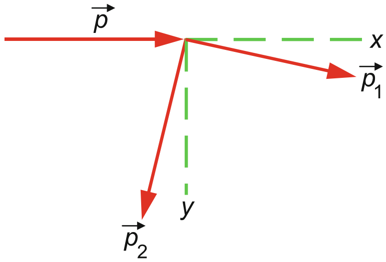


Fig. 2.29b

Fig. 2.30



(b) Suppose one of the objects (say 2) is scattered in the backward direction then the momenta would appear as in Fig. 2.30, and because by (a) the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ has to be a right angle, both the objects will be scattered on the same side of the incident direction (x-axis). In that case, the y-component of momentum cannot be conserved as initially $\sum p_y = 0$. Both the objects cannot be scattered in the backward direction. In that case the x-component of momentum cannot be conserved.

2.29 We work out in the CM system. The total kinetic energy available in the CMS is

$$= K^* = \frac{1}{2}\mu v^2 \tag{1}$$

where
$$\mu = \frac{m\,M}{m+M} \tag{2}$$

is the reduced mass.

If the compression of the spring is x then the spring energy would be $\frac{1}{2}kx^2$. Equating the total kinetic energy available in the CM–system to the spring energy

$$\frac{1}{2}\mu v^2 = \frac{1}{2}kx^2$$

$$x = v\sqrt{\frac{\mu}{k}} = v\sqrt{\frac{mM}{k(m+M)}}$$

2.30 After the first collision (head-on) with the sphere 2 on the right-hand side, sphere 1 moves with a velocity

$$v_1 = \frac{2m_2u_2 + u_1(m_1 - m_2)}{m_1 + m_2} = \frac{0 + u_1(m - 4m)}{m + 4m} = -0.6u_1 \tag{1}$$

and the sphere 2 moves with a velocity

$$v_2 = \frac{2m_1u_1 + u_2(m_2 - m_1)}{m_1 + m_2} = \frac{2mu_1 + 0}{m + 4m} = 0.4u_1 \tag{2}$$

where u and v with appropriate subscripts refer to the initial and final velocities.
The negative sign shows that the ball 1 moves toward left after the collision and hits ball 3. After the second collision with ball 3, ball 2 acquires a velocity v_2 and moves toward right

$$v_1' = \frac{2m_3u_3 + v_1(m_1 - m_3)}{m_1 + m_3} = \frac{0 - u_1(m_1 - 4m)}{m + 4m} = 0.36u_1 \tag{3}$$

But $v_1' < v_2$. Therefore ball 1 will not undergo the third collision with ball 2. Thus in all there will be only two collisions.

2.31 Momentum conservation gives $m_1 \boldsymbol{u}_1 + m_2 \boldsymbol{u}_2 = m_2 \boldsymbol{v}_2$ (1)

Conservation of kinetic energy in elastic collision gives

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_2v_2^2 \tag{2}$$

By the problem $\frac{1}{2}m_1u_1^2 = \frac{1}{2}m_2u_2^2$ (3)

$$u_2 = \alpha u_1 \tag{4}$$

$$\therefore \quad \alpha^2 = \frac{m_1}{m_2} \tag{5}$$

Using (3) in (2)

$$m_2u_2^2 = \frac{1}{2}m_2v_2^2$$

$$\therefore \quad v_2 = \sqrt{2}u_2 \tag{6}$$

Using (4) and (6) in (1)

$$m_1u_1 + m_2\alpha u_1 = \sqrt{2}m_2u_2 = \sqrt{2}m_2\alpha u_1$$

$$\text{or} \qquad m_1 + m_2\alpha = \sqrt{2}\alpha m_2$$

Dividing by m_2 and using (5) and rearranging

$$\alpha\left[\alpha - \left(\sqrt{2} - 1\right)\right] = 0$$

since $\alpha \neq 0$, $\alpha = \sqrt{2} - 1$

$$\therefore \quad \alpha = \frac{u_2}{u_1} = \sqrt{2} - 1$$

$$\therefore \quad \frac{u_1}{u_2} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$$

From (5)

$$\frac{m_1}{m_2} = \alpha^2 = \left(\sqrt{2} - 1\right)^2 = 3 - 2\sqrt{2}$$

2.32 (**i**) If the common velocity of the merged bodies is v then momentum conservation gives

$$(m_A + m_B)\boldsymbol{v} = m_A \boldsymbol{v}_A + m_B \boldsymbol{v}_B$$

$$\therefore \quad \boldsymbol{v} = \frac{m_A \boldsymbol{v}_A + m_B \boldsymbol{v}_B}{m_A + m_B}$$

(**ii**) $$\boldsymbol{v} = \frac{\frac{3}{2} m_B (5\hat{i} + 3\hat{j}) + m_B(-\hat{i} + 4\hat{j})}{\frac{3}{2} m_B + m_B} = 2.6\hat{i} + 3.4\hat{j}$$

(**iii**) $$\Delta \boldsymbol{p}_A = m_A(\boldsymbol{v} - \boldsymbol{v}_A) = m_A \left[2.6\hat{i} + 3.4\hat{j} - (5\hat{i} + 3\hat{j})\right] = m_A\left[-2.4\hat{i} + 0.4\hat{j}\right]$$

$$\Delta p_A = 1200\sqrt{(-2.4)^2 + (0.4)^2} = 2920\,\text{N m}$$

$$F_A = \frac{\Delta p_A}{\Delta t} = \frac{2920}{0.2} = 14{,}600\,\text{N}$$

$$\Delta \vec{\boldsymbol{p}}_B = m_B(\vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_B) = m_B\left[2.6\hat{i} + 3.4\hat{j} - (-\hat{i} + 4\hat{j})\right] = m_B\left[3.6\hat{i} - 0.6\hat{j}\right]$$

$$m_B = \frac{2}{3} m_A = \frac{2}{3} \times 1200 = 800\,\text{kg}$$

$$\Delta p_B = 800\sqrt{(3.6)^2 + (-0.6)^2} = 2920\,\text{N m}$$

$$F_B = \frac{\Delta p_B}{\Delta t} = \frac{2920}{0.2} = 14{,}600\,\text{N}$$

(**iv**) $$K' = \frac{1}{2}(m_A + m_B)v^2 = \frac{1}{2}(1200 + 800)\left[(2.6)^2 + (3.4)^2\right] = 18,320\,\text{J}$$

2.33 Let the velocity of the particle moving below the x-axis be v'. Momentum conservation along x- and y-axis gives

$$mv_0 = mv\cos\theta + mv'\cos\beta \tag{1}$$

$$0 = mv\sin\theta - mv'\sin\beta \tag{2}$$

Cancelling m and reorganizing (1) and (2)

$$v' \cos\beta = v_0 - v\cos\theta \tag{3}$$

$$v' \sin\beta = v\sin\theta \tag{4}$$

Dividing (4) by (3)

$$\tan\beta = \frac{v\sin\theta}{v_o - v\cos\theta} \tag{5}$$

If the collision is elastic, kinetic energy must be conserved:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}mv'^2 \tag{6}$$

or $$v_0^2 = v^2 + v'^2 \tag{7}$$

Squaring (3) and (4) and adding

$$v'^2 = v^2 + v_0^2 - 2v_0 v\cos\theta \tag{8}$$

Eliminating v'^2 between (7) and (8) and simplifying

$$v = v_0\cos\theta$$

2.34 The momenta of β and ^{14}N are indicated in both magnitude and direction in Fig. 2.31. The resultant R of these momenta is given from the diagonal AC of the rectangle (parallelogram law). The momentum of u is obtained by protruding CA to E such that AE = AC:

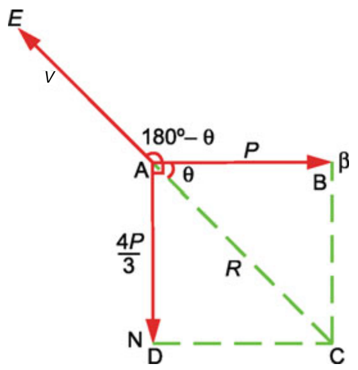


Fig. 2.31 Decay of ^{14}C at rest

$$R = \sqrt{\mathrm{AB}^2 + \mathrm{BC}^2} = \sqrt{p^2 + (4p/3)^2} = \frac{5p}{3}$$
$$\tan\theta = \frac{\mathrm{BC}}{\mathrm{AB}} = \frac{4p/3}{p} = \frac{4}{3}$$
$$\therefore \quad \theta = 53^\circ$$

Thus the neutrino is emitted with momentum $5p/3$ at an angle $(180 - 53^\circ)$ or 127° with respect to the β particle.

2.35 Denoting the angles with (*) for the CM system transformation of angles for CMS to LS is given by

$$\tan\theta = \frac{\sin\theta^*}{\cos\theta^* + \dfrac{m}{M}} \tag{1}$$

But $\theta^* = \pi - \phi^* = \pi - 2\phi$

$$\therefore \quad \sin\theta^* = \sin(\pi - 2\phi) = \sin 2\phi$$
$$\cos\theta^* = \cos(\pi - 2\phi) = -\cos 2\phi$$

(**i**) becomes

$$\tan\theta = \frac{\sin 2\phi}{\dfrac{m}{M} - \cos 2\phi}$$

Furthermore $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin 2\phi}{\dfrac{m}{M} - \cos 2\phi}$

Cross-multiplying and rearranging

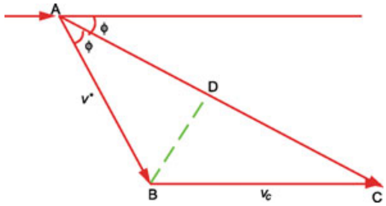
$$\frac{m}{M}\sin\theta = \sin\theta\cos 2\phi + \cos\theta\sin 2\phi = \sin(\theta + 2\phi)$$
$$\therefore \quad \frac{m}{M} = \frac{\sin(2\phi + \theta)}{\sin\theta}$$

2.36 In the lab system let M be projected at an angle ϕ with velocity v. In the CMS the velocity v^* for the struck nucleus will be numerically equal to v_c, the centre of mass velocity. Therefore, M is projected at an angle 2ϕ with velocity $v^* = \dfrac{mv}{M+m}$. The CM system velocity $v_c = \dfrac{mv}{M+m}$. The velocities v^* and v_c must be combined vectorially to yield v, Fig. 2.32. Since $v_c = v^*$ the velocity triangle ABC is an isosceles triangle. If BD is perpendicular on AC, then

$$AC = 2AD = 2AB\cos\phi$$

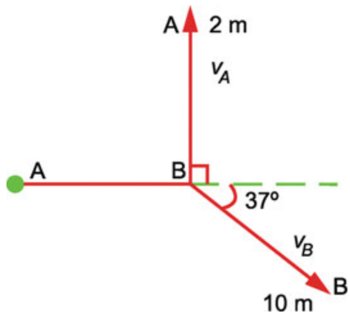
$$\therefore \quad v = 2v^* \cos\phi = \frac{2mu\cos\phi}{M+m}$$

Fig. 2.32



2.37 **(a)** Kinetic energy of A before collision $K_A = \frac{1}{2}(2m)u^2 = mu^2$. Since B is initially stationary, its kinetic energy $K_B = 0$. Hence before collision, total kinetic energy $K_0 = mu^2 + 0 = mu^2$.
Let A and B move with velocity v_A and v_B, respectively, after the collision, Fig. 2.33. Total kinetic energy after the collision,

Fig. 2.33



$$K' = K'_A + K'_B = \frac{1}{2}(2m)v_A^2 + \frac{1}{2}(10m)v_B^2 = mv_A^2 + 5mv_B^2$$

If an energy Q is lost in the collision process, conservation of total energy gives

$$mu^2 = mv_A^2 + 5mv_B^2 + Q \tag{1}$$

Applying momentum conservation along the incident direction and perpendicular to it

$$2mu = 10mv_B\cos 37° = 8mv_B \tag{2}$$

$$2mv_B = 10mv_B\sin 37° = 6mv_B \tag{3}$$

From (2) and (3) we find

$$v_A = \frac{3u}{4}; \quad v_B = \frac{u}{4}$$

(b) From (1), $Q = m(u^2 - v_A^2 - 5v_B^2)$

$$= m\left(u^2 - \frac{9u^2}{16} - \frac{5u^2}{16}\right) = \frac{mu^2}{8}$$

$$\therefore \quad \frac{Q}{k_A} = \frac{mu^{2/8}}{mu^2} = \frac{1}{8}$$

Since Q is positive, energy is lost in the collision process.

2.38 **(a)** In the elastic collision (head-on) of a particle of mass m_1 and kinetic energy K_1 with a particle of mass m_2 initially at rest, the fraction of kinetic energy imparted to m_2 is

$$\frac{K_2}{K_0} = \frac{4m_1m_2}{(m_1+m_2)^2} = \frac{4 \times 1 \times 12}{(1+12)^2} = \frac{48}{169}$$

(b) $$\frac{\frac{1}{2}m_2v_2^2}{\frac{1}{2}m_1v_1^2} = \frac{12}{1}\frac{v_2^2}{v_0^2} = \frac{48}{169}$$

$$\therefore \quad v_2 = \frac{2}{13}v_0$$

$$K_1 = K_0 - K_2 = K_0 - \frac{48}{169}K_0 = \frac{121}{169}K_0$$

$$\therefore \quad \frac{1}{2}m_1v_1^2 = \frac{121}{169} \times \frac{1}{2}m_1v_0^2$$

$$\therefore \quad v_1 = -\frac{11}{13}v_0$$

Negative sign is introduced because neutron being lighter then the carbon nucleus will bounce back.

2.39 Let the heavy body of mass M with momentum P_0 collide elastically with a very light body of mass m be initially at rest. After the collision both the bodies will be moving in the direction of incidence, the heavier one with velocity v_H and the lighter one with velocity v_L.
Momentum conservation gives

$$p_0 = p_L + p_H \tag{1}$$

Energy conservation gives

$$\frac{p_0^2}{2M} = \frac{p_L^2}{2m} + \frac{p_H^2}{2M} \tag{2}$$

Eliminating p_{H} between (1) and (2) and simplifying

$$P_{\mathrm{L}} = \frac{2p_0 m}{M+m}$$

$$mv_{\mathrm{L}} = \frac{2Mum}{M+m}$$

$$\text{or} \quad v_{\mathrm{L}} = \frac{2uM}{M+m} = 2u \quad (\because m << M)$$

2.40 Let a body of mass m_1 moving with velocity u make a completely inelastic collision with the body of mass m_2 initially at rest. Let the combined mass moves with a velocity v_c given by

$$v_{\mathrm{c}} = \frac{m_1 u}{m_1 + m_2} = \frac{u}{2} \quad (\because m_1 = m_2)$$

$$\text{Energy lost} \quad = \frac{1}{2}mu^2 - \frac{1}{2}(2m)\left(\frac{u}{2}\right)^2 = \frac{1}{4}mu^2 = \frac{1}{2}K_0$$

where $K_0 = \frac{1}{2}mu^2$ is the initial kinetic energy.

2.41 Let the speed of the bullet be u. Let the block + bullet system be travelling with initial speed v. If m and M are the masses of the bullet and the block, respectively, then momentum conservation gives

$$mu = (M+m)v \tag{1}$$

$$\therefore \quad v = \frac{mu}{M+m} \tag{2}$$

The initial kinetic energy of the block + bullet system

$$K = \frac{1}{2}(M+m)v^2 = \frac{1}{2}\frac{m^2u^2}{(M+m)}$$

Work done to bring the block + bullet system to rest in distance s is

$$W = \mu(M+m)gs = \frac{1}{2}\frac{m^2u^2}{(M+m)}$$

$$\therefore \quad u = \frac{(M+m)}{m}\sqrt{2\mu\, gs} = \frac{(2.000+0.005)}{0.005}\sqrt{2\times 0.2\times 9.8\times 2}$$
$$= 1123\,\mathrm{m/s}$$

2.42 **(a)** Let $m_1 = m$ with velocity u collide with $m_2 = M$, initially at rest. For elastic collision the final velocities will be

$$v_1 = \frac{(m_1 - m_2)}{m_1 + m_2}u = \frac{(m-M)}{m+M}u \qquad (m < M) \tag{1}$$

$$v_2 = \frac{2m_1}{m_1 + m_2} u = \frac{2mu}{m + M} \tag{2}$$

$$\text{By problem} - v_1 = v_2 \tag{3}$$

Combining (1), (2) and (3)

$$\frac{M}{m} = 3 \tag{4}$$

(b) $$v_c = \frac{mu}{M + m} = \frac{mu}{3m + m} = \frac{u}{4} \tag{5}$$

(c) $$K^* = K_1{}^* + K_2{}^* = \frac{1}{2} m v_1{}^{*2} + \frac{1}{2} M v_2{}^{*2}$$

$$\text{But} \quad v_1{}^* = \frac{Mu}{M + m} = \frac{3mu}{3m + m} = \frac{3u}{4}$$

$$v_2{}^* = -v_c = -\frac{u}{4}$$

$$\therefore \quad K^* = \frac{1}{2} m \left(\frac{3u}{4}\right)^2 + \frac{1}{2} 3m \left(\frac{u}{4}\right)^2 = \frac{3}{8} mu^2$$

(d) $$K_1(\text{final}) = \frac{1}{2} m v_1^2 = \frac{1}{8} mu^2$$

where we have used (1) and (4).

2.43 We can work out this problem in the lab system. But we prefer to use the centre of mass system. The CMS and LS scattering angles are related by

$$\tan\theta = \frac{\sin\theta^*}{\cos\theta^* + \dfrac{M}{m}} \tag{1}$$

$\theta_{\max}$ is obtained from the condition

$$\frac{d\tan\theta}{d\theta^*} = 0 \tag{2}$$

$$\text{This gives} \quad \cos\theta^* = \frac{m}{M} \tag{3}$$

$$\therefore \quad \sin\theta^* = \frac{\sqrt{M^2 - m^2}}{M} \tag{4}$$

Use (3) and (4) in (1) to get

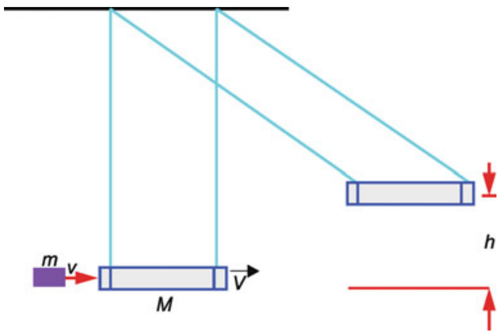
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta = \frac{M^2}{m^2}$$

$$\therefore \quad \sin\theta_{\max} = \frac{m}{M}$$

$$\text{or} \quad \theta_{\max} = \sin^{-1}\left(\frac{m}{M}\right)$$

2.44 Momentum of the bullet before collision = momentum of the block + bullet system immediately after collision, Fig. 2.34:

Fig. 2.34



$$mv = (m + M)V \tag{1}$$

where V is the initial speed of the block + bullet system. The kinetic energy of the system immediately after the impact is

$$K = \frac{1}{2}(m + M)V^2 \tag{2}$$

Due to the impact, the pendulum would swing to the right and would be raised through the maximum height h vertically above the rest position of the pendulum, Fig. 2.34. At this point, the kinetic energy of the pendulum is entirely converted into gravitational potential energy:

$$\frac{1}{2}(m + M)V^2 = (m + M)gh \tag{3}$$

$$\therefore \quad V = \sqrt{2gh} \tag{4}$$

Using (4) in (1)

$$v = \left(1 + \frac{M}{m}\right)\sqrt{2gh} \tag{5}$$

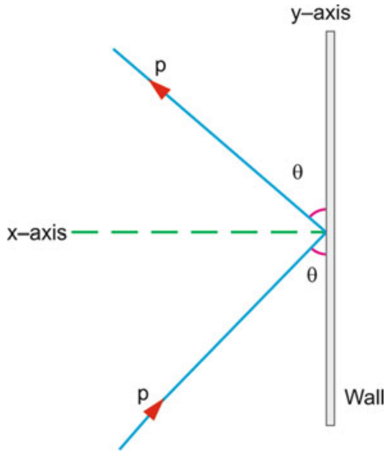
By measuring h and knowing m and M, the original velocity of the bullet can be calculated.

2.45 Let the area of the nozzle through where the jet comes be A m^2. The mass of water in the jet per second is $\rho A\ v$, where ρ is the density of water and v the get velocity.
The momentum associated with this volume of water is

$$p = (\rho A v)v = \rho A v^2 \tag{1}$$

The momentum after hitting the wall will also be equal to $\rho A v^2$ since the collision is assumed to be elastic. Resolve the momentum along the x-axis and y-axis, Fig. 2.35.

Fig. 2.35



The change of the x-component of momentum is

$$\Delta p_x = p \sin\theta - (-p \sin\theta) = 2p \sin\theta \tag{2}$$

The change in the y-component of momentum is

$$\Delta p_y = p \cos\theta - p \cos\theta = 0 \tag{3}$$

$$\text{Then } \Delta p = \Delta p_x = 2p \sin\theta = 2\rho A v^2 \sin\theta \tag{4}$$

Pressure exerted on the wall will be

$$P = \frac{\Delta p}{A} = 2\rho v^2 \sin\theta \tag{5}$$

For normal incidence, $\theta = 90°$ and

$$P = 2\rho v^2 \tag{6}$$

2.46 For completely inelastic collision there is no rebounding of the jet. The pressure on the wall is given by

$$P = \rho v^2 \sin\theta \tag{1}$$

For normal incidence, $\theta = 90°$ and

$$P = \rho v^2 \tag{2}$$

2.47 Resolve the momentum mv_1 and mv_2 along the original line of motion and in a direction perpendicular to it. Along the original line of motion, the initial momentum must be equal to the sum of the components of momentum after the collision:

$$mv_0 = mv_1 \cos 30° + mv_2 \cos 30° \tag{1}$$

In the direction perpendicular to the original direction of motion, the sum of components of momentum after the collision must be equal to zero because before collision the balls do not have any component of momentum in the perpendicular direction:

$$mv_1 \sin 30° - mv_2 \sin 30° = 0$$
$$\text{or} \quad v_1 = v_2 \tag{2}$$

This result could have been anticipated from symmetry.
Using (2) in (1)

$$v_0 = 2v_1 \cos 30° = \sqrt{3}v_1$$
$$\text{or} \quad v_1 = v_2 = \frac{v_0}{\sqrt{3}} = \frac{9}{\sqrt{3}} = 5.19\,\text{m/s}$$

Total kinetic energy of the two balls before collision

$$K_0 = \frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_0^2 \tag{3}$$

Total kinetic energy after the collision

$$K' = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = mv_1^2 = \frac{1}{3}mv_0^2 \tag{4}$$

On comparing (3) and (4) we conclude that kinetic energy is not conserved. The collision is said to be inelastic.

2.48 Time taken for the ball to reach the plane in the initial fall

$$t_0 = \sqrt{\frac{2h}{g}} \tag{1}$$

Velocity with which it reaches the plane

$$u_1 = \sqrt{2gh} \tag{2}$$

The velocity with which it rebounds from the plane

$$v_1 = eu_1 = e\sqrt{2gh} \tag{3}$$

Time to reach the plane again

$$t_1 = \frac{2v_1}{g} = 2e\sqrt{\frac{2h}{g}} = 2et_0$$

If this process is repeated indefinitely the total time

$$\begin{aligned} T &= t_0 + t_1 + t_2 + \cdots + t_\infty = t_0 + 2et_0 + 2e^2 t_0 + \cdots \\ &= t_0[1 + 2e(1 + e + e^2 + \cdots)] \end{aligned}$$

$$= t_0\left[1 + \frac{2e}{1-e}\right] = \sqrt{\frac{2h}{g}}\frac{1+e}{1-e}$$

where we have used the formula for the sum of infinite number of terms of a geometric series.

2.49 Total distance traversed

$$\begin{aligned} S &= h + 2h_1 + 2h_2 + \cdots \quad = h + 2e^2 h + 2e^4 h + 2e^6 h + \cdots \\ &= h\left[1 + \frac{2e^2}{1-e^2}\right] = h\frac{(1+e^2)}{1-e^2} \end{aligned}$$

2.50 On the first bounce, $v_1 = e\sqrt{2gh}$

On the second bounce, $v_2 = e^2\sqrt{2gh}$

On the nth bounce, $v_n = e^n\sqrt{2gh}$

$$h_n = \frac{v_n^2}{2g} = e^{2n} h$$

2.51 Let the two bodies of mass m_1 and m_2 be travelling with the velocities u_1 and u_2 before the impact, and the combined body of mass $m_1 + m_2$ with velocity v after the impact is

$$v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \tag{1}$$

Energy wasted = total kinetic energy before the collision minus total kinetic energy after the collision

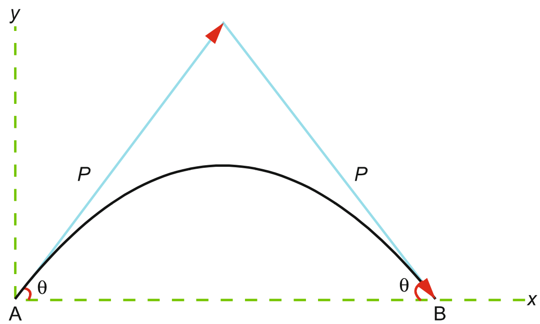
$$W = \left(\frac{1}{2}m_1 {u_1}^2 + \frac{1}{2}m_2 {u_2}^2\right) - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \frac{1}{2}m_1 {u_1}^2 + \frac{1}{2}m_2 {u_2}^2 - \frac{1}{2}\frac{(m_1 u_1 + m_2 u_2)^2}{m_1 + m_2} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(u_1 - u_2)^2$$

2.52 Resolve the momentum along x- and y-axes at points A and B, Fig. 2.36. Take the downward direction as positive:

$p_x(\mathrm{A}) = p\cos\theta \quad p_x(\mathrm{B}) = p\cos\theta$
$p_y(\mathrm{A}) = -p\sin\theta \quad p_y(\mathrm{B}) = p\sin\theta$
Then $\Delta p_x = p_x(\mathrm{B}) - p_x(\mathrm{A}) = p\cos\theta - p\cos\theta = 0$
$\Delta p_y = p_y(\mathrm{B}) - p_y(\mathrm{A}) = p\sin\theta - (-p\sin\theta) = 2p\sin\theta$
$\therefore \quad \Delta p = \Delta p_y = 2p\sin\theta$

Fig. 2.36



2.53 Let the shell of mass $2m$ explode into two pieces each of mass m. At the highest point the entire velocity consists of the horizontal component ($v\cos\theta$) alone. Since one of the components retraces its path, it follows that it has velocity $-v\cos\theta$, and therefore a momentum $-mv\cos\theta$. Let the momentum of the other pieces be p. Now, the momentum of the shell just before the explosion was $2mv\cos\theta$ momentum conservation gives

$$p - mv\cos\theta = 2mv\cos\theta$$

$$\therefore \quad p = 3mv\cos\theta$$

$$\therefore \quad \text{velocity} = \frac{p}{m} = 3v\cos\theta$$

2.54 Volume of air moving down per second = Av, where v is the air velocity moving down through an area A.
Mass of air moving down per second = ρAv

$$F = \frac{\Delta p}{\Delta t} = \left(\frac{\text{mass}}{\text{sec}}\right)(\Delta v) = \rho Av^2$$

Reaction force upward = Helicopter's weight

$$\rho Av^2 = Mg$$

$$v = \sqrt{\frac{Mg}{\rho A}} = \sqrt{\frac{500 \times 9.8}{1.3 \times 45}} = 9.15\,\text{m/s}$$

2.55 If v is the velocity of each bullet of mass m and n the number of bullets that can be fired per second then rate of change of momentum will be

$$\frac{\Delta p}{\Delta t} = mnv \tag{1}$$

$$\therefore \quad \frac{\Delta p}{\Delta t} = F = mnv \tag{2}$$

$$n = \frac{F}{mv} = \frac{150}{(0.1)(1000)} = 1.5/\text{s}$$

Thus the number of bullets that can be fired per minute will be $60 \times 1.5 = 90$.

2.56 If v is the velocity with which a particle of mass m falls on the balance pan, momentum before impact is mv and after impact $-mv$ so that

$$\Delta p = -mv - mv = -2mv \tag{1}$$

If height of fall is h then

$$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 1.6} = 5.6\text{ m/s} \tag{2}$$

If n particles fall per second, the force exerted on the pan is

$$F = +2mnv = (2)(0.1)\left(\frac{441}{60}\right)(5.6) = 8.232\,\text{N}$$
$$= \frac{8.232}{9.8}\text{kg wt} = 0.84\,\text{kg wt}$$

2.57 In this case, the particles will stick to the pan. Therefore the scale reading will increase due to the weight of the particles that get accumulated in the pan. For complete inelastic collision $\Delta p = mv$ as the final momentum is zero. Net force on the scale = weight of the particle + force of impact. At time t, scale reading (in newtons)

$$= mngt + mn\sqrt{2gh}$$
$$= mng\left[t + \sqrt{\frac{2h}{g}}\right]$$

Scale reading in kg wt $= mn\left[t + \sqrt{\frac{2h}{g}}\right]$

2.58 Let a sphere of mass m_1 travelling with velocity u_1 collide with the second sphere of mass m_2 at rest, with their centres in straight line. After the collision let the final velocities be v_1 and v_2, respectively, for m_1 and m_2. By definition the coefficient of restitution e is given by the ratio

$$e = \frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}} = \frac{v_2 - v_1}{v_1} \tag{1}$$

Momentum conservation requires that total momentum before collision = total momentum after collision:

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \tag{2}$$

Eliminating v_2 between (1) and (2),

$$v_1 = \frac{(m_1 - em_2)u_1}{m_1 + m_2} \tag{3}$$

$$v_2 = \frac{m_1(1+e)u_1}{m_1 + m_2} \tag{4}$$

(i) Putting $u_1 = u, \quad m_1 = m$ and $m_2 = \frac{m}{2}$

$$v_1 = \frac{u}{3}(2 - e) \tag{5}$$

$$v_2 = \frac{2u}{3}(1 + e) \tag{6}$$

Total energy after the collision

$$K' = K_1 + K_2 = \frac{1}{2}mv_1^2 + \frac{1}{2}\left(\frac{m}{2}\right)v_2^2 \tag{7}$$

Using (5) and (6) in (7) and simplifying

$$K' = \frac{mu^2}{6}(2 + e^2) \tag{8}$$

(ii) Kinetic energy lost during the collision

$$\Delta K = K_0 - K' = \frac{1}{2}mu^2 - \frac{mu^2}{6}(2 + e^2) = \frac{mu^2}{6}(1 - e^2)$$

2.59 **(a)** Distance traversed by the car before it falls off, $s = 18 - 2 = 16\,\text{m}$:

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 16}{4}} = 2\sqrt{2}\,\text{s}$$

(b) By Newton's third law, the force exerted by the car is equal to that by boat + car

$$(M + m)a_B = ma$$

where $M = 8000\,\text{kg}$, $m = 1200$, $a = 4\,\text{m/s}^2$
The acceleration of the boat $a_B = \frac{ma}{M+m} = 0.26\,\text{m/s}^2$
The distance travelled by the boat in the opposite direction

$$s_B = \frac{1}{2}a_B t^2 = \frac{1}{2} \times 0.26 \times \left(2\sqrt{2}\right)^2 = 104\,\text{m}$$

(c) Momentum conservation gives

$$mv_c = (M + m)v_B$$

$$\frac{v_B}{v_C} = \frac{m}{M + m} = \frac{1200}{8000 + 1200} = 0.13$$

which is independent of the car's acceleration.

2.3.5 Variable Mass

2.60 **(a)** Resultant force on rocket = (upward thrust on rocket) − (weight of rocket)

$$\therefore \quad m\frac{dv}{dt} = -v_r\frac{dm}{dt} - g \tag{1}$$

Setting $\alpha = -\dfrac{\mathrm{d}m}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 0$, minimum exhaust velocity

$$v_\mathrm{r} = \frac{g}{\alpha}$$

(b) Dividing (1) by m

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v_\mathrm{r}}{m}\frac{\mathrm{d}m}{\mathrm{d}t} - g \tag{2}$$

$$\therefore \quad \mathrm{d}v = -v_\mathrm{r}\frac{\mathrm{d}m}{m} - g\,\mathrm{d}t$$

Assuming that v_r and g remain constant, and at $t = 0$, $v = 0$ and $m = m_0$,

$$\int_0^{v_\mathrm{B}} \mathrm{d}v = -v_\mathrm{r}\int_{m_0}^{m_\mathrm{B}} \frac{\mathrm{d}m}{m} - g\int_0^t \mathrm{d}t$$

$$\therefore \quad v_\mathrm{B} = -v_\mathrm{r}\ln\left(\frac{m_0}{m_\mathrm{B}}\right) - gt \tag{3}$$

where m_0 is the initial mass of the system and m_B the mass at burn-out velocity v_B

(c) Setting $g = 0$ in (2)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v_\mathrm{r}}{m}\frac{\mathrm{d}m}{\mathrm{d}t} \tag{4}$$

$$-\frac{\mathrm{d}m}{m} = \alpha = \text{Positive constant}$$

$$\mathrm{d}m = -\alpha\mathrm{d}t$$

$$\therefore \quad \int \mathrm{d}m = -\alpha\int \mathrm{d}t + C$$

where C is the constant of integration

$$m = -\alpha t + C$$

When $t = 0$, $m = m_0$. Therefore, $C = m_0$

$$m(t) = m_0 - \alpha t \tag{5}$$

Using (5) in (4)

$$\frac{dv}{dt} = -\frac{\alpha\, v_r}{m_o - \alpha t}$$

$$dv = \frac{(v_r\alpha/m_0)dt}{1 - \dfrac{\alpha}{m_0}t}$$

Integrating between $v = 0$ and v

$$v = -v_r \ln\left(1 - \frac{\alpha t}{m_0}\right) \tag{6}$$

Writing $\left(1 - \dfrac{\alpha t}{m_0}\right) = \dfrac{m}{m_0}$ in (6) with the aid of (5), the rocket equation simplifies to

$$v = -v_r \ln\frac{m}{m_0}$$

$$\text{or } m = m_0 e^{-v/v_r} \tag{7}$$

(d) Time taken for the rocket to reach the burn-out velocity is given by (5):

$$t = t_0 = \frac{m_0 - m}{\alpha} \tag{8}$$

2.61 $a = 0.5\, g = \dfrac{v_r}{m}\dfrac{dm}{dt} - g$

$$\frac{dm}{dt} = 1.5\frac{mg}{v_r} = \frac{1.5 \times 10^6 \times 9.8}{2000} = 7350\,\text{kg/s}$$

2.62 **(a)** Rocket thrust $= v_r\dfrac{dm}{dt} = 55 \times 10^3 \times 1290 = 71 \times 10^6$ N.

(b) Net acceleration $a = \dfrac{v_r}{m}\dfrac{dm}{dt} - g = \dfrac{71 \times 10^6}{2.72 \times 10^6} - 9.8 = 16.3\,\text{m/s}^2$.

(c) Time to reach the burn-out velocity $t = \dfrac{m_0 - m_B}{\alpha}$

$$= \frac{2.72 \times 10^6 - 2.52 \times 10^6}{1290} = 155\,\text{s}.$$

(d) Burn-out velocity $v_B = v_i + v_r \ln\dfrac{m_0}{m_\beta} - gt$

$$= 0 + 55,000 \ln\frac{2.72 \times 10^6}{2.52 \times 10^6} - (9.8 \times 155) = 2714\,\text{m/s} = 2.7\,\text{km/s}$$

2.63 **(a)** Weight of the rocket

$$M_0 g = 5000 \times 9.8 = 49,000\,\text{N}$$

let x kg of gas be ejected per second. Then

$$x v_e = M_0 g$$
$$\therefore \quad x = \frac{M_0 g}{v_e} = \frac{49,000}{1000} = 49\,\text{kg/s}$$

(b) Upward acceleration required, $a = 2g$. Upward thrust required

$$F = M_0 a = (M_0)(2g) = 2\,M_0 g$$

Weight of the rocket $W = M_0 g$

Total force required $= F + W = 2\,M_0 g + M_0 g = 3\,M_0 g$

Let x' kg gas be ejected per second with $v_e = 1000\,\text{m/s}$

$$1000x' = 147,000 \quad x' = 147\,\text{kg/s}$$

2.64 At any time, the total kinetic energy of the system is

$$K = \frac{1}{2}(\mu L)\left(\frac{dy}{dt}\right)^2 \tag{1}$$

Let the potential energy at the surface of the table be zero. The potential energy of the portion of the rope hanging down is

$$U = -(\mu y)g\left(\frac{y}{2}\right) = \frac{1}{2}\mu\ g\ y^2 \tag{2}$$

Total mechanical energy

$$E = K + U = \text{constant}$$
$$\frac{1}{2}\mu L\left(\frac{dy}{dt}\right)^2 - \frac{1}{2}\mu\ g\ y^2 = \text{constant}$$

Differentiating with respect to time

$$\frac{1}{2}\mu\ L2\frac{d^2y}{dt^2}\frac{dy}{dt} - -\frac{1}{2}\mu g 2y\frac{dy}{dt} = 0$$

Cancelling the common factors,

$$\frac{d^2y}{dt^2} - \frac{g}{L}y = 0 \tag{3}$$

Calling $\beta^2 = g/L$, (3) becomes

$$\frac{d^2y}{dt^2} - \beta^2 y = 0 \tag{4}$$

which has the solution

$$y = Ce^{\beta t} + De^{-\beta t} \tag{5}$$

where C and D are constants. When $t = 0$, $y = y_0$

$$\therefore \quad y_0 = C + D \tag{6}$$

Further $\dfrac{dy}{dt} = \beta\left(Ce^{\beta t} - De^{-\beta t}\right)$

When $t = 0$, $\dfrac{dy}{dt} = 0$

$$\therefore \quad 0 = C - D$$

$$\therefore \quad C = D = \frac{y_0}{2} \tag{7}$$

Using (7) in (5)

$$y = \frac{y_0}{2}\left(e^{\beta t} + e^{-\beta t}\right) \tag{8}$$

Thus the complete solution is

$$y = y_0 \cosh(\beta t) \tag{9}$$

Note that initially both the terms in the parenthesis of (8) are important. As t increases, the second term becomes vanishingly small and the first term alone dominates. Thus y (length of the rope hanging down) increases exponentially with time.
From (3) the acceleration

$$\frac{d^2y}{dt^2} = \frac{g}{L}y$$

Thus acceleration continuously increases with increasing value of y. This then is the case of non-uniform acceleration.

2.65 Consider the equation for the variable mass

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} = F = 0 \tag{1}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = k \tag{2}$$

$$\therefore \quad m\frac{\mathrm{d}v}{\mathrm{d}t} + kv = 0 \tag{3}$$

Integrating (2)

$$m = \int \mathrm{d}m = \int k\mathrm{d}t = kt + C_1$$

where $C_1 =$ constant.

$$\text{At} \quad t = 0, m = W$$

$$\therefore \quad m = kt + W \tag{4}$$

Using (4) in (3)

$$\frac{\mathrm{d}v}{v} = -\frac{k}{m}\mathrm{d}t = -\frac{k\mathrm{d}t}{kt + W} \tag{5}$$

$$\therefore \quad \int \frac{\mathrm{d}v}{v} = -\int \frac{k\mathrm{d}t}{kt + W}$$

$$\therefore \quad \ln \ v = -\ln(kt + W) + C_2$$

where $C_2 =$ constant.

$$\text{At} \quad t = 0, v = v_0, C_2 = \ln v_0 + \ln W$$

$$\therefore \quad \ln\left(\frac{v_0}{v}\right) = \ln\left(1 + \frac{kt}{W}\right)$$

$$\therefore \quad \frac{v_0}{v} = 1 + \frac{kt}{W}$$

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{v_0}{1 + \frac{kt}{W}}$$

The distance travelled in time t

$$S = \int_0^s \mathrm{d}s = v_0 \int_0^t \frac{\mathrm{d}t}{1 + \frac{kt}{W}} = \frac{Wv_0}{k} \ln\left(1 + \frac{kt}{W}\right)$$

2.66 The pressure on the table consists of two parts:

(a) The weight of the coil on the table producing the pressure and
(b) the destruction of momentum producing the pressure.

First consider part (b).
Let a length x be coiled up on the table. Since the chain is falling freely under gravity, the velocity of the chain will be $\sqrt{2gx}$. In a small time interval δt, the length which reaches the table is $\delta t\sqrt{2gx}$.
$\therefore$ The momentum destroyed in time δt is

$$\delta p = \delta t \frac{M}{L}\sqrt{2gx}\sqrt{2gx} = \delta t \frac{M}{L} 2gx$$

$\therefore$ The rate of destruction of momentum is

$$\frac{\delta p}{\delta t} = \frac{M}{L} 2gx$$

Pressure due to part (a) will be $\frac{Mg}{L}x$
$\therefore$ Total pressure on the table $= \frac{M}{L}2gx + \frac{Mgx}{L} = \frac{3Mgx}{L} =$ three times the weight of the coil on the table.

2.67 Measuring x vertically down, the equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}x}{\mathrm{d}t}\right) = mg \tag{1}$$

where m is the mass of the rain drop after time t and x the distance through which the drop has fallen. If ρ is the density and r the radius after time t:

$$m = \frac{4}{3}\pi r^3 \rho \tag{2}$$

$$\therefore \quad \frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi\rho r^2 \frac{\mathrm{d}r}{\mathrm{d}t} \tag{3}$$

By problem $\frac{\mathrm{d}m}{\mathrm{d}t} = k\rho 4\pi r^2$ (4)

Comparing (3) and (4) $\frac{\mathrm{d}r}{\mathrm{d}t} = k$ (5)

Integrating $\quad r = kt + C_1$ (6)

where $C_1 =$ constant.

At $t = 0, r = R$

$\therefore \quad C_1 = R$

$$\therefore \quad r = kt + R \tag{7}$$

Using (2) and (7) in (1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{4}{3}\pi\rho(R+kt)^3\frac{\mathrm{d}x}{\mathrm{d}t}\right\} = \frac{4}{3}\pi\rho(R+kt)^3 g$$

Integrating $(R+kt)^3\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{(R+kt)^4 g}{4k} + C_2$

But $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$. $C_2 = -\dfrac{R^4}{4k}g$

The velocity after time t is therefore

$$v(t) = \frac{g}{4k}\left\{a + kt - \frac{R^4}{(R+kt)^3}\right\}$$

Chapter 3
Rotational Kinematics

Abstract *Chapter 3* is devoted to rotational motion on horizontal and vertical planes and on loop-the-loop.

3.1 Basic Concepts and Formulae

If s be the length of the arc and r the radius of the circle, the angle in radians is given by

$$\theta = s/r \tag{3.1}$$

If the angular displacement $\Delta\theta = \theta_2 - \theta_1$ in the time interval $\Delta t = t_2 - t_1$, then the average angular velocity

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \tag{3.2}$$

The instantaneous angular velocity ω is defined as the limiting value of the ratio as Δt approaches zero.

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{3.3}$$

In case the angular speed is not constant a particle would undergo an angular acceleration α. If ω_1 and ω_2 are the angular speeds at time t_1 and t_2, respectively, then the average acceleration of the particle is defined as

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \tag{3.4}$$

The instantaneous angular acceleration is the limiting value of the ratio as Δt approaches zero.

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{3.5}$$

Rotation with Constant Angular Acceleration

$$\omega = \omega_0 + \alpha t \tag{3.6}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \tag{3.7}$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \tag{3.8}$$

$$\theta = \frac{(\omega_0 + \omega)t}{2} \tag{3.9}$$

Linear and angular variables for circular motion, scalar form

$$s = r\theta \tag{3.10}$$

$$v = \omega t \tag{3.11}$$

$$a_{\mathrm{T}} = \alpha r \tag{3.12}$$

where a_{T} is the tangential component of acceleration. The radial component of acceleration is

$$a_{\mathrm{R}} = v^2/R = \omega^2 R \tag{3.13}$$

Total acceleration

$$a = \sqrt{a_{\mathrm{T}}^2 + a_{\mathrm{R}}^2} \tag{3.14}$$

Vector form:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{3.15}$$

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{v} \tag{3.16}$$

Motion in a Horizontal Plane

Typical problems are as follows:

(i) A coin is placed at a distance r from the centre of a gramophone record rotating with angular frequency $\omega = 2\pi f$. Find the maximum frequency for which the coin will not slip if μ is the coefficient of friction.
This problem is solved by equating the centripetal force to the frictional force.

(ii) An object of mass m attached to a string is whirled around in a horizontal circle of radius r with a constant speed v. Find the tension in the string.
The problem is solved by equating the centripetal force to the tension in the string.
$T = mv^2/r$

(iii) An elastic cord of length L_0 and elastic constant k is attached to an object of mass m. If it is swung around with a constant frequency f, find the new length L.
Solution: Equate the centripetal force to the stretching force

$$m\omega^2 r = m(2\pi f)^2 L = (L - L_0)k$$

Solve for k.

(iv) Find the difference in the level of the bob of a conical pendulum when the number of steady revolutions per second is increased from n_1 to n_2.
Solution: A conical pendulum is a simple pendulum which is allowed to execute rotations about the vertical axis. Equating the horizontal and vertical components of the tension T in the thread

$$T \sin\alpha = m\omega^2 R$$
$$T \cos\alpha = mg$$

whence $\tan\alpha = R/H = \omega^2 R/g$

$$\text{or } \omega = \sqrt{\frac{R}{H}}$$

$$\therefore\ \Delta H = H_1 - H_2 = g\left[\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right]$$

Substitute $\omega_1 = 2\pi n_1$ and $\omega_2 = 2\pi n_2$.

(v) Find the maximum speed of a vehicle that can safely negotiate a circular curve on a banked road.

$$\textit{Solution}:\ v_{\text{max}} = \sqrt{gr\tan\theta}$$

where r is the radius of curvature of the curve.

(vi) Find the condition that a carriage speeding with v negotiating a circular curve of radius r on a level road may not overturn. Assume that a is half of the distance between the wheels and h is the height of the centre of gravity (CG) of the carriage above the ground.
Solution: The centripetal force mv^2/r produces a torque on the inner rear wheel tending to overturn the vehicle. This is countered by an opposite torque caused by the weight of the carriage acting vertically down through the centre of gravity. The condition for the maximum safe speed is given by equating these two torques:

$$\frac{mv_{\text{max}}^2}{r}h = mg\,a$$

$$\text{or } v_{\text{max}} = \sqrt{\frac{gra}{h}}$$

Motion in a Vertical Plane

Typical problems are of the following type:

(i) An object of mass m tied to a string is whirled in a vertical plane such that at the top (A) of the circular path its speed is v_A and at the bottom (B) it is v_B. Calculate v_A and v_B given the tension $T_B = xT_A$, where x is a number.
Solution: At the top the weight acts down while the centrifugal force acts up. Therefore,

$$T_A = \frac{mv_A^2}{r} - mg$$

while at the bottom both weight and centrifugal force act downwards. Therefore,

$$T_B = \frac{mv_B^2}{r} + mg$$

We get one another equation from the conservation of mechanical energy:

$$mg2r = ½mv_B^2 - ½mv_A^2$$

Finally, by problem

$$T_B = xT_A$$

The four equations can be solved to permit the determination of v_A and v_B.

(ii) The bob of a simple pendulum of length L is drawn on one side such that it makes an angle θ with the vertical passing through the equilibrium position. If the bob is released from rest it passes through the equilibrium position with velocity v. Find v.
Here we use the principle gain in kinetic energy = loss in potential energy

$$\frac{1}{2}mv^2 = mgL(1 - \cos\theta)$$

whence $v = \sqrt{2gL(1 - \cos\theta)}$

(iii) A particle of mass m is placed at A, the highest point of a smooth sphere of radius R with the centre at O. If it is gently pushed, it will slide down along the arc of a great circle and leave the surface at B, at depth h below A, Fig. 3.16. Determine the position where the particle leaves the sphere.
Here we balance the radial component of g at B with the centripetal force.

$$mg\cos\theta = mv^2/R$$

Energy conservation gives another equation:

$$mgh = ½mv^2$$

Eliminating v between the two equations and noting that

$$\cos\theta = 1 - h/R$$

we find $h = R/3$.

(iv) A motorcyclist goes around in a vertical circle inside a spherical cage. Find the minimum speed at the top so that he may successfully complete the circular ride.
Here we equate the reaction on the cage to the total weight of the rider plus motorcycle

$$mg = mv^2/R$$
$$\text{or } v = \sqrt{gR}$$

(v) *Loop-the-Loop* is a track which consists of a frictionless slide connected to a vertical loop of radius R, Fig. 3.1. Let a particle start at a height h on the slide and acquire a velocity v at the bottom of the loop.
If $v < \sqrt{2gR}$, the particle will not be able to climb up beyond the point B. It will oscillate in the lower semicircle about the point D.

$$\text{If } \sqrt{2gR} < v < \sqrt{5gR}$$

the particle will be able to climb up the arc BC and leave at some point E and describe a parabolic path. If $v > \sqrt{5gR}$, the particle will be able to execute a complete circle. This corresponds to a height $h = 2.5R$.

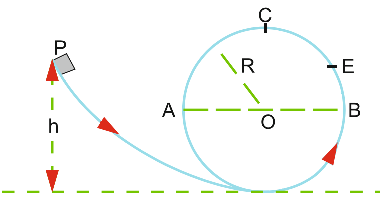


Fig. 3.1 Loop-the-loop

3.2 Problems

3.2.1 Motion in a Horizontal Plane

3.1 Show that a particle with coordinates $x = a\cos t$, $y = a\sin t$ and $z = t$ traces a path in time which is a helix.

[Adapted from Hyderabad Central University 1988]

3.2 A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration a varies with time t as $a = k^2rt^2$, where k is a constant. Show that the power delivered to the particle by the forces acting on it is $mk^4r^2t^5/3$

[Adapted from Indian Institute of Technology 1994]

3.3 A particle is moving in a plane with constant radial velocity of magnitude $\dot{r} = 5\,\mathrm{m/s}$ and a constant angular velocity of magnitude $\dot{\theta} = 4\,\mathrm{rad/s}$. Determine the magnitude of the velocity when the particle is 3 m from the origin.

3.4 A point moves along a circle of radius 40 cm with a constant tangential acceleration of $10\,\mathrm{cm/s^2}$. What time is needed after the motion begins for the normal acceleration of the point to be equal to the tangential acceleration?

3.5 A point moves along a circle of radius 4 cm. The distance x is related to time t by $x = ct^3$, where $c = 0.3\,\mathrm{cm/s^3}$. Find the normal and tangential acceleration of the point at the instant when its linear velocity is $v = 0.4\,\mathrm{m/s}$.

3.6 **(a)** Using the unit vectors $\hat{i}$ and $\hat{j}$ write down an expression for the position vector in the polar form. **(b)** Show that the acceleration is directed towards the centre of the circular motion.

3.7 Find the angular acceleration of a wheel if the vector of the total acceleration of a point on the rim forms an angle 30° with the direction of linear velocity of the point in 1.0 s after uniformly accelerated motion begins.

3.8 A wheel rotates with a constant angular acceleration $\alpha = 3\,\mathrm{rad/s^2}$. At time $t = 1.0\,\mathrm{s}$ after the motion begins the total acceleration of the wheel becomes $a = 12\sqrt{10}\,\mathrm{cm/s^2}$. Determine the radius of the wheel.

3.9 A car travels around a horizontal bend of radius R at constant speed V.

(i) If the road surface has a coefficient of friction μ_s, what is the maximum speed, V_{max}, at which the car can travel without sliding?

(ii) Given $\mu_s = 0.85$ and $R = 150\,\mathrm{m}$, what is V_{max}?

(iii) What is the magnitude and direction of the car's acceleration at this speed?

(iv) If $\mu_s = 0$, at what angle would the bend need to be banked in order for the car to still be able to round it at the same maximum speed found in part (ii)?

[University of Durham 2000]

3.10 The conical pendulum consists of a bob of mass m attached to the end of an inflexible light string tied to a fixed point O and swung around so that it describes a circle in a horizontal plane; while revolving the string generates a conical surface around the vertical axis ON, the height of the cone being ON $= H$, the projection of OP on the vertical axis (Fig. 3.2). Show that the angular velocity of the bob is given by $\omega = \sqrt{g/H}$, where g is the acceleration due to gravity.

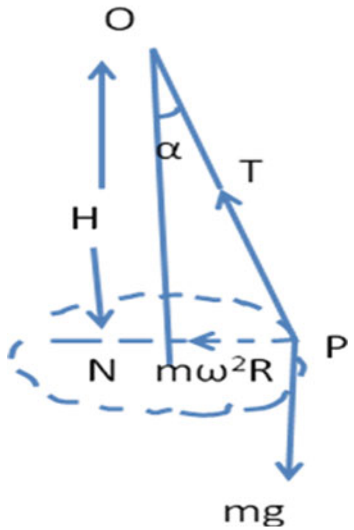


Fig. 3.2 Conical pendulum

3.11 If the number of steady revolutions per minute of a conical pendulum is increased from 70 to 80, what would be the difference in the level of the bob?

3.12 A central wheel can rotate about its central axis, which is vertical. From a point on the rim hangs a simple pendulum. When the wheel is caused to rotate uniformly, the angle of inclination of the pendulum to the vertical is θ_0. If the radius of the wheel is R cm and the length of the pendulum is 1 cm, obtain an expression for the number of rotations of the wheel per second.

[University of Newcastle]

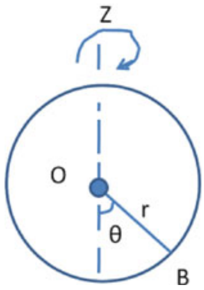
3.13 A coin is placed at a distance r from the centre of a gramophone record rotating with angular frequency $\omega = 2\pi f$. Find the maximum frequency for which the coin will not slip if μ is the coefficient of friction.

3.14 A particle of mass m is attached to a spring of initial length L_0 and spring constant k and rotated in a horizontal plane with an angular velocity ω. What is the new length of the spring and the tension in the spring?

3.15 A hollow cylinder drum of radius r is placed with its axis vertical. It is rotated about an axis passing through its centre and perpendicular to the face and a coin is placed on the inside surface of the drum. If the coefficient of friction is μ, what is the frequency of rotation so that the coin does not fall down?

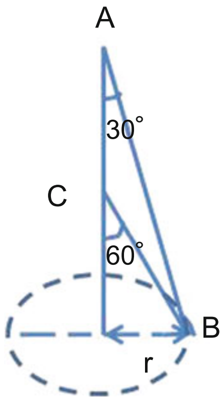
3.16 A bead B is threaded on a smooth circular wire frame of radius r, the radius vector $\vec{r}$ making an angle θ with the negative z-axis (see Fig. 3.3). If the frame is rotated with angular velocity ω about the z-axis then show that the bead will be in equilibrium if $\omega = \sqrt{\dfrac{g}{r\cos\theta}}$.

Fig. 3.3



3.17 A wire bent in the form ABC passes through a ring B as in Fig. 3.4. The ring rotates with constant speed in a horizontal circle of radius r. Show that the speed of rotation is $\sqrt{gr}$ if the wires are to maintain the form.

Fig. 3.4



3.18 A small cube placed on the inside of a funnel rotates about a vertical axis at a constant rate of f rev/s. The wall of the funnel makes an angle θ with the horizontal (Fig. 3.5). If the coefficient of static friction is μ and the centre of the cube is at a distance r from the axis of rotation, show that the largest frequency for which the block will not move with respect to the funnel is

$$f_{\max} = \frac{1}{2\pi}\sqrt{\frac{g(\sin\theta + \mu\cos\theta)}{r(\cos\theta - \mu\sin\theta)}}$$

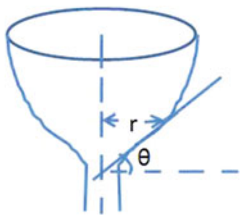


Fig. 3.5

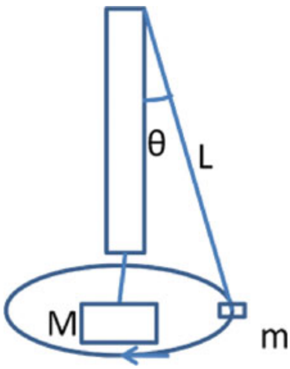
3.19 In prob. (3.18), show that the minimum frequency for which the block will not move with respect to the funnel will be

$$f_{\min} = \frac{1}{2\pi}\sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{r(\cos\theta + \mu\sin\theta)}}$$

3.20 A large mass M and a small mass m hang at the two ends of a string that passes through a smooth tube as in Fig. 3.6. The mass m moves around in a circular path which lies in a horizontal plane. The length of the string from the mass m to the top of the tube is L, and θ is the angle this length makes with the vertical. What should be the frequency of rotation of the mass m so that the mass M remains stationary?

[Indian Institute of Technology 1978]

Fig. 3.6



3.21 An object is being weighed on a spring balance going around a curve of radius 100 m at a speed of 7 m/s. The object has a weight of 50 kg wt. What reading is registered on the spring balance?

3.22 A railway carriage has its centre of gravity at a height of 1 m above the rails, which are 1.5 m apart. Find the maximum safe speed at which it could travel round the unbanked curve of radius 100 m.

3.23 A curve on a highway has a radius of curvature r. The curved road is banked at θ with the horizontal. If the coefficient of static friction is μ,

(a) Obtain an expression for the maximum speed v with which a car can go over the curve without skidding.
(b) Find v if $r = 100$ m, $\theta = 30°$, $g = 9.8\,\text{m/s}^2$, $\mu = 0.25$

3.24 Determine the linear velocity of rotation of points on the earth's surface at latitude of 60°.

3.25 With what speed an aeroplane on the equator must fly towards west so that the passenger in the plane may see the sun motionless?

3.2.2 Motion in a Vertical Plane

3.26 A particle is placed at the highest point of a smooth sphere of radius R and is given an infinitesimal displacement. At what point will it leave the sphere?
[University of Cambridge]

3.27 A small sphere is attached to a fixed point by a string of length 30 cm and whirls round in a vertical circle under the action of gravity at such a speed that the tension in the string when the sphere is at the lowest point is three times the tension when the sphere is at its highest point. Find the speed of the sphere at the highest point.
[University of Cambridge]

3.28 A light rigid rod of length L has a mass m attached to its end, forming a simple pendulum. The pendulum is put in the horizontal position and released from rest. Show that the tension in the suspension will be equal to the magnitude of weight at an angle $\theta = \cos^{-1}(1/3)$ with the vertical.

3.29 In a hollow sphere of diameter 20 m in a circus, a motorcyclist rides with sufficient speed in the vertical plane to prevent him from sliding down. If the coefficient of friction is 0.8, find the minimum speed of the motorcyclist.

3.30 The bob of a pendulum of mass m and length L is displaced 90° from the vertical and gently released. What should be the minimum strength of the string in order that it may not break upon passing through the lowest point?

3.31 The bob of a simple pendulum of length L is deflected through a small arc s from the equilibrium position and released. Show that when it passes through the equilibrium position its velocity will be $s\sqrt{g/L}$, where g is the acceleration due to gravity.

3.32 A simple pendulum of length 1.0 metre with a bob of mass m swings with an angular amplitude of 60°. What would be the tension in the string when its angular displacement is 45°?

3.33 The bob of a pendulum is displaced through an angle θ with the vertical line and is gently released so that it begins to swing in a vertical circle. When it passes through the lowest point, the string experiences a tension equal to double the weight of the bob. Determine θ.

3.2.3 Loop-the-Loop

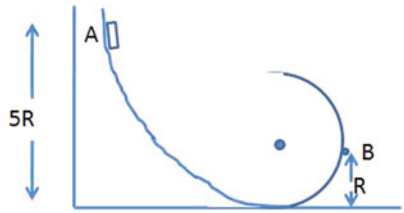
3.34 The bob of a simple pendulum of length 1.0 m has a velocity of 6 m/s when it is at the lowest point. At what height above the centre of the vertical circle will the bob leave the path?

3.35 A block of 2 g when released on an inclined plane describes a circle of radius 12 cm in the vertical plane on reaching the bottom. What is the minimum height of the incline?

3.36 A particle slides down an incline from rest and enters the loop-the-loop. If the particle starts from a point that is level with the highest point on the circular track then find the point where the particle leaves the circular groove above the lowest point.

3.37 A small block of mass m slides along the frictionless loop-the-loop track as in Fig. 3.7. If it starts at A at height $h = 5R$ from the bottom of the track then show that the resultant force acting on the track at B at height R will be $\sqrt{65}$ mg.

Fig. 3.7



3.38 In prob. (3.37), the block is released from a height h above the bottom of the loop such that the force it exerts against the track at the top of the loop is equal to its weight. Show that $h = 3R$.

3.39 A particle of mass m is moving in a vertical circle of radius R. When m is at the lowest position, its speed is $0.8944\sqrt{5gR}$. The particle will move up the track to some point p at which it will lose contact with the track and travel along a path shown by the dotted line (Fig. 3.8). Show that the angular position of θ will be 30°.

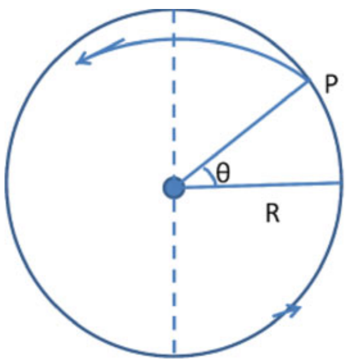


Fig. 3.8

3.40 A block is allowed to slide down a frictionless track freely under gravity. The track ends in a circular loop of radius R. Show that the minimum height from which the block must start is $2.5R$ so that it completes the circular track.

3.41 A nail is located at a certain distance vertically below the point of suspension of a simple pendulum. The pendulum bob is released from a position where the string makes an angle 60° with the vertical. Calculate the distance of the nail from the point of suspension such that the bob will just perform revolutions with the nail as centre. Assume the length of the pendulum to be 1 m.

[Indian Institute of Technology 1975]

3.42 A test tube of mass 10 g closed with a cork of mass 1 g contains some ether. When the test tube is heated, the cork flies out under the pressure of the ether gas. The test tube is suspended by a weightless rigid bar of length 5 cm. What is the minimum velocity with which the cork would fly out of the test tube so that the test tube describes a full vertical circle about the point of suspension? Neglect the mass of ether.

[Indian Institute of Technology 1969]

3.43 A car travels at a constant speed of 14.0 m/s round a level circular bend of radius 45 m. What is the minimum coefficient of static friction between the tyres and the road in order for the car to go round the bend without skidding?

[University of Manchester 2008]

3.3 Solutions

3.3.1 Motion in a Horizontal Plane

3.1 $x = a\cos t \quad (1)$

$y = a\sin t \quad (2)$

$z = t \quad (3)$

Squaring (1) and (2) and adding

$$x^2 + y^2 = a^2(\cos^2 t + \sin^2 t) = a^2$$

which is the equation of a circle.
Since $z = t$, the circular path drifts along the z-axis so that the path is a helix.

3.2 $$a = \frac{\mathrm{d}v}{\mathrm{d}t} = k^2 r t^2$$

$$v = \int \mathrm{d}v = k^2 r \int t^2 \mathrm{d}t = \frac{k^2 r\, t^3}{3}$$

Power, $$P = Fv = mav = mk^2 r\, t^2 \frac{k^2 r\, t^3}{3} = \frac{mk^4 r^2 t^5}{3}$$

3.3 $v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$

$$= \sqrt{5^2 + (3 \times 4)^2} = 13\,\mathrm{m/s}$$

3.4 $a_\mathrm{N} = \omega^2 r = a_\mathrm{T} = 10$

$$\omega = \sqrt{\frac{10}{40}} = 0.5\,\mathrm{rad/s}$$

$$\omega = \omega_0 + \alpha t = 0 + \frac{a_\mathrm{T}\, t}{r}$$

$$t = \frac{\omega r}{a_\mathrm{T}} = \frac{0.5 \times 40}{10} = 2\,\mathrm{s}$$

3.5 $x = ct^3$

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = 3ct^2 = 3 \times 0.3 \times 10^{-2} t^2 = 0.4$$

$$t = \frac{20}{3}\mathrm{s}$$

$$a_\mathrm{N} = \frac{v^2}{r} = \frac{(0.4)^2}{0.04} = 4\,\mathrm{m/s^2}$$

$$a_\mathrm{T} = \frac{\mathrm{d}v}{\mathrm{d}t} = 6ct = 6 \times 0.3 \times 10^{-2} \times \frac{20}{3} = 0.12\,\mathrm{m/s^2}$$

3.6 **(a)** $x = r\cos\theta$

$$y = r\sin\theta$$

$$\vec{r} = \hat{i}x + \hat{j}y$$

$$\theta = \omega t$$

where θ is the angle which the radius vector makes with the x-axis and ω is the angular speed.

$$\vec{r} = \hat{i}(r\ \cos\ \omega t) + \hat{j}(r\ \sin\ \omega t)$$

(b) $\dot{\vec{r}} = -\hat{i}(\omega r\ \sin\ \omega t) + \hat{j}(\omega r\ \cos\ \omega t)$

$$\vec{a} = \ddot{\vec{r}} = -\hat{i}(\omega^2 r\ \cos\ \omega t) - \hat{j}(\omega^2 r\ \sin\ \omega t)$$

$$= -\omega^2 r\,(\hat{i}\cos\ \omega t + \hat{j}\ \sin\ \omega t)$$

$$= -\omega^2(\hat{i}x + \hat{j}y)$$

$$\vec{a} = -\omega^2\vec{r}$$

where we have used the expression for the position vector $\vec{r}$. The last relation shows that by virtue of minus sign $\vec{a}$ is oppositely directed to $\vec{r}$, i.e. $\vec{a}$ is directed radially inwards.

3.7 $a = \sqrt{a_\mathrm{N}^2 + a_\mathrm{T}^2}$ (Fig. 3.9)

$$\frac{a_\mathrm{N}}{a_\mathrm{T}} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$a_\mathrm{N} = \frac{a_\mathrm{T}}{\sqrt{3}} \tag{1}$$

$$a_\mathrm{T} = \alpha R \tag{2}$$

$$\omega = \alpha t \tag{3}$$

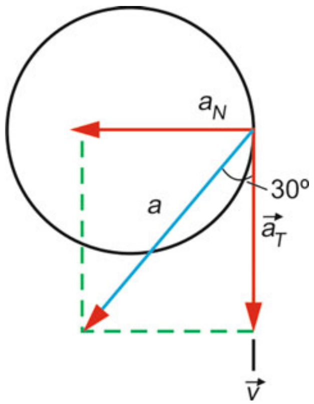
$$\therefore \quad a_\mathrm{N} = \omega^2 R = \frac{a_\mathrm{T}}{\sqrt{3}} = \frac{\alpha\, R}{\sqrt{3}}$$

$$\omega^2 = \frac{\alpha}{\sqrt{3}}$$

$$\alpha^2 t^2 = \frac{\alpha}{\sqrt{3}}$$

$$\alpha = \frac{1}{\sqrt{3}t^2} = \frac{1}{\sqrt{3}.1^2} = 0.577\,\mathrm{rad/s^2}$$

Fig. 3.9



3.8 $a = \sqrt{a_\mathrm{N}^2 + a_\mathrm{T}^2}$

$$12\sqrt{10} = \sqrt{(\omega^2 R)^2 + \alpha^2 R^2} = \sqrt{(\alpha^2 t^2 R)^2 + \alpha^2 R^2}$$

$$= \alpha R\,\sqrt{\alpha^2\, t^4 + 1} = 3R\sqrt{3^2 \times 1^2 + 1}$$

$$R = 4\,\mathrm{cm}$$

3.9 **(i)** Equating the centripetal force to the frictional force

$$\frac{m v_\mathrm{max}^2}{R} = \mu\, m g$$

$$\therefore \quad v_\mathrm{max} = \sqrt{\mu\, g R}$$

(ii) $v_{\max} = \sqrt{0.85 \times 9.8 \times 150} = 35.35\,\text{m/s}$

(iii) $a_N = \dfrac{v^2}{R} = \dfrac{(35.35)^2}{150} = 8.33\,\text{m/s}$ towards the centre of the circle

(iv) $\tan\theta = \dfrac{v^2}{gR} = \dfrac{(35.35)^2}{9.8 \times 150} = 0.85$

$\therefore \quad \theta = 40.36°$

3.10 Equating the horizontal component of the tension to the centripetal force

$$T \sin\alpha = m\omega^2 R \tag{1}$$

Furthermore, the bob has no acceleration in the vertical direction.

$$T \cos\alpha = mg \tag{2}$$

$$\tan\alpha = \frac{R}{H} = \frac{\omega^2 R}{g}$$

$$\therefore \quad \omega = \sqrt{\frac{g}{H}}$$

3.11 Using the results of prob. (3.10), the difference in the level of the bob

$$\Delta H = H_1 - H_2 = g\left[\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right] \tag{1}$$

$$\omega_1 = 2\pi f_1 = \frac{140}{60}\pi \tag{2}$$

$$\omega_2 = 2\pi f_2 = \frac{160}{60}\pi \tag{3}$$

Using (2) and (3) in (1) and $g = 980\,\text{cm}$, $\Delta H = 31.95\,\text{cm}$.

3.12 The centripetal force acting on the bob of the pendulum $= m\omega^2 r$, where r is the distance of the bob from the axis of rotation, Fig. 3.10. For equilibrium, the vertical component of the tension in the string of the pendulum must balance the weight of the bob

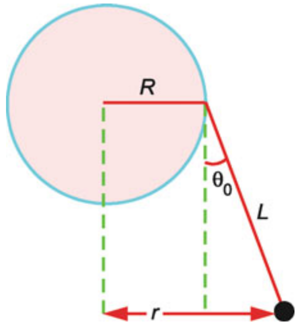
$$\therefore \quad T\cos\theta_0 = m\omega^2 r \tag{1}$$

Further, the horizontal component of the tension in the string must be equal to the centripetal force.

$$\therefore \quad T\sin\theta_0 = m\omega^2 r \tag{2}$$

Dividing (2) by (1)

Fig. 3.10



$$\tan\theta_0 = \frac{\omega^2 r}{g} = \frac{4\pi^2 n^2 r}{g} \tag{3}$$

where n = number of rotations per second. From the geometry of Fig. 3.10,

$$r = R + L\sin\theta_0$$

$$n = \frac{1}{2\pi}\sqrt{\frac{g\ \tan\theta_0}{R + L\ \sin\theta_0}}$$

3.13 The equilibrium condition requires that the centripetal force = the frictional force, $m\omega^2 r = \mu\, mg$

$$\therefore \quad f_{\max} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\mu g}{r}}$$

3.14 Let the spring length be stretched by x. Equating the centripetal force to the spring force

$$m\omega^2(L_0 + x) = kx$$

$$\therefore \quad x = \frac{m\omega^2 L_0}{k - m\omega^2}$$

Therefore, the new length L will be

$$L = L_0 + x = \frac{kL_0}{k - m\omega^2}$$

and the tension in the spring will be

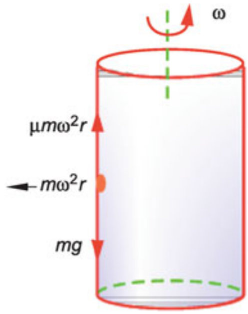
$$m\,\omega^2 L = \frac{m\,\omega^2 k L_0}{k - m\omega^2}$$

3.15 As the drum rotates with angular velocity ω, the normal reaction on the coin acting horizontally would be equal to $m\omega^2 r$, (Fig. 3.11). As the coin tends to slip down under gravity a frictional force would act vertically up.
If the coin is not to fall, the minimum frequency of rotation is given by the condition
Frictional force = weight of the coin

$$\mu m\omega^2 r = mg$$

$$\therefore \quad \omega = \frac{1}{2\pi}\sqrt{\frac{g}{\mu r}}$$

Fig. 3.11



3.16 The bead is to be in equilibrium by the application of three forces, the weight mg acting down, the centrifugal force $m\omega^2 R$ acting horizontally and the normal force acting radially along NO. Balancing the x- and z-components of forces (Fig. 3.12)

$$N \sin\theta = m\omega^2 R$$

$$N \cos\theta = mg$$

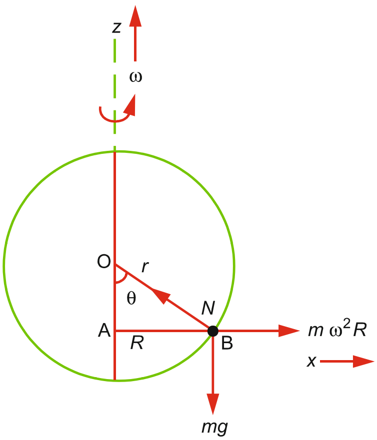


Fig. 3.12

Dividing the two equations

$$\tan\theta = \frac{\omega^2 R}{g} = \frac{\omega^2 r\sin\theta}{g}$$
$$\omega = \sqrt{\frac{g}{r\cos\theta}}$$

3.17 Since the wire is continuous, tension in the parts *AB* and *BC* will be identical. Equating the horizontal and vertical components of forces separately

$$\frac{mv^2}{r} = T\sin 30^\circ + T\sin 60^\circ \tag{1}$$
$$mg = T\cos 30^\circ + T\cos 60^\circ \tag{2}$$

As the right-hand sides of (1) and (2) are identical

$$\frac{mv^2}{r} = mg$$
$$\text{or} \quad v = \sqrt{gr}$$

3.18 Resolve the centripetal force along and normal to the funnel surface, Fig. 3.13. When the funnel rotates with maximum frequency, the cube tends to move up the funnel, and both the weight (mg) and the frictional force (μN) will act down the funnel surface, Fig. 3.13. Now

$$N = mg\cos\theta + m\omega^2 r\sin\theta$$

Taking the upward direction as positive, equation of motion is

$$m\omega^2 r\cos\theta - mg\sin\theta - \mu(mg\cos\theta + m\omega^2 r\sin\theta) = 0$$
$$\therefore \quad f_{\max} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{r}\frac{(\sin\theta + \mu\cos\theta)}{(\cos\theta - \mu\sin\theta)}}$$

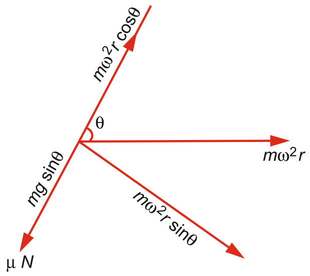


Fig. 3.13

3.19 At the minimum frequency of rotation, the cube tends to go down the surface and therefore the frictional force acts up the funnel. The equation of motion becomes

$$m\omega^2 r\cos\theta - mg\sin\theta + \mu(mg\cos\theta + m\omega^2 r\sin\theta) = 0$$

$$f_{\min} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{r}\frac{(\sin\theta - \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)}}$$

3.20 Tension is provided by the weight Mg

$$T = Mg \tag{1}$$

Three forces, weight (Mg), tension (T) and normal reaction ($m\omega^2 r$), are to be balanced:

$$T\sin\theta = m\omega^2 r \tag{2}$$

$$\text{Further } r = L\sin\theta \tag{3}$$

Combining (1), (2) and (3)

$$\omega^2 = \frac{Mg}{mL}$$

Frequency of rotation

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Mg}{mL}}$$

3.21 The two forces acting at right angles are (i) weight (mg) and (ii) reaction (mv^2/r).

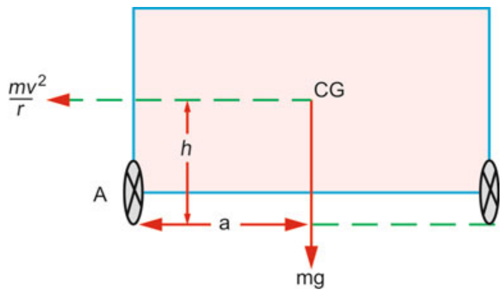
$$F = \sqrt{(mg)^2 + (mv^2/r)^2} = mg\sqrt{1 + (v^2/gr)^2}$$

Using $v = 7\,\text{m/s}$, $g = 9.8\,\text{m/s}^2$, $r = 100\,\text{m}$ and $m = 60\,\text{kg}$,

$$F = 60.075\,\text{kg wt.}$$

3.22 Figure 3.14 shows the rear of the carriage speeding with v, negotiating a circular curve of radius r. 'a' is half of the distance between the wheels and h is the height of the centre of gravity (CG) of the carriage above the ground. The centripetal force mv^2/r produces a counterclockwise torque about the left wheel at A. The weight of the carriage acting vertically down through the

Fig. 3.14



centre of gravity produces a clockwise torque. The condition for the maximum speed $\upsilon_{\max}$ is given by equating these two torques:

$$\frac{m\upsilon_{\max}^2}{r}h = mga$$

$$\text{or} \quad \upsilon_{\max} = \sqrt{\frac{gra}{h}} = \sqrt{\frac{9.8 \times 100 \times 0.75}{1.0}} = 27.11\,\text{m/s}$$

3.23 **(a)** When a vehicle takes a turn on a level road, the necessary centripetal force is provided by the friction between the tyres and the road. However, this results in a lot of wear and tear of tyres. Further, the frictional force may not be large enough to cause a sharp turn on a smooth road.
If the road is constructed so that it is tilted from the horizontal, the road is said to be banked. Figure 3.15 shows the profile of a banked road at an angle θ with the horizontal. The necessary centripetal force is provided by the horizontal component of the normal reaction N and the horizontal component of frictional force.
Three external forces act on the vehicle, and they are not balanced, the weight W, the normal reaction N, and the frictional force. Balancing the horizontal components

$$\frac{m\upsilon^2}{r} = \mu\, mg\cos^2\theta + N\sin\theta$$

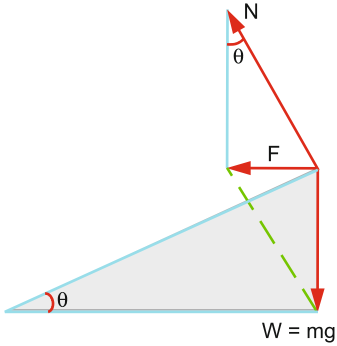
$$\text{or } N\sin\theta = \frac{m\upsilon^2}{r} - \mu\, mg\cos^2\theta \qquad (1)$$

Balancing the vertical components

$$mg = N\cos\theta - \mu mg\cos\theta\sin\theta$$

$$\text{or } N\cos\theta = mg + \mu mg\cos\theta\sin\theta \qquad (2)$$

Fig. 3.15



Dividing (1) by (2)

$$\tan\theta = \frac{v^2/r - \mu\, g \cos^2\theta}{g + \mu\, g\, \cos\theta \sin\theta}$$

$$v_{\max} = \sqrt{gr\,(\mu + \tan\theta)}$$

(b) For $\theta = 30°$, $\mu = 0.25$, $g = 9.8\,\text{m/s}^2$ and $r = 100\,\text{m}$, $v_{\max} = 28.47\,\text{m/s}$.

3.24 At latitude λ the distance r of a point from the axis of rotation will be $r = R\cos\lambda$
where R is the radius of the earth.
The angular velocity, however, is the same as for earth's rotation

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86,400} = 7.27 \times 10^{-5}\,\text{rad/s}$$

The linear velocity

$$v = \omega r = \omega R\cos\lambda = 7.27 \times 10^{-5} \times 6.4 \times 10^{6} \times \cos 60°$$
$$= 232.64\,\text{m/s}$$

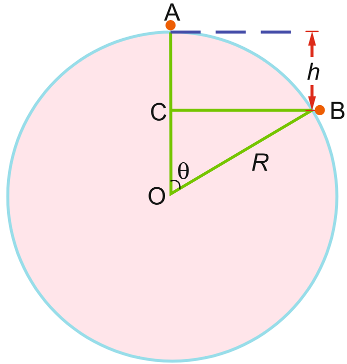
3.25 The speed of the plane must be equal to the linear velocity of a point on the surface of the earth. Suppose the plane is flying close to the earth's surface, $\omega = 7.27 \times 10^{-5}$ rad/s (*see* prob. 3.24)

$$v = \omega R = 7.27 \times 10^{-5} \times 6.4 \times 10^{6} = 465.28\,\text{m/s}$$
$$= 1675\,\text{km/h.}$$

3.3.2 Motion in a Vertical Plane

3.26 Let a particle of mass m be placed at A, the highest point on the sphere of radius R with the centre of O. Let it slide down from rest along the arc of the great circle and leave the surface at B, at depth h below A, Fig. 3.16. Let the radius OB make an angle θ with the vertical line OA. The centripetal

Fig. 3.16



force experienced by the particle at B is mv^2/R, where v is the velocity of the particle at this point. Now the weight mg of the particle acts vertically down so that its component along the radius BO is $mg\cos\theta$. So long as $mg\cos\theta >$ mv^2/R the particle will stick to the surface. The condition that the particle will leave the surface is

$$mg\cos\theta = \frac{mv^2}{R} \tag{1}$$

$$\text{or} \quad \cos\theta = \frac{v^2}{gR} \tag{2}$$

Now, in descending from A to B, the potential energy is converted into kinetic energy

$$mgh = \frac{1}{2}mv^2 \tag{3}$$

$$\text{or} \quad \frac{v^2}{g} = 2h \tag{4}$$

using (4) in (2)

$$\cos\theta = \frac{2h}{R} \tag{5}$$

Drop a perpendicular BC on AO.

$$\text{Now } \cos\theta = \frac{\text{OC}}{\text{OB}} = \frac{R-h}{R} \tag{6}$$

Combining (5) and (6), $\quad h = \dfrac{R}{3}$

Thus the particle will leave the sphere at a point whose vertical distance below the highest point is $\dfrac{R}{3}$.

3.27 At the highest point A the tension T_A acts vertically up, the centrifugal force also acts vertically up but the weight acts vertically down. We can then write

$$T_A = \frac{mv_A^2}{r} - mg \tag{1}$$

where m is the mass of the sphere, v_A is its speed at the point A and r is the radius of the vertical circle.
At the lowest point B both the centrifugal force and the weight act vertically down and both add up to give the tension T_B. If v_B is the speed at B, then we can write

$$T_B = \frac{mv_B^2}{r} + mg \tag{2}$$

By problem

$$T_B = 3T_A \tag{3}$$

Combining (1), (2) and (3), we get

$$v_B^2 = 3v_A^2 - 4gr \tag{4}$$

Conservation of mechanical energy requires that loss in potential energy = gain in kinetic energy. Therefore, in descending from A to B,

$$mg2r = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \tag{5}$$

$$\text{or} \quad v_B^2 = v_A^2 + 4gr \tag{6}$$

From (4) and (6) we get

$$v_A = \sqrt{4gr} = \sqrt{4 \times 980 \times 30} = 343\,\text{cm/s}$$

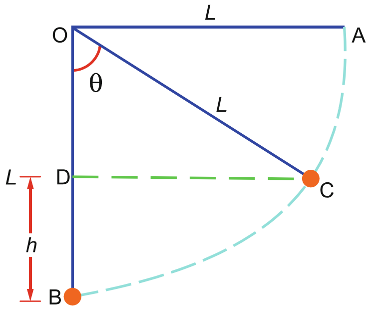
3.28 Measure potential energy from the equilibrium position B, Fig. 3.17. At A the total mechanical energy $E = mgL$ as the pendulum is at rest. As it passes through C let its speed be v. The potential energy will be mgh, where $h =$ BD and CD is perpendicular on the vertical OB. Now

$$h = L - L\cos\theta = L(1 - \cos\theta) \tag{1}$$

Energy conservation gives

$$mgL = mgL\,(1 - \cos\theta) + \frac{1}{2}mv^2 \tag{2}$$

Fig. 3.17



The tension in the string at C will be

$$T = \frac{mv^2}{L} + mg\cos\theta \tag{3}$$

By problem, $T = mg$ (4)

Combining (2), (3) and (4) we get $\cos\theta = \frac{1}{3}$ or $\theta = \cos^{-1}\left(\frac{1}{3}\right)$.

3.29 At the top of the sphere, v is in the horizontal direction and the frictional force acts upwards. The condition that the motorcyclist may not fall is
Friction force = Weight

$$\mu\frac{mv^2}{r} = mg$$

$$v = \sqrt{\frac{gr}{\mu}} = \sqrt{\frac{9.8 \times 10}{0.8}} = 11\,\text{m/s}$$

3.30 At the lowest point A, Fig. 3.18, the tension in the string is

$$T_A = \frac{mv_A^2}{L} + mg \tag{1}$$

where v_A is the velocity at point A.

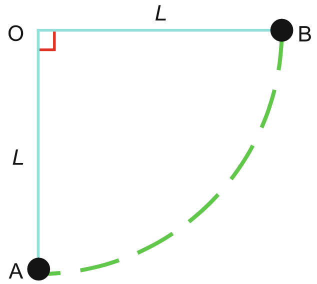


Fig. 3.18

Measure potential energy with respect to A, the equilibrium position. At the point B, at height L the mechanical energy is entirely potential energy as the bob is at rest. As the vertical height is L, the potential energy will be mgL.
When the bob is released, at the point A, the energy is entirely kinetic, potential energy being zero, and is equal to $\frac{1}{2}mv_A^2$.
Conservation of mechanical energy requires that

$$\frac{1}{2}mv_A^2 = mgL$$
$$\text{or} \quad v_A^2 = 2\,gL \tag{2}$$

Using (2) in (1)

$$T_A = 2mg + mg = 3mg$$

Thus the minimum strength of the string that it may not break upon passing through the lowest point is three times the weight of the bob.

3.31 Let the ball be deflected through a small angle θ from the equilibrium position A, Fig. 3.19.

$$\theta = \frac{s}{L} \tag{1}$$

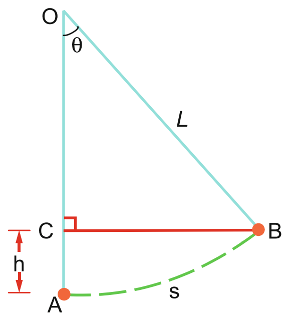


Fig. 3.19

where s is the corresponding arc. Drop a perpendicular BC on AO, so that the height through which the bob is raised is AC $= h$.
Now, $h = \text{AC} = \text{OA} - \text{OC} = L - L\cos\theta = L(1 - \cos\theta)$

$$= L\left[1 - 1 + \frac{\theta^2}{2!} + \cdots\right]$$
$$\therefore \quad h = \frac{L\theta^2}{2} = \frac{s^2}{2L} \tag{2}$$

where we have used (1).

From energy conservation, $mgh = \frac{1}{2}mv^2$

$$\therefore \quad v = \sqrt{2gh} = s\sqrt{\frac{g}{L}}$$

3.32 In coming down from angular displacement of 60° to 45°, loss of potential energy is given by

$$mg(h_1 - h_2) = mgL(1 - \cos 60°) - mgL(1 - \cos 45°)$$
$$= 0.207\, mgL$$

Gain in kinetic energy $= \frac{1}{2}mv^2$

$$\therefore \quad \frac{1}{2}mv^2 = 0.207\, mgL = 0.207\, mg \quad (\because \quad L = 1\text{ m})$$

or $\quad v = 2.014\,\text{m/s}$

The tension in the string would be

$$T = \frac{mv^2}{L} + mg\cos 45°$$
$$= mg\left[\frac{4.056}{gL} + 0.707\right]\text{N} = 1.12\, mg\text{ N}$$

3.33 When the bob is displaced through angle θ, the potential energy is $mgL(1 - \cos\theta)$. At the lowest position the energy is entirely kinetic

$$\frac{1}{2}mv^2 = mg\,L\,(1 - \cos\theta) \tag{1}$$

The tension in the string will be

$$T = mg + \frac{mv^2}{L} = mg + 2mg\,(1 - \cos\theta) \tag{2}$$

where we have used (1)
By problem

$$T = 2mg \tag{3}$$

From (2) and (3) we find $\cos\theta = \frac{1}{2}$ or $\theta = 60°$

3.3.3 Loop-the-Loop

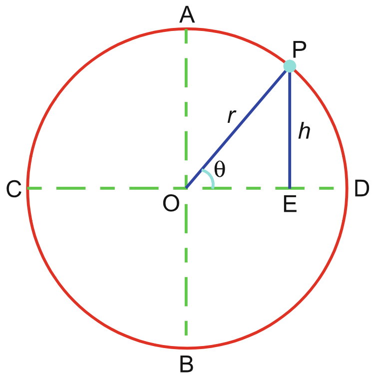
3.34 If the bob of the pendulum has velocity u at B, the bottom of the vertical circle of radius r such that

$$\sqrt{2gr} < u < \sqrt{5gr} \tag{1}$$

then the bob would leave some point P on the are DA (Fig. 3.20). Here

$$\sqrt{2gr} = \sqrt{2 \times 9.8 \times 1} = 4.427\,\text{m/s and}\,\sqrt{5gr} = \sqrt{5 \times 9.8 \times 1} = 7\,\text{m/s}$$

Fig. 3.20



Therefore (1) is satisfied for $u = 6\,\text{m/s}$.
Drop a perpendicular PE on the horizontal CD. Let PE $= h$ and PO make an angle θ with OD. When the bob leaves the point P, the normal reaction must vanish.

$$\frac{mv^2}{r} - mg\sin\theta = 0 \tag{2}$$

Loss in kinetic energy = gain in potential energy

$$\frac{1}{2}m\,u^2 - \frac{1}{2}mv^2 = mg(h + r) \tag{3}$$

$$\sin\theta = \frac{h}{r} \tag{4}$$

Eliminating v^2 between (2) and (3) and using (3), with $u = 6\,\text{m/s}$ and $r = 1.0\,\text{m}$,

$$h = \frac{1}{3}\left[\frac{u^2}{g} - 2r\right] = 0.558\,\text{m}$$

3.35 Let the height of the incline be h. Then the velocity of the block at the bottom of the vertical circle will be $v = \sqrt{2gh}$. Minimum height is given by the condition that $v = \sqrt{5gr}$ which is barely needed for the completion of the loop.

$$\sqrt{2gh} = \sqrt{5gr}$$

$$\text{or } h = \frac{5}{2}r = \frac{5}{2} \times 12 = 30\,\text{cm}$$

3.36 The analysis is similar to that of prob. (2.34). The velocity of the particle at the bottom of the circular groove will be given by

$$v = \sqrt{(2g)(2r)} = \sqrt{4gr} \tag{1}$$

which satisfies the condition

$$\sqrt{2gr} < u < \sqrt{5gr}$$

The particle leaves the circular groove at a height h above the centre of the circle, Fig. 3.20.

$$h = \frac{1}{3}\left[\frac{u^2}{g} - 2r\right] \tag{2}$$

$$\text{But } u^2 = 4gr \qquad (1)$$

$$\therefore \quad h = \frac{2}{3}r$$

Thus, the particle leaves the circular groove at a height of $h + r = \frac{5}{3}r$ above the lowest point.

3.37 Let the velocity at B be v.

Kinetic energy gained = potential energy lost

$$\frac{1}{2}mv^2 = mg(5R - R)$$

$$\therefore \quad m\frac{v^2}{R} = 8\,mg$$

which is the centrifugal force acting on the track horizontally. The weight acts vertically down. Hence the resultant force

$$F = \sqrt{(8\,mg)^2 + (mg)^2} = \sqrt{65}\,mg$$

3.38 Gain in kinetic energy = loss of potential energy

$$\frac{1}{2}mv^2 = mg(h - 2R)$$

$$v^2 = 2g\,(h - 2R) \tag{1}$$

The force exerted on the track at the top

$$F = \frac{mv^2}{R} - mg \tag{2}$$

By problem

$$F = mg \tag{3}$$

$$\therefore \quad \frac{mv^2}{R} - mg = mg$$

$$\text{or} \quad v^2 = 2gR \tag{4}$$

Using (4) in (1) we find $h = 3R$.

3.39 Let the particle velocity at the lowest position be $u = 0.8944\sqrt{5gR}$ and v at point P.
Loss in kinetic energy = gain in potential energy

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mg(R + R\sin\theta)$$

$$\text{or} \quad v^2 = \left(0.8944\sqrt{5gR}\right)^2 - 2gR(1 + \sin\theta) \tag{1}$$

The particle would leave at P (Fig. 3.7) when

$$\frac{mv^2}{R} = mg\sin\theta$$

$$\text{or} \quad v^2 = gR\sin\theta \tag{2}$$

Using (2) in (1) and solving
$\sin\theta = 2/3$ or $\theta = 41.8°$

3.40 Let the minimum height be h. The velocity of the block at the beginning of the circular track will be

$$v = \sqrt{2gh} \tag{1}$$

For completing the circular track

$$v = \sqrt{5gR} \tag{2}$$

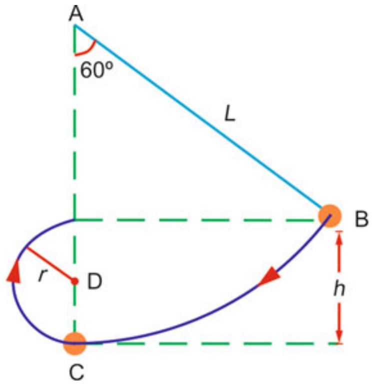
From (1) and (2)

$$h = 2.5R$$

3.41 Let the nail be located at D at distance x vertically below A, the point of suspension, Fig. 3.21. Initially the bob of the pendulum is positioned at B at height h above the equilibrium position C.

$$h = L(1 - \cos\theta) = L(1 - \cos 60^\circ) = \frac{1}{2}L \tag{1}$$

Fig. 3.21



where $L = 1$ m is the length of the pendulum. When the pendulum is released its velocity at C will be

$$v = \sqrt{2gh} = \sqrt{gL} \tag{2}$$

The velocity needed at C to make complete revolution in the vertical circle centred at the nail and radius r is

$$v = \sqrt{5gr} \tag{3}$$

From (2) and (3)

$$r = \frac{1}{5}L \tag{4}$$

Therefore $x = \mathrm{AD} = \mathrm{AC} - \mathrm{DC} = L - \dfrac{L}{5} = 0.8\,L$

$$= 0.8 \times 1\,\mathrm{m} = 80\,\mathrm{cm}$$

3.42 If M and m are the mass of the test tube and cork, respectively, and their velocity V and υ respectively, momentum conservation gives

$$MV = mv \tag{1}$$

$$\text{or} \quad v = \frac{M}{m}V = \frac{10}{1}V = 10\,V \tag{2}$$

Condition for describing a full vertical circle is that the minimum velocity of the test tube should be

$$V = \sqrt{5gr} = \sqrt{5 \times 980 \times 5} = 156.5\,\text{cm/s}$$

Therefore the minimum velocity of the cork which flies out ought to be

$$v = 10\,V = 1565\,\text{cm/s} = 15.65\,\text{m/s}$$

3.43 Equating centripetal force to frictional force

$$\frac{mv^2}{r} = \mu mg$$

$$\mu = \frac{v^2}{gr} = \frac{(14)^2}{9.8 \times 45} = \frac{4}{9}$$

Chapter 4
Rotational Dynamics

Abstract *Chapter 4* is concerned with the moment of inertia and rotational motion on horizontal and inclined planes and Coriolis acceleration.

4.1 Basic Concepts and Formulae

Moment of Inertia/Rotational Inertia (M.I.) or (I)

Table 4.1 Moments of inertia (M.I.) of some regular bodies

Body	Axis	M.I.
Thin rod of length L	Perpendicular to length through centre	$mL^2/12$
	Perpendicular to length at one end	$mL^2/3$
Thin rectangular sheet of sides a and b	Through centre parallel to side b	$ma^2/12$
	Through centre perpendicular to sheet	$m(a^2 + b^2)/12$
Thin hoop	Through centre perpendicular to plane of ring	mr^2
	Through centre along diameter	$mr^2/4$
Thin circular disc	Through centre perpendicular to disc	$mr^2/2$
Solid sphere of radius r	About any diameter	$2mr^2/5$
Thin spherical shell	About any diameter	$2mr^2/3$
Right cone of radius of base r	Along axis of cone	$3mr^2/10$
Circular cylinder of length L and radius R	Through centre perpendicular to axis	$m\left(\frac{R^2}{4} + \frac{L^2}{12}\right)$

Table 4.2 Translational and rotational analogues

Quantity	Translation	Rotation
Displacement	s	θ
Velocity	$v = \mathrm{d}s/\mathrm{d}t$	$\omega = \mathrm{d}\theta/\mathrm{d}t$
Acceleration	$a = \mathrm{d}v/\mathrm{d}t$	$\alpha = \mathrm{d}\omega/\mathrm{d}t$
Mass/inertia	m	I
Momentum	$p = mv$	$J = I\omega$
Impulse	$J = Ft$	$J = \tau t$
Work	$W = Fs$	$W = \tau\theta$

Table 4.2 (continued)

Quantity	Translation	Rotation
Kinetic energy	$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
Power	$P = Fv$	$P = \tau\omega$
Newton's second law	$F = ma$	$\tau = I\alpha$
Equilibrium condition	$\sum \boldsymbol{F}_{\text{ext}} = 0$	$\sum \boldsymbol{\tau}_{\text{ext}} = 0$
Kinematics	$v = u + at$	$\omega = \omega_0 + \alpha t$
	$v^2 = u^2 + 2as$	$\omega^2 = {\omega_0}^2 + 2\alpha\theta$
	$s = ut + \frac{1}{2}at^2$	$\theta = \theta_0 t + \frac{1}{2}\alpha t^2$
	$s = \frac{1}{2}(u + v)t$	$\theta = \frac{1}{2}(\omega_0 + \omega)t$

The Perpendicular Axes Theorem

The sum of the moments of inertia of a plane lamina about any two perpendicular axes in its plane is equal to its moment of inertia about an axis perpendicular to its plane and passing through the point of intersection of the first two axes:

$$I_z = I_x + I_y \tag{4.1}$$

The theorem is valid for plane lamina only.

The Parallel Axes Theorem

The M.I. of a body about any axis is equal to the sum of its M.I. about a parallel axis through the centre of mass and the product of its mass and the square of the distance between the two axes.

Conservation of angular momentum (J) implies

$$J = I_1\omega_1 = I_2\omega_2 \tag{4.2}$$

Motion of a body rolling down an incline of angle θ:

$$a = \frac{g \sin\theta}{1 + \frac{k^2}{r^2}} \tag{4.3}$$

where $I = Mk^2$ and k is known as the radius of gyration.

$$t = \sqrt{\frac{2s}{a}} \tag{4.4}$$

$$K_{\text{total}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}mv^2\left(1 + \frac{k^2}{r^2}\right) \tag{4.5}$$

Table 4.3 k^2/r^2 values for various bodies

	Hollow cylinder	Hollow sphere	Solid cylinder	Solid sphere
k^2/r^2	1	2/3	1/2	2/5

The smaller the value of k^2/r^2, the greater will be the acceleration and smaller the travelling time.

Coriolis Force

The Coriolis force arises because of the motion of the particle in the rotating frame of reference (non-inertial frame) and is given by the term $2m\boldsymbol{\omega} \times v_R$. It vanishes if $v_R = 0$. It is directed at right angles to the axis of rotation, similar to the centrifugal force. Note that the Coriolis force would reverse if the direction of $\boldsymbol{\omega}$ is reversed. However, the direction of the centrifugal force remains unchanged.

(i) *Cyclonic motion* is affected by the Coriolis force resulting from the earth's rotation. The cyclonic motion is found to be mostly in the counterclockwise direction in the northern hemisphere and clockwise direction in the southern hemisphere. The radius of curvature r for mass of air moving north or south is approximately given by

$$r = v/(2\omega \sin \lambda) \tag{4.6}$$

where v is the wind velocity, ω is the earth's angular velocity and λ is the latitude.

(ii) *Free fall* on the rotating earth:
A body in its free fall through a height h in the northern hemisphere undergoes an eastward deviation through a distance

$$d = \frac{1}{8}\omega \cos \lambda \sqrt{\frac{8h^3}{g}} \tag{4.7}$$

(iii) *Foucault's pendulum*:
Foucault's pendulum is a simple pendulum suspended by a long string from a high ceiling. The effect of Coriolis force on the motion of the pendulum is to produce a precession or rotation of the plane of oscillation with time. The plane of oscillation rotates clockwise in the northern hemisphere and counterclockwise in the southern hemisphere. The period of rotation of the plane of oscillation T' is given by

$$T' = 2\pi/\omega \sin \lambda = 24/\sin \lambda \text{ hours} \tag{4.8}$$

4.2 Problems

4.2.1 Moment of Inertia

4.1 Calculate the moment of inertia of a solid sphere about an axis through its centre.

4.2 Two particles of masses m_1 and m_2 are connected by a rigid massless rod of length r to constitute a dumbbell which is free to move in a plane. Show that the moment of inertia of the dumbbell about an axis perpendicular to the plane passing through the centre of mass is μr^2 where μ is the reduced mass.

4.3 Show that the moment of inertia of a right circular cone of mass M, height h and radius 'a' about its axis is $3Ma^2/10$.

4.4 Calculate the moment of inertia of a right circular cylinder of radius R and length h about a line at right angles to its axis and passing through the middle point.

4.5 Show that the radius of gyration about an axis through the centre of a hollow cylinder of external radius 'a' and internal radius 'b' is $\sqrt{\frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right)}$.

4.6 Calculate the moment of inertia of a thin rod **(a)** about an axis passing through its centre and perpendicular to its length **(b)** about an end perpendicular to the rod.

4.7 Show that the moment of inertia of a rectangular plate of mass m and sides $2a$ and $2b$ about the diagonal is $\frac{2}{3}\frac{ma^2b^2}{(a^2 + b^2)}$

4.8 Lengths of sides of a right angle triangular lamina are 3, 4 and 5 cm, and the moment of inertia of the lamina about the sides I_1, I_2 and I_3, respectively (Fig. 4.1). Show that $I_1 > I_2 > I_3$.

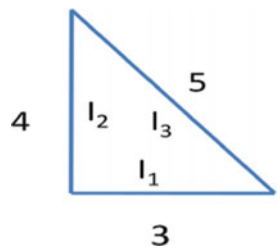


Fig. 4.1

4.9 A circular disc of radius R and thickness $R/6$ has moment of inertia I about the axis perpendicular to the plane and passing through its centre. The disc is melted and recasted into a solid sphere. Show that the moment of inertia of the sphere about its diameter is $I/5$.

4.10 Calculate the moment of inertia of a hollow sphere of mass M and radius R about its diameter.

4.11 Use the formula for moment of inertia of a uniform sphere about its diameter $\left(I = \frac{2}{5}MR^2\right)$ to deduce the moment of inertia of a thin hollow sphere about the axis passing through the centre.

4.2.2 Rotational Motion

4.12 A solid cylinder of mass m and radius R rolls down an inclined plane of height h without slipping. Find the speed of its centre of mass when the cylinder reaches the bottom.

4.13 A star has initially a radius of 6×10^8 m and a period of rotation about its axis of 30 days. Eventually it evolves into a neutron star with a radius of only 10^4 m and a period of 0.1 s. Assuming that the mass has not changed, find the ratio of initial and final **(a)** angular momentum and **(b)** kinetic energy.

4.14 A uniform solid ball rolls down a slope. If the ball has a diameter of 0.5 m and a mass of 0.1 kg, find the following:

(a) The equation which describes the velocity of the ball at any time, given that it starts from rest. Clearly state any assumptions you make.
(b) If the slope has an incline of 30° to the horizontal, what is the speed of the ball after it travels 3 m?
(c) At this point, what is the angular momentum of the ball?
(d) If the coefficient of friction between the ball and the slope is 0.26, what is the maximum angle of inclination the slope could have which still allows the ball to roll?

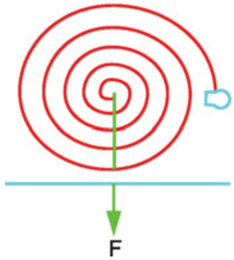
[University of Durham 2000]

4.15 **(a)** Show that the least coefficient of friction for an inclined plane of angle θ in order that a solid cylinder will roll down without slipping is $\frac{1}{3}\tan\theta$. **(b)** Show that for a hoop the least coefficient of friction is $\frac{2}{3}\tan\theta$.

4.16 A small mass m tied to a non-stretchable thread moves over a smooth horizontal plane. The other end of the thread is drawn through a hole with constant velocity, Fig. 4.2. Show that the tension in the thread is inversely proportional to the cube of the distance from the hole.

[Osmania University]

Fig. 4.2



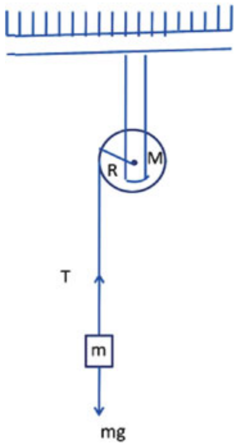
4.17 An ice skater spins at 4π rad/s with her arms extended.

(a) If her moment of inertia with arms folded is 80% of that with arms extended, what is her angular velocity when she folds her arms?
(b) Find the fractional change in kinetic energy.

4.18 A sphere of radius R and mass M rolls down a horizontal plane. When it reaches the bottom of an incline of angle θ it has velocity v_0. Assuming that it rolls without slipping, how far up the incline would it travel?

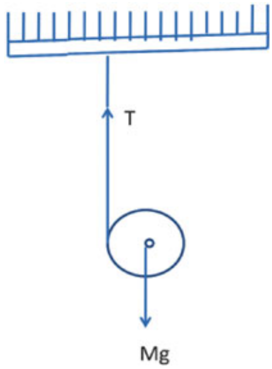
4.19 A body of mass m is attached to a light string wound around a pulley of mass M and radius R mounted on an axis supported by fixed frictionless bearings (Fig. 4.3). Find the linear acceleration 'a' of m and the tension T in the string.

Fig. 4.3



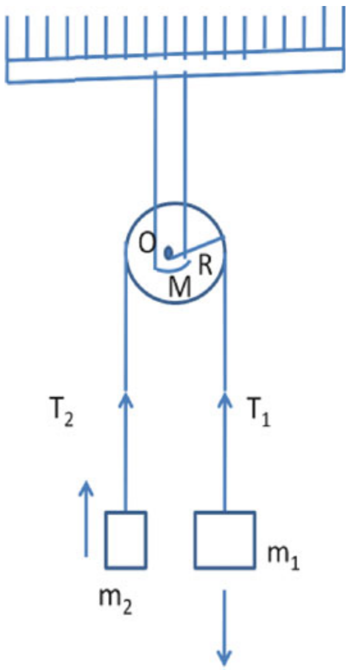
4.20 A light string is wound several times around a spool of mass M and radius R. The free end of the string is attached to a fixed point and the spool is held so that the part of the string not in contact with it is vertical (see Fig. 4.4). If the spool is let go, find the acceleration and the tension of the string.

Fig. 4.4



4.21 Two unequal masses m_1 and $m_2 (m_1 > m_2)$ are suspended by a light string over a pulley of mass M and radius R as in Fig. 4.5. Assuming that slipping does not occur and the friction of the axle is negligible, **(a)** find the acceleration with which the masses move; **(b)** angular acceleration of the pulley; **(c)** ratio of tensions T_1/T_2 in the process of motion.

Fig. 4.5



4.22 Two wheels of moment of inertia I_1 and I_2 are set in rotation with angular speed ω_1 and ω_2. When they are coupled face to face they rotate with a common angular speed ω due to frictional forces. Find **(a)** ω and **(b)** work done by the frictional forces.

4.23 Consider a uniform, thin rod of length l and mass M.

(a) The rod is held vertically with one end on the floor and is then allowed to fall. Use energy conservation to find the speed of the other end just before it hits the floor, assuming the end on the floor does not slip.
(b) You have an additional point mass m that you have to attach to the rod. Where do you have to attach it, in order to make sure that the speed of the falling end is not altered if the experiment in (a) is repeated?

[University of Durham 2005]

4.24 A thin circular disc of mass M and radius R is rotated with a constant angular velocity ω in the horizontal plane. Two particles each of mass m are gently attached at the opposite end of the diameter of the disc. What is the new angular velocity of the disc?

4.25 If the velocity is $\mathbf{v} = 2\hat{i} - 3\hat{j} + \hat{k}$ and the position vector is $\mathbf{r} = \hat{i} + 2\hat{j} - 3\hat{k}$, find the angular momentum for a particle of mass m.

4.26 A ball of mass 0.2 kg and radius 0.5 m starting from rest rolls down a 30° inclined plane. **(a)** Find the time it would take to cover 7 m. **(b)** Calculate the torque acting at the end of 7 m.

4.27 A string is wrapped around a cylinder of mass m and radius R. The string is pulled vertically upwards to prevent the centre of mass from falling as the cylinder unwinds the string. Find

(a) the tension in the string.
(b) the work done on the cylinder when it acquires angular velocity ω.
(c) the length of the string unwound in the time the angular speed reaches ω.

4.28 Two cords are wrapped around the cylinder, one near each end and the cord ends which are vertical are attached to hooks on the ceiling (Fig. 4.6). The cylinder which is held horizontally has length L, radius R and weight W. If the cylinder is released find

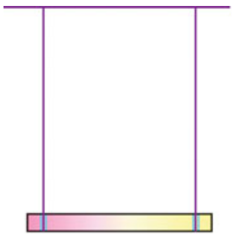
(a) the tension in the cords.
(b) acceleration of the cylinder.

[Osmania University]

4.29 A body of radius R and mass M is initially rolling on a level surface with speed u. It then rolls up an incline to a maximum height h. If $h = 3u^2/4g$, figure out the geometrical shape of the body.

4.30 A solid cylinder, a hollow cylinder, a solid sphere and a hollow sphere of the same mass and radius are placed on an incline and are released simultaneously from the same height. In which order would these bodies reach the bottom of the incline?

Fig. 4.6

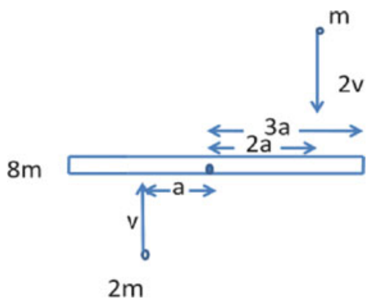


4.31 A tube of length L is filled with an incompressible liquid of mass M and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. Show that the force exerted by the liquid at the other end is $F = \frac{1}{2} m\omega^2 L$.

4.32 A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses m and $2m$ moving in the same horizontal plane with speed $2v$ and v, respectively, strike the bar and stick to the bar (Fig. 4.7). The bar is set in rotation. Show that

(a) the centre of mass velocity $v_c = 0$
(b) the angular momentum $J = 6mva$
(c) the angular velocity $\omega = v/5a$
(d) the rotational energy $E = 3mv^2/5$

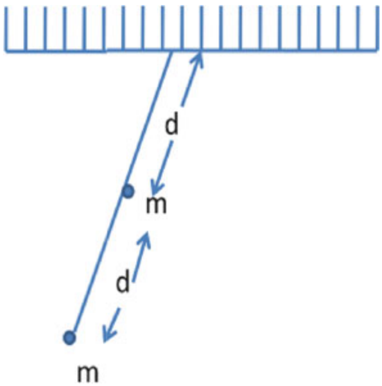
Fig. 4.7



4.33 A thin rod of negligible weight and of length $2d$ carries two point masses of m each separated by distance d, Fig. 4.8. If the rod is released from a horizontal position show that the speed of the lower mass when the rod is in the vertical position will be $v = \sqrt{\frac{24\,gd}{5}}$.

4.34 If the radius of the earth suddenly decreases to half its present value, the mass remaining constant, what would be the duration of day?

Fig. 4.8



4.35 A tall pole cracks and falls over. If θ is the angle made by the pole with the vertical, show that the radial acceleration of the top of the pole is $a_\mathrm{R} = \frac{3}{2}g(1 - \cos\theta)$ and its tangential acceleration is $a_\mathrm{T} = \frac{3}{2}g\sin\theta$.

4.36 The angular momentum of a particle of a point varies with time as $\mathbf{J} = at^2\hat{i} + b\hat{j}$, where a and b are constants. When the angle between the torque about the point and the angular momentum is 45°, show that the magnitude of the torque and angular momentum will be $2\sqrt{ab}$ and $\sqrt{2}b$, respectively.

4.37 A uniform disc of radius R is spun about the vertical axis and placed on a horizontal surface. If the initial angular speed is ω and the coefficient of friction μ show that the time before which the disc comes to rest is given by $t = 3\omega R/4\mu g$.

4.38 A small homogeneous solid sphere of mass m and radius r rolls without slipping along the loop-the-loop track, Fig. 4.9. If the radius of the circular part of the track is R and the sphere starts from rest at a height $h = 6R$ above the bottom, find the horizontal component of the force acting on the track at Q at a height R from the bottom.

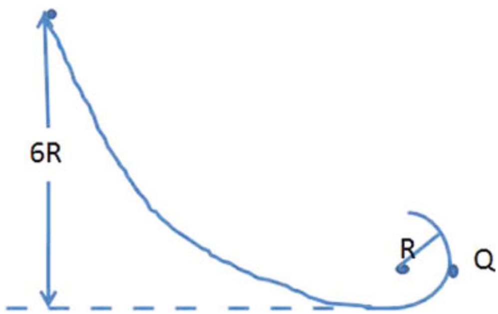
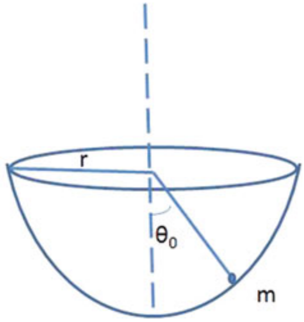


Fig. 4.9

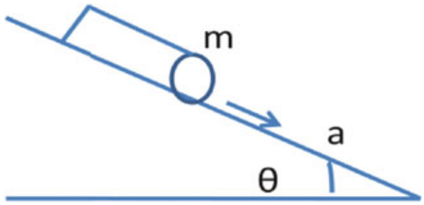
4.39 A particle is projected horizontally along the interior of a smooth hemispherical bowl of radius r. If the initial angular position of the particle is θ_0, find the initial velocity required by the particle just to reach the top of the bowl (Fig. 4.10).

Fig. 4.10



4.40 A spool of mass m, with a thread wound on it, is placed on an incline of 30° to the horizontal. The free end of the thread is attached to a nail, Fig. 4.11. Find the acceleration of the spool.

Fig. 4.11



4.41 A flywheel with initial angular velocity ω_0 undergoes deceleration due to frictional forces, with the torque on the axle being proportional to the square root of its angular velocity. Calculate the mean angular velocity of the wheel averaged over the total deceleration time.

4.42 A conical pendulum consisting of a thin uniform rod of length L and mass m with the upper end of the rod freely hanging rotates about a vertical axis with angular velocity ω. Find the angle which the rod makes with the vertical.

4.43 A billiard ball is initially struck such that it slides across the snooker table with a linear velocity V_0. The coefficient of friction between the ball and table is μ. At the instant the ball begins to roll without sliding calculate

(a) its linear velocity
(b) the time elapsed after being struck
(c) the distance travelled by the ball

State clearly what assumptions you have made about the forces acting on the ball throughout.

4.44 Consider a point mass m with momentum p rotating at a distance r about an axis. Starting from the definition of the angular momentum $L \equiv r \times p$ of this point mass, show that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tau,$$

where τ is the torque.
A uniform rod of length l and mass M rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet of mass m travelling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its centre and becomes embedded in it. Using the result above, show that the angular momentum of the rod after the collision is given by

$$|L| = \frac{1}{2}lv$$

Is $L = (l/2)\mathrm{m}v$ also correct?
What is the final angular speed of the rod?
Assuming $M = 5m$, what is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?
[University of Durham 2008]

4.45 A uniform sphere of radius r initially at rest rolls without slipping down from the top of a sphere of radius R. Find the angular velocity of the ball at the instant it breaks off the sphere and show that the angle $\theta = \cos^{-1}(10/17)$ with the vertical.

4.46 A uniform rod of mass m and length $2a$ is placed vertically with one end in contact with a smooth horizontal floor. When it is given a small displacement, it falls. Show that when the rod is about to strike, the reaction is equal to $mg/4$.
[courtesy from R.W. Norris and W. Seymour, Mechanics via Calculus, Longmans & Co.]

4.47 The double pulley shown in Fig. 4.12 consists of two wheels which are fixed together and turn at the same rate on a frictionless axle. A rope connected to mass m_1 is wound round the circumference of the larger wheel and a second rope connected to mass m_2 is wound round the circumference of the smaller wheel. Both ropes are of negligible mass. The moment of inertia, I, of the double pulley is $38\,\mathrm{kg\,m^2}$. The radii of the wheels are $R_1 = 1.2\,\mathrm{m}$ and $R_2 = 0.5\,\mathrm{m}$.

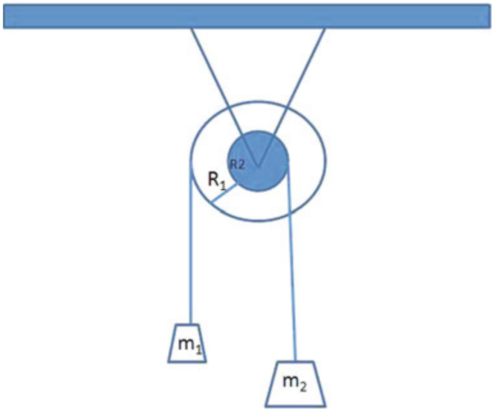
(a) If $m_1 = 25\,\mathrm{kg}$, what should the value of m_2 be so that there is no angular acceleration of the double pulley?
(b) The mass m_1 is now increased to $35\,\mathrm{kg}$ and the system released from rest.

(i) For each mass, write down the relationship between its linear acceleration and the angular acceleration of the pulley. Which mass has the greater linear acceleration?

(ii) Determine the angular acceleration of the double pulley and the tensions in both ropes.

[University of Manchester 2008]

Fig. 4.12



4.48 Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of this system of particles is independent of origin.

4.49 A small sphere of mass m and radius r rolls without slipping on the inside of a large hemisphere of radius R, the axis of symmetry being vertical. It starts from rest. When it arrives at the bottom show that

(a) the fraction $K(\text{rot})/K(\text{total}) = 2/7$

(b) the normal force exerted by the small sphere is given by $N = 17mg/7$

4.50 A solid sphere, a hollow sphere, a solid disc and a hoop with the same mass and radius are spinning freely about a diameter with the same angular speed on a table. For which object maximum work will have to be done to stop it?

4.51 In prob. (4.50) the four objects have the same angular momentum. For which object maximum work will have to be done to stop it?

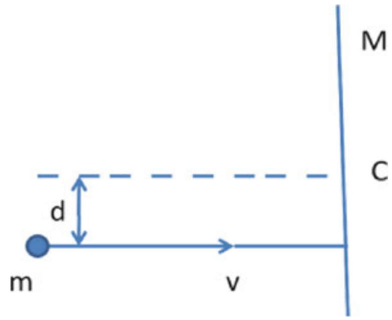
4.52 In prob. (4.50) the four objects have the same angular speed and same angular momentum. Compare the work to be done to stop them.

4.53 A solid sphere, a hollow sphere, a solid cylinder and a hollow cylinder roll down an incline. For which object the torque will be least?

4.54 A particle moves with the position vector given by $\mathbf{r} = 3t\hat{i} + 2\hat{j}$. Show that the angular momentum about the origin is constant.

4.55 A metre stick of length l and mass M is placed on a frictionless horizontal table. A hockey ball of mass m sliding along the table perpendicular to the stick with speed v strikes the stick elastically at distance d from the centre of the metre stick. Find d if the ball is to be brought to rest immediately after the collision (Fig. 4.13).

Fig. 4.13

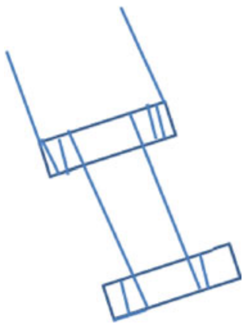


4.56 A uniform solid cylinder of mass m and radius R is set in rotation about its axis and lowered with the lateral surface on to the horizontal plane with initial centre of mass velocity v_0. If the coefficient of friction between the cylinder and the plane is μ, find

(a) how long the cylinder will move with sliding friction.
(b) the total work done by the sliding friction force on the cylinder.

4.57 Two identical cylinders, each of mass m, on which light threads are wound symmetrically are arranged as in Fig. 4.14. Find the tension of each thread in the process of motion. Neglect the friction in the axle of the upper cylinder.

Fig. 4.14



4.58 A uniform circular disc of radius r and mass m is spinning with uniform angular velocity ω in its own plane about its centre. Suddenly a point on its circumference is fixed. Find the new angular speed ω' and the impulse of the blow at the fixed point.

4.59 A uniform thin rod of mass m and length L is rotating on a smooth horizontal surface with one end fixed. Initially it has an angular velocity Ω and the motion slows down only because of air resistance which is $k\,\mathrm{d}x$ times the square of the velocity on each element of the rod of length $\mathrm{d}x$. Find the angular velocity ω after time t.

4.60 A sphere of radius a oscillates at the bottom of a hollow cylinder of radius b in a plane at right angles to the axis which is horizontal. If the cylinder is fixed and the sphere does not slide, find T, the time period of oscillations in terms of a, b and g, the acceleration due to gravity.

4.61 **(a)** Show that the moment of inertia of a disc of radius R and mass M about an axis through the centre perpendicular to its plane is

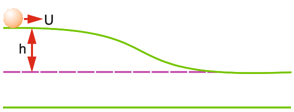
$$I = \frac{1}{2}MR^2$$

(b) A disc rolls without slipping along a horizontal surface with velocity u. The disc then encounters a smooth drop of height h, after which it continues to move with velocity v. At all times the disc remains in a vertical plane (Fig. 4.15).

Show that $v = \sqrt{u^2 + \dfrac{4gh}{3}}$

[University of Manchester 2008]

Fig. 4.15



4.62 A circular ring of mass m and radius r lies on a smooth horizontal surface. An insect of mass m sits on it and crawls round the ring with a uniform speed v relative to the ring. Obtain an expression for the angular velocity of the ring.

[With courtesy from R.W. Norris and W. Seymour, Longmans, Green and Co., 1923]

4.2.3 Coriolis Acceleration

4.63 **(a)** Given that earth rotates once every 23 h 56 min around the axis from the North to South Pole, calculate the angular velocity, ω, of the earth. When viewed from above the North Pole, the earth rotates counterclockwise (west to east). Which way does ω point?

(b) Foucault's pendulum is a simple pendulum suspended by a long string from a high ceiling. The effect of Coriolis force on the motion of the pendulum is to produce a precession or rotation of the plane of oscillation with time. Find the time for one rotation for the plane of oscillation of the Foucault pendulum at 30° latitude.

4.64 An object is dropped at the equator from a height of 400 m. How far does it hit the earth's surface from a point vertically below?

4.65 An object at the equator is projected upwards with a speed of 20 m/s. How far from its initial position will it land?

4.66 With what speed must an object be thrown vertically upwards from the surface of the earth on the equator so that it returns to the earth 1 m away from its original position?

4.67 A body is dropped from a height at latitude λ in the northern hemisphere. Show that it strikes the ground a distance $d = \frac{1}{3}\omega \cos\lambda \sqrt{\frac{8h^3}{g}}$ to the west, where ω is the earth's angular velocity.

4.68 An iceberg of mass 5×10^5 tons near the North Pole moves west at the rate of 8 km/day. Neglecting the curvature of the earth, find the magnitude and direction of the Coriolis force.

4.69 A tidal current is running due north in the northern latitude λ with velocity v in a channel of width b. Prove that the level of water on the east coast is raised above that on the western coast by $(2bv\omega \sin\lambda)g$ where ω is the earth's angular velocity.

4.70 If an object is dropped on the earth's surface, prove that its path is a semicubical parabola, $y^2 = z^3$.

4.71 A train of mass 1000 tons moves in the latitude 60° north. Find the magnitude and direction of the lateral force that the train exerts on the rails if it moves with a velocity of 15 m/s.

4.72 A train of mass m is travelling with a uniform velocity v along a parallel latitude. Show that the difference between the lateral force on the rails when it travels towards east and when it travels towards west is $4mv\omega \cos\lambda$, where λ is latitude and ω is the angular velocity of the earth.

4.73 A body is thrown vertically upwards with a velocity of 100 m/s at a 60° latitude. Calculate the displacement from the vertical in 10 s.

4.3 Solutions

4.3.1 Moment of Inertia

4.1 Imagine the sphere of mass M and radius R to be made of a series of circular discs, a typical one being of thickness dx at distance x from the centre, Fig. 4.16. The area of the disc is $\pi(R^2 - x^2)$, and if the density of the sphere is ρ, the mass of the disc is $\rho\,\pi(R^2 - x^2)\,\mathrm{d}x$. The elementary moment of inertia of the disc about the axis OX is $\frac{1}{2}$ (mass)(radius)^2

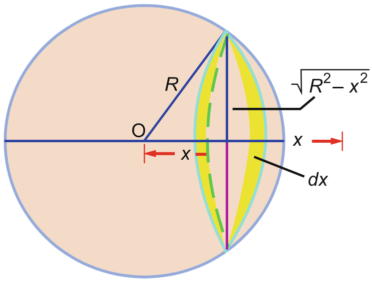
$$\therefore \quad \mathrm{d}I = \frac{1}{2}\pi\rho\,(R^2 - x^2)\mathrm{d}x(R^2 - x^2)$$

Hence the moment of inertia of the sphere is

$$I = \int \mathrm{d}I = \int_0^R \frac{\pi\rho}{2}(R^2 - x^2)^2\mathrm{d}x = \frac{8\pi\rho}{15}R^5 = \frac{2}{5}MR^2$$

as $\rho = \dfrac{3M}{4\pi R^3}$

Fig. 4.16



4.2 Let the mass m_1 and m_2 be at distance r_1 and r_2, respectively, from the centre of mass. Then

$$r_1 = \frac{m_2 r}{m_1 + m_2}, \quad r_2 = \frac{m_1 r}{m_1 + m_2}$$

Moment of inertia of the masses about the centre of mass is given by

$$I = m_1 r_1^2 + m_2 r_2^2$$

$$= m_1\left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2\left(\frac{m_1 r}{m_1 + m_2}\right)^2 = \frac{m_1 m_2}{m_1 + m_2}r^2 = \mu r^2$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

4.3 Consider the cone to be made up of a series of discs, a typical one of radius r and of thickness dz at distance z from the apex. Volume of the disc is $\mathrm{d}V = \pi r^2 \mathrm{d}z$. Its mass will be $\mathrm{d}m = \rho \mathrm{d}V = \pi \rho r^2 \mathrm{d}z$, where ρ is the mass density of the cone. The moment of inertia dI of the disc about the z-axis is given by (Fig. 4.17)

$$\mathrm{d}I = \frac{1}{2} r^2 \mathrm{d}m = \frac{\pi}{2} \rho r^4 \mathrm{d}z$$

But $\dfrac{r}{a} = \dfrac{z}{h}$ (from the geometry of the figure)

or $r = \dfrac{az}{h}$

where h is the height of the cone and a is the radius of the base

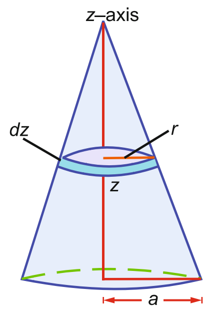
$$\therefore \quad \mathrm{d}I = \frac{\pi}{2} \rho \frac{a^4}{h^4} z^4 \mathrm{d}z$$

$$I = \int \mathrm{d}I = \frac{\pi}{2} \rho \frac{a^4}{h^4} \int_0^h z^4 \mathrm{d}z = \frac{\pi}{10} \rho\, a^4 h$$

But $\rho = \dfrac{3M}{\pi a^2 h}$

$$\therefore \quad I = \frac{3Ma^2}{10}$$

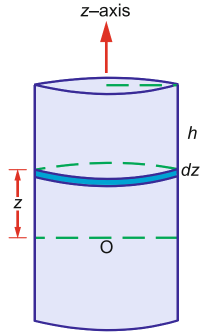
Fig. 4.17



4.4 Consider a slice of the cylinder of thickness dz at distance z from the centre of mass of cylinder O. The moment of inertia about an axis passing through the centre of the slice and perpendicular to z-axis will be

$$\mathrm{d}I = \frac{1}{4} \mathrm{d}m R^2$$

Fig. 4.18



Then the moment of inertia about an axis parallel to the slice and passing through the centre of mass is given by the parallel axis theorem, Fig. 4.18.

$$
\begin{aligned}
\mathrm{d}I_C &= \frac{1}{4}\,\mathrm{d}m\,R^2 + \mathrm{d}m\,z^2 \\
&= \pi R^2 \rho \left(\frac{R^2}{4} + z^2\right)\mathrm{d}z
\end{aligned}
$$

where $\mathrm{d}m = \pi R^2 \rho \mathrm{d}z$ is the mass of the slice and ρ is the density.

$$
\begin{aligned}
\therefore \quad I_C &= \int \mathrm{d}I_C = \pi\,R^2\rho \int_{-h/2}^{+h/2} \left(\frac{R^2}{4} + z^2\right)\mathrm{d}z \\
&= \pi\,R^2\rho\left[\frac{R^2}{4}h + \frac{h^3}{12}\right]
\end{aligned}
$$

But $\quad \rho = \dfrac{M}{\pi\,R^2 h}$

$$\therefore \quad I_C = \frac{M}{12}\left(3R^2 + h^2\right)$$

4.5 The moment of inertia of the larger solid sphere of mass M

$$I_1 = \frac{2}{5}\,Ma^2 \tag{1}$$

The moment of inertia of the smaller solid sphere of mass m, which is removed to hollow the sphere, is

$$I_2 = \frac{2}{5}\,mb^2 \tag{2}$$

As the axis about which the moment of inertia is calculated is common to both the spheres, the moment of inertia of the hollow sphere will be

$$I = I_1 - I_2 = \frac{2}{5}(Ma^2 - mb^2) = (M - m)k^2 \tag{3}$$

where $(M - m)$ is the mass of the hollow sphere and k is the radius of gyration.

$$\text{Now} \quad M = \frac{4}{3}\pi\, a^3 \rho \text{ and } m = \frac{4}{3}\pi\, b^3 \rho \tag{4}$$

Using (4) in (3) and simplifying we get

$$k = \sqrt{\frac{2}{5}\frac{(a^5 - b^5)}{(a^3 - b^3)}}$$

4.6 **(a)** Let AB represent a thin rod of length L and mass M, Fig. 4.19. Choose the x-axis along length of the rod and y-axis perpendicular to it and passing through its centre of mass O. Consider a differential element of length dx at a distance x from O. The mass associated with it is $M\,(\mathrm{d}x/L)$. The contribution to moment of inertia about the y-axis by this element of length will be $M\,(\mathrm{d}x/L)\,x^2$. The moment of inertia of the rod about y-axis passing through the centre of mass is

$$I_\mathrm{C} = \int \mathrm{d}I_\mathrm{C} = \int_{-L/2}^{+L/2} M\frac{\mathrm{d}x}{L}x^2 = \frac{ML^2}{12}$$

(b) Moment of inertia about y-axis passing through the end of the rod (A or B) is given by the parallel axis theorem:

$$I_\mathrm{A} = I_\mathrm{B} = I_\mathrm{C} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$$

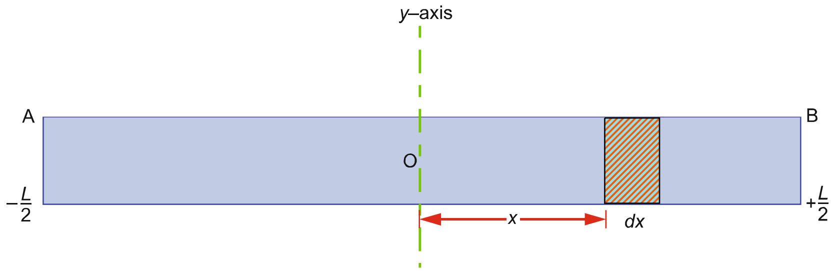


Fig. 4.19

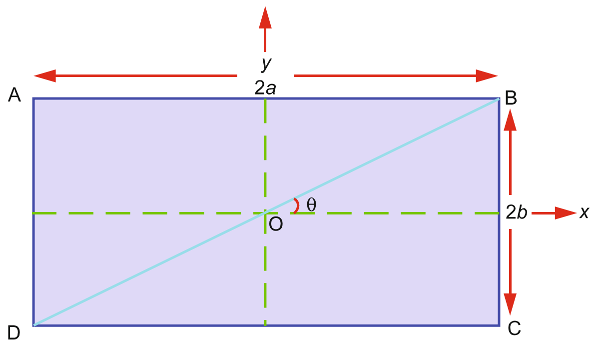
4.7 The moment of inertia of the plate about x-axis is $I_x = (1/3)\ Mb^2$ and about y-axis is $I_y = (1/3)\ Ma^2$. It can be shown that the moment of inertia about the line BD is

$$I_{\text{BD}} = \frac{1}{3}Mb^2\cos^2\theta + \frac{1}{3}Ma^2\sin^2\theta \tag{1}$$

where θ is the angle made by BD with the x-axis. From Fig. 4.20, $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$.

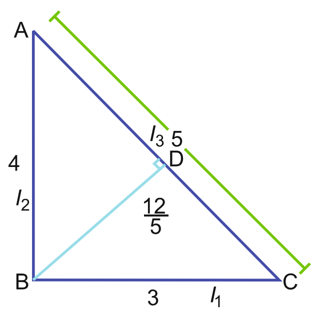
$$\therefore \quad I_{\text{BD}} = \frac{1}{3}\frac{M\,b^2a^2}{(a^2+b^2)} + \frac{1}{3}\frac{M\,a^2b^2}{(a^2+b^2)} = \frac{2}{3}M\frac{a^2\,b^2}{(a^2+b^2)}$$

Fig. 4.20



4.8 The moment of inertia about any side of a triangle is given by the product of the one-sixth mass m of the triangle and the square of the distance (p) from the opposite vertex, i.e. $I = mp^2/6$. The perpendicular BD on AC is found to be equal to 12/5 from the geometry of Fig. 4.21.

Fig. 4.21



$$I_1 = \frac{m}{6}\,(\mathrm{AB})^2 = \frac{m}{6}4^2 = \frac{8m}{3}$$
$$I_2 = \frac{m}{6}\,(\mathrm{BC})^2 = \frac{m}{6}3^2 = \frac{3m}{2}$$
$$I_3 = \frac{m}{6}\,(\mathrm{BD})^2 = \frac{m}{6}\left(\frac{12}{5}\right)^2 = \frac{24m}{25}$$
$$\therefore I_1 > I_2 > I_3$$

4.9 If the radius of the sphere is r then the volume of the sphere must be equal to that of the disc:

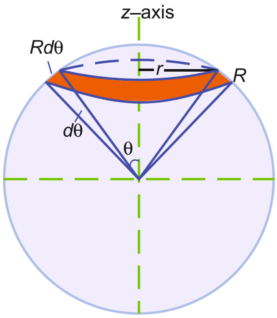
$$\frac{4}{3}\pi r^3 = \pi R^2 \frac{R}{6}$$
$$\therefore \quad r = \frac{R}{2}$$

The moment of inertia of the disc $I = I_{\mathrm{D}} = (1/2)\ m\,R^2$
The moment of inertia of the sphere

$$I_{\mathrm{S}} = \frac{2}{5}mr^2 = \frac{2}{5}m\,\frac{R^2}{4} = \frac{1}{5} \times \frac{1}{2}m\,R^2 = \frac{1}{5}I_{\mathrm{D}}$$

4.10 Consider a strip of radius r on the surface of the sphere symmetrical about the z-axis and width $R\mathrm{d}\theta$, where R is the radius of the hollow sphere, Fig. 4.22.

Fig. 4.22



Area of the strip is $2\pi r \cdot R\mathrm{d}\theta = 2\pi R^2 \sin\theta\,\mathrm{d}\theta$. If σ is the surface mass density (mass per unit area) then the mass of the strip is $\mathrm{d}m = 2\pi R^2\sigma\sin\theta\,\mathrm{d}\theta$. Moment of inertia of the elementary strip about the z-axis

$$\mathrm{d}I = \mathrm{d}m\,r^2 = 2\pi\ R^4\sigma\ \sin^3\theta\,\mathrm{d}\theta$$

Moment of inertia contributed by the entire surface will be

$$I = \int \mathrm{d}I = 2\pi\ R^4\,\sigma \int_0^{\pi} \sin^3\theta\,\mathrm{d}\theta$$

$$= 2\pi\ R^4 \sigma \frac{4}{3}$$

But $\sigma = \dfrac{M}{4\pi\ R^2}$

$$\therefore \quad I = \frac{2}{3} M R^2$$

4.11 By prob. (4.5), the radius of gyration of a hollow sphere of external radius a and internal radius b is

$$k = \sqrt{\frac{2}{5}\,\frac{(a^5 - b^5)}{(a^3 - b^3)}} \tag{1}$$

The derivation of (1) is based on the assumed value of moment of inertia for a solid sphere about its diameter $\left(I = \dfrac{2}{5} M R^2\right)$. Squaring (1) and multiplying by M, the mass of the hollow cylinder is

$$I = Mk^2 = \frac{2}{5}\,M\,\frac{(a^5 - b^5)}{(a^3 - b^3)} \tag{2}$$

Let $a = b + \Delta$ (3)

where Δ is a small quantity. Then (2) becomes

$$I = \frac{2}{5}\,M\,\frac{\left[(b+\Delta)^5 - b^5\right]}{\left[(b+\Delta)^3 - b^3\right]} = \frac{2}{5}\,M\,\frac{\left[b^5 + 5b^4\Delta + \cdots - b^5\right]}{\left[b^3 + 3b^3\Delta + \cdots - b^3\right]}$$

where we have neglected higher order terms in Δ. Thus

$$I = \frac{2}{5}\,M\,\frac{5b^4\Delta}{3b^2\Delta} = \frac{2}{3} M b^2 = \frac{2}{3} M R^2$$

where $b = a = R$ is the radius of the hollow sphere.

4.3.2 Rotational Motion

4.12 Potential energy at height h is mgh and kinetic energy is zero. At the bottom the potential energy is assumed to be zero. The kinetic energy (K) consists of translational energy $\left(\dfrac{1}{2} m v^2\right)$ + rotational energy $\left(\dfrac{1}{2} I \omega^2\right)$:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{1}{2}m\,R^2\frac{v^2}{R^2} = \frac{3}{4}mv^2$$

Gain in kinetic energy = loss of potential energy

$$\frac{3}{4}mv^2 = mgh$$

$$\text{or} \quad v = \sqrt{\frac{4g\,h}{3}}$$

4.13 **(a)** Initial angular momentum

$$L_1 = I_1\omega_1 = \frac{2}{5}MR_1^2\frac{2\pi}{T_1}$$

Final angular momentum $L_2 = I_2\omega_2 = \frac{2}{5}MR_2^2\frac{2\pi}{T_2}$

$$\frac{L_1}{L_2} = \frac{R_1^2}{R_2^2}\frac{T_2}{T_1} = \left(\frac{6 \times 10^8}{10^4}\right)^2\left(\frac{0.1}{30 \times 86{,}400}\right) = 138.9$$

(b) Initial kinetic energy (rotational)

$$K_1 = \frac{1}{2}I_1\omega_1^2 = \frac{1}{2} \times \frac{2}{5}MR_2^2\left(\frac{2\pi}{T_1}\right)^2$$

Final kinetic energy $K_2 = \frac{1}{2}I_2\omega_2^2 = \frac{1}{2} \times \frac{2}{5}MR_2^2\left(\frac{2\pi}{T_2}\right)^2$

$$\frac{K_1}{K_2} = \left(\frac{R_1}{R_2}\frac{T_2}{T_1}\right)^2 = \left(\frac{6 \times 10^8}{10^4} \times \frac{0.1}{30 \times 86{,}400}\right)^2 = 5.36 \times 10^{-6}$$

4.14 **(a)** Let M be the mass of the sphere, R its radius, θ the angle of incline. Let F and N be the friction and normal reaction at A, the point of contact, Fig. 4.23. Denoting the acceleration dx^2/dt^2 by $\ddot{x}$, the equations of motion are

$$M\ddot{x} = Mg\sin\theta - F \tag{1}$$

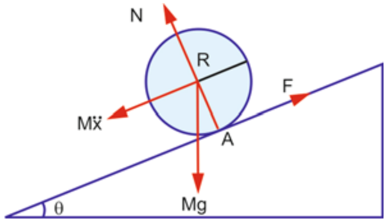
$$Mg\cos\theta - N = 0 \tag{2}$$

$$\text{Torque } I\alpha = FR \tag{3}$$

$$\text{or} \quad \frac{2}{5}MR^2\frac{a}{R} = FR$$

$$\text{or} \quad F = \frac{2}{5}M\ddot{x} \tag{4}$$

Fig. 4.23



Using (4) in (1)

$$a = \ddot{x} = \frac{5}{7} g \sin\theta \tag{5}$$

Thus the centre of the sphere moves with a constant acceleration. The assumption made in the derivation is that we have pure rolling without sliding

(b) $v = \sqrt{2as} = \sqrt{2 \times \frac{5}{7} \times 9.8 \times \sin 30^\circ \times 3} = 4.58\,\mathrm{m/s}$

(c) $L = I\omega = \frac{2}{5} MR^2 \frac{v}{R} = \frac{2}{5} MvR$

$$= \frac{2}{5} \times 0.1 \times 4.58 \times 0.25 = 0.0458\,\mathrm{kg\ m^2/T}$$

(d) Using (5) in (1)

$$F = \frac{2}{7} Mg \sin\theta \tag{4.9}$$

$$\therefore \quad \frac{F}{N} = \frac{2}{7} \tan\theta \tag{4.10}$$

For no slipping F/N must be less than μ, the coefficient of friction between the surfaces in contact. Therefore, the condition for pure rolling is that μ must exceed $(2/7)\tan\theta$.

$$\mu = \frac{2}{7} \tan\theta \tag{4.11}$$

$$\therefore \quad \tan\theta = \frac{7\mu}{2} = \frac{7}{2} \times 0.26 = 0.91 \tag{4.12}$$

$$\text{or} \quad \theta = 42.3^\circ \tag{4.13}$$

4.15 **(a)** Equation of motion of the cylinder for sliding down the incline is

$$ma_s = mg\sin\theta - \mu mg\cos\theta \tag{1}$$

$$\text{or} \quad a_s = g(\sin\theta - \mu\cos\theta) \tag{2}$$

When the cylinder rolls down without slipping, the linear acceleration is given by

$$a_{\mathrm{R}} = R\alpha = R\,\frac{\tau}{I_{\mathrm{CM}}} = R\frac{(\mu\, mg\ \cos\theta\ R)}{(1/2)mR^2} = 2\mu\, g\ \cos\theta \tag{3}$$

The least coefficient of friction when the cylinder would roll down without slipping is obtained by setting

$$a_{\mathrm{R}} = a_{\mathrm{s}}$$

$$\therefore \quad 2\mu g \cos\theta = g(\sin\theta - \mu\cos\theta)$$

$$\text{or} \quad \mu = \frac{1}{3}\tan\theta$$

(b) For the loop (2) is the same for sliding. But for rolling

$$a_{\mathrm{R}} = \frac{R\tau}{I_{\mathrm{CM}}} = R\frac{(\mu\, mg\ \cos\theta\ R)}{mR^2} = \mu\, g\ \cos\theta$$

Setting $a_{\mathrm{R}} = a_{\mathrm{s}}$

$$\mu g\cos\theta = g(\sin\theta - \mu\cos\theta) \tag{4.14}$$

$$\mu = \frac{1}{2}\tan\theta \tag{4.15}$$

4.16 Since the thread is being drawn at constant velocity v_0, angular momentum of the mass may be assumed to be constant. Further the particle velocities $\boldsymbol{v}$ and $\boldsymbol{r}$ are perpendicular. The angular momentum

$$J = mvr = \text{constant}$$

$$\therefore \quad v\,\alpha\,\frac{1}{r}$$

Now the tension T arises from the centripetal force

$$T = \frac{mv^2}{r}$$

$$\therefore \quad T\,\alpha\,\frac{1}{r^2}\frac{1}{r} \quad \text{or} \quad \alpha\,\frac{1}{r^3}$$

4.17 **(a)** Conservation of angular momentum gives

$$I_1\omega_1 = I_2\omega_2 \tag{4.16}$$

$$(I_1)(4\pi) = \frac{80}{100}\,I_1\omega_2 \tag{4.17}$$

$$\therefore \quad \omega_2 = 5\pi \tag{4.18}$$

(b) $$\frac{\Delta K}{K_1} = \frac{K_2 - K_1}{K_1} = \frac{K_2}{K_1} - 1$$
$$= \frac{(1/2)\, I_2\omega_2^2}{(1/2)\, I_1\omega_1^2} - 1 = (0.8)\left(\frac{5\pi}{4\pi}\right)^2 - 1 = \frac{1}{4}$$

4.18 At the bottom of the incline translational energy is $(1/2)\, Mv_0^2$ while the rotational energy is

$$\frac{1}{2}I\omega^2 = \frac{1}{2} \times \frac{2}{5}MR^2\frac{v_0^2}{R^2} = \frac{1}{5}Mv_0^2$$
$$\text{Total initial kinetic energy} = \frac{1}{2}Mv_0^2 + \frac{1}{5}Mv_0^2 = \frac{7}{10}Mv_0^2$$

Let the sphere reach a distance s up the incline or a height h above the bottom of the incline. Taking potential energy at the bottom of the incline as zero, the potential energy at the highest point reached is Mgh. Since the entire kinetic energy is converted into potential energy, conservation of energy gives

$$\frac{7}{10}Mv_0^2 = Mgh$$

But $h = s\sin\theta$, so that

$$s = \frac{7}{10}\frac{v_0^2}{g\sin\theta}$$

4.19 Equation of motion is

$$Ma = mg - T \tag{1}$$

The resultant torque τ on the wheel is TR and the moment of inertia is $(1/2)\, MR^2$.

Now $\quad \tau = I\alpha$
$$\therefore \quad TR = \frac{1}{2}MR^2\frac{a}{R}$$
$$\text{or} \quad T = \frac{1}{2}Ma \tag{2}$$

Solving (1) and (2)

$$a = \frac{2mg}{M + 2m} \quad T = \frac{Mmg}{M + 2m}$$

4.20 Equation of motion is

$$Ma = Mg - T \tag{1}$$

$$\text{Torque } \tau = TR = I\alpha = \frac{1}{2}MR^2\frac{a}{R} \tag{2}$$

$$\therefore \quad T = \frac{1}{2}Ma \tag{3}$$

Solving (1) and (3) $a = \frac{2}{3}g \quad T = \frac{Mg}{3}$

4.21 **(a)** Obviously m_1 moves down and m_2 up with the same acceleration 'a' if the string is taut. Let the tension in the string be T_1 and T_2 (Fig. 4.5). The equations of motion are

$$m_1 a = m_1 g - T_1 \tag{1}$$

$$m_2 a = T_2 - m_2 g \tag{2}$$

Taking moments about the axis of rotation O

$$T_1 R - T_2 R = I\alpha = \frac{MR^2}{2}\alpha \tag{3}$$

where α is the angular acceleration of the pulley and I is the moment of inertia of the pulley about the axis through O.

But $\alpha = \frac{a}{R}$

$$\therefore \quad T_1 - T_2 = \frac{Ma}{2} \tag{4}$$

Adding (1) and (2)

$$(m_1 + m_2)a = T_2 - T_1 + (m_1 - m_2)g \tag{5}$$

Using (4) in (5) and solving for 'a', we find

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + (1/2)\,M} \tag{6}$$

(b) $\alpha = \frac{a}{R} = \frac{(m_1 - m_2)g}{(m_1 + m_2 + (1/2)\,M)\,R}$

(c) Using (5) in (1) and (2), the values of T_1 and T_2 can be obtained from which the ratio T_1/T_2 can be found.

$$\frac{T_1}{T_2} = \frac{m_1(4m_2 + M)}{m_2(4m_1 + M)}$$

4.22 **(a)** Conservation of angular momentum gives

$$I_1\omega_1 + I_2\omega_2 = I\omega = (I_1 + I_2)\omega$$

The two moments of inertia I_1 and I_2 are additive because of common axis of rotation.

$$\therefore \quad \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

(b) Work done = loss of energy

$$W = \frac{1}{2}(I_1 + I_2)\omega^2 - \left(\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2\right)$$
$$= \frac{1}{2}(I_1 + I_2)\frac{(I_1\omega_1 + I_2\omega_2)^2}{(I_1 + I_2)^2} - \frac{1}{2}I_1\omega_1^2 - \frac{1}{2}I_2\omega_2^2$$
$$= -\frac{1}{2}\frac{I_1 I_2(\omega_1 - \omega_2)^2}{(I_1 + I_2)}$$

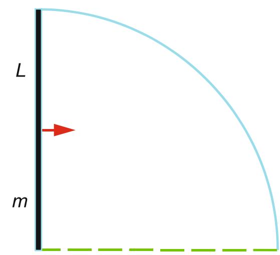
4.23 **(a)** Measure the potential energy from the bottom of the rod in the upright position, the height through which it falls is the distance of the centre of mass from the ground, i.e. $(1/2)\,L$ (Fig. 4.24). When it falls on the ground the potential energy is converted into kinetic energy (rotational).

$$mg\frac{1}{2}L = \frac{1}{2}I\omega^2 = \frac{1}{2} \times \frac{1}{3}mL^2\omega^2 = \frac{1}{6}mv^2$$

where I is the moment of inertia of the rod about one end and $v = \omega L$ is the linear velocity of the top end of the pole, $v = \sqrt{3gL}$.

(b) The additional mass has to be attached at the bottom of the rod.

Fig. 4.24



4.24 If I_1 and I_2 are the initial and final moments of inertia, ω_1 and ω_2 the initial and final angular velocity, respectively, the conservation of angular momentum gives

$$L = I_1\omega_1 = I_2\omega_2$$
$$MR^2\omega_1 = (MR^2 + 2mR^2)\omega_2$$
$$\therefore \quad \omega_2 = \frac{\omega_1 M}{M + 2m}$$

4.25 $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v})$

$$= m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & -3 & 1 \end{vmatrix} = (-7\hat{i} - 7\hat{j} - 7\hat{k})m = -7m(\hat{i} + \hat{j} + \hat{k})$$

4.26 **(a)** $a = \dfrac{g\sin\theta}{1 + \left(k^2/r^2\right)} = \dfrac{9.8\sin 30^\circ}{1 + (2/5)} = 3.5\,\mathrm{m/s^2}$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 7}{3.5}} = 2\,\mathrm{s}$$

(b) $\tau = I\alpha = \dfrac{2}{5}mR^2\dfrac{a}{R} = \dfrac{2}{5}mRa = \dfrac{2}{5} \times 0.2 \times 0.5 \times 3.5 = 0.14\,\mathrm{kg\ m^2/s^2}.$

4.27 **(a)** The equation of motion is

$$ma = mg - T \tag{1}$$
$$\tau = TR = I\alpha = \frac{1}{2}mR^2\frac{a}{R}$$
$$\therefore \quad T = \frac{1}{2}ma \tag{2}$$

Solving (1) and (2), $a = \dfrac{2g}{3}$ · (3)

(b) Work done = increase in the kinetic energy

$$W = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 = \frac{1}{4}mR^2\omega^2 \tag{4}$$

(c) $W = \displaystyle\int \tau\,\mathrm{d}\theta = \tau\theta = mgR\theta$ (5)

(where θ is the angular displacement) is an alternative expression for the work done. Equating (4) and (5) and simplifying

$$\theta = \frac{1}{4}\omega^2\frac{R}{g} \tag{6}$$

Length of the string unwound $= \theta R = \dfrac{1}{4}\dfrac{\omega^2 R^2}{g}$ (7)

4.28 As there are two strings, the equation of motion is

$$ma = mg - 2T \tag{1}$$

The net torque

$$\begin{aligned}\tau &= \tau_1 + \tau_2 = 2TR = I\alpha \\ &= \frac{1}{2}mR^2\frac{a}{R} = \frac{1}{2}maR \\ \therefore \quad T &= \frac{ma}{4} \end{aligned}\tag{2}$$

Solving (1) and (2)
(a) $T = \dfrac{mg}{6}$ **(b)** $a = \dfrac{2}{3}g$

4.29 The total kinetic energy (translational + rotational) at the bottom of the incline is

$$K = \frac{1}{2}mu^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mu^2 + \frac{1}{2}mk^2\frac{u^2}{R^2} = \frac{1}{2}mu^2\left(1 + \frac{k^2}{R^2}\right) \tag{1}$$

where k is the radius of gyration.
At the maximum height the kinetic energy is transformed into potential energy.

$$\frac{1}{2}mu^2\left(1 + \frac{k^2}{R^2}\right) = mgh = mg\frac{3u^2}{4g}$$

Solving we get $k = R/\sqrt{2}$. Therefore the body can be either a disc or a solid cylinder.

4.30 Time taken for a body to roll down an incline of angle θ over a distance s is given by

$$t = \sqrt{\frac{2s}{a}}$$

where $a = \dfrac{g\sin\theta}{1 + \left(k^2/R^2\right)}$. The quantity k^2/R^2 for various bodies is as follows:

Solid cylinder $\dfrac{1}{2}$ hollow cylinder 1

Solid sphere $\dfrac{2}{5}$ hollow sphere $\dfrac{2}{3}$

These bodies reach the bottom of the incline in the ascending order of acceleration 'a' or equivalently ascending order of k^2/R^2. Therefore the order in which the bodies reach is solid sphere, solid cylinder, hollow sphere and hollow cylinder. The physical reason is that the larger the value of k the greater will be I, and larger fraction of kinetic energy will go into rotational motion. Consequently less energy will be available for the translational motion and greater will be the travelling time.

4.31 Consider an element of length dx at distance x from the axis of rotation (Fig. 4.25). The corresponding mass will be

$$\mathrm{d}m = \rho A \mathrm{d}x$$

where ρ is the liquid density and A is the area of cross-section of the tube. The centrifugal force arising from the rotation of dm will be

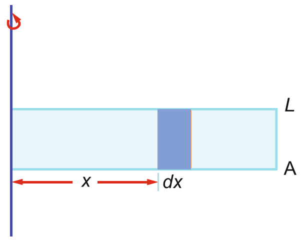
$$\mathrm{d}F = (\mathrm{d}m)\omega^2 x = \omega^2 \rho A x \mathrm{d}x$$

The total force exerted at A, the other end of the tube, will be

$$F = \int \mathrm{d}F = \omega^2 \rho A \int_0^L x \ \mathrm{d}x = \frac{1}{2}\omega^2 \rho A L^2; \ \rho = \frac{M}{LA}$$

$$\therefore \quad F = \frac{1}{2} M\omega^2 L$$

Fig. 4.25



4.32 **(a)** Total initial momentum

$$= (2m)v - m(2v) = 0$$

Therefore the centre of mass system is the laboratory system and $v_c = 0$

(b) $J = (2m)(v)(a) + (m)(2v)(2a) = 6mva$

(c) $J = I\omega$ (conservation of angular momentum)

$$6mva = \left[\frac{1}{12} 8m(6a)^2 + 2ma^2 + m(2a)^2\right]\omega = 30ma^2\omega$$

The first term in square brackets is the M.I. of the bar, the second and the third terms are for the M.I. of the particles which stick to the bar.

Thus $\omega = \dfrac{v}{5a}$

(d) $E = \frac{1}{2}I\omega^2 = \frac{1}{2}30\,ma^2\left(\frac{v}{5a}\right)^2 = \frac{3}{5}mv^2$

4.33 Let the potential energy be zero when the rod is in the horizontal position. In the vertical position the loss in potential energy of the system will be $mg(d + 2d) = 3mgd$. The gain in rotational kinetic energy will be

$$\frac{1}{2}I\omega^2 = \frac{1}{2}(I_1 + I_2)\omega^2 = \frac{1}{2}\left[md^2 + m(2d)^2\right]\omega^2 = \frac{5}{2}md^2\omega^2$$

Gain in kinetic energy = loss of potential energy

$$\frac{5}{2}md^2\omega^2 = 3mgd$$

$$\therefore \quad \omega = \sqrt{\frac{6g}{5\alpha}}$$

The linear velocity of the lower mass in the vertical position will be

$$v = (\omega)(2d) = \sqrt{\frac{24}{5}gd}$$

4.34 Conservation of J gives

$$I_1\omega_1 = I_2\omega_2$$

$$\frac{2}{5}MR^2\frac{2\pi}{T_1} = \frac{2}{5}M\left(\frac{R}{2}\right)^2\frac{2\pi}{T_2}$$

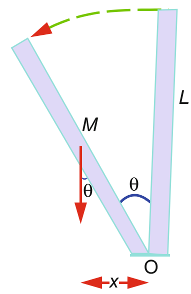
$$\therefore \quad T_2 = \frac{T_1}{4} = \frac{24}{4} = 6\text{ h}$$

4.35 The moment of inertia of the pole of length L and mass M about O is (Fig. 4.26)

$$I = \frac{ML^2}{3} \tag{1}$$

The torque $\tau = I\alpha = Mgx$ (2)

Fig. 4.26



where x is the projection of the centre of mass on the ground from the point O and α is the angular acceleration.

$$\text{Now } x = \frac{L}{2}\sin\theta \tag{3}$$

Using (1) and (3) in (2)

$$\alpha = \frac{3}{2}\frac{g}{L}\sin\theta \tag{4}$$

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = \frac{3g}{2L}\sin\theta$$

Integrating

$$\int \omega\,\mathrm{d}\omega = \frac{3}{2}\frac{g}{L}\int \sin\theta\,\mathrm{d}\theta + C$$

where $C =$ constant.

$$\frac{\omega^2}{2} = -\frac{3}{2}\frac{g}{L}\cos\theta + C \tag{5}$$

When $\theta = 0,\ \omega = 0$

$$\therefore \quad C = \frac{3}{2}\frac{g}{L} \tag{6}$$

Using (6) in (5) $\quad \omega^2 = \dfrac{3g}{2L}(1 - \cos\theta)$

Radial acceleration $a_{\mathrm{R}} = \omega^2 L = \dfrac{3}{2}g(1 - \cos\theta)$

Tangential acceleration of the top of the pole $a_{\mathrm{T}} = \alpha L = \dfrac{3}{2}g\sin\theta$

4.36 $$\boldsymbol{J} = at^2\hat{i} + b\hat{j} \tag{1}$$

$$\therefore \quad \boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = 2at\hat{i} \tag{2}$$

Take the scalar product of $\boldsymbol{J}$ and $\boldsymbol{\tau}$.

$$\boldsymbol{J} \cdot \boldsymbol{\tau} = 2a^2t^3 = \left(\sqrt{a^2t^4 + b^2}\right)(2at)\cos 45^\circ$$

Simplify and solve for t. We get

$$t = \sqrt{\frac{b}{a}} \tag{3}$$

Using (3) in (2), $|\boldsymbol{\tau}| = 2\sqrt{ab}$

Using (3) in (1), $|\boldsymbol{J}| = \sqrt{2}b$

4.37 Consider a ring of radii r and $r + \mathrm{d}r$, concentric with the disc ($r < R$). If the surface density is σ, the mass of the ring is $\mathrm{d}m = 2\pi r \mathrm{d}r\sigma$. The moment of inertia of the ring about the central axis will be

$$\mathrm{d}I = (2\pi r\ \mathrm{d}r\sigma)r^2 = 2\pi\sigma r^3\mathrm{d}r \tag{1}$$

and the corresponding torque will be

$$\mathrm{d}\tau = \alpha\mathrm{d}I = 2\pi\sigma\alpha r^3\ \mathrm{d}r \tag{2}$$

The frictional force on the ring is $\mu\mathrm{d}m\ g = \mu(2\pi r\ \mathrm{d}r\sigma)g$ and the corresponding torque will be

$$\mathrm{d}\tau = \mu(2\pi r\ \mathrm{d}r\sigma)gr = 2\pi\sigma\mu g r^2\mathrm{d}r \tag{3}$$

Calculating the torques from (2) and (3) for the whole disc and equating them

$$\int_0^R 2\pi\sigma\alpha r^3\mathrm{d}r = \int_0^R 2\pi\sigma\mu g r^2\mathrm{d}r$$

$$\therefore \quad \alpha = \frac{4\mu g}{3R} \tag{4}$$

but $\quad 0 = \omega - \alpha t$

$$\therefore \quad t = \frac{\omega}{\alpha} = \frac{3\omega R}{4\mu g}$$

4.38 The horizontal component of force at Q is mv^2/R. The drop in height in coming down to Q is

$$(6R - R) = 5R$$

Gain in kinetic energy = loss in potential energy

$$\frac{7}{10}mv^2 = (mg)(5R)$$

$$\therefore \quad \frac{mv^2}{R} = \frac{50}{7}mg$$

4.39 Let the velocity on the top be v. Energy conservation gives

$$\frac{1}{2}mv_0^2 = mgr\cos\theta_0 + \frac{1}{2}mv^2 \tag{1}$$

where $r\cos\theta_0$ is the height to which the particle is raised. Angular momentum conservation gives

$$mvr = mv_0 r\sin\theta_0 \tag{2}$$

Eliminating v between (1) and (2) and simplifying

$$v_0 = \sqrt{\frac{2gr}{\cos\theta_0}}$$

4.40 Equation of motion is

$$ma = mg\sin\theta - T \tag{1}$$

$$\text{Torque } \tau = TR = I\alpha = \frac{1}{2}mR^2\frac{a}{R}$$

$$\therefore \quad T = \frac{1}{2}ma \tag{2}$$

Using (2) in (1)

$$a = \frac{2}{3}g\sin\theta = \frac{2}{3}g\sin 30^\circ = \frac{g}{3}$$

4.41 $$< \omega > = \frac{\int \omega \mathrm{d}t}{\int \mathrm{d}t} \tag{1}$$

$$\tau = I\alpha = C\sqrt{\omega}$$

where C is a constant.

$$\therefore \quad \alpha = C_1\sqrt{\omega}$$

where $C_1 =$ constant

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = C_1\sqrt{\omega}$$

$$\therefore \quad \mathrm{d}t = \frac{\mathrm{d}\omega}{C_1\sqrt{\omega}} \tag{2}$$

Using (2) in (1)

$$< \omega >= \frac{\int_0^{\omega_0} \sqrt{\omega}\mathrm{d}\omega}{\int_0^{\omega_0} \frac{\mathrm{d}\omega}{\sqrt{\omega}}} = \frac{\omega_0}{3}$$

4.42 OC $= L$ is the length of the rod with the centre of mass G at the midpoint, Fig. 4.27. As the rod rotates with angular velocity ω it makes an angle θ with the vertical OA through O. Drop a perpendicular GD $= r$ on the vertical OA and a perpendicular GB on OC.

$$r = \frac{L}{2}\sin\theta$$

The acceleration of the rod at G at any instant is $\omega^2 r = \omega^2\,(L/2)\sin\theta$, horizontally and in the plane containing the rod and OA. The component at right angles to OG is $\omega^2\,(L/2)\sin\theta\cos\theta$ and the angular acceleration α about O in the vertical plane containing the rod and OA will be $\omega^2\sin\theta\cos\theta$

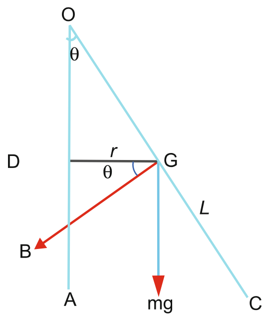
$$\text{Torque } m\,g\,r = mg\frac{L}{2}\sin\theta = I\alpha = m\frac{L^2}{3}\omega^2\sin\theta\cos\theta$$

$$\frac{mL}{2}\sin\theta\left[g - \frac{2L}{3}\omega^2\cos\theta\right] = 0$$

$$\theta = 0 \text{ or } \cos^{-1}\left(\frac{3g}{2\omega^2 L}\right)$$

If $3g > 2\omega^2 L$, i.e. $\omega^2 < \dfrac{3g}{2L}$, the only possible solution is $\theta = 0$, i.e. the rod hangs vertically. If $\omega^2 > \dfrac{3g}{2L}$, then $\theta = \cos^{-1}\dfrac{3g}{2\omega^2 L}$.

Fig. 4.27



4.43 **(a)** For pure sliding equation of motion is

$$\text{ma} = -\mu mg$$

$$\text{or} \quad a = -\mu g \tag{1}$$

$$v = v_0 - \mu gt \tag{2}$$

At the instant pure rolling sets in

$$\text{Torque } I\alpha = FR \tag{3}$$

$$\frac{2}{5}mR^2\alpha = \mu mgR$$

$$\therefore \quad \alpha = \frac{5}{2}\frac{\mu g}{R} \tag{4}$$

$$\omega = \alpha t = \frac{5}{2}\frac{\mu gt}{R} \tag{5}$$

Using (5) in (2)

$$v = v_0 - \frac{2}{5}\omega R = v_0 - \frac{2}{5}v$$

$$\therefore \quad v = \frac{5}{7}v_0$$

(b) $v = v_0 - \mu gt$

$$t = \frac{v_0 - v}{\mu g} = \frac{v_0 - (5/7)\,v_0}{\mu g} = \frac{2v_0}{7\mu g}$$

(c) $v^2 = v_0^2 + 2as$

$$= v_0^2 - 2\mu g s$$

$$\left(\frac{5}{7}v_0\right)^2 = v_0^2 - 2\mu g s$$

$$s = \frac{12}{49}\frac{v_0^2}{\mu g}$$

The assumption made is that we have either pure sliding or pure rolling. Actually in the transition both may be present.

4.44 $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$

Differentiating

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} + \boldsymbol{p} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{r} \times \boldsymbol{F} + \boldsymbol{p} \times \boldsymbol{v} = \boldsymbol{\tau} + 0 = \boldsymbol{\tau}$$

because the momentum and velocity vectors are in the same direction.
Angular momentum conservation requires that

$$|\boldsymbol{L}_\mathrm{i}| = |\boldsymbol{L}_\mathrm{f}| = mv\left(\frac{l}{2}\right)$$

$\boldsymbol{L} = (l/2)m\boldsymbol{v}$ is not correct because $\boldsymbol{L}$ is perpendicular to $\boldsymbol{v}$.

L conservation gives

$$mv\frac{l}{2} = \frac{1}{3}Ml^2\omega + m\omega\frac{l^2}{4}$$

$$\therefore \quad \omega = \frac{6mv}{(4M + 3m)l} \tag{1}$$

$$K_\mathrm{rot} = \frac{1}{2}I\omega^2 + \frac{1}{2}m\left(\frac{\omega l}{2}\right)^2 = \frac{3m^2v^2}{2(4M + 3m)}$$

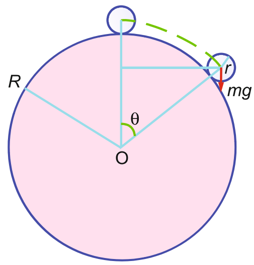
where we have used (1).

$$\therefore \quad \frac{K_\mathrm{rot}}{\frac{1}{2}mv^2} = \frac{3m}{4M + 3m} = \frac{3}{23}$$

where we have used $M = 5m$ (by problem).

4.45 Let the small sphere break off from the large sphere at angle θ with the vertical, Fig. 4.28. At that point the component of (gravitational force) – (centrifugal force) = reaction = 0

Fig. 4.28



$$mg\cos\theta = \frac{mv^2}{R+r} \tag{1}$$

Loss in potential energy = gain in kinetic energy

$$mg(R+r)(1-\cos\theta) = \frac{7}{10}mv^2 \tag{2}$$

Solving (1) and (2)

$$g(R+r) = \frac{17}{10}v^2 = \frac{17}{10}\omega^2 r^2$$

$$\therefore \quad \omega = \sqrt{\frac{10}{17}g\frac{(R+r)}{r^2}} \text{ and } \theta = \cos^{-1}\left(\frac{10}{17}\right)$$

4.46 Let N be the reaction of the floor and θ the angle which the rod makes with the vertical after time t, Fig. 4.29. The only forces acting on the rod are the weight and the reaction which act vertically and consequently the centre of mass moves in a straight line vertically downwards.
Equation of motion for the centre of mass is

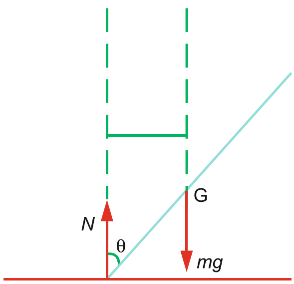


Fig. 4.29

$$mg - N = m\frac{d^2}{dt^2}(a - a\cos\theta)$$

$$\text{or} \quad mg - N = ma\left[\cos\theta\left(\frac{d\theta}{dt}\right)^2 + \sin\theta\frac{d^2\theta}{dt^2}\right] \tag{1}$$

The work–energy theorem gives

$$mga(1 - \cos\theta) = \frac{1}{2}m\left(\frac{d\theta}{dt}\right)^2\left[\frac{a^2}{3} + a^2\sin^2\theta\right]$$

where the square bracket has been written using the parallel axis theorem.

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{6g(1 - \cos\theta)}{a(1 + 3\sin^2\theta)} \tag{2}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{3g}{a}\left[\frac{\sin\theta(7 - 6\cos\theta - 3\sin^2\theta)}{(1 + 3\sin^2\theta)^2}\right] \tag{3}$$

By substituting $\left(\frac{d\theta}{dt}\right)^2$ and $\left(\frac{d^2\theta}{dt^2}\right)$ from (2) and (3) in (1), the reaction N is obtained as a function of θ. When the rod is about to strike the floor,

$$\theta = \frac{\pi}{2}; \ \left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{2a} \text{ and } \frac{d^2\theta}{dt^2} = \frac{3g}{4a}$$

Thus the reaction from (1) will be

$$N = m\left[g - \frac{3g}{4}\right] \text{ or } \frac{1}{4}mg$$

4.47 **(a)** For $\alpha_{net} = 0$, the two torques which act in the opposite sense must be equal (Fig. 4.30), i.e.

$$\tau_1 = \tau_2$$

$$\text{or} \quad m_1 g R_1 = m_2 g R_2$$

$$m_2 = \frac{m_1 R_1}{R_2} = \frac{25 \times 1.2}{0.5} = 60\,\text{kg}$$

(b) **(i)** $a_1 = \alpha R_1, \quad a_2 = \alpha R_2$ (1)

as $R_1 > R_2, a_1 > a_2$

(ii) Equations of motion are

$$m_1 a_1 = m_1 g - T_1 \tag{2}$$
$$m_2 a_2 = T_2 - m_2 g \tag{3}$$
$$T_1 R_1 - T_2 R_2 = I\alpha \tag{4}$$

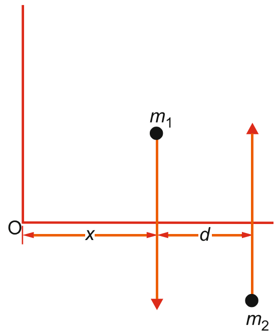
Combining (1), (2), (3) and (4) and substituting $m_1 = 35\,\text{kg}$, $m_2 = 60\,\text{kg}$, $R_1 = 1.2\,\text{m}$, $R_2 = 0.5\,\text{m}$, $I = 38\,\text{kg m}^2$ and $g = 9.8\,\text{m/s}^2$, we find

$$a_1 = \frac{(m_1 R_1 - m_2 R_2) R_1 g}{m_1 R_1^2 + m_2 R_2^2 + I}$$
$$= \frac{(35 \times 1.2 - 60 \times 0.5)1.2g}{35 \times 1.2^2 + 60 \times 0.5^2 + 38} = 0.139g$$
$$a_2 = a_1 \frac{R_2}{R_1} = 0.139 \times \frac{0.5}{1.2} = 0.058\,g$$
$$T_1 = m_1(g - a_1) = 35g(1 - 0.139) = 295.3\,\text{N}$$
$$T_2 = m_2(g + a_2) = 60g(1 + 0.058) = 622.1\,\text{N}$$
$$\alpha = \frac{T_1 R_1 - T_2 R_2}{I} = \frac{295.3 \times 1.2 - 622.1 \times 0.5}{38} = 1.14\,\text{rad/s}^2$$

4.48 $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$
$$= x\hat{i} \times (-mv\hat{j}) + (x + d)\hat{i} \times (mv\hat{j})$$
$$= mvd\hat{i} \times \hat{j} = mvd\hat{k}$$

which is independent of x and therefore independent of the origin.

Fig. 4.30



4.49 **(a)** $$K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2} \cdot \frac{2}{5}mr^2\frac{v^2}{r^2} = \frac{1}{5}mv^2$$
$$K_{\text{total}} = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2$$
$$\therefore \quad \frac{K_{\text{rot}}}{K_{\text{total}}} = \frac{(1/5)\,mv^2}{(7/10)\,mv^2} = \frac{2}{7}$$

(b) In coming down to the bottom of the hemisphere loss of potential energy $= mgh = mgR$. Gain in kinetic energy $= (7/10)\,mv^2$.

$$\therefore \quad \frac{7}{10}mv^2 = mgR$$

$$\text{or} \quad \frac{mv^2}{R} = \frac{10mg}{7}$$

The normal force exerted by the small sphere at the bottom of the large sphere will be

$$N = mg + \frac{mv^2}{R} = mg + \frac{10mg}{7} = \frac{17mg}{7}$$

4.50 Work done $W = \tau\theta = I\alpha\theta = \dfrac{I\omega^2}{2}$

Along the diameter for hoop, $I = mR^2/2$, while for the solid sphere, hollow sphere and the disc, $I = (2/5)\,mR^2$, $(2/3)\,mR^2$ and $(1/4)\,mR^2$, respectively, maximum work will have to be done to stop the hollow sphere, ω being identical as it has the maximum moment of inertia.

4.51 Work done $W = \tau\theta = I\alpha\theta = \dfrac{I\omega^2}{2} = \dfrac{J^2}{2I}$

where we have used the formula $J = I\omega$. Maximum work will have to be done for the disc since I is the least, τ being identical.

4.52 $W = \dfrac{1}{2}I\omega^2 = \dfrac{1}{2}(I\omega)\omega = \dfrac{1}{2}J\omega$

Since J and ω are the same for all the four objects, work done is the same.

4.53 $\tau = I\alpha = I\dfrac{a}{R} = \dfrac{MgR\sin\theta}{1 + \left(R^2/k^2\right)}$

For solid sphere, hollow sphere, solid cylinder and hollow cylinder the quantity $1 + (R^2/k^2)$ is 7/2, 5/2, 3, 2, respectively. Therefore τ will be least for solid sphere.

4.54 $\boldsymbol{r} = 3t\hat{i} + 2\hat{j}$

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = 3\hat{i}$$

$$L = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v}) = m(3t\hat{i} + 2\hat{j}) \times 3\hat{i}$$

$$= 6\,m(\hat{j} \times \hat{i}) = -6\,m\hat{k} \quad \text{(constant)}$$

4.55 Angular momentum conservation gives

$$J = mvd = I\omega \tag{1}$$

Linear momentum conservation gives

$$mv = Mv_{\mathrm{c}} \tag{2}$$

Energy conservation gives

$$\frac{1}{2}mv^2 = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_{\mathrm{c}}^2 \tag{3}$$

$$I = \frac{Ml^2}{12} \tag{4}$$

Eliminating ω and v_{c} from (1) and (2) and using (3)

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2d^2}{I} + \frac{1}{2}\frac{m^2v^2}{M} \tag{5}$$

Simplifying and using (4) in (5)

$$d = \frac{l}{2}\sqrt{\frac{M-m}{3m}}$$

4.56 **(a)** Let the initial velocity be v_0, then at instant t the velocity

$$v = v_0 - at = v_0 - \mu gt \tag{1}$$

Torque $\tau = I\alpha = FR$

$$\frac{1}{2}mR^2\alpha = \mu\,mgR$$

$$\alpha = \frac{2\mu g}{R}$$

$$\mu gt = \frac{\alpha Rt}{2} = \frac{\omega R}{2}$$

Therefore (1) becomes

$$v = v_0 - \frac{\omega R}{2} = v_0 - \frac{v}{2}$$

$$\therefore \quad v = \frac{2}{3} v_0 \tag{2}$$

Using (2) in (1)

$$\frac{2}{3} v_0 = v_0 - \mu\, gt$$

$$\text{or} \quad t = \frac{v_0}{3\,\mu g}$$

(b) Work done $W = \Delta K = \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2$

$$= \frac{1}{2} m \left[\frac{4}{9} v_0^2 - v_0^2 \right] = -\frac{5}{18} m v_0^2$$

4.57 Equation of motion is

$$ma = mg - 2T \tag{1}$$

where 'a' is the linear acceleration and T the tension in each thread.

Torque $\quad I\alpha = 2Tr \qquad (\because$ there are two threads)

$$\frac{1}{2} m r^2 \alpha = 2Tr$$

$$\text{or} \quad \alpha = \frac{4T}{mr} \tag{2}$$

As both the cylinders are rotating,

$$a = 2\alpha r = \frac{8T}{m} \tag{3}$$

$$\text{or} \quad ma = 8T \tag{4}$$

Using (4) in (1) we get

$$T = \frac{1}{10} mg$$

Note that if the lower cylinder is not wound then

$$a = \frac{4T}{m} \quad \text{and} \quad T = \frac{1}{6}mg$$

4.58 C is the centre of the disc and A the point which is fixed, Fig. 4.31. The forces acting at A have no torque at A, so that the angular momentum is conserved. Initially the moment of inertia of the disc about the axis passing through its centre and perpendicular to its plane is

$$I_c = I = \frac{1}{2}mr^2 \tag{1}$$

When the point A is fixed the moment of inertia about an axis parallel to the central axis and passing through A will be

$$I_A = I_c + mr^2 = m\left(\frac{1}{2}r^2 + r^2\right) = \frac{3}{2}mr^2 \tag{2}$$

by parallel axis theorem.
Angular momentum conservation requires

$$I_A\omega' = I_c\omega \tag{3}$$

Substituting (1) and (2) in (3) we obtain

$$\omega' = \frac{\omega}{3} \tag{4}$$

If X and Y are the impulses of the forces at A perpendicular and along CA, then

$$X = m\,r\omega' = mr\frac{\omega}{3} \quad \text{and} \quad Y = 0$$

Thus the impulse of the blow at A is $mr\dfrac{\omega}{3}$ at right angles to CA.

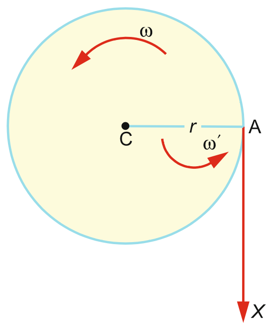


Fig. 4.31

4.59 The torque of the air resistance on an element dx at distance x from the fixed end, about this end, will be

$$d\tau = k(\omega x)^2 x \, dx = k\omega^2 x^3 dx$$

$$\tau = \int d\tau = -k\omega^2 \int_0^L x^3 dx = I\alpha$$

i.e. $$-\frac{k\omega^2 L^4}{4} = \frac{1}{3} m L^2 \frac{d\omega}{dt}$$

$$\therefore \quad -3kL^2 dt = 4m \frac{d\omega}{\omega^2}$$

$$\therefore \quad -3kL^2 t = -\frac{4m}{\omega} + C$$

where C is the constant of integration. Initial condition: when $t = 0$, $\omega = \Omega$.

Therefore $C = \dfrac{4m}{\Omega}$

$$\therefore \quad -3kL^2 t = 4m \left(\frac{1}{\Omega} - \frac{1}{\omega} \right)$$

$$\therefore \quad \omega = \frac{4m\Omega}{4m + 3\Omega k L^2 t}$$

4.60 OA is the vertical radius b of the cylinder and a the radius of the sphere which is vertical in the lowest position and shown as CA, Fig 4.32.
In the time the centre of mass of the sphere C has moved to C′ through an angle θ, the sphere has rotated through ϕ so that the reference lime CA has gone into the place of C′D.

If there is no slipping

$$a(\phi + \theta) = b\theta \tag{1}$$

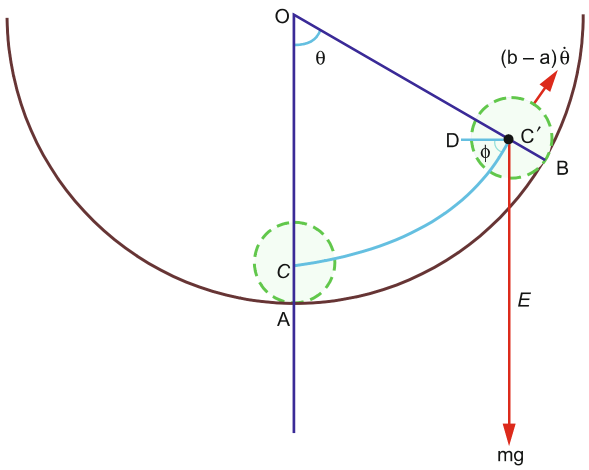
The velocity of the centre of mass is $(b - a)\dot{\theta}$ and the angular velocity of the sphere about its centre is

$$\dot{\phi} = \frac{(b-a)}{a}\dot{\theta} \tag{2}$$

Taking A as zero level, the potential energy

$$U = mg(b-a)(1 - \cos\theta) \tag{3}$$

Fig. 4.32



The kinetic energy $= T\,(\text{trans}) + T\,(\text{rot})$

$$T = \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}I\,\dot{\phi}^2$$

$$= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}\cdot\frac{2}{5}ma^2\frac{(b-a)^2}{a^2}\dot{\theta}^2$$

$$= \frac{7}{10}m(b-a)^2\dot{\theta}^2 \tag{4}$$

where we have used (2).
Total energy

$$E = T + U = \frac{7}{10}m(b-a)^2\dot{\theta}^2 + mg(b-a)(1-\cos\theta) = \text{constant} \tag{5}$$

Differentiating with respect to time and cancelling common factors

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{7}{5}m(b-a)\ddot{\theta}\cdot\dot{\theta} + g\sin\theta\cdot\dot{\theta} = 0 \tag{6}$$

$$\text{or}\quad \ddot{\theta} + \frac{5g}{7(b-a)}\sin\theta = 0 \tag{7}$$

For small oscillation angles $\sin\theta \to \theta$.

$$\therefore\quad \ddot{\theta} + \frac{5g\,\theta}{7(b-a)} = 0 \tag{8}$$

which is the equation for simple harmonic motion with frequency

$$\omega = \sqrt{\frac{5g}{7(b-a)}} \tag{9}$$

and time period

$$T = 2\pi\sqrt{\frac{7(b-a)}{5g}} \tag{10}$$

4.61 **(a)** Let the disc be composed of a number of concentric rings of infinitesimal width. Consider a ring of radius r, width dr and surface density σ (mass per unit area). Then its mass will be $(2\pi r \mathrm{d}r)\sigma$. The moment of inertia of the ring about an axis passing through the centre of the ring and perpendicular to its plane will be

$$\mathrm{d}I = (2\pi r\,\mathrm{d}r)\sigma r^2$$

Then the moment of inertial of the disc

$$I = \int \mathrm{d}I = 2\pi\sigma \int_0^R r^3 \mathrm{d}r = \frac{1}{2}\pi\,\sigma\,R^4 \tag{1}$$

If M is the mass of the disc, then

$$\sigma = \frac{M}{\pi R^2} \tag{2}$$

$$\therefore \quad I = \frac{1}{2}MR^2 \tag{3}$$

(b) The total kinetic energy T of the disc on the horizontal surface is

$$T(\text{initial}) = \frac{1}{2}Mu^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}Mu^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\frac{u^2}{R^2} = \frac{3}{4}Mu^2 \tag{4}$$

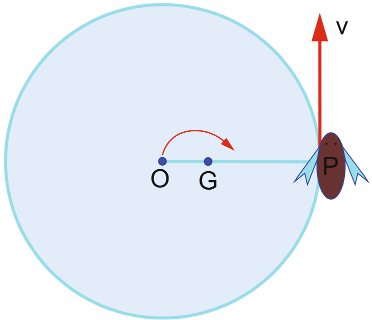
$$T(\text{final}) = \frac{3}{4}Mv^2 = \frac{3}{4}Mu^2 + Mgh$$

by energy conservation

$$\text{Solving,}\ \ v = \sqrt{u^2 + \frac{4}{3}gh}$$

4.62 In Fig. 4.33, O is the centre of the ring, P the instantaneous position of the insect and G the centre of mass of the system. Suppose the insect crawls

Fig. 4.33



around the ring in the counterclockwise sense. The only forces acting in a horizontal plane are the reactions at P which are equal and opposite. Consequently G will not move and the angular momentum about G which was zero initially will remain zero throughout the motion due to its conservation.

$$m \cdot \mathrm{PG}(v - \mathrm{PG}\omega) - I_{\mathrm{G}}\omega = 0 \tag{1}$$

where ω is the angular velocity of the ring.

$$\text{Now} \quad \mathrm{PG} = \frac{Mr}{M+m}, \quad \mathrm{OG} = \frac{mr}{M+m} \tag{2}$$

$$I_{\omega} = I_{\mathrm{CM}} + M(\mathrm{OG})^2 = Mr^2 + M\frac{m^2r^2}{(M+m)^2} \tag{3}$$

Using (2) and (3) in (1) and simplifying we obtain

$$\omega = \frac{mv}{(M+2m)r} \tag{4}$$

4.3.3 Coriolis Acceleration

4.63 **(a)** $\boldsymbol{\omega}$ points in the south to north direction along the rotational axis of the earth.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86,160} = 7.292 \times 10^{-5}\ \mathrm{rad/s}$$

(b) The period of rotation of the plane of oscillation is given by

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\omega \sin\lambda} = \frac{T_0}{\sin\lambda} = \frac{24}{\sin 30°} = 48\ \mathrm{h}$$

4.64 The object undergoes an eastward deviation through a distance

$$d = \frac{1}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}} = \frac{1}{3} \times 7.29 \times 10^{-5} \times \cos 0^\circ\sqrt{\frac{8 \times 400^3}{9.8}} = 0.1756\,\text{m}$$
$$= 17.56\,\text{cm}$$

4.65 $$y' = \frac{4}{3}\frac{u^3}{g^2}\omega\cos\lambda$$
$$= \frac{4}{3} \times \frac{(20)^3}{(9.8)^2} \times 7.29 \times 10^{-5}\cos 0^\circ = 0.0081\,\text{m} = 8.1\,\text{mm}$$

4.66 $$y' = \frac{4}{3}\frac{u^3}{g^2}\omega\cos\lambda, \lambda = 0^\circ$$
$$u = \left[\frac{3y'g^2}{4\omega\cos\lambda}\right]^{1/3} = \left[\frac{3}{4} \times \frac{1 \times (9.8)^2}{7.27 \times 10^{-5}}\right]^{1/3} = 99.7\,\text{m/s}$$

4.67 Consider two coordinate systems, one inertial system S and the other rotating one S', which are rotating with constant angular velocity ω

Acceleration in inertial frame		Acceleration in rotating frame		Coriolis acceleration		centrifugal acceleration	
$\frac{d^2\boldsymbol{r}}{dt^2}$	=	$\frac{d^2\boldsymbol{r}'}{dt^2}$	+	$2\vec{\omega} \times \frac{d\boldsymbol{r}'}{dt}$	+	$\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{r}$	(1)

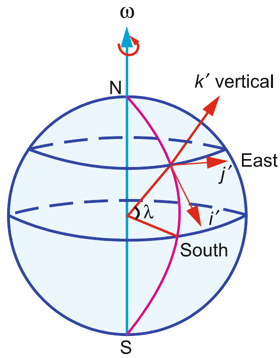
Let the k axis in the inertial frame S be directed along the earth's axis. Let the rotating frame S' be rigidly attached to the earth at a geographical latitude λ in the northern hemisphere. Let the k' axis be directed outwards at the latitude λ along the plumb line, whose direction is that of the resultant passing through the earth's centre. With the choice of a right-handed system, the i'-axis is in the southward direction and the j'-axis in the eastward direction, Fig. 4.34. Assume g the acceleration due to gravity to be constant. It includes the centrifugal term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ since g is supposed to represent the resultant acceleration of a falling body at the given place.

$$\frac{d^2r'}{dt^2} = g - 2\boldsymbol{\omega} \times \boldsymbol{v}_R \tag{2}$$

Since we are considering the fall of a body in the northern hemisphere, the components of angular velocity are

$$\left.\begin{aligned} \omega_x &= -\omega\cos\lambda \\ \omega_y &= 0 \\ \omega_z &= \omega\sin\lambda \end{aligned}\right\} \tag{3}$$

Fig. 4.34



$$\boldsymbol{\omega} \times \boldsymbol{v}_R = \begin{vmatrix} i' & j' & k' \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix}$$

$$= -\omega\sin\lambda\,\dot{y}'i' + (\omega\sin\lambda\,\dot{x}' + \omega\cos\lambda\,\dot{z}')\,j' - (\omega\cos\lambda\,\dot{y}')k'$$

But $\dfrac{d^2\boldsymbol{r}'}{dt^2} = g - 2(\boldsymbol{\omega} \times \boldsymbol{v})$

$$\therefore \quad \ddot{x}'\,i' + \ddot{y}'\,j' + \ddot{z}'\,k' = -g\,k' + 2\omega\sin\lambda\,\dot{y}'i' - 2(\omega\sin\lambda\,\dot{x}' + \omega\cos\lambda\,\dot{z}')\,j' + 2\omega\cos\lambda\,\dot{y}'k' \tag{4}$$

Equating coefficients of i', j' and k'on both sides of (4), we obtain the equations of motion

$$\ddot{x}' = 2\omega\ \sin\lambda\,\dot{y}' \tag{5}$$
$$\ddot{y}' = -2(\omega\ \sin\lambda\,\dot{x}' + \omega\cos\lambda\,\dot{z}') \tag{6}$$
$$\ddot{z}' = -g + 2\omega\cos\lambda \tag{7}$$

Now the quantities $\dot{x}'$ and $\dot{y}'$ are small compared to $\dot{z}'$. To the first approximation we can write

$$(v_R)_x = 0; \quad (v_R)_y = 0; \quad (v_R)_z = \dot{z}' = -g \tag{8}$$

Setting $\dot{x}' = \dot{y}' = 0$ in (5), (6) and (7), we obtain the equations for the components of a_R:

$$(a_R)_x = \ddot{x}' = 0 \tag{9}$$
$$(a_R)_y = \ddot{y}' = -2\omega\,\dot{z}'\cos\lambda \tag{10}$$
$$(a_R)_z = \ddot{z}' = -g \tag{11}$$

Equation (9) shows that no deviation occurs in the north–south direction.

Integrating (11)

$$\dot{z}' = -gt \tag{12}$$

and $$z' = -\frac{1}{2}gt^2 \tag{13}$$

with the initial condition that at $t = 0$, $\dot{z}' = 0$, $z' = 0$.

Using (12) in (10) and integrating twice

$$\dot{y}' = \omega gt^2 \cos\lambda \tag{14}$$

because $(\dot{y}')_0 = 0$.

$$y' = \frac{1}{3}\omega gt^3 \cos\lambda \tag{15}$$

because $(y')_0 = 0$.

Setting $-z' = h = (1/2)\,gt^2$, or $t = \sqrt{2h/g}$, in (15) the body undergoes eastward deviation through a distance

$$d = y' = \frac{1}{3}\omega \cos\lambda \sqrt{\frac{8h^3}{g}} \tag{16}$$

4.68 $\boldsymbol{F}_{\text{coriolis}} = -2m\boldsymbol{\omega} \times \boldsymbol{v}_{\text{R}}$

$$F_{\text{cor}} = 2m\omega v_{\text{R}} \sin\theta = 2 \times 5 \times 10^8 \times 7.27 \times 10^{-5} \times \frac{8000}{86,400} \quad (\because \theta = 90°)$$

$$= 6730\,\text{N due north}$$

4.69 Coriolis action on a mass m of water towards the eastern side (Fig. 4.35) is

$$m\ddot{y}' = 2mv\omega \sin\lambda \tag{1}$$

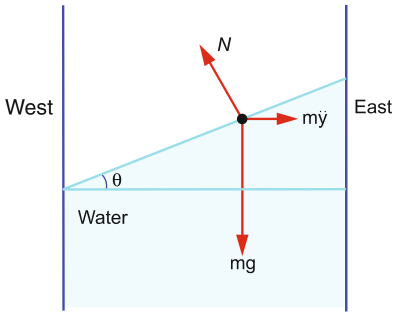


Fig. 4.35

Let N be the normal reaction and let the water level be tilted through an angle θ. Resolve N into horizontal and vertical components and balance them with the Coriolis force and the weight, respectively.

$$N \sin\theta = 2m\, v\omega\, \sin\lambda$$

$$N \cos\theta = mg$$

Dividing the equations, $\tan\theta = \dfrac{d}{b} = 2\, v\, \omega \sin\lambda$

or $\quad d = \dfrac{2bv\omega}{g} \sin\lambda$

4.70 By eqn. (15) prob. (4.67)

$$y' = \frac{1}{3}\omega g t^3 \cos\lambda \tag{1}$$

$$z' = \frac{1}{2} g t^2 \tag{2}$$

Eliminate t between (1) and (2) to find

$$\frac{y'^2}{z'^3} = \frac{8}{9}\,\frac{\omega^2 \cos^2\lambda}{g}$$

or $\quad y'^2 = C z'^3 \quad$ (semi-cubical parabola)

where $C =$ constant.

4.71 $F_{\text{cor}} = 2m\, v\, \omega\, \sin\lambda$

$$= 2 \times 10^6 \times 15 \times 7.27 \times 10^{-5} \sin 60^\circ$$

$$= 1889\,\text{N on the right rail.}$$

4.72 The difference between the lateral forces on the rails arises because when the train reverses its direction of motion Coriolis force also changes its sign, the magnitude remaining the same. Therefore, the difference between the lateral force on the rails will be equal to $2m\, v\, \omega\, \cos\lambda - (-2m\, v\, \omega \cos\lambda)$ or $4mv\omega \cos\lambda$.

4.73 The displacement from the vertical is given by

$$y' = \left(\frac{1}{3} g t^3 - u t^2\right) \omega \cos\lambda$$

$$= \left(\frac{1}{3} \times 9.8 \times 10^3 - 100 \times 10^2\right) \times 7.27 \times 10^{-5} \cos 60^\circ$$

$$= -0.245\,\text{m} = -24.5\,\text{cm}$$

Thus the body has a displacement of 24.5 cm on the west.

Chapter 5
Gravitation

Abstract *Chapter 5* involves problems on gravitational field and potential for various situations variation of g, rocket motion, orbital motion of planets, satellites and meteorites, circular and elliptic motion, bound and unbound orbits, Kepler's laws, equation of motion under various types of forces.

5.1 Basic Concepts and Formulae

$$F = -Gm_1m_2/r^2 \text{ (gravitational force)} \tag{5.1}$$

The negative sign shows that the force is attractive.

When SI units are used the gravitational constant

$$G = 6.67 \times 10^{-11}\ \mathrm{kg}^{-1}\ \mathrm{m}^3\ \mathrm{s}^{-2}$$

The intensity or field strength g of a gravitational field is equal to the force exerted on a unit mass placed at that point.

$$g = -Gm/r^2 \tag{5.2}$$

The (negative) gravitational potential at a given point, due to any system of masses, is the work done in bringing a unit mass from infinity up to that point. The zero potential is chosen conventionally at infinity. Symbolically

$$g = -\frac{\partial V}{\partial r} \tag{5.3}$$

$$V = -Gm/r \tag{5.4}$$

The potential energy

$$U = -GMm/r \tag{5.5}$$

Spherical Shell

The gravitational intensity due to a spherical shell of radius a.

$$\begin{aligned} g(r) &= 0 \; (r < a) \\ &= -GM/r^2 \quad (r > a) \end{aligned} \tag{5.6}$$

where r is measured from the centre of the shell. The potential

$$\begin{aligned} V(r) &= -GM/a \quad (r < a) \\ &= -GM/r \quad (r > a) \end{aligned} \tag{5.7}$$

Uniform Solid Sphere

$$\begin{aligned} g(r) &= -GMr/a \quad (r \leq a) \\ &= -GM/r^2 \quad (r \geq a) \end{aligned} \tag{5.8a}$$

$$\begin{aligned} V(r) &= -\frac{GM}{2a}\left(3 - \frac{r^2}{a^2}\right) (r \leq a) \\ &= -GM/r \quad (r \geq a) \end{aligned} \tag{5.8b}$$

Potential energy of a uniform sphere

$$U = -3GM^2/5a \tag{5.9}$$

Variation of *g* on Earth

(a) Altitude: $g = g_0/(1 + h/R)^2$ (5.10)

$$g = g_0\left(1 - \frac{2h}{R}\right) \; (h << R) \tag{5.10a}$$

(b) Latitude (λ) (at sea level)

$$g_0 = 9.83215 - 0.05178 \cos^2 \lambda \tag{5.11}$$

Formula (5.11) is accurate to better than two parts in a million.

(c) Rotation of earth:

$$g' = g - R\omega^2 \cos^2 \lambda \tag{5.12}$$

where $\omega = 7.27 \times 10^{-5}$ /s and $R = 6.4 \times 10^6$ m

(d) Depth (d) (constant density model)

$$g = g_0\,(1 - d/R) \tag{5.13}$$

Relation Between g and G

$$g = GM/r^2$$

where M is the mass of parent body and $r \geq R$.

Kepler's Laws

(i) All planets move in an elliptic path, with the sun at one focus. This is a consequence of inverse square law of gravitation and the constancy of total energy and angular momentum.
(ii) A line drawn from the sun to the planet sweeps out equal areas in equal times. This is a consequence of the constancy of angular momentum.
(iii) The squares of the period of rotation of planets about the sun are proportional to the cubes of the semi-major axes of the ellipses. This is a consequence of the inverse square law of gravitation for circular orbits.

Central force is a conservative force which acts along a line connecting the centres of particles.

If $\mathbf{F}$ is a central force then curl $\mathbf{F} = 0$.

Areal velocity (C) and the angular momentum (J):

$$J = 2mC \tag{5.14}$$

where m is the mass of the orbiting body.

Orbits of Planets and Satellites

Circular orbits:

$$\text{Orbital velocity } v_0 = \sqrt{GM/r} = \sqrt{gr} \tag{5.15}$$

$$\text{Escape velocity } v_e = \sqrt{2}v_0 = \sqrt{2GM/R} = \sqrt{2g_0 R} \tag{5.16}$$

Time period

$$T = 2\pi\sqrt{r^3/GM} \tag{5.17}$$

If the planet's mass cannot be ignored in comparison with the sun's mass then (5.17) is modified as

$$T = 2\pi\sqrt{r^3/G(M+m)} \tag{5.18}$$

Total energy:

$$E = -GMm/2r \tag{5.19}$$

Elliptic Orbits

Orbital velocity

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{5.20}$$

where r is the distance of the planet/satellite from the centre of parent body and a is the semi-major axis.

The eccentricity

$$\varepsilon = \sqrt{1 + \frac{2E\,J^2}{G^2M^2m^3}} \tag{5.21}$$

Total energy

$$E = -\frac{G^2M^2m^3}{2J^2}(1 - \varepsilon^2) \tag{5.22}$$

$$E = -GMm/2a \tag{5.23}$$

When the orbiting body is at the maximum distance from the parent body then $r = r_{\max}$ is called aphelion and the minimum distance $r = r_{\min}$ is called perihelion for the planetary motion. For the satellites the corresponding terms are apogee and perigee. At both perigee (perihelion) and apogee (aphelion) the velocity of the orbiting body is perpendicular to the radius vector and they constitute the turning points.

$$\varepsilon = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} \tag{5.24}$$

$$\varepsilon = \frac{v_{\max} - v_{\min}}{v_{\max} + v_{\min}} \tag{5.25}$$

Classification of Orbits

Circle: $\varepsilon = 0 \quad E < 0$
Ellipse: $0 < \varepsilon < 1 \quad E < 0$
Parabola: $\varepsilon = 1 \quad E = 0$
Hyperbola: $\varepsilon > 1 \quad E > 0$
To determine the law of force, given the orbit by (r, θ) equation. Let f represent force per unit mass. Using the formula

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \tag{5.26}$$

$$f = -\frac{h^2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} \tag{5.27}$$

where p is the impact parameter and h is the angular momentum per unit mass.

5.2 Problems

5.2.1 Field and Potential

5.1 Calculate the gravitational force between two lead spheres of radius 10 cm in contact with one another, $G = 6.67 \times 10^{-11}$ MKS units. Density of lead $= 11{,}300\,\mathrm{kg/m^3}$.

[University of Dublin]

5.2 Considering Fig. 5.1, what is the magnitude of the net gravitational force exerted on the uniform sphere, of mass 0.010 kg, at point P by the other two uniform spheres, each of mass 0.260 kg, that are fixed at points A and B as shown.

[The University of Wales, Aberystwyth 2005]

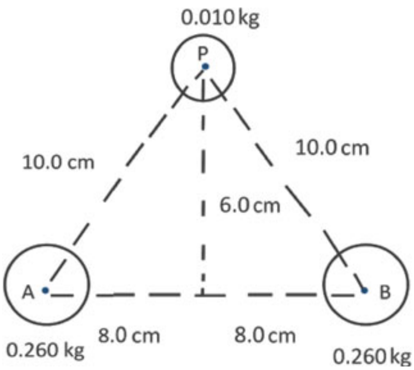


Fig. 5.1

5.3 Two bodies of mass m and M are initially at rest in an inertial reference frame at a great distance apart. They start moving towards each other under gravitational attraction. Show that as they approach a distance d apart ($d << r$), their relative velocity of approach will be $\sqrt{\dfrac{2G(M+m)}{d}}$, where G is the gravitational constant.

5.4 If the earth suddenly stopped in its orbit assumed to be circular, find the time that would elapse before it falls into the sun.

5.5 Because of the rotation of the earth a plumb bob when hung may not point exactly in the direction of the earth's gravitational force on the plumb bob. It may slightly deviate through a small angle.

(a) Show that at latitude λ, the deflection angle θ in radians is given by

$$\theta = \left(\frac{2\pi^2 R}{gT^2}\right)\sin 2\lambda$$

where R is the radius of earth and T is the period of the earth's rotation.
(b) At what latitude is the deflection maximum?
(c) What is the deflection at the equator?

5.6 Show that the gravitational energy of earth assumed to be the uniform sphere of radius R and mass M is $3GM^2/5R$. What is the potential energy of earth assuming it to be a uniform sphere of radius $R = 6.4 \times 10^6$ m and of mass $M = 6.0 \times 10^{24}$ kg.

5.7 Assuming that the earth has constant density, at what distance d from the earth's surface the gravity above the earth is equal to that below the surface.

5.8 Assuming the radius of the earth to be 6.38×10^8 cm, the gravitational constant to be $6.67 \times 10^{-8}\ \text{cm}^3\ \text{g/m/s}^2$, acceleration due to gravity on the surface to be $980\ \text{cm/s}^2$, find the mean density of the earth.

[University of Cambridge]

5.9 How far from the earth must a body be along a line towards the sun so that the sun's gravitational pull balances the earth? The sun is about 9.3×10^7 km away and its mass is $3.24 \times 10^5\ M_e$, where M_e is the mass of the earth.

5.10 Assuming the earth to be a perfect sphere of radius 6.4×10^8 cm, find the difference due to the rotation of the earth in the value of g at the poles and at the equator.

[Northern Universities of UK]

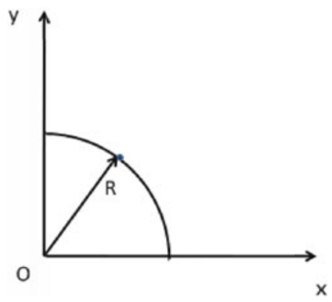
5.11 Derive an expression for the gravitational potential $V(r)$ due to a uniform solid sphere of mass M and radius R when $r < R$.

5.12 Derive an expression for the potential due to a thin uniform rod of mass M and length L at a point distant d from the centre of the rod on the axial line of the rod.

5.13 Show that for a satellite moving close to the earth's surface along the equator, moving in the western direction will require launching speed 11% higher than that moving in the eastern direction.

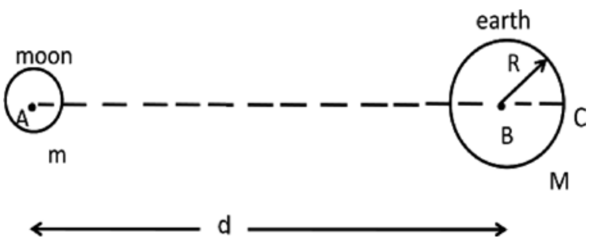
5.14 A thin wire of linear mass density λ is bent in the form of a quarter circle of radius R (Fig. 5.2). Calculate the gravitational intensity at the centre O.

Fig. 5.2



5.15 A tidal force is exerted on the ocean by the moon. This is estimated by the differential (Δg) which is the difference of the acceleration at B and that at C due to the moon (Fig. 5.3). If R is the radius of the earth, d the distance of separation of the centre of earth and moon, M and m the mass of the earth and moon, respectively, show that $\Delta g \approx \dfrac{2GmR}{d^3}$.

Fig. 5.3



5.16 Assume that a star has uniform density. Show that the gravitational pressure $P \propto V^{-4/3}$, where V is the volume.

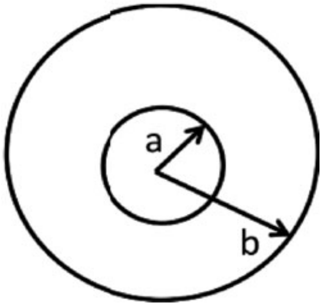
5.17 Find the gravitational field due to an infinite line mass of linear density λ, at distance R.

5.18 If the earth–moon distance is d and the mass of earth is 81 times that of the moon, locate the neutral point on the line joining the centres of the earth and moon.

5.19 A particle of mass m was taken from the centre of the base of a uniform hemisphere of mass M and radius R to infinity. Calculate the work done in overcoming gravitational force due to the hemisphere.

5.20 The cross-section of a spherical shell of uniform density and mass M and of radii a and b is shown in Fig. 5.4. How does the gravitational field vary in the region $a < r < b$?

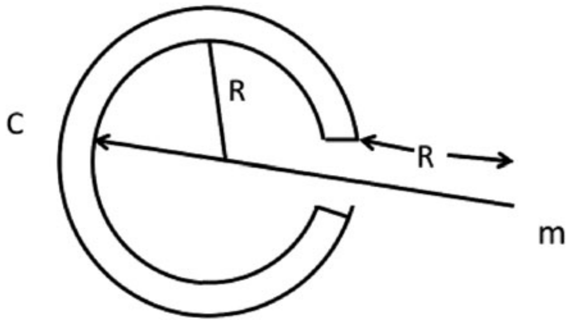
Fig. 5.4



5.21 Find the variation of the magnitude of gravitational field along the z-axis due to a disc of radius 'a' and surface density σ, lying in the xy-plane.

5.22 Figure 5.5 shows a spherical shell of mass M and radius R in a force-free region with an opening. A particle of mass m is released from a distance R in front of the opening. Calculate the speed with which the particle will hit the point C on the shell, opposite to the opening.

Fig. 5.5



5.2.2 *Rockets and Satellites*

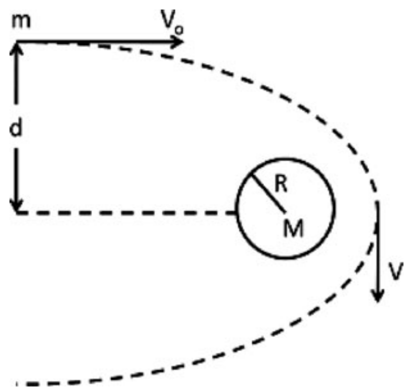
5.23 A particle of mass m is fired upwards from the surface of a planet of mass M and radius R with velocity $v = \sqrt{\dfrac{GM}{2R}}$. Show that the maximum height which the particle attains is $R/3$.

5.24 Consider a nebula in the form of a ring of radius R and mass M. A star of mass $m(m << M)$ is located at distance r from the centre of the ring on its axis, initially at rest. Show that the speed with which it crosses the centre of the ring will be $v = \sqrt{(2-\sqrt{2})\dfrac{GM}{R}}$.

5.25 If W_1 is the work done in taking the satellite from the surface of the earth of radius R to a height h, and W_2 the extra work required to put the satellite in the orbit at altitude h, and if $h = R/2$ then show that the ratio $\dfrac{W_1}{W_2} = 1.0$.

5.26 An asteroid is moving towards a planet of mass M and radius R, from a long distance with initial speed v_0 and impact parameter d (Fig. 5.6). Calculate the minimum value of v_0 such that the asteroid does not hit the planet.

Fig. 5.6



5.27 The orbits of earth and Venus around the sun are very nearly circular with mean radius of the earth's orbit $r_E = 1.50 \times 10^{11}$ m and mean radius of Venus' orbit $r_V = 1.08 \times 10^{11}$ m. If the earth's period of orbit round the sun is 365.3 days and Venus is 224.7 days

(i) Show that these figures are approximately consistent with Kepler's third law.
(ii) Derive a formula to estimate the mass of the sun ($G = 6.67 \times 10^{-11}$ N m^2/kg^2).

[The University of Aberystwyth, Wales]

5.28 The greatest and least velocities of a certain planet in its orbit around the sun are 30.0 and 29.2 km/s. Find the eccentricity of the orbit.

5.29 A binary star is formed when two stars bound by gravity move around a common centre of mass. Each component of a binary star has period of revolution about their centre of mass, equal to 14.4 days and the velocity of each component of 220 km/s. Further, the orbit is nearly circular. Calculate **(a)** the separation of the two components and **(b)** the mass of each component.

5.30 A satellite is fired from the surface of the moon of mass M and radius R with speed v_0 at 30° with the vertical. The satellite reaches a maximum distance of $5R/2$ from the centre of the planet. Show that $v_0 = (5GM/4R)^{1/2}$.

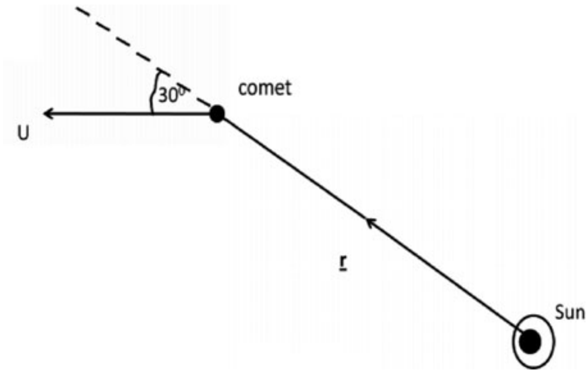
5.31 If a satellite has its largest and smallest speeds given by $v_{\max}$ and $v_{\min}$, respectively, and has time period equal to T, then show that it moves on an elliptic path of semi-major axis $\dfrac{T}{2\pi}\sqrt{v_{\max} v_{\min}}$.

5.32 A satellite of radius 'a' revolves in a circular orbit about a planet of radius b with period T. If the shortest distance between their surfaces is c, prove that the mass of the planet is $4\pi^2(a + b + c)/GT^2$.

5.33 When a comet is at a distance 1.75 AU from the sun, it is moving with velocity $u = 30\,\text{km/s}$ and its velocity vector is at an angle of 30° relative to its radius vector **r** centred on the sun (see Fig. 5.7).
What is the angular momentum per unit mass of the comet about the sun?
The closest distance from the sun that the comet reaches is 0.39 AU. What is the speed of the comet at this point?
Is the comet's orbit bound or unbound?
(1 AU = 1.5×10^{11} m, mass of the sun = 2×10^{30} kg)

[University of Durham 2002]

Fig. 5.7



5.34

(a) Assuming that the earth (mass M_E) orbits the sun (mass M_S) in a circle of radius R and with a speed v, write down the equation of motion for the earth. Hence show that $GM_S = v^2 R$

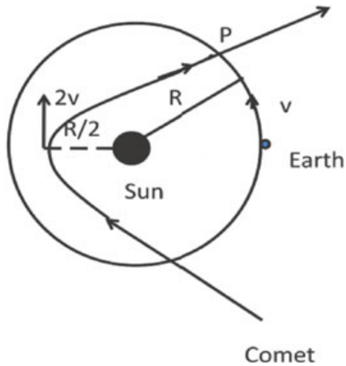
(b) A comet is in orbit around the sun in the same plane as the earth's orbit, as shown in Fig. 5.8. Its distance of closest approach to the sun's centre is $R/2$, at which point it has speed $2v$.
Using the condition for the Earth's orbit given in **(a)**, show that the comet's total energy is zero. (Neglect the effect of the earth on the comet.)

(c) Use conservation of angular momentum to determine the component of the comet's velocity which is tangential to the earth's orbit at the point P, where the comet's orbit crosses that of the earth.

(d) Use conservation of energy to find its speed at the point P. Hence show that the comet crosses the earth's orbit at an angle of 45°.

[University of Manchester 2008]

Fig. 5.8



5.35 The geocentric satellite 'Apple' was first launched into an elliptic orbit with the perigee (nearest point) of $r_p = 6570\,\text{km}$ and apogee (farthest point) at $r_A = 42{,}250\,\text{km}$. The respective velocities were $v_p = 10.25\,\text{km/s}$ and $v_A = 1.594\,\text{km/s}$. Show that the above data are consistent with the conservation of angular momentum of the satellite about the centre of the earth.

5.36 **(a)** Assuming that the earth is a sphere of radius 6400 km, with what velocity must a projectile be fired from the earth's surface in order that its subsequent path be an ellipse with major axis equal to 80,000 km?

(b) If the projectile is fired upwards at an angle 45° to the vertical, what would be the eccentricity of this ellipse?

5.37 A satellite of mass m is orbiting in a circular orbit of radius r and velocity v around the earth of mass M. Due to an internal explosion, the satellite breaks into two fragments each of mass $m/2$. In the frame of reference of the satellite, the two fragments appears to move radially along the line joining the original satellite and the centre of the earth, each with the velocity $v_0/2$. Show that immediately after the explosion each fragment has total energy $-3GM/16r$ and angular momentum $\frac{m}{2}\sqrt{GMr}$, with reference to the centre of the earth.

5.38 A particle describes an ellipse of eccentricity e under a force to a focus. When it approaches the nearer apse (turning point) the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is $\varepsilon(3+\varepsilon)/(1-\varepsilon)$.

5.39 A particle of mass m describes an elliptical orbit of semi-major axis 'a' under a force mk/r^2 directed to a focus. Prove that

(a) the time average of reciprocal distance

$$\frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{1}{a}$$

(b) the time average of square of the speed $\frac{1}{T}\int v^2 \mathrm{d}t = \frac{GM}{a}$

5.40 A small meteor of mass m falls into the sun when the earth is at the end of the minor axis of its orbit. If M is the mass of the sun, find the changes in the major axis and in the time period of the earth.

5.41 A particle is describing an ellipse of eccentricity 0.5 under the action of a force to a focus and when it arrives to an apse (turning point) the velocity is doubled. Show that the new orbit will be a parabola or hyperbola accordingly as the apse is the farther or nearer one.

5.42 When a particle is at the end of the minor axis of an ellipse, the force is increased by half. Prove that the axes of the new orbit are $3a/2$ and $\sqrt{2}b$, where $2a$ and $2b$ are the old axes.

5.43 A satellite is placed in a circular orbit of radius R around the earth.

(a) What are the forces acting on the satellite? Write down the equilibrium condition.
(b) Derive an expression for the time period of the satellite.
(c) What conditions must be satisfied by a geocentric satellite?
(d) What is the period of a geosynchronous satellite?
(e) Calculate the radius of orbit of a geocentric satellite from the centre of the earth.

5.44 A satellite moves in an elliptic path with the earth at one focus. At the perigee (nearest point) its speed is v and its distance from the centre of the earth is r. What is its speed at the apogee (farthest point)?

5.45 A small body encounters a heavy body of mass M. If at a great distance the velocity of the small body is v and the impact parameter is p, and φ is the angle of encounter, prove that $\tan(\varphi/2) = GM/v^2 p$.

5.46 Obtain an expression for the time required to describe an arc of a parabola under the action of the force k/r^2 to the focus, starting from the end of the axis.

5.47 A comet describes a parabolic path in the plane of the earth's orbit, assumed to be circular. Show that the maximum time the comet is able to remain inside the earth's orbit is $2/3\pi$ of a year.

5.48 Find the law of force for the orbit $r = a \sin n\theta$.

5.49 Find the law of force to the pole when the orbit described by the cardioid $r = a(1 - \cos\theta)$.

5.50 In prob. (5.49) prove that if Q be the force at the apse and v the velocity, $3v^2 = 4aQ$.

5.51 A particle moves in a plane under an attractive force varying as the inverse cube of the distance. Find the equation of the orbit distinguishing three cases which may arise.

5.52 Show that the central force necessary to make a particle describe the lemniscate $r^2 = a^2 \cos 2\theta$ is inversely proportional to r^7.

5.53 Show that if a particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle, then the force varies as the inverse fifth power of distance.

5.54 If the sun's mass suddenly decreased to half its value, show that the earth's orbit assumed to be originally circular would become parabolic.

5.3 Solutions

5.3.1 Field and Potential

5.1 $F = \dfrac{GM_1M_2}{r^2}$

If R is the radius of either sphere, the distance between the centre of the spheres in contact is $r = 2R$:

$$M_1 = M_2 = M = \frac{4}{3}\pi R^3 \rho$$

$$F = \frac{GM^2}{4R^2} = \frac{4\pi^2 G R^4 \rho^2}{9}$$

$$= \frac{4\pi^2}{9} \times 6.67 \times 10^{-11} \times (0.2)^4 (11300)^2 = 5.98 \times 10^{-5}\,\text{N}$$

5.2 As the mass of A and B are identical and the distance PA = PB, the magnitude of the force $F_{\text{PA}} = F_{\text{PB}}$. Resolve these forces in the horizontal and vertical direction. The horizontal components being in opposite direction get cancelled. The vertical components get added up.

$$F_{\text{PA}} = F_{\text{PB}} = G\frac{(0.01)(0.26)}{(0.1)^2} = 0.26G$$

Each vertical component $= 0.26\,G \times \dfrac{6}{10} = 0.156\,G$

Therefore $F_{\text{Net}} = 2\times 0.156\,G = 2\times 0.156\times 6.67\times 10^{-11}\,\text{N} = 2.08\times 10^{-1}\,\text{N}$

5.3 At distance r, $F_{\text{M}} = F_{\text{m}} = \dfrac{GMm}{r^2}$

Acceleration of mass m $\quad a_{\text{m}} = \dfrac{F_{\text{m}}}{m} = \dfrac{GM}{r^2}$

Acceleration of mass M $\quad a_{\text{M}} = \dfrac{F_{\text{m}}}{M} = \dfrac{Gm}{r^2}$

$$a_{\text{rel}} = a_{\text{m}} + a_{\text{M}} = \frac{G(M+m)}{r^2}$$

$$a_{\text{rel}} = \frac{\mathrm{d}v_{\text{rel}}}{\mathrm{d}t} = v_{\text{rel}}\frac{\mathrm{d}v_{\text{rel}}}{\mathrm{d}r} = \frac{G(M+m)}{r^2}$$

Integrating $\displaystyle\int v_{\text{rel}}\,\mathrm{d}v_{\text{rel}} = \frac{v_{\text{rel}}^2}{2} = G(M+m)\int_d^{\infty}\frac{\mathrm{d}r}{r^2} = \frac{G(M+m)}{d}$

$$\therefore \qquad v_{\text{rel}} = \sqrt{\frac{2G(M+m)}{d}}$$

5.4 Gravitational force $F = -\dfrac{GMm}{x^2}$
where M and m are the masses of the sun and the earth which are a distance x apart.
Earth's acceleration

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{F}{m} = -\frac{GM}{x^2}$$

$$\therefore \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v\mathrm{d}v}{\mathrm{d}x} = -\frac{GM}{x^2}$$

$$v\mathrm{d}v = -GM\frac{\mathrm{d}x}{x^2}$$

Integrating $\displaystyle\int v\mathrm{d}v = \frac{v^2}{2} = -Gm\int\frac{\mathrm{d}x}{x^2} + C$

where $C =$ constant

$$\frac{v^2}{2} = \frac{GM}{x} + C$$

Initially $v = 0,\ x = r$

$$\therefore \quad C = -\frac{GM}{r}$$

$$\therefore \quad v = \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{r}}$$

$$\mathrm{d}t = \frac{1}{\sqrt{2GM}} \frac{\mathrm{d}x}{\sqrt{\frac{1}{x} - \frac{1}{r}}}$$

Integrating

$$t = \int \mathrm{d}t = \frac{1}{\sqrt{2GM}} \int_0^r \frac{\mathrm{d}x}{\sqrt{\frac{1}{x} - \frac{1}{r}}}$$

Put $x = r\cos^2\theta$, $\mathrm{d}x = -2r\ \sin\theta\cos\theta\mathrm{d}\theta$

$$t = -2\sqrt{\frac{r^3}{2GM}} \int_{\pi/2}^{0} \cos^2\theta\mathrm{d}\theta = 2\sqrt{\frac{r^3}{2GM}} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{2\sqrt{2}}\sqrt{\frac{r^3}{GM}}$$

But the period of earth's orbit is

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\therefore \quad t = \frac{T}{4\sqrt{2}} = \frac{365}{4\sqrt{2}} = 64.53 \text{ days}$$

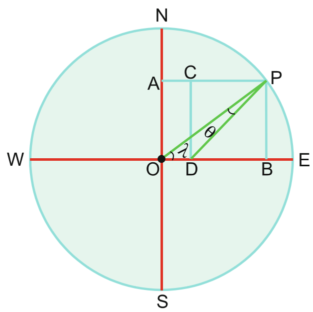


Fig. 5.9

5.5 If the earth were at rest, then the gravitational force on a body of mass at P would be in the direction PO, i.e. towards the centre of the earth, Fig. 5.9.

However, due to the rotation of the earth about the polar axis NS, a part of the gravitational force is used up to provide the necessary centripetal force to enable the mass m at P in the latitude λ to describe a circular radius PA $= r = R\cos\lambda$, where PO $= R$ is the earth's radius. This is equal to $m\omega^2 r$, or $m\omega^2 R\cos\lambda$ towards the centre and is represented by CA, ω being the angular velocity of earth's diurnal rotation. In the absence of rotation the gravitational force mg acts radially towards the centre O and is represented by PO. Resolve this into two mutually perpendicular components, one along PA given by mg $\cos\lambda$ and the other along PB given by mg $\sin\lambda$ and is represented by PB. Drop CD perpendicular on the EW-axis. Then the resultant force mg' is given by PD both in magnitude and direction. A plumb line at P will make a small angle $\theta(\mathrm{O}\hat{P}D)$ with line PO.

$$\begin{aligned} mg' &= \sqrt{(mg\cos\lambda - m\omega^2 R\cos\lambda)^2 + (mg\sin\lambda)^2} \\ &= m\sqrt{g^2 - 2gR\omega^2\cos^2\lambda + \omega^4 R^2\cos^2\lambda} \end{aligned} \tag{1}$$

The third term in the radical is much smaller than the second term and is neglected.

$$\begin{aligned} \therefore\quad g' &\simeq (g^2 - 2gR\omega^2\cos^2\lambda)^{1/2} \\ &= g\left(1 - \frac{2R}{g}\omega^2\cos^2\lambda\right)^{1/2} \\ &\simeq g\left(1 - \frac{R}{g}\omega^2\cos^2\lambda\right)^{1/2} \end{aligned} \tag{2}$$

where we have expanded binomially and retained only the first two terms.
Now in ΔOPD

$$\frac{\mathrm{PD}}{\sin \mathrm{P}\hat{O}D} = \frac{\mathrm{OD}}{\sin\theta} \tag{3}$$

$$\text{or}\quad \frac{g - R\omega^2\cos^2\lambda}{\sin\lambda} = \frac{\omega^2 R\cos\lambda}{\sin\theta} \tag{4}$$

$$\sin\theta \simeq \theta = \frac{\omega^2 R\cos\lambda\sin\lambda}{g - R\omega^2\cos^2\lambda} \tag{5}$$

$\theta \simeq \dfrac{\omega^2}{g} R\cos\lambda\sin\lambda$ ($\because$ the second term in the denominator of (5) is much smaller than the first term)

$$\simeq \frac{2\pi^2 R}{gT^2}\sin 2\lambda$$

(a) θ will be maximum when $\sin 2\lambda$ is maximum, i.e. $2\lambda = 90°$ or $\lambda = 45°$.
(b) At the poles $\lambda = 90°$ and so $\theta = 0°$.
(c) At the equator $\lambda = 0°$ and so $\theta = 0°$.

5.6 Consider a spherical shell of radius r and thickness $\mathrm{d}r$ concentric with the sphere of radius R. If ρ is the density, then

$$\rho = \frac{3M}{4\pi R^3} \tag{1}$$

The mass of the shell $= 4\pi r^2 \mathrm{d}r\rho$.
The mass of the sphere of radius r which is equal to $4\pi r^3/3$ may be considered to be concentrated at the centre.
The gravitational potential energy between the spherical shell and the sphere of radius r is

$$\mathrm{d}U = -\frac{G(4\pi r^2 \mathrm{d}r\rho)\left(\frac{4\pi}{3}r^3\rho\right)}{r} = -\frac{16\pi^2 G\rho^2 r^4 \mathrm{d}r}{3} \tag{2}$$

The total gravitational energy of the earth

$$U = \int \mathrm{d}U = -\frac{16\pi^2 G\rho^2}{3}\int_0^R r^4 \mathrm{d}r = -\frac{16\pi^2 G\rho^2 R^5}{15}$$
$$= -\frac{3}{5}\frac{GM^2}{R} \tag{3}$$

where we have used (1).

$$U = -\frac{6.67 \times 10^{-11} \times 0.6 \times (6 \times 10^{24})^2}{6.4 \times 10^6} = 2.25 \times 10^{32}\,\mathrm{J}$$

5.7 If g_0 is the gravity at the earth's surface, g_h at height h and g_d at depth d, then

$$g_\mathrm{h} = g_0 \frac{R^2}{(R+h)^2} \tag{1}$$
$$g_\mathrm{d} = g_0\left(1 - \frac{d}{R}\right) \tag{2}$$

By problem, $g_\mathrm{d} = g_\mathrm{h}$ at $h = d$, (3)

From (1) and (2) we get

$$d^2 + dR - R^2 = 0$$

$$d = \frac{\left(\sqrt{5}-1\right)}{2} R = 0.118R = 0.118 \times 6400 = 755\,\text{km}$$

5.8 Weight $mg = \dfrac{GMm}{R^2}$

$$M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore \quad \rho = \frac{3g}{4\pi GR} = \frac{3 \times 980}{4\pi \times 6.67 \times 10^{-8} \times 6.38 \times 10^8} = 5.5\,\text{g/cm}^3$$

5.9 Let M_s and M_E be the masses of the sun and earth, respectively. Let the body of mass m be at distance x from the centre of the earth and d the distance between the centres of the sun and the earth. The forces are balanced if

$$\frac{G_m M_E}{x^2} = \frac{G_m M_s}{(d-x)^2}$$

Given that $M_s = 3.24 \times 10^5 M_E$

$$x = \frac{d}{570.2} = \frac{9.3 \times 10^7}{570.2} = 1.631 \times 10^5\,\text{km}$$

5.10 By problem (5.5) $g' = g - R\omega^2 \cos^2 \lambda$
Set $\lambda = 0, \; \omega = 7.27 \times 10^{-5}\,\text{rad/s}, \; R = 6.4 \times 10^8\,\text{cm}$

$$\Delta g = g - g' = R\omega^2 = 6.4 \times 10^8 \times (7.27 \times 10^{-5})^2 = 3.38\,\text{cm/s}^2$$

5.11 Figure 5.10 shows the cross-section of a solid sphere of mass M and radius 'a' with constant density ρ, its centre being at O. It is required to find the potential $V(r)$ at the point P, at distance r from the centre. The contribution to $V(r)$ comes from two regions, one V_1 from mass lying within the sphere of radius r and the other V_2 from the region outside it. Thus

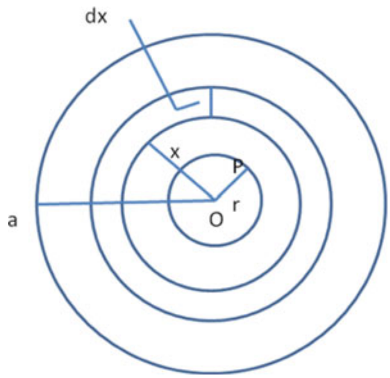


Fig. 5.10

$$V(r) = V_1 + V_2 \tag{1}$$

The potential V_1 at P is the same as due to the mass of the sphere of radius r concentrated at the centre O and is given by

$$V_1 = -G\frac{4\pi r^3}{3}\frac{\rho}{r} = -\frac{4}{3}\pi G r^2 \rho \tag{2}$$

For the mass outside r, consider a typical shell at distance x from the centre O and of thickness dx.
Volume of the shell $= 4\pi x^2\, dx$

Mass of the shell $= (4\pi x^2\, dx)\rho$

Potential due to this shell at the centre or at any point inside the shell, including at P, will be

$$dV_2 = -\frac{4\pi\rho x^2\, dx}{x} = -4\pi G\rho x\, dx \tag{3}$$

Potential V_2 at P due to the outer shells $(x > r)$ is obtained by integrating (3) between the limits r and a.

$$V_2 = \int dV_2 = -4\pi G\rho \int_r^a x\, dx = -2\pi G\rho(a^2 - r^2) \tag{4}$$

Using (2) and (4) in (1) and using $\rho = \dfrac{3M}{4\pi a^3}$

$$V(r) = -\frac{GM}{2a}\left(3 - \frac{r^2}{a^2}\right) \tag{5}$$

The potential (5) is that of a simple harmonic oscillator as the force

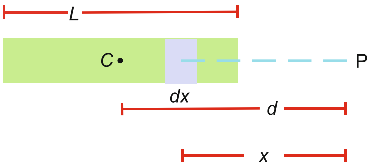
$$F = -\frac{dV}{dr} = -\frac{GMr}{a^3}$$

i.e. the force is opposite and proportional to the distance.

5.12 Consider a length element dx of a thin rod of length L, at distance x from P (Fig. 5.11). The mass element is $(M/L)\, dx$. The potential at P due to this mass length will be

$$dV = -\frac{GM}{L}\frac{dx}{x}$$

Fig. 5.11



The potential at p from the entire rod is given by

$$V = \int \mathrm{d}V = -\frac{GM}{L} \int_{d-\frac{L}{2}}^{d+\frac{L}{2}} \frac{\mathrm{d}x}{x} = -\frac{GM}{L} \ln \frac{2d+L}{2d-L}$$

5.13 The linear speed of an object on the equator

$$v = \omega R = (7.27 \times 10^{-5})(6.4 \times 10^{6}) = 465.3\,\mathrm{m/s}$$

The orbital velocity of a surface satellite is

$$v_0 = \sqrt{gr} = \sqrt{9.8 \times 6.4 \times 10^6} = 7920\,\mathrm{m/s}$$

When launched in the westerly direction the launching speed v_0 will be added to v as the earth rotates from west to east, while in the easterly direction it will be subtracted.

$$\frac{\text{westerly launching speed}}{\text{easterly launching speed}} = \frac{7920 + 465}{7920 - 465} = 1.125$$

or 11%.

5.14 Consider an element of arc of length $\mathrm{d}s = R\,\mathrm{d}\theta$, Fig. 5.12. The corresponding mass element $\mathrm{d}m = \lambda \mathrm{d}s = \lambda R\,\mathrm{d}\theta$.

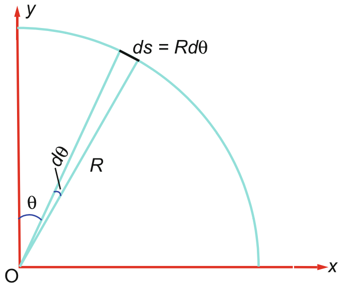
The intensity at the origin where λ is the linear density (mass per unit length) due to $\mathrm{d}m$ will be

$$\frac{G\lambda R\,\mathrm{d}\theta}{R^2} \text{ or } \frac{G\lambda\,\mathrm{d}\theta}{R}$$

The x-component of intensity at the origin due to $\mathrm{d}m$ will be

$$\mathrm{d}E_x = \frac{G\lambda}{R}\mathrm{d}\theta \sin\theta$$

Fig. 5.12



Therefore, the x-component of intensity due to the quarter of circle at the origin will be

$$E_x = \frac{G\lambda}{R}\int_0^{\pi/2} \sin\theta \, \mathrm{d}\theta = \frac{G\lambda}{R}$$

Similarly, the y-component of intensity due to the quarter of circle at the origin will be

$$E_y = \frac{G\lambda}{R}\int_0^{\pi/2} \cos\theta \, \mathrm{d}\theta = \frac{G\lambda}{R}$$

$$\therefore \quad E = \sqrt{E_x^2 + E_y^2} = \sqrt{2}\frac{G\lambda}{R}$$

5.15 $g_B = \frac{F_B}{M} = \frac{GmM}{d^2M} = \frac{Gm}{d^2}$

$$g_C = \frac{F_c}{M} = \frac{GmM}{(d+R)^2M} = \frac{Gm}{(d+R)^2}$$

$$\Delta g = g_B - g_C = \frac{Gm}{d^2} - \frac{Gm}{(d+R)^2} = \frac{Gm(2Rd + R^2)}{d^2(d+R)^2}$$

Since $d >> R$, $\Delta g \approx \frac{2GmR}{d^3}$

5.16 By problem (5.6) the gravitational energy is given by

$$U = -\frac{3}{5}\frac{GM^2}{R} \tag{1}$$

The volume of the star

$$V = \frac{4\pi R^3}{3} \tag{2}$$

$$\therefore \quad R = \left(\frac{3V}{4\pi}\right)^{1/3} \tag{3}$$

$$\therefore \quad U = -\frac{3}{5}\left(\frac{4\pi}{3V}\right)^{1/3} GM^2 \tag{4}$$

$$P = -\frac{\partial U}{\partial V} = -\frac{1}{5}\left(\frac{4\pi}{3}\right)\frac{GM^2}{V^{4/3}}$$

$$\therefore \quad P \propto V^{-4/3}$$

5.17 Consider a line element dx at A distance x from O, Fig. 5.13. The field point P is at a distance R from the infinite line. Let PA $= r$. The x-component of gravitational field at P due to this line element will get cancelled by a symmetric line element on the other side. However, the y-component will add up. If λ is the linear mass density, the corresponding mass element is λdx

$$\mathrm{d}E = \mathrm{d}E_y = -\frac{G\lambda \mathrm{d}x \sin\theta}{r^2} \tag{1}$$

$$\text{Now,} \quad r^2 = x^2 + R^2 \tag{2}$$

$$x = R\cot\theta \tag{3}$$

$$\therefore \quad r^2 = R^2 \operatorname{cosec}^2\theta \tag{4}$$

$$dx = R \operatorname{cosec}^2\theta \mathrm{d}\theta \tag{5}$$

Using (4) and (5) in (1)

$$\mathrm{d}E = -\frac{G\lambda}{R}\sin\theta \mathrm{d}\theta$$

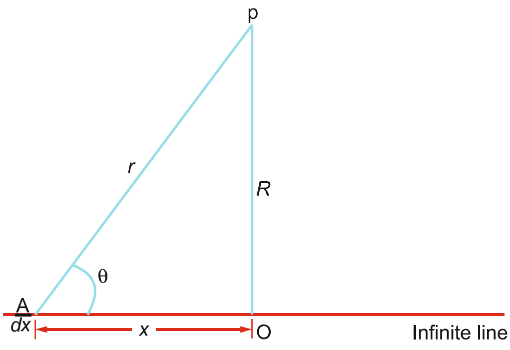


Fig. 5.13

Integrating from 0 to $\pi/2$ for the contribution from the line elements on the left-hand side of O and doubling the result for taking into account contributions on the right-hand side

$$E = -\frac{2\lambda}{R}\int_0^{\pi/2} \sin\theta \mathrm{d}\theta = -\frac{2G\lambda}{R}$$

5.18 Let the neutral point be located at distance x from the earth's centre on the line joining the centres of the earth and moon. If M_e and M_m are the masses of the earth and the moon, respectively, and m the mass of the body placed at the neutral point, then the force exerted by M_e and M_m must be equal and opposite to that of M_m on m.

$$\frac{GM_\mathrm{e}m}{x^2} = \frac{GM_\mathrm{m}m}{(d-x)^2}$$

$$\therefore \quad \frac{M_\mathrm{e}}{M_\mathrm{m}} = 81 = \frac{x^2}{(d-x)^2}$$

Since $d > x$, there is only one solution

$$\frac{x}{d-x} = +9$$

$$\text{or} \quad x = \frac{9}{10}d$$

5.19 For a homogeneous sphere of mass M the potential for $r \leq R$ is given by $V(r) = -\frac{1}{2}\frac{GM}{R}\left(3 - \frac{r^2}{R^2}\right)$. At the centre of the sphere $V(0) = -\frac{3}{2}\frac{GM}{R}$. For a hemisphere at the centre of the base $V(0) = -\frac{3}{4}\frac{GM}{R}$. The work done to move a particle of mass m to infinity will be $\frac{3}{4}\frac{GMm}{R}$.

5.20 Let the point P be at distance r from the centre of the shell such that $a < r < b$. The gravitational field at P will be effective only from matter within the sphere of radius r. The mass within the shell of radii a and r is $\frac{4\pi}{3}(r^3 - a^3)\rho$. Assume that this mass is concentrated at the centre. Then the gravitational field at a point distance r from the centre will be

$$g(r) = -\frac{4\pi}{3}\frac{(r^3 - a^3)}{r^2}\rho G$$

$$\text{But} \quad \rho = \frac{3M}{4\pi(b^3 - a^3)}$$

$$\therefore \quad g(r) = -\frac{GM(r^3 - a^3)}{r^2(b^3 - a^3)}$$

5.21 Let the disc be located in the xy-plane with its centre at the origin. P is a point on the z-axis at distance z from the origin. Consider a ring of radii r and $r + \mathrm{d}r$ concentric with the disc, Fig. 5.14. The mass of the ring will be

$$\mathrm{d}m = 2\pi r \mathrm{d}r \sigma \tag{1}$$

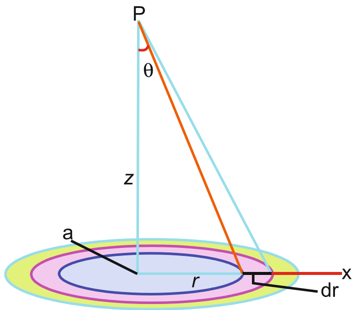
The horizontal component of the field at P will be zero because for each point on the ring there will be another point symmetrically located on the ring which will produce an opposite effect. The vertical component of the field at P will be

$$\mathrm{d}g_z = -\frac{G \times 2\pi r \, \mathrm{d}r \, \sigma \cos\theta}{(r^2 + z^2)} \tag{2}$$

But $$\cos\theta = \frac{z}{\sqrt{r^2 + z^2}} \tag{3}$$

$$g = g_z = \int \mathrm{d}g_z = -2\pi\sigma G \int_0^a \frac{zr \, \mathrm{d}r}{(r^2 + z^2)^{3/2}} \tag{4}$$

Fig. 5.14



Put $r = z\tan\theta$, $\mathrm{d}r = z\sec^2\theta \mathrm{d}\theta$

$$g = -2\pi G = \int_0^{z/\sqrt{a^2+z^2}} \sin\theta d\theta$$

$$= -2\pi\sigma G \left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right]$$

5.22 Initially the particle is located at a distance $2R$ from the centre of the spherical shell and is at rest. Its potential energy is $-GMm/2R$. When the particle

arrives at the opening the potential energy will be $-GMm/R$ and kinetic energy $\frac{1}{2}mv^2$.
Kinetic energy gained = potential energy lost

$$\frac{1}{2}mv^2 = -\frac{GMm}{2R} - \left(-\frac{GMm}{R}\right) = \frac{1}{2}\frac{GMm}{R}$$

$$\therefore \quad v = \sqrt{\frac{GM}{R}}$$

After passing through the opening the particle traverses a force-free region inside the shell. Thus, within the shell its velocity remains unaltered. Therefore, it hits the point C with velocity $v = \sqrt{\frac{GM}{R}}$.

5.3.2 Rockets and Satellites

5.23 Energy conservation gives

$$\frac{1}{2}mv^2 - \frac{GmM}{R} = -\frac{GmM}{r} + 0$$

where r is the distance from the earth's centre.
Using $v = \sqrt{\frac{GM}{2R}}$, we find $r = \frac{4}{3}R$
Maximum height attained

$$h = r - R = \frac{R}{3}$$

5.24 In Fig. 5.15, total energy at P = total energy at O.

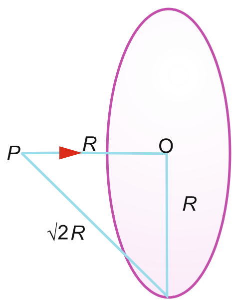


Fig. 5.15

$$-\frac{GmM}{\sqrt{2}R} = \frac{1}{2}mv^2 - \frac{GmM}{R}$$

$$\therefore \quad v = \sqrt{(2-\sqrt{2})\frac{GM}{R}}$$

5.25 The potential energy of the satellite on the earth's surface is

$$U(R) = -\frac{GMm}{R} \tag{1}$$

where M and m are the mass of the earth and the satellite, respectively, and R is the earth's radius.
The potential energy at a height $h = 0.5R$ above the earth's surface will be

$$U(R+h) = -\frac{GMm}{R+h} = -\frac{GMm}{1.5R} \tag{2}$$

Gain in potential energy

$$\Delta U = -\frac{GMm}{1.5R} - \left[-\frac{GMm}{R}\right] = \frac{GMm}{3R} \tag{3}$$

Thus the work done W_1 in taking the satellite from the earth's surface to a height $h = 0.5\,r$

$$W_1 = \frac{GMm}{3R} \tag{4}$$

Extra work W_2 required to put the satellite in the orbit at an attitude $h = 0.5R$ is equal to the extra energy that must be supplied:

$$W_2 = \frac{1}{2}mv_0 = \frac{1}{2}m\left[\frac{GM}{R+h}\right] = \frac{GMm}{3R} \tag{5}$$

where v_0 is the satellite's orbital velocity.
Thus from (4) and (5), $\dfrac{W_1}{W_2} = 1.0$.

5.26 The initial angular momentum of the asteroid about the centre of the planet is $L = mv_0 d$.
At the turning point the velocity $\boldsymbol{v}$ of the asteroid will be perpendicular to the radial vector. Therefore the angular momentum $L' = mvR$ if the asteroid is to just graze the planet. Conservation of angular momentum requires that $L' = L$. Therefore

$$mvR = mv_0 d \tag{1}$$

$$\text{or} \quad v = \frac{v_0 d}{R} \tag{2}$$

Energy conservation requires

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{GMm}{R} \tag{3}$$

$$\text{or} \quad v^2 = v_0^2 + \frac{2GM}{R} \tag{4}$$

Eliminating v between (2) and (4) the minimum value of v_0 is obtained.

$$v_0 = \sqrt{\frac{2GMR}{d^2 - R^2}}$$

5.27 According to Kepler's third law

$$T^2 \propto r^3$$

(i) $$\frac{T_E^2}{r_E^3} = \frac{(365.3)^2}{(1.5 \times 10^{11})^3} = 3.9539 \times 10^{-29} \quad \text{days}^2/\text{m}^3$$

$$\frac{T_v^2}{r_v^2} = \frac{(224.7)^2}{(1.08 \times 10^{11})^3} = 4.0081 \times 10^{-29} \quad \text{days}^2/\text{m}^3$$

Thus Kepler's third law is verified

(ii) $$T = 2\pi\sqrt{\frac{r^3}{GM}} \tag{1}$$

where M is the mass of the parent body.

$$M = \frac{4\pi^2}{G}\frac{r^3}{T^2} \tag{2}$$

From (i) the mean value, $\left\langle \frac{T^2}{r^3} \right\rangle = 3.981 \times 10^{-29}$ days^2/m^3 $= 2.972 \times 10^{-19}$ s^2/m^3

$$M = \frac{4\pi^2}{6.67 \times 10^{-11}} \times \frac{1}{2.972 \times 10^{-19}} = 1.99 \times 10^{30}\,\text{kg}$$

5.28 At the perihelion (nearest point from the focus) the velocity (v_p) is maximum and at the aphelion (farthest point) the velocity (v_A) is minimum. At both these points the velocity is perpendicular to the radius vector. Since the angular momentum is constant

$$mv_A r_A = mv_p r_p$$
$$\text{or} \quad r_A = \frac{v_p r_p}{v_A} \tag{1}$$

where $r_A = r_{max}$ and $r_p = r_{min}$
The eccentricity

$$\varepsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{r_A - r_p}{r_A + r_p} = \frac{v_p - v_A}{v_p + v_A} \tag{2}$$

where we have used (1)

$$\varepsilon = \frac{30.0 - 29.2}{30.0 + 29.2} = 0.0135$$

A small value of eccentricity indicates that the orbit is very nearly circular.

5.29 **(a)** For circular orbit,

$$T = \frac{2\pi a}{v}$$
$$T = 14.4 \text{ days} = 1.244 \times 10^6 \text{s}$$
$$2a = \frac{vT}{\pi} = \frac{2.2 \times 10^5 \times 1.244 \times 10^6}{3.1416} = 8.7 \times 10^{10} \text{ m}$$

(b) Since the velocity of each component is the same, the masses of the components are identical.

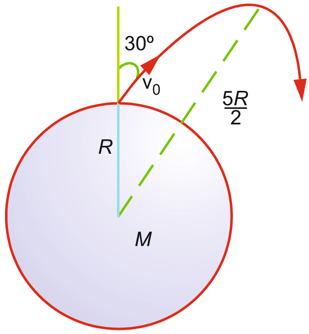
$$v^2 = \frac{G(M+m)}{a} = \frac{2GM}{a} \quad (\because m = M)$$
$$\therefore \quad M = \frac{av^2}{2G} = \frac{(4.35 \times 10^{10})(2.2 \times 10^5)^2}{2 \times 6.67 \times 10^{-11}} = 1.58 \times 10^{31} \text{ kg}$$

5.30 At the surface, the component of velocity of the satellite perpendicular to the radius R is

$$v_0 \sin 30° = \frac{v_0}{2} \quad \text{(Fig. 5.16)}$$

Therefore, the angular momentum at the surface $= \dfrac{mv_0 R}{2}$

Fig. 5.16



At the apogee (farthest point), the velocity of the satellite is perpendicular to the radius vector. Therefore, the angular momentum at the apogee $= (mv)(5R/2)$.
Conservation of angular momentum gives

$$\frac{5}{2}mvR = \frac{m}{2}v_0R$$

$$\text{or} \quad v = \frac{v_0}{5} \tag{1}$$

The kinetic energy at the surface $K_0 = \frac{1}{2}mv_0^2$ and potential energy $U_0 = -\frac{GMm}{R}$.
Therefore, the total mechanical energy at the surface is

$$E_0 = \frac{1}{2}mv_0^2 - \frac{GMm}{R} \tag{2}$$

At the apogee kinetic energy $K = \frac{1}{2}mv^2$ and the potential energy $U = -\frac{2GMm}{5R}$.
Therefore, the total mechanical energy at the apogee is

$$E = \frac{1}{2}mv^2 - \frac{2}{5}\frac{GMm}{R} \tag{3}$$

Conservation of total energy requires that $E = E_0$. Eliminating v in (3) with the aid of (1) and simplifying we get

$$v_0 = \sqrt{\frac{5GM}{4R}}$$

5.31 For an elliptic orbit

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$\therefore \quad v_{\max} = \sqrt{GM\left(\frac{2}{r_{\min}} - \frac{1}{a}\right)} = \sqrt{\frac{GM(2a - r_{\min})}{ar_{\min}}} = \sqrt{\frac{GMr_{\max}}{ar_{\min}}} \tag{1}$$

as $r_{\max} + r_{\min} = 2a$

$$v_{\min} = \sqrt{GM\left(\frac{2}{r_{\max}} - \frac{1}{a}\right)} = \sqrt{\frac{GM(2a - r_{\max})}{ar_{\max}}} = \sqrt{\frac{GMr_{\min}}{ar_{\max}}} \tag{2}$$

Multiplying (1) and (2)

$$v_{\max} v_{\min} = \frac{GM}{a}$$

or $\sqrt{v_{\max} v_{\min}} = \sqrt{\frac{GM}{a}} = a\sqrt{\frac{GM}{a^3}} = \frac{2\pi a}{T}$

as $T = 2\pi\sqrt{\frac{a^3}{GM}}$

$$\therefore \quad a = \frac{T}{2\pi}\sqrt{v_{\max} v_{\min}}$$

5.32 $$T = 2\pi\sqrt{\frac{r^3}{GM}} \tag{1}$$

But $r = a + b + c$ (2)

Combining (1) and (2)

$$M = \frac{4\pi^2(a + b + c)^3}{GT^2}$$

5.33 $L = |r \times p| = rp \sin\theta$

L per unit mass $= rv \sin\theta$

$= (1.75 \times 1.5 \times 10^{11})(3 \times 10^4) \sin 30°$

$= 3.9375 \times 10^{15}$ m^2/s

When the comet is closest to the sun its velocity will be perpendicular to the radius vector. The angular momentum $L' = r'v'$. Angular momentum conservation requires

$$L' = L$$

$$\therefore \quad v' = \frac{L'}{r'} = \frac{L}{r} = \frac{3.9375 \times 10^{15}}{0.39 \times 1.5 \times 10^{11}} = 6.73 \times 10^4\ \text{m/s} = 67.3\ \text{km/s}$$

Total energy per unit mass

$$E = \frac{1}{2}v^2 - \frac{GM}{r} = \frac{1}{2}(3 \times 10^4)^2 - \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{1.75 \times 10^{11}} = -3.12 \times 10^8\,\text{J}$$

a negative quantity. Therefore the orbit is bound.

5.34 **(a)** The centripetal force is provided by the gravitational force.

$$\frac{GM_E M_S}{R^2} = \frac{M_E v^2}{R}$$

$$\text{or} \quad GM_S = v^2 R \tag{1}$$

(b) Total energy of the comet when it is closest to the sun

$$E = \frac{1}{2}M_C(2v)^2 - \frac{GM_C M_S}{R/2} \tag{2}$$

Using (1) in (2) we find $E = 0$.

(c) At the distance of the closest approach, the comet's velocity is perpendicular to the radius vector. Therefore the angular momentum

$$L = M_C(2v)\left(\frac{R}{2}\right) = M_C v R \tag{3}$$

Let v_t be the comet's velocity which is tangential to the earth's orbit at P. Then the angular momentum at P will be

$$L' = M_c v_t R \tag{4}$$

Angular momentum conservation gives

$$M_C v_t R = M_C v R \tag{5}$$

$$\text{or} \quad v_t = v \tag{6}$$

(d) The total energy of the comet at P is

$$E' = \frac{1}{2}M_C(v')^2 - \frac{GM_S M_C}{R} = 0 \tag{7}$$

where v' is the comet's velocity at P, because $E' = E = 0$, by energy conservation.

Using (1) in (7) we find

$$v' = \sqrt{2}v \tag{8}$$

If θ is the angle between $\boldsymbol{v}'$ and the radius vector $\boldsymbol{R}$ angular momentum conservation gives

$$M_C vR = M_C v' R \sin\theta = M_C \sqrt{2} vR \sin\theta$$

$$\text{or} \quad \sin\theta = \frac{1}{\sqrt{2}} \quad \theta = 45^\circ$$

5.35 At both perigee and apogee the velocity of the satellite is perpendicular to the radius vector. In order to show that the angular momentum is conserved we must show that

$$m v_p r_p = m v_A r_A$$

$$\text{or} \quad v_p r_p = v_A r_A$$

where m is the mass of the satellite.

$$v_p r_p = 10.25 \times 6570 = 67342.5$$

$$v_A r_A = 1.594 \times 42250 = 67346.5$$

The data are therefore consistent with the conservation of angular momentum.

5.36 **(a)** $$v_0 = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{1}$$

$$GM = (6.67 \times 10^{-11})(6 \times 10^{24}) = 4 \times 10^{14}$$

$$r = R = 6.4 \times 10^6$$

$$a = 8 \times 10^7\ m$$

$$v_0 = 1.095 \times 10^4\ \text{m/s} = 10.095\ \text{km/s}$$

(b) $$\varepsilon = \sqrt{1 + \frac{2EJ^2}{G^2M^2m^3}} \tag{2}$$

$$J = mRv_0 \sin 45^\circ = \frac{mRv_0}{\sqrt{2}} \tag{3}$$

$$E = -\frac{GMm}{2a} \tag{4}$$

Combining (1), (2), (3) and (4)

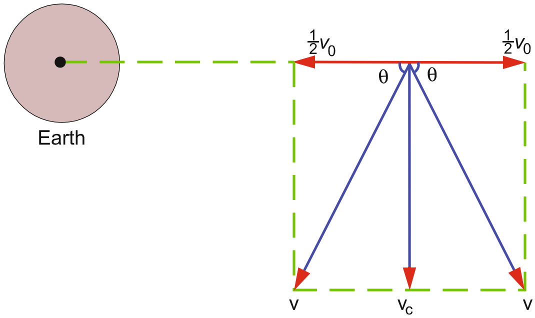
$$\varepsilon = \sqrt{1 - \frac{R}{a} + \frac{1}{2}\frac{R^2}{a^2}} \tag{5}$$

Now $\quad \dfrac{R}{a} = \dfrac{6400}{80000} = 0.08$

$\therefore \quad \varepsilon = 0.96$

5.37 The resultant velocity v of each fragment is obtained by combining the velocities $\dfrac{1}{2}v_0$ and v_0 vectorially, Fig. 5.17.

Fig. 5.17



$$v = \sqrt{\left(\frac{1}{2}v_0\right)^2 + v_0^2} = \frac{1}{2}\sqrt{5}v_0$$

Kinetic energy of each fragment

$$K = \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{\sqrt{5}}{2}v_0\right)^2 = \frac{5}{16}mv_0^2 = \frac{5}{16}m\frac{GM}{r}$$

Potential energy of each fragment $U = -\dfrac{GM\left(\frac{1}{2}m\right)}{r}$

$\therefore \quad$ Total energy $E = K + U = \dfrac{5GMm}{16r} - \dfrac{1}{2}\dfrac{GMm}{r} = -\dfrac{3}{16}\dfrac{GMm}{r}$

If $\boldsymbol{v}$ makes on angle θ with the radius vector $\boldsymbol{r}$, then $v\sin\theta = v_0$. The angular momentum of either fragment about the centre of the earth is

$$J = \frac{1}{2}mv_0r = \frac{mr}{2}\sqrt{\frac{GM}{r}} = \frac{1}{2}m\sqrt{GMr}$$

5.38 Velocity at the nearer apse is given by

$$v^2 = GM\left[\frac{2}{a(1-\varepsilon)} - \frac{1}{a}\right] = \frac{GM}{a}\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{1}$$

as there is no instantaneous change of velocity. If a_1 is the semi-major axis for the new orbit

$$v^2 = GM\left[\frac{2}{a(1-\varepsilon)} - \frac{1}{a_1}\right] \tag{2}$$

As the nearer and farther apses are inter-changed

$$a_1(1-\varepsilon_1) = a(1+\varepsilon) \tag{3}$$

Equating the right-hand side of (1) and (2) and eliminating a_1 from (3) and solving for ε_1 we get

$$\varepsilon_1 = \frac{\varepsilon(3+\varepsilon)}{1-\varepsilon}$$

5.39 **(a)** $$\frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{1}{T}\int \frac{\mathrm{d}\theta}{r\dot{\theta}} \tag{1}$$

Now, $J = mr^2\dot{\theta}$ (constant) (2)

$$\therefore \quad \frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{m}{TJ}\int r\,\mathrm{d}\theta \tag{3}$$

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta} \tag{4}$$

Using (4) in (3)

$$\frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{ma(1-\varepsilon^2)}{TJ}\int_0^{2\pi} \frac{\mathrm{d}\theta}{1+\varepsilon\cos\theta}$$

$$= \frac{ma(1-\varepsilon^2)}{TJ}\frac{2\pi}{\sqrt{1-\varepsilon^2}} \tag{5}$$

where we have used the integral $\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Further $$T = \frac{2\pi ma^2\sqrt{1-\varepsilon^2}}{J} \tag{6}$$

Using (6) in (5)

$$\frac{1}{T}\int\frac{\mathrm{d}t}{r}=\frac{1}{a} \tag{7}$$

(b) $$\frac{1}{T}\int v^2\mathrm{d}t=\frac{GM}{T}\int\left(\frac{2}{r}-\frac{1}{a}\right)\mathrm{d}t$$
$$=2GM\frac{1}{T}\int\frac{\mathrm{d}t}{r}-\frac{GM}{Ta}\int\mathrm{d}t=\frac{2GM}{a}-\frac{GM}{a}=\frac{GM}{a}$$

where we have used (7) and put $\int\mathrm{d}t=T$.

5.40 The distance between the focus and the end of minor axis is a. Let the new semi-major axis be a_1. Since the instantaneous velocity does not change

$$GM\left(\frac{2}{a}-\frac{1}{a}\right)=G(M+m)\left(\frac{2}{a}-\frac{1}{a_1}\right)$$

or $$a_1=\frac{a\left(1+\frac{m}{M}\right)}{1+\frac{2m}{M}}\approx a\left(1+\frac{m}{M}\right)\left(1-\frac{2m}{M}\right)$$

$$a_1=a\left(1-\frac{m}{M}\right) \tag{1}$$

The new time period

$$T_1=\frac{2\pi a_1^{3/2}}{\sqrt{G(M+m)}}=\frac{2\pi a^{3/2}}{\sqrt{GM}}\left(1-\frac{m}{M}\right)^{3/2}\left(1+\frac{m}{M}\right)^{-1/2}$$
$$\approx T\left(1-\frac{3m}{2M}\right)\left(1-\frac{m}{2M}\right)\approx T\left(1-\frac{2m}{M}\right)$$

where we have used binomial expansion and the value of the old time period.

5.41 *Case 1: Apse is farther*
It is sufficient to show that the total energy is zero.

$$r_1=a(1+\varepsilon)=a(1+0.5)=1.5a$$
$$v_1^2=GM\left(\frac{2}{r_1}-\frac{1}{a}\right)=GM\left(\frac{2}{1.5a}-\frac{1}{a}\right)=\frac{GM}{3a}$$

New velocity $v_1'=2v_1$.
New kinetic energy

$$K_1'=\frac{1}{2}m(v_1')^2=\frac{1}{2}m(2v_1)^2=\frac{2GMm}{3a}$$

The potential energy is unaltered and is therefore

$$U_1 = -\frac{GMm}{r_1} = -\frac{2GMm}{3a}$$

Total energy $E_1' = K' + U_1 = \frac{2GMm}{3a} - \frac{2GMm}{3a} = 0$
Case 2: Apse is nearer
It is sufficient to show that the total energy is positive.

$$r_2 = a(1-\varepsilon) = a(1-0.5) = 0.5a$$
$$v_2^2 = GM\left(\frac{2}{r_2} - \frac{1}{a}\right) = GM\left(\frac{2}{0.5a} - \frac{1}{a}\right) = \frac{3GM}{a}$$

New velocity $v_2' = 2v_2$.
New kinetic energy $K_2' = \frac{1}{2}m\left(v_2'\right)^2 = \frac{1}{2}m(2v_2)^2 = \frac{6GMm}{a}$
Potential energy is unaltered and is given by

$$U_2 = -\frac{GMm}{r_2} = -\frac{GMm}{0.5a} = -\frac{2GMm}{a}$$

Total energy $E_2 = K_2' + U_2 = \frac{6GMm}{a} - \frac{2GMm}{a} = +\frac{4GMm}{a}$,

which is a positive quantity.

5.42 The velocity of the particle in the orbit is given by

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

When the particle is at one extremity of the minor axis, $r = a$

$$v^2 = GM\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{GM}{a}$$

Let the new axes be $2a_1$ and $2b_1$. By problem the force is increased by half, but the velocity at $r = a$ is unaltered.

$$v^2 = 1.5\,GM\left(\frac{2}{a} - \frac{1}{a_1}\right) = \frac{GM}{a}$$
$$\therefore \quad 2a_1 = \frac{3a}{2}$$

As v is unaltered in both magnitude and direction, the semi-latus rectum $l = \frac{b^2}{a} = a(1-\varepsilon^2)$. The constant $h^2 = (\mathrm{GM})$ (semi-latus rectum) is unchanged.

$$\therefore \quad GM\frac{b^2}{a} = \frac{3}{2}GM\frac{b_1^2}{a_1}$$

$$\therefore \quad b_1^2 = \frac{2b^2}{3}\frac{a_1}{a} = \frac{2}{3}\cdot\frac{3}{4}b^2$$

$$\therefore \quad 2b_1 = \sqrt{2}b$$

5.43 **(a)** The forces acting on the satellite are gravitational force and centripetal force.

(b) Equating the centripetal force and gravitational force

$$\frac{mv^2}{R} = mg$$

$$\therefore \quad v = \sqrt{gR} = \frac{2\pi R}{T}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{1}$$

(c) The geocentric satellite must fly in the equatorial plane so that its centripetal force is entirely used up by the gravitational force. Second, it must fly at the right altitude so that its time period is equal to that of the diurnal rotation of the earth.

(d) 24 h.

(e) Using (1)

$$r = \left[\frac{T^2GM}{4\pi^2}\right]^{1/3}$$

Using $T = 86,400$ s, $G = 6.67\times10^{-11}\,\mathrm{kg^{-1}\,m^3/s^2}$, $M = 6.4\times10^{24}$ kg, we find $r = 4.23\times10^7$ m or 42,300 km.

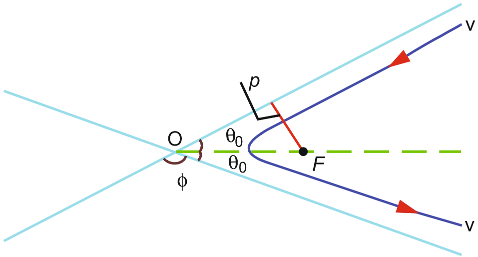
5.44 At both perigee and apogee $\boldsymbol{v}$ is perpendicular to $\boldsymbol{r}$. Angular momentum conservation gives $mv_\mathrm{A}r_\mathrm{A} = mv_\mathrm{p}r_\mathrm{p}$

$$r_\mathrm{A} = 2a - r_\mathrm{p} = 4r - r = 3r$$

$$v_\mathrm{A} = \frac{vr}{3r} = \frac{v}{3}$$

5.45 The orbit of the small body will be a hyperbola with the heavy body at the focus F, Fig. 5.18.

Fig. 5.18



$$r = \frac{a(\varepsilon^2 - 1)}{\varepsilon \cos\theta - 1} \tag{1}$$

As $r \to \infty$, the denominator on the right-hand side of (1) becomes zero and the limiting angle θ_0 is given by

$$\cos\theta_0 = \frac{1}{\varepsilon}$$

$$\text{or} \quad \cot\theta_0 = \frac{1}{\sqrt{\varepsilon^2 - 1}}$$

The complete angle of deviation

$$\phi = \pi - 2\theta_0$$

$$\text{or} \quad \frac{\phi}{2} = \frac{\pi}{2} - \theta_0$$

$$\tan\frac{\phi}{2} = \cot\theta_0 = \frac{1}{\sqrt{\varepsilon^2 - 1}}$$

$$\text{But} \quad \varepsilon = \sqrt{1 + \frac{2Eh^2}{G^2M^2}}$$

where $h = pv$ and $E = \frac{1}{2}v^2$

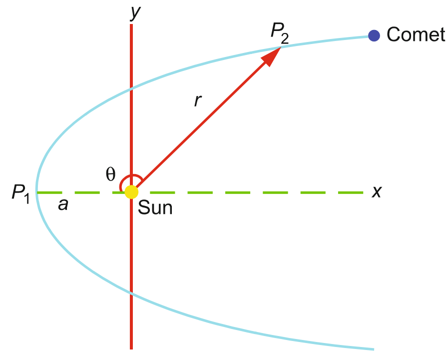
$$\therefore \quad \tan\frac{\phi}{2} = \frac{1}{\sqrt{\varepsilon^2 - 1}} = \frac{GM}{h\sqrt{2E}} = \frac{GM}{pv^2}$$

5.46 In Fig. 5.19

$$r = \frac{2a}{1 + \cos\theta}$$

$$r^2\dot{\theta} = h \quad \text{(constant, law of areas)}$$

Fig. 5.19



$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{r^2}{h}$$

Time taken for the object to move from P_1 to P_2 (Fig. 5.19) is given by

$$t = \int \mathrm{d}t = \int \frac{r^2 \mathrm{d}\theta}{h} = \frac{4a^2}{h} \int_0^\theta \frac{\mathrm{d}\theta}{(1+\cos\theta)^2}$$

$$= \frac{a^2}{h} \int_0^\theta \sec^4\left(\frac{\theta}{2}\right) \mathrm{d}\theta = \frac{2a^2}{h} \int_0^\theta \left(1 + \tan^2\frac{\theta}{2}\right) \mathrm{d}\left(\tan\frac{1}{2}\theta\right)$$

$$= \frac{2a^2}{h}\left(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta\right)$$

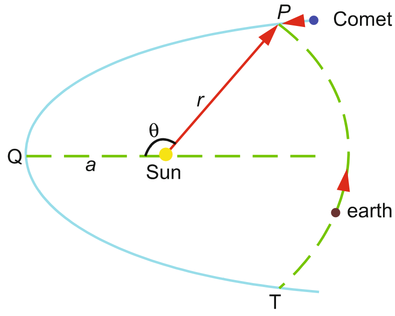
But $h = \sqrt{GM \times \text{semi - latus rectum}} = \sqrt{2aGM}$

$$\therefore \quad t = \sqrt{\frac{2a^3}{GM}}\left(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta\right)$$

5.47 Required time for traversing the arc PQT is obtained by the formula derived in problem (5.46), Fig. 5.20

$$t_0 = 2t = 2\sqrt{\frac{2a^3}{GM}}\left(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta\right) \tag{1}$$

Fig. 5.20



For parabola

$$r = \frac{2a}{1+\cos\theta} \tag{2}$$

$$\text{or} \quad \cos\theta = \frac{2a}{r} - 1 \tag{3}$$

$$\therefore \quad \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{R}{a} - 1} \tag{4}$$

where we have put $r = R$, the radius of earth's orbit. Using (4) in (1)

$$t_0 = \frac{2}{3}\sqrt{\frac{2}{GM}}(2a + R)\sqrt{R - a} \tag{5}$$

t_0 is maximized by setting $\frac{dt_0}{da} = 0$. This gives

$$a = \frac{R}{2} \tag{6}$$

Using (6) in (5) gives

$$t_0(\max) = \frac{4}{3}\sqrt{\frac{R^3}{GM}} = \frac{2}{3\pi}2\pi\sqrt{\frac{R^3}{GM}} = \frac{2}{3\pi}T$$

where $T = 2\pi\sqrt{\frac{R^3}{GM}} = 1$ year is the time period of the earth.

Thus $t_0(\max) = \frac{2}{3\pi}$ years.

5.48 $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \tag{1}$$

$$r=a\sin n\theta \tag{2}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2=n^2a^2(1-\sin^2 n\theta)=n^2a^2\left(1-\frac{r^2}{a^2}\right) \tag{3}$$

Using (3) in (1)

$$\frac{1}{p^2}=\frac{n^2a^2}{r^4}+\frac{1-n^2}{r^2}$$

Differentiating

$$-\frac{2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r}=-\frac{4n^2a^2}{r^5}-\frac{2(1-n^2)}{r^3}$$

or $$\frac{1}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r}=\frac{2n^2a^2}{r^5}+\frac{1-n^2}{r^3}$$

Force per unit mass

$$f=-\frac{h^2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r}=-h^2\left(\frac{2n^2a^2}{r^5}+\frac{1-n^2}{r^3}\right)$$

5.49 $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \tag{1}$$

$$r=a(1-\cos\theta)$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta}=a\sin\theta$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2=a^2\sin^2\theta=a^2\left[1-\left(1-\frac{r}{a}\right)^2\right]=2ra-r^2 \tag{2}$$

Using (2) in (1)

$$\frac{1}{P^2}=\frac{2a}{r^3}$$

or $$p^2=\frac{r^3}{2a} \tag{3}$$

Force per unit mass

$$f=-\frac{h^2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} \tag{4}$$

Differentiating (3)

$$2p\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{3r^2}{2a} \tag{5}$$

Using (5) in (4)

$$f = -\frac{3}{4}\frac{h^2r^2}{ap^4} = -\frac{3ah^2}{r^4} \tag{6}$$

where we have used (3). Thus the force is proportional to the inverse fourth power of distance.

5.50 If $u = \frac{1}{r}$, then at the apse

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = 0$$

$$\text{or} \quad -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$$

$$\therefore \quad -\frac{1}{r^2}a\sin\theta = 0$$

from which either $\sin\theta = 0$ or r is infinite, the latter case being inadmissible so long the particle is moving along the cardioid.

$$\text{Thus} \quad \theta = \pi \text{ or } 0$$

$$\text{When} \quad \theta = \pi,\ r = 2a$$

$$\text{and} \quad Q = \frac{3ah^2}{r^4} = \frac{3ah^2}{16a^4} = \frac{3h^2}{16a^3}$$

$$\text{Also} \quad p^2 = \frac{r^2}{2a} = \frac{8a^3}{2a} = 4a^2$$

$$\text{and} \quad v^2 = \frac{h^2}{p^2} = \frac{h^2}{4a^2}$$

$$\text{Thus} \quad 4a\ Q = \frac{3h^2}{4a^2} = 3v^2$$

When $\theta = 0, r = 0$ and $p = 0$ and the particle is moving with infinite velocity along the axis of the cardioid and continues to move in a straight line.

5.51 Let the force $f = -\frac{k}{r^3} = -ku^3$

where $u = \frac{1}{r}$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{f}{h^2u^2} = \frac{ku}{h^2}$$

Case (i): $\dfrac{k}{h^2} > 1$

Let $\dfrac{k}{h^2} - 1 = n^2$

$$\frac{d^2u}{d\theta^2} - n^2u = 0$$

which has the solution $u = Ae^{n\theta} + Be^{-n\theta}$, where the constants A and B depend on the initial conditions of projection. If these are such that either A or B is zero then the path is an equiangular spiral

Case (ii): $\dfrac{k}{h^2} = 1$, the equation becomes $\dfrac{d^2u}{d\theta^2} = 0$, whose solution is $v = A\theta + B$, a curve known as the reciprocal spiral curve.

Case (iii): $\dfrac{k}{h^2} < 1$. Let $1 - \dfrac{k}{h^2} = n^2$, the equation becomes $\dfrac{d^2u}{d\theta^2} + n^2u = 0$ whose solution is $u = A\cos n\theta + B\sin n\theta$, a curve with infinite branches.

5.52 $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \tag{1}$$

$$r^2 = a^2\cos^2\theta \tag{2}$$

$$r\frac{dr}{d\theta} = -a^2\sin^2\theta \tag{3}$$

$$\therefore \quad \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6}\sin^2 2\theta = \frac{a^4}{r^6}\left(1 - \frac{r^4}{a^4}\right) = \frac{a^4}{r^6} - \frac{1}{r^2} \tag{4}$$

From (1) and (4)

$$\frac{1}{p^2} = \frac{a^4}{r^6}$$

$$\text{or} \quad p = \frac{r^3}{a^2} \tag{5}$$

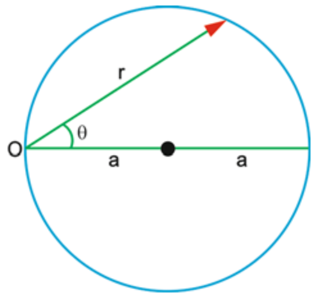
$$\therefore \quad \frac{dp}{dr} = \frac{3r^2}{a^2} \tag{6}$$

$$f = -\frac{h^2}{p^3}\frac{dp}{dr} = -\frac{3h^2a^4}{r^7}$$

where we have used (5) and (6).

5.53 The polar equation of a circle with the origin on the circumference is $r = 2a\cos\theta$ where a is the radius of the circle, Fig. 5.21.

Fig. 5.21



$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \tag{1}$$

$$r = 2a\cos\theta \tag{2}$$

$$\frac{dr}{d\theta} = -2a\sin\theta \tag{3}$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2 = 4a^2(1-\cos^2\theta) = 4a^2 - r^2 \tag{4}$$

$$\therefore \quad \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{4a^2}{r^4} - \frac{1}{r^2} \tag{5}$$

Using (5) in (1) and simplifying

$$p = \frac{r^2}{2a} \tag{6}$$

$$\frac{dp}{dr} = \frac{r}{a} \tag{7}$$

$$f = -\frac{h^2}{p^3}\frac{dp}{dr} = -\frac{h^2a^4}{r^5}$$

5.54 Initially the earth's orbit is circular and its kinetic energy would be equal to the modulus of potential energy

$$\frac{1}{2}mv_0^2 = \frac{1}{2}\frac{mGM}{r} \tag{1}$$

Suddenly, sun's mass becomes half and the earth is placed with a new quantity of potential energy, its instantaneous value of kinetic energy remaining unaltered.

New total energy = new potential energy + kinetic energy

$$= -G\left(\frac{M}{2}\right)\frac{m}{r} + \frac{1}{2}mv_0^2$$
$$= -\frac{GMm}{2r} + \frac{1}{2}\frac{GMm}{r} = 0$$

As the total energy $E = 0$ the earth's orbit becomes parabolic.

Chapter 6
Oscillations

Abstract *Chapter 6* deals with simple harmonic motion and its application to various problems, physical pendulums, coupled systems of masses and springs, the normal coordinates and damped vibrations.

6.1 Basic Concepts and Formulae

Simple Harmonic Motion (SHM)

In SHM the restoring force (F) is proportional to the displacement but is oppositely directed.

$$F = -kx \tag{6.1}$$

where k is a constant, known as force constant or spring constant. The negative sign in (6.1) implies that the force is opposite to the displacement.

When the mass is released, the force produces acceleration a given by

$$a = F/m = -k/m = -\omega^2 x \tag{6.2}$$

$$\text{where } \omega^2 = k/m \tag{6.3}$$

$$\text{and } \omega = 2\pi f \tag{6.4}$$

is the angular frequency.

Differential equation for SHM:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0 \tag{6.5}$$

Most general solution for (6.5) is

$$x = A \sin(\omega t + \varepsilon) \tag{6.6}$$

where A is the amplitude, $(\omega t + \varepsilon)$ is called the phase and ε is called the phase difference.

The velocity v is given by

$$v = \pm\omega\sqrt{A^2 - x^2} \tag{6.7}$$

The acceleration is given by

$$a = -\omega^2 x \tag{6.8}$$

The frequency of oscillation is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{6.9}$$

where m is the mass of the particle.

The time period is given by

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{6.10}$$

Total energy (E) of the oscillator:

$$E = {}^1\!/_2\, m A^2 \omega^2 \tag{6.11}$$

$$K_{av} = U_{av} = {}^1\!/_4\, m A^2 \omega^2 \tag{6.12}$$

Loaded spring:

$$T = 2\pi\sqrt{\frac{\left(M + \frac{m}{3}\right)}{k}} \tag{6.13}$$

where M is the load and m is the mass of the spring.

If v_1 and v_2 are the velocities of a particle at x_1 and x_2, respectively, then

$$T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}} \tag{6.14}$$

$$A = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}} \tag{6.15}$$

Pendulums

Simple Pendulum (Small Amplitudes)

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{6.16}$$

T is independent of the mass of the bob. It is also independent of the amplitude for small amplitudes.

Seconds pendulum is a simple pendulum whose time period is 2 s.

Simple Pendulum (Large Amplitude)

For large amplitude θ_0, the time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\theta_0}{2}\right) + \left(\frac{1.3}{2.4}\right)^2 \sin^4\left(\frac{\theta_0}{2}\right) + \left(\frac{1.3.5}{2.4.6}\right)^2 \sin^6\left(\frac{\theta_0}{2}\right)\right] \tag{6.17}$$

where we have dropped higher order terms.

Simple pendulum on an elevator/trolley moving with acceleration a. Time period of the stationary pendulum is T and that of moving pendulum T'.

(a) Elevator has upward acceleration a

$$T' = T\sqrt{\frac{g}{g+a}} \tag{6.18}$$

(b) Elevator has downward acceleration a

$$T' = T\sqrt{\frac{g}{g-a}} \tag{6.19}$$

(c) Elevator has constant velocity, i.e. $a = 0$

$$T' = T \tag{6.20}$$

(d) Elevator falls freely or is kept in a satellite, $a = g$

$$T' = \infty \tag{6.21}$$

The bob does not oscillate at all but assumes a fixed position.

(a) Trolley moving horizontally with acceleration a

$$T' = T\sqrt{\frac{g}{\sqrt{g^2 + a^2}}} \tag{6.22}$$

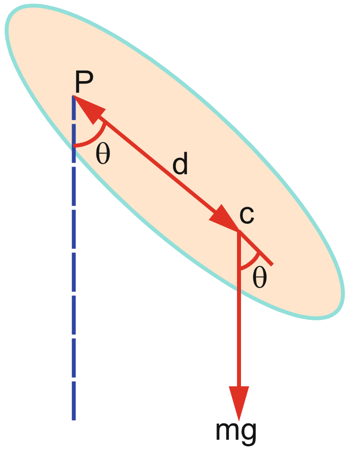
(b) Trolley rolls down on a frictionless incline at an angle θ to the horizontal plane

$$T' = T/\cos\theta \tag{6.23}$$

Physical Pendulum

Any rigid body mounted such that it can swing in a vertical plane about some axis passing through it is called a physical pendulum, Fig. 6.1.

Fig. 6.1



The body is pivoted to a horizontal frictionless axis through P and displaced from the equilibrium position by an angle θ. In the equilibrium position the centre of mass C lies vertically below the pivot P. If the distance from the pivot to the centre of mass be d, the mass of the body M and the moment of inertia of the body about an axis through the pivot I, the time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{1}{Mgd}} \tag{6.24}$$

The equivalent length of simple pendulum is

$$L_{\text{eq}} = I/Md \tag{6.25}$$

The torsional oscillator consists of a flat metal disc suspended by a wire from a clamp and attached to the centre of the disc. When displaced through a small angle about the vertical wire and released the oscillator would execute oscillations in the horizontal plane. For small twists the restoring torque will be proportional to the angular displacement

$$\tau = -C\theta \tag{6.26}$$

where C is known as torsional constant. The time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \tag{6.27}$$

Coupled Harmonic Oscillators

Two equal masses connected by a spring and two other identical springs fixed to rigid supports on either side, Fig. 6.2, permit the masses to jointly undergo SHM along a straight line, so that the system corresponds to two coupled oscillators. The equation of motion for mass m_1 is

$$m\ddot{x}_1 + k(2x_1 - x_2) = 0 \tag{6.28}$$

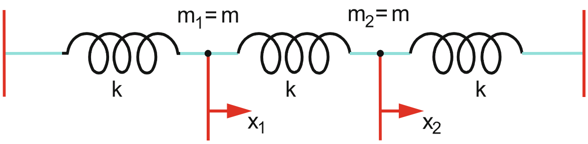


Fig. 6.2

and that for m_2 is

$$m\ddot{x}_2 + k(2x_2 - x_1) = 0 \tag{6.29}$$

Equations (6.28) and (6.29) are coupled equations.

Assuming $x_1 = A_1 \sin\omega t$ and $x_2 = A_2 \sin\omega t$

(6.28) and (6.29) become

$$\ddot{x}_1 = -\omega^2 A_1 \sin\omega t = -\omega^2 x_1 \tag{6.30}$$

$$\ddot{x}_2 = -\omega^2 A_2 \sin\omega t = -\omega^2 x_2 \tag{6.31}$$

Inserting (6.30) and (6.31) in (6.28) and (6.29), we get on rearrangement

$$(2k - m\omega^2)x_1 - kx_2 = 0 \tag{6.32}$$

$$-kx_1 + (2k - m\omega^2)x_2 = 0 \tag{6.33}$$

For a non-trivial solution, the determinant formed from the coefficients of x_1 and x_2 must vanish.

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0$$

The expansion of the determinant gives a quadratic equation in ω whose solutions are

$$\omega_1 = \sqrt{k/m} \tag{6.35}$$

$$\omega_2 = \sqrt{3k/m} \tag{6.36}$$

Normal coordinates: It is always possible to define a new set of coordinates called normal coordinates which have a simple time dependence and correspond to the excitation of various oscillation modes of the system. Consider a pair of coordinates defined by

$$\eta_1 = x_1 - x_2, \eta_2 = x_1 + x_2 \tag{6.37}$$

$$\text{or} \quad x_1 = \frac{1}{2}(\eta_1 + \eta_2), x_2 = \frac{1}{2}(\eta_2 - \eta_1) \tag{6.38}$$

Substituting (6.38) in (6.28) and (6.29) we get

$$m(\ddot{\eta}_1 + \ddot{\eta}_2) + k(3\eta_1 + \eta_2) = 0$$

$$m(\ddot{\eta}_1 - \ddot{\eta}_2) + k(3\eta_1 - \eta_2) = 0$$

which can be solved to yield

$$m\ddot{\eta}_1 + 3k\eta_1 = 0$$

$$m\ddot{\eta}_2 + k\eta_2 = 0 \tag{6.39}$$

The coordinates η_1 and η_2 are now uncoupled and are therefore independent unlike the old coordinates x_1 and x_2 which were coupled.

The solutions of (6.39) are

$$\eta_1(t) = B_1 \sin \omega_1 t, \quad \eta_2(t) = B_2 \sin \omega_2 t \tag{6.40}$$

where the frequencies are given by (6.35) and (6.36).

A deeper insight is obtained from the energies expressed in normal coordinates as opposed to the old coordinates. The potential energy of the system

$$U = \frac{1}{2}k{x_1}^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}\,k{x_2}^2$$

$$= k({x_1}^2 - x_1 x_2 + {x_2}^2) \tag{6.41}$$

The term proportional to the cross-product $x_1 x_2$ is the one which expresses the coupling of the system. The kinetic energy of the system is

$$K = \tfrac{1}{2} m\dot{x}_1^2 + \tfrac{1}{2} m\dot{x}_2^2 \tag{6.42}$$

In terms of normal coordinates defined by (6.38)

$$U = \frac{k}{4}(\eta_1^2 + 3\eta_2^2) \tag{6.43}$$

$$K = \frac{m}{4}(\dot{\eta}_1^2 + \dot{\eta}_2^2) \tag{6.44}$$

Thus, the cross-product term has disappeared and the kinetic and potential energies appear in quadratic form. Each normal coordinate corresponds to an independent mode of vibration of the system, with its own characteristic frequency and the general vibratory motion may be regarded as the superposition of some or all of the independent normal vibrations.

Damped Vibrations

For small velocities the resisting force f_r (friction) is proportional to the velocity:

$$f_\mathrm{r} = -r\frac{\mathrm{d}x}{\mathrm{d}t} \tag{6.45}$$

where r is known as the resistance constant or damping constant. The presence of the dissipative forces results in the loss of energy in heat motion leading to a gradual decrease of amplitude. The equation of motion is written as

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + r\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0 \tag{6.46}$$

where m is the mass of the body and k is the spring constant.

Putting $r/m = 2b$ and $k/m = \omega_0{}^2$, (6.46) becomes on dividing by m

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2b\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0 \tag{6.47}$$

Let $x = \mathrm{e}^{\lambda t}$ so that $\mathrm{d}x/\mathrm{d}t = \lambda \mathrm{e}^{\lambda t}$ and $\mathrm{d}^2x/\mathrm{d}t^2 = \lambda^2 \mathrm{e}^{\lambda t}$

The corresponding characteristic equation is

$$\lambda^2 + 2b\lambda + \omega_0{}^2 = 0 \tag{6.48}$$

The roots are

$$\lambda = -b \pm \sqrt{b^2 - \omega_0^2} \tag{6.49}$$

Calling $R = \sqrt{b^2 - \omega_0^2}$

$$\lambda_1 = -b + R \quad \lambda_2 = -b - R$$

Using the boundary conditions, at $t = 0$, $x = x_0$ and $\mathrm{d}x/\mathrm{d}t = 0$ the solution to (6.47) is found to be

$$x = \frac{1}{2} x_0 \mathrm{e}^{-bt} \left[(1 + b/R)\mathrm{e}^{Rt} + (1 - b/R)\mathrm{e}^{-Rt} \right] \tag{6.50}$$

The physical solution depends on the degree of damping.

Case 1: Small frictional forces: $b < \omega_0$ (underdamping)

$b^2 < k/m$ or $(r/2m)^2 < k/m$

$$R \text{ is imaginary. } R = j\omega', \text{ where } j = \sqrt{-1}$$

$$\omega'^2 = \omega_0^2 - b^2 \tag{6.51}$$

$$x = A\mathrm{e}^{-bt} \cos(\omega' t + \varepsilon) \tag{6.52}$$

$$\text{where } A = \omega_0 x_0/\omega' \text{ and } \varepsilon = \tan^{-1}(-b/\omega') \tag{6.53}$$

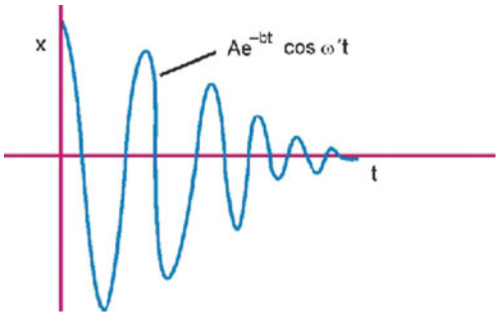


Fig. 6.3 Underdamped motion

Equation (6.52) represents damped harmonic motion of period

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} \tag{6.54}$$

$T = 1/b$ is the time in which the amplitude is reduced to $1/e$.

The logarithmic decrement Δ is

$$\Delta = \ln\left(\frac{A'}{Ae^{-bT'}}\right) = bT' \tag{6.55}$$

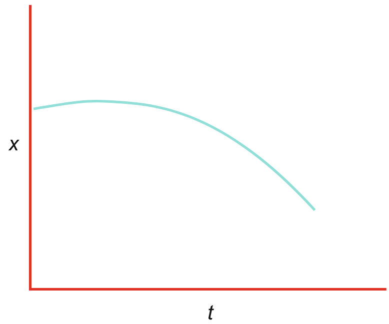
Case 2: Large frictional forces (overdamping)

$b > \omega_0$. Distinct real roots.

Both the exponential terms in (6.50) are negative and they correspond to exponential decrease. The motion is not oscillatory. The general solution is of the form

$$x = \mathrm{e}^{-bt}(A\mathrm{e}^{Rt} + B\mathrm{e}^{-Rt}) \tag{6.56}$$

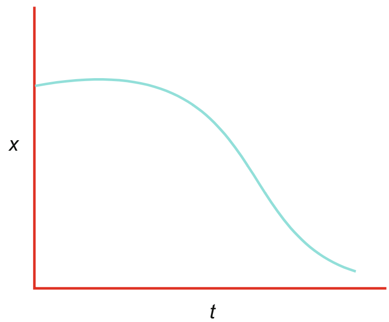
Fig. 6.4 Overdamped motion



Case 3: Critical damping

$$b = \omega, \ \ R = 0$$

Fig. 6.5 Criticallydamped motion



The exponentials in the square bracket may be expanded to terms linear in Rt. The solution is of the form

$$x = x_0\, \mathrm{e}^{-bt}(1 + bt) \tag{6.57}$$

The motion is not oscillatory and is said to be critically damped. It is a transition case and the motion is just aperiodic or non-oscillatory. There is an initial rise in the displacement due to the factor $(1 + bt)$ but subsequently the exponential term dominates.

Energy and Amplitude of a Damped Oscillator

$$E(t) = E_0\mathrm{e}^{-t/t_c} \tag{6.58}$$

where $t_c = m/r$

$$A(t) = A_0 e^{-t/2t_c} \tag{6.59}$$

Quality factor

$$Q = \omega t_c = \omega m/r \tag{6.60}$$

$$\omega' = \omega_0 \sqrt{1 - \frac{1}{4Q^2}} \tag{6.61}$$

The value of quality factor indicates the sharpness of resonance.

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} \tag{6.62}$$

where ω_0 is the resonance angular frequency and ω_2 and ω_1 are, respectively, the two angular frequencies above and below resonance at which the average power has dropped to one-half its resonance value. (Fig. 6.6).

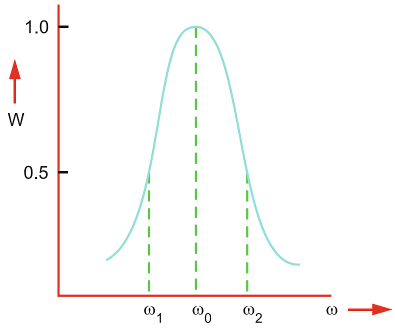


Fig. 6.6 Resonance frequency curve, ω_0 is the resonance angular frequency. ω_1 and ω_2 are defined in the text

Forced vibrations are set up by a periodic force $F\ \cos\ \omega t$.

Equation of motion of a particle of mass m

$$\frac{m\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{r\mathrm{d}x}{\mathrm{d}t} + kx = F\cos\omega t \tag{6.63}$$

or

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2b\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = p\cos\omega t \tag{6.64}$$

where

$$k/m = \omega_0^2,\ r/m = 2b \text{ and } F/m = p \tag{6.65}$$

ω_0 being the resonance frequency.

$$x = A\cos(\omega t - \varepsilon) \tag{6.66}$$

$$\tan\varepsilon = \frac{2b\omega}{\omega_0^2 - \omega^2} \tag{6.67}$$

Mechanical impedance

$$Z_m = \sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} \tag{6.68}$$

$$A = \frac{p}{Z_m} \tag{6.69}$$

$$Q = \frac{\omega_0}{2b} \tag{6.70}$$

Power

$$W = \frac{F^2 - \sin\varepsilon}{2Z_m} \tag{6.71}$$

6.2 Problems

6.2.1 Simple Harmonic Motion (SHM)

6.1 The total energy of a particle executing SHM of period 2π s is 0.256 J. The displacement of the particle at $\pi/4$ s is $8\sqrt{2}$ cm. Calculate the amplitude of motion and mass of the particle.

6.2 A particle makes SHM along a straight line and its velocity when passing through points 3 and 4 cm from the centre of its path is 16 and 12 cm/s, respectively. Find **(a)** the amplitude; **(b)** the time period of motion.

[Northern Universities of UK]

6.3 A small bob of mass 50 g oscillates as a simple pendulum, with amplitude 5 cm and period 2 s. Find the velocity of the bob and the tension in the supporting thread when velocity of the bob is maximum.

[University of Aberystwyth, Wales]

6.4 A particle performs SHM with a period of 16 s. At time $t = 2$ s, the particle passes through the origin while at $t = 4$ s, its velocity is 4 m/s. Show that the amplitude of the motion is $32\sqrt{2}/\pi$.

[University of Dublin]

6.5 Show that given a small vertical displacement from its equilibrium position a floating body subsequently performs simple harmonic motion of period $2\pi\sqrt{V/Ag}$ where V is the volume of displaced liquid and A is the area of the plane of floatation. Ignore the viscous forces.

6.6 Imagine a tunnel bored along the diameter of the earth assumed to have constant density. A box is thrown into the tunnel (chute). **(a)** Show that the box executes SHM inside the tunnel about the centre of the earth. **(b)** Find the time period of oscillations.

6.7 A particle which executes SHM along a straight line has its motion represented by $x = 4\sin(\pi t/3 + \pi/6)$. Find **(a)** the amplitude; **(b)** time period; **(c)** frequency; **(d)** phase difference; **(e)** velocity; **(f)** acceleration, at $t = 1$ s, x being in cm.

6.8 **(a)** At what distance from the equilibrium position is the kinetic energy equal to the potential energy for a SHM?
(b) In SHM if the displacement is one-half of the amplitude show that the kinetic energy and potential energy are in the ratio 3:1.

6.9 A mass M attached to a spring oscillates with a period 2 s. If the mass is increased by 2 kg, the period increases by 1 s. Assuming that Hooke's law is obeyed, find the initial mass M.

6.10 A particle vibrates with SHM along a straight line, its greatest acceleration is $5\pi^2\ \mathrm{cm/s^2}$, and when its distance from the equilibrium is 4 cm the velocity of the particle is 3π cm/s. Find the amplitude and the period of oscillation of the particle.

6.11 If the maximum acceleration of a SHM is α and the maximum velocity is β, show that the amplitude of vibration is given by β^2/α and the period of oscillation by $2\pi\beta/\alpha$.

6.12 If the tension along the string of a simple pendulum at the lowest position is 1% higher than the weight of the bob, show that the angular amplitude of the pendulum is 0.1 rad.

6.13 A particle executes SHM and is located at $x = a$, b and c at time t_0, $2t_0$ and $3t_0$, respectively. Show that the frequency of oscillation is $\dfrac{1}{2\pi t_0}\cos^{-1}\dfrac{a+c}{2b}$.

6.14 A 4 kg mass at the end of a spring moves with SHM on a horizontal frictionless table with period 2 s and amplitude 2 m. Determine **(a)** the spring constant; **(b)** maximum force exerted on the spring.

6.15 A particle moves in the xy-plane according to the equations $x = a \sin \omega t$; $y = b \cos \omega t$. Determine the path of the particle.

6.16 **(a)** Prove that the force $\boldsymbol{F} = -kx\acute{i}$ acting in a SHO is conservative. **(b)** Find the potential energy of an SHO.

6.17 A 2 kg weight placed on a vertical spring stretches it 5 cm. The weight is pulled down a distance of 10 cm and released. Find **(a)** the spring constant; **(b)** the amplitude; **(c)** the frequency of oscillations.

6.18 A mass m is dropped from a height h on to a scale-pan of negligible weight, suspended from a spring of spring constant k. The collision may be considered to be completely inelastic in that the mass sticks to the pan and the pan begins to oscillate. Find the amplitude of the pan's oscillations.

6.19 A particle executes SHM along the x-axis according to the law $x = A \sin \omega t$. Find the probability $\mathrm{d}p(x)$ of finding the particle between x and $x + \mathrm{d}x$.

6.20 Using the probability density distribution for the SHO, calculate the mean potential energy and the mean kinetic energy over an oscillation.

6.21 A cylinder of mass m is allowed to roll on a smooth horizontal table with a spring of spring constant k attached to it so that it executes SHM about the equilibrium position. Find the time period of oscillations.

6.22 Two simple pendulums of length 60 and 63 cm, respectively, hang vertically one in front of the other. If they are set in motion simultaneously, find the time taken for one to gain a complete oscillation on the other.

[Northern Universities of UK]

6.23 A pendulum that beats seconds and gives correct time on ground at a certain place is moved to the top of a tower 320 m high. How much time will the pendulum lose in 1 day? Assume earth's radius to be 6400 km.

6.24 Taking the earth's radius as 6400 km and assuming that the value of g inside the earth is proportional to the distance from the earth's centre, at what depth below the earth's surface would a pendulum which beats seconds at the earth's surface lose 5 min in a day?

[University of London]

6.25 A U-tube is filled with a liquid, the total length of the liquid column being h. If the liquid on one side is slightly depressed by blowing gently down, the levels of the liquid will oscillate about the equilibrium position before finally coming to rest. **(a)** Show that the oscillations are SHM. **(b)** Find the period of oscillations.

6.26 A gas of mass m is enclosed in a cylinder of cross-section A by means of a frictionless piston. The gas occupies a length l in the equilibrium position and is at pressure P. **(a)** If the piston is slightly depressed, show that it will execute SHM. **(b)** Find the period of oscillations (assume isothermal conditions).

6.27 A SHM is given by $y = 8\sin\left(\frac{2\pi t}{\tau} + \varphi\right)$, the time period being 24 s. At $t = 0$, the displacement is 4 cm. Find the displacement at $t = 6$ s.

6.28 In a vertical spring-mass system, the period of oscillation is 0.89 s when the mass is 1.5 kg and the period becomes 1.13 s when a mass of 1.0 kg is added. Calculate the mass of the spring.

6.29 Consider two springs A and B with spring constants k_A and k_B, respectively, A being stiffer than B, that is, $k_A > k_B$. Show that

(a) when two springs are stretched by the same amount, more work will be done on the stiffer spring.
(b) when two springs are stretched by the same force, less work will be done on the stiffer spring.

6.30 A solid uniform cylinder of radius r rolls without sliding along the inside surface of a hollow cylinder of radius R, performing small oscillations. Determine the time period.

6.2.2 Physical Pendulums

6.31 Consider the rigid plane object of weight Mg shown in Fig. 6.7, pivoted about a point at a distance D from its centre of mass and displaced from equilibrium by a small angle φ. Such a system is called a physical pendulum. Show that the oscillatory motion of the object is simple harmonic with a period given by $T = 2\pi\sqrt{\frac{I}{MgD}}$ where I is the moment of inertia about the pivot point.

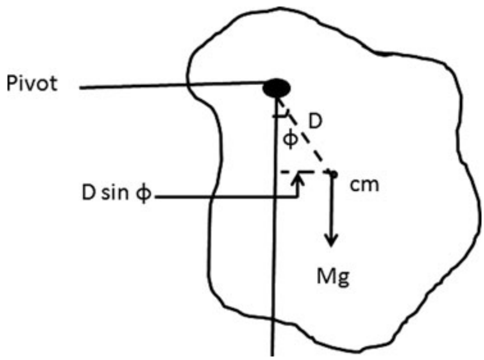
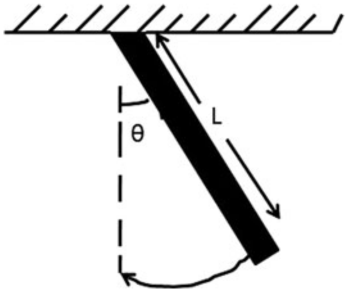


Fig. 6.7

6.32 A thin, uniform rod of mass M and length L swings from one of its ends as a physical pendulum (see Fig. 6.8). Given that the moment of inertia of a

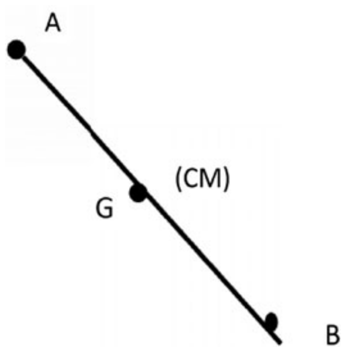
Fig. 6.8



uniform rod about one end is $I = \frac{1}{3}ML^2$, obtain an equation for the period of the oscillatory motion for small angles. What would be the length l of a simple pendulum that has the same period as the swinging rod?

6.33 The physical pendulum has two possible pivot points A and B, distance L apart, such that the period of oscillations is the same (Fig. 6.9). Show that the acceleration due to gravity at the pendulum's location is given by $g = 4\pi^2 L/T^2$.

Fig. 6.9



6.34 A semi-circular homogeneous disc of radius R and mass m is pivoted freely about the centre. If slightly tilted through a small angle and released, find the angular frequency of oscillations.

6.35 A ring is suspended on a nail. It can oscillate in its plane with time period T_1 or it can oscillate back and forth in a direction perpendicular to the plane of the ring with time period T_2. Find the ratio T_1/T_2.

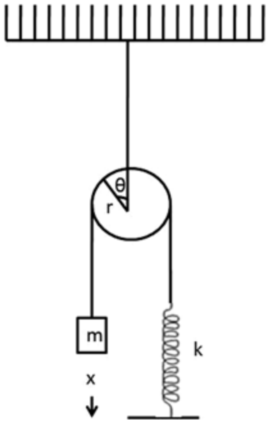
6.36 A torsional oscillator consists of a flat metal disc suspended by a wire. For small angular displacements show that time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

where I is the moment of inertia about its axis and C is known as torsional constant given by $\tau = -C\theta$, where τ is the torque.

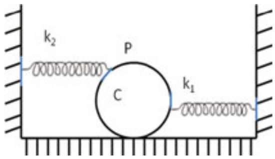
6.37 In the arrangement shown in Fig. 6.10, the radius of the pulley is r, its moment of inertia about the rotation axis is I and k is the spring constant. Assuming that the mass of the thread and the spring is negligible and that the thread does not slide over the frictionless pulley, calculate the angular frequency of small oscillations.

Fig. 6.10



6.38 Two unstretched springs with spring constants k_1 and k_2 are attached to a solid cylinder of mass m as in Fig. 6.11. When the cylinder is slightly displaced and released it will perform small oscillations about the equilibrium position. Assuming that the cylinder rolls without sliding, find the time period.

Fig. 6.11



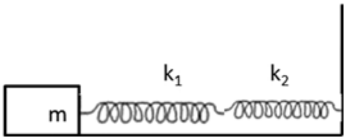
6.39 A particle of mass m is located in a one-dimensional potential field $U(x) = \dfrac{a}{x^2} - \dfrac{b}{x}$ where a and b are positive constants. Show that the period of small oscillations that the particle performs about the equilibrium position will be $T = 4\pi\sqrt{\dfrac{2a^3m}{b^4}}$

[Osmania University 1999]

6.2.3 Coupled Systems of Masses and Springs

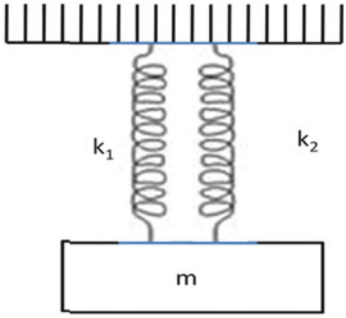
6.40 Two springs of constants k_1 and k_2 are connected in series, Fig. 6.12. Calculate the effective spring constant.

Fig. 6.12



6.41 A mass m is connected to two springs of constants k_1 and k_2 in parallel, Fig. 6.13. Calculate the effective (equivalent) spring constant.

Fig. 6.13



6.42 A mass m is placed on a frictionless horizontal table and is connected to fixed points A and B by two springs of negligible mass and of equal natural length with spring constants k_1 and k_2, Fig. 6.14. The mass is displaced along x-axis and released. Calculate the period of oscillation.

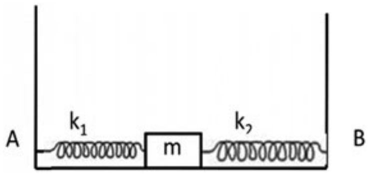


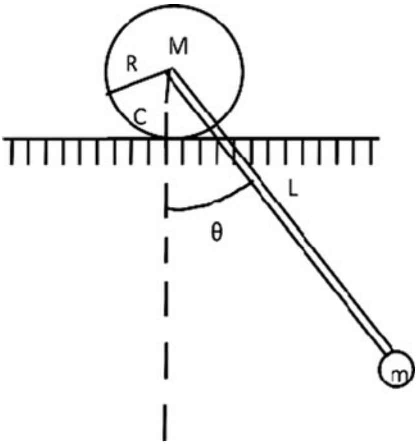
Fig. 6.14

6.43 One end of a long metallic wire of length L is tied to the ceiling. The other end is tied to a massless spring of spring constant k. A mass m hangs freely from the free end of the spring. The area of cross-section and the Young's modulus of the wire are A and Y respectively. The mass is displaced down and released. Show that it will oscillate with time period $T = 2\pi\sqrt{\dfrac{m(YA + kL)}{YAk}}$.

[Adapted from Indian Institute of Technology 1993]

6.44 The mass m is attached to one end of a weightless stiff rod which is rigidly connected to the centre of a uniform cylinder of radius R, Fig. 6.15. Assuming that the cylinder rolls without slipping, calculate the natural frequency of oscillation of the system.

Fig. 6.15



6.45 Find the natural frequency of a semi-circular disc of mass m and radius r which rolls from side to side without slipping.

6.46 Determine the eigenfrequencies and describe the normal mode motion for two pendula of equal lengths b and equal masses m connected by a spring of force constant k as shown in Fig. 6.16. The spring is unstretched in the equilibrium position.

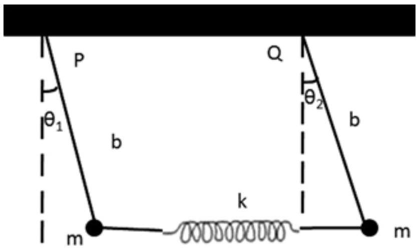


Fig. 6.16

6.47 In prob. (6.46) express the equations of motion and the energy in terms of normal coordinates. What are the characteristics of normal coordinates?

6.48 The superposition of two harmonic oscillations in the same direction leads to the resultant displacement $y = A\cos 6\pi t \sin 90\pi$, where t is expressed in seconds. Find the frequency of the component vibrations and the beat frequency.

6.49 Find the fundamental frequency of vibration of the HCl molecule. The masses of H and Cl may be assumed to be 1.0 and 36.46 amu.

$1\,\text{amu} = 1.66 \times 10^{-27}\,\text{kg}$ and $k = 480\,\text{N/m}$

6.50 Find the resultant of the vibrations $y_1 = \cos\omega t$, $y_2 = \frac{1}{2}\cos(\omega t + \pi/2)$ and $y_3 = \frac{1}{3}\cos(\omega t + \pi)$, acting in the same straight line.

6.2.4 Damped Vibrations

6.51 A mass attached to a spring vibrates with a natural frequency of 20 c/s while its frequency for damped vibrations is 16 c/s. Determine the logarithmic decrement.

6.52 The equation of motion for a damped oscillator is given by

$$4\mathrm{d}^2x/\mathrm{d}t^2 + r\mathrm{d}x/\mathrm{d}t + 32x = 0$$

For what range of values for the damping constant will the motion be **(a)** underdamped; **(b)** overdamped; **(c)** critically damped?

6.53 A mass of 4 kg attached to the lower end of a vertical spring of constant 20 N/m oscillates with a period of 10 s. Find **(a)** the natural period; **(b)** the damping constant; **(c)** the logarithmic decrement.

6.54 Solve the equation of motion for the damped oscillator $\mathrm{d}^2x/\mathrm{d}t^2 + 2\mathrm{d}x/\mathrm{d}t + 5x = 0$, subject to the condition $x = 5$, $\mathrm{d}x/\mathrm{d}t = -3$ at $t = 0$.

6.55 A 1 kg weight attached to a vertical spring stretches it 0.2 m. The weight is then pulled down 1.5 m and released. **(a)** Is the motion underdamped, overdamped or critically damped? **(b)** Find the position of the weight at any time if a damping force numerically equal to 14 times the instantaneous speed is acting.

6.56 A periodic force acts on a 6 kg mass suspended from the lower end of a vertical spring of constant 150 N/m. The damping force is proportional to the instantaneous speed of the mass and is 80 N when $v = 2\,\text{m/s}$. find the resonance frequency.

6.57 The equation of motion for forced oscillations is $2\,\mathrm{d}^2x/\mathrm{d}t^2 + 1.5\mathrm{d}x/\mathrm{d}t + 40x = 12\cos 4t$. Find **(a)** amplitude; **(b)** phase lag; **(c)** Q factor; **(d)** power dissipation.

6.58 An electric bell has a frequency 100 Hz. If its time constant is 2 s, determine the Q factor for the bell.

6.59 An oscillator has a time period of 3 s. Its amplitude decreases by 5% each cycle **(a)** By how much does its energy decrease in each cycle? **(b)** Find the time constant **(c)** Find the Q factor.

6.60 A damped oscillator loses 3% of its energy in each cycle. **(a)** How many cycles elapse before half its original energy is dissipated? **(b)** What is the Q factor?

6.61 A damped oscillator has frequency which is 9/10 of its natural frequency. By what factor is its amplitude decreased in each cycle?

6.62 Show that for small damping $\omega' \approx (1 - r^2/8mk)\omega_0$ where ω_0 is the natural angular frequency, ω' the damped angular frequency, r the resistance constant, k the spring constant and m the particle mass.

6.63 Show that the time elapsed between successive maximum displacements of a damped harmonic oscillator is constant and equal to $4\pi m/\sqrt{4km - r^2}$, where m is the mass of the vibrating body, k is the spring constant, $2b = r/m$, r being the resistance constant.

6.64 A dead weight attached to a light spring extends it by 9.8 cm. It is then slightly pulled down and released. Assuming that the logarithmic decrement is equal to 3.1, find the period of oscillation.

6.65 The position of a particle moving along x-axis is determined by the equation $\mathrm{d}^2x/\mathrm{d}t^2 + 2\mathrm{d}x/\mathrm{d}t + 8x = 16\cos 2t$.

(a) What is the natural frequency of the vibrator?
(b) What is the frequency of the driving force?

6.66 Show that the time $t_{1/2}$ for the energy to decrease to half its initial value is related to the time constant by $t_{1/2} = t_c \ln 2$.

6.67 The amplitude of a swing drops by a factor $1/e$ in 8 periods when no energy is pumped into the swing. Find the Q factor.

6.3 Solutions

6.3.1 Simple Harmonic Motion (SHM)

6.1 $x = A\sin\omega t \qquad \text{(SHM)}$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\ \text{rad/s}$$

$$8\sqrt{2} = A\sin\left(\frac{1\cdot\pi}{4}\right)$$

$$A = 16\ \text{cm} = 0.16\ \text{m}$$

$$E = \frac{1}{2}mA^2\omega^2$$

$$\therefore \quad m = \frac{2E}{A^2\omega^2} = \frac{2\times 0.256}{(0.16)^2\times 1^2} = 20.0\ \text{kg}$$

6.2 **(a)** $$v = \omega\sqrt{A^2 - x^2} \quad (1)$$

$$16 = \omega\sqrt{A^2 - 3^2} \quad (2)$$

$$12 = \omega\sqrt{A^2 - 4^2} \quad (3)$$

Solving (2) and (3) A $= 5\,\text{cm}$ and $\omega = 4\,\text{rad/s}$

(b) Therefore $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{4} = 1.57\,\text{s}$

6.3 $x = A\sin\omega t$

$$v = \frac{dx}{dt} = \omega A\cos\,\omega\text{t}$$

$$v_{\max} = A\omega = \frac{2\pi A}{T} = \frac{2\pi \times 5}{2} = 5\pi\ \text{cm/s}$$

At the equilibrium position the weight of the bob and the tension act in the same direction

$$\text{Tension} = mg + \frac{mv_{\max}^2}{L}$$

Now the length of the simple pendulum is calculated from its period T.

$$L = \frac{gT^2}{4\pi^2} = \frac{980 \times 2^2}{4\pi^2} = 99.29\,\text{cm}$$

$$\text{Tension} = m\left(1 + \frac{v_{\max}^2}{gL}\right)g = 50\left(1 + \frac{25\pi^2}{980 \times 99.29}\right)g$$

$$= 50.13\ \text{g dynes} = 50.13\ \text{g wt}$$

6.4 The general equation of SHM is

$$x = A\sin(\omega t + \varepsilon)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$$

When $t = 2\,\text{s}$, $x = 0$.

$$0 = A\sin\left(\frac{\pi}{8} \times 2 + \varepsilon\right)$$

Since $A \neq 0$, $\sin\left(\dfrac{\pi}{4} + \varepsilon\right) = 0$

$$\therefore \quad \frac{\pi}{4} + \varepsilon = 0 \quad \varepsilon = -\frac{\pi}{4}$$

Now $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = A\omega\cos(\omega t + \varepsilon)$

When $t = 4$, $v = 4$.

$$\therefore \quad 4 = \frac{A\pi}{8}\cos\left(\frac{\pi}{8}4 - \frac{\pi}{4}\right)$$

$$\therefore \quad A = \frac{32\sqrt{2}}{\pi}$$

6.5 Let the body with uniform cross-section A be immersed to a depth h in a liquid of density D. Volume of the liquid displaced is $V = Ah$. Weight of the liquid displaced is equal to VDg or $AhDg$. According to Archimedes principle, the weight of the liquid displaced is equal to the weight of the floating body Mg.

$Mg = Ahdg$ or $\mathrm{M} = AhD$

The body occupies a certain equilibrium position. Let the body be further depressed by a small amount x. The body now experiences an additional upward thrust in the direction of the equilibrium position. When the body is released it moves up with acceleration

$$a = -\frac{AxDg}{M} = -\frac{AxDg}{AhD} = -\frac{gx}{h} = -\omega^2 x$$

with $\quad \omega^2 = \dfrac{g}{h}$

Time period $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{h}{g}} = 2\pi\sqrt{\dfrac{V}{Ag}}$

6.6 The acceleration due to gravity g at a depth d from the surface is given by

$$g = g_0\left(1 - \frac{d}{R}\right) \tag{1}$$

where g_0 is the value of g at the surface of the earth of radius R.

Writing $x = R - d$ (2)

Equation (1) becomes $g = g_0 \dfrac{x}{R}$ (3)

where x measures the distance from the centre. The acceleration g points opposite to the displacement x. We can therefore write

$$a = g = -\frac{g_0 x}{R} = -\omega^2 x \tag{4}$$

with $\omega^2 = \dfrac{g_0}{R}$

Equation (4) shows that the box performs SHM. The period is calculated from

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g_0}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} = 5074\,\text{s or } 84.6\,\text{min}$$

6.7 Standard equation for SHM is

$x = A\sin(\omega t + \varepsilon)$

$x = 4\sin\left(\dfrac{\pi t}{3} + \dfrac{\pi}{6}\right)$

(a) A = 4 cm

(b) $\omega = \dfrac{\pi}{3}$. Therefore $T = \dfrac{2\pi}{\omega} = 6\,\text{s}$

(c) $f = \dfrac{1}{T} = \dfrac{1}{6}/\,\text{s}$

(d) $\varepsilon = \dfrac{\pi}{6}$

(e) $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi t}{3} + \dfrac{\pi}{6}\right) = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi}{3} \times 1 + \dfrac{\pi}{6}\right) = 0$

(f) $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\dfrac{4\pi^2}{9}\sin\left(\dfrac{\pi}{3} \times 1 + \dfrac{\pi}{6}\right) = -\dfrac{4\pi^2}{9}$

6.8 **(a)** $K = \dfrac{1}{2}m\omega^2(A^2 - x^2) \quad U = \dfrac{1}{2}m\omega^2 x^2 \quad K = U$

$$\therefore \quad \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}m\omega^2 x^2$$

$$\therefore \quad x = \frac{A}{\sqrt{2}}$$

(b) $K = \dfrac{1}{2}m\omega^2\left(A^2 - \dfrac{A^2}{4}\right) = \dfrac{1}{2}m\omega^2\dfrac{3}{4}A^2$

$$U = \frac{1}{2}m\omega^2 \frac{A^2}{4}$$

$$\therefore \quad K : U = 3 : 1$$

6.9 $$T = 2\pi\sqrt{\frac{M}{k}} \tag{1}$$

$$2 = 2\pi\sqrt{\frac{M}{k}} \tag{2}$$

$$3 = 2\pi\sqrt{\frac{M+2}{k}} \tag{3}$$

Dividing (2) by (3) and solving for M, we get $M = 1.6\,\text{kg}$.

6.10 $$a_{\text{max}} = \omega^2 A$$

$$5\pi^2 = \omega^2 A \tag{1}$$

$$v = \omega\sqrt{A^2 - x^2}$$

$$3\pi = \omega\sqrt{A^2 - 16} \tag{2}$$

Solving (1) and (2), we get $A = 5\,\text{cm}$ and $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi} = 2\,\text{s}$.

6.11 $$\alpha = \omega^2 A \tag{1}$$

$$\beta = \omega A \tag{2}$$

$$\therefore \quad \beta^2 = \omega^2 A^2 = \alpha A$$

$$\text{or} \quad A = \frac{\beta^2}{\alpha}$$

Dividing (2) by (1)

$$\frac{\beta}{\alpha} = \frac{1}{\omega}$$

$$\text{or} \quad T = \frac{2\pi}{\omega} = \frac{2\pi\beta}{\alpha}$$

6.12 By problem $\dfrac{mg + mv^2/L}{mg} = 1.01$

$$\therefore \quad \frac{v^2}{gL} = 0.01$$

Conservation of energy gives

$$\frac{1}{2}mv^2 = mgh = mgL(1-\cos\theta) \simeq mgL\frac{\theta^2}{2} \quad \text{for small } \theta$$

$$\theta^2 = \frac{v^2}{gL} = 0.01$$

$$\therefore \quad \theta = \sqrt{0.01} = 0.1 \text{ rad}$$

6.13 $a = A \sin \omega t_0$

$$b = A \sin 2\omega t_0$$

$$c = A \sin 3\omega t_0$$

$$a + c = 2A \sin 2\omega t_0 \cos \omega t_0$$

$$\frac{a+c}{2b} = \cos \omega t_0$$

$$\omega = \frac{1}{t_0} \cos^{-1}\left(\frac{a+c}{2b}\right)$$

$$f = \frac{1}{2\pi t_0} \cos^{-1}\left(\frac{a+c}{2b}\right)$$

6.14 **(a)** $\omega = \sqrt{\dfrac{k}{m}}$

$$k = m\omega^2 = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 \times 4}{2^2} = 39.478\,\text{N/m}$$

(b) $F_{\max} = m\omega^2 A = kA = 39.478 \times 2 = 78.96\,\text{N}$

6.15 $x = a \sin \omega t$

$$y = b \cos \omega t$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2 \omega t + \cos^2 \omega t = 1$$

Thus the path of the particle is an ellipse.

6.16 **(a)** To show that $\nabla \times \boldsymbol{F} = 0$.

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -Kx & 0 & 0 \end{vmatrix} = 0$$

(b) $U = -\int F \mathrm{d}x = -\int (K\ ix)\,(-\hat{\imath}\,\mathrm{d}x) = \dfrac{1}{2} K x^2$

6.17 **(a)** $F = kx$

$$\therefore \quad k = \frac{F}{x} = \frac{2 \times 9.8}{5 \times 10^{-2}} = 392\,\mathrm{N/m}$$

(b) 10 cm

(c) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{392}{2 \times 9.8}} = 0.712/\,\mathrm{s}$

6.18 Let x_0 be the extension of the spring. Deformation energy = gravitational potential energy

$$\frac{1}{2}kx_0^2 = mgh + mgx_0$$

Rearranging

$$x_0^2 - \frac{2mg}{k}x_0 - mgh = 0$$

The quadratic equation has the solutions

$$x_{01} = \frac{mg}{k} + \sqrt{\frac{m^2g^2}{k^2} + \frac{2mgh}{k}}$$

$$x_{02} = \frac{mg}{k} - \sqrt{\frac{m^2g^2}{k^2} + \frac{2mgh}{k}}$$

The equilibrium position is depressed by $x_0 = \dfrac{mg}{k}$ below the initial position. The amplitude of the oscillations as measured from the equilibrium position is equal to $\sqrt{\dfrac{m^2g^2}{k^2} + \dfrac{2mgh}{k}}$.

6.19 It is reasonable to assume that the probability density $\dfrac{\mathrm{d}p(x)}{\mathrm{d}x}$ for finding the particle is proportional to the time spent at a given point and is therefore inversely proportional to its speed v.

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} = \frac{C}{v} \tag{1}$$

where C = constant of proportionality.

$$\text{But } v = \omega\sqrt{A^2 - x^2} \tag{2}$$

The probability density

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} = \frac{C}{\omega\sqrt{A^2 - x^2}} \tag{3}$$

C can be found by normalization of distribution

$$\int_{-A}^{A} \mathrm{d}p(x) = \frac{C}{\omega} \int_{-A}^{A} \frac{\mathrm{d}x}{\sqrt{A^2 - x^2}} = 1$$

or $\dfrac{C\pi}{\omega} = 1 \rightarrow \dfrac{C}{\omega} = \dfrac{1}{\pi}$

$$\therefore \quad \frac{\mathrm{d}p(x)}{\mathrm{d}x} = \frac{1}{\pi\sqrt{A^2 - x^2}}$$

6.20 $U = \dfrac{1}{2}kx^2$

Using the result of prob. (6.19)

$$\langle U \rangle = \int U \mathrm{d}p(x) = \int_{-A}^{A} \frac{1}{2}kx^2 \frac{\mathrm{d}x}{\pi\sqrt{A^2 - x^2}}$$

Put $x = A\sin\theta$, $\quad \mathrm{d}x = A\cos\theta\,\mathrm{d}\theta$

$$\langle U \rangle = \left(\frac{kA^2}{2\pi}\right) \int_{-\pi/2}^{\pi/2} \sin^2\theta\,\mathrm{d}\theta = \frac{1}{4}kA^2$$

Also, $\langle K \rangle = \langle E - U \rangle = \dfrac{1}{2}kA^2 - \dfrac{1}{4}kA^2 = \dfrac{1}{4}kA^2$

6.21 $K_{\mathrm{trans}} + K_{\mathrm{rot}} + U = \text{constant}$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = \text{constant}$$

But $I = \dfrac{1}{2}mR^2$ and $\omega = \dfrac{v}{R}$

$$\therefore \quad \frac{3}{4}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}kx^2 = 0 \text{ constant}$$

Differentiating

$$\frac{3}{2}m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\frac{\mathrm{d}x}{\mathrm{d}t} + kx\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

Cancelling $\mathrm{d}x/\mathrm{d}t$ throughout and simplifying

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{2k}{3m}\right)x = 0$$

This is the equation for SHM

$$\text{with } \omega^2 = \left(\frac{2k}{3m}\right)$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3m}{2k}}$$

6.22 The time period of the pendulums is

$$T_1 = 2\pi\sqrt{\frac{60}{g}} \tag{1}$$

$$T_2 = 2\pi\sqrt{\frac{63}{g}} \tag{2}$$

Let the time be t in which the longer length pendulum makes n oscillations while the shorter one makes $(n+1)$ oscillations. Then

$$t = (n+1)T_1 = nT_2 \tag{3}$$

Using (1) and (2) in (3), we find $n = 40.5$ and $t = 64.49$ s.

6.23 Let g_0 be the acceleration due to gravity on the ground and g at height above the ground. Then

$$g = \frac{g_0R^2}{(R+h)^2}$$

At the ground, $T_0 = 2\pi\sqrt{\frac{L}{g_0}}$. At height h, $T = 2\pi\sqrt{\frac{L}{g}}$

$$T = T_0\sqrt{\frac{g_0}{g}} = T_0\left(1 + \frac{h}{R}\right) = 2\left(1 + \frac{320}{6.4\times10^6}\right) = 2.0001\,\text{s}$$

Time lost in one oscillation on the top of the tower $= 2.0001 - 2.0000 = 0.0001$ s. Number of oscillations in a day for the pendulum which beats seconds on the ground

$$= \frac{86400}{2.0} = 43{,}200$$

Therefore, time lost in 43,200 oscillations

$= 42{,}300 \times 0.0001 = 4.32\,\text{s}$

6.24 $$g = g_0\left(1 - \frac{d}{R}\right) \tag{1}$$

where g and g_0 are the acceleration due to gravity at depth d and surface, respectively, and R is the radius of the earth.

$$T = T_0\sqrt{\frac{g_0}{g}} = T_0\left(1 - \frac{d}{R}\right)^{-1/2} = T_0\left(1 + \frac{d}{2R}\right)$$

Time registered for the whole day will be proportional to the time period. Thus

$$\frac{T}{T_0} = \frac{t}{t_0} = 1 + \frac{d}{2R}$$
$$\frac{86{,}400}{86{,}400 - 300} = 1 + \frac{d}{2R}$$

Substituting $R = 6400\,\text{km}$, we find $d = 44.6\,\text{km}$.

6.25 **(a)** Let the liquid level in the left limb be depressed by x, so that it is elevated by the same height in the right limb (Fig. 6.17). If ρ is the density of the liquid, A the cross-section of the tube, M the total mass, and m the mass of liquid corresponding to the length $2x$, which provides the unbalanced force,

$$\frac{M\mathrm{d}^2x}{\mathrm{d}t^2} = -mg = -(2xA\rho)g$$
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{2A\rho g}{M}x = -\frac{2A\rho g x}{hA\rho} = -\frac{2gx}{h} = -\omega^2 x$$

This is the equation of SHM.

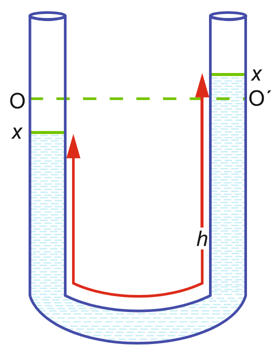


Fig. 6.17

(b) The time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{2g}}$$

6.26 **(a)** Let the gas at pressure P and volume V be compressed by a small length x, the new pressure being p' and new volume V' (Fig. 6.18) under isothermal conditions.

$$P'V' = PV$$
$$\text{or} \quad P'(l - x)A = PlA$$

where A is the cross-sectional area.

$$P' = \frac{Pl}{l - x} - P\left(1 - \frac{x}{l}\right)^{-1} \simeq P\left(1 + \frac{x}{l}\right)$$

where we have expanded binomially up to two terms since $x << l$. The change in pressure is

$$\Delta P = P' - P = \frac{Px}{h}$$

The unbalanced force

$$F = -\Delta PA = -\frac{APx}{l}$$

and the acceleration

$$a = \frac{F}{m} = -\frac{APx}{ml} = -\omega^2 x$$

which is the equation for SHM.

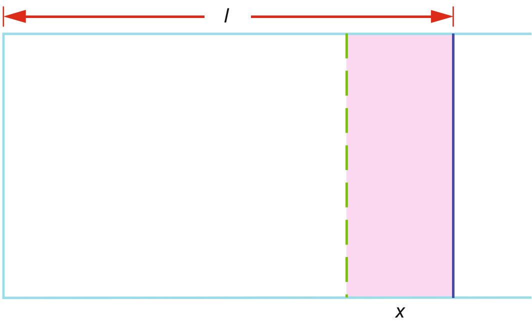


Fig. 6.18

(b) The time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{AP}}$$

6.27 $y = 8\sin\left(\frac{2\pi t}{T} + \phi\right)$

At $t = 0$; $\quad 4 = 8\sin\phi$

$\therefore \quad \phi = 30° = \frac{\pi}{6}$

$y = 8\sin\left(\frac{2\pi \times 6}{24} + \frac{\pi}{6}\right) = 8\sin 120 = 4\sqrt{3}\,\text{cm}$

6.28 Time period of a loaded spring

$$T = 2\pi\sqrt{\frac{M + \frac{m}{3}}{k}} \tag{1}$$

where M is the suspended mass, m is the mass of the spring and k is the spring constant

$$0.89 = 2\pi\sqrt{\frac{1.5 + \frac{m}{3}}{k}} \tag{2}$$

$$1.13 = 2\pi\sqrt{\frac{2.5 + \frac{m}{3}}{k}} \tag{3}$$

Dividing the two equations and solving for m, we get $m = 0.39\,\text{kg}$.

6.29 **(a)** $k_A > k_B$

Let the springs be stretched by the same amount. Then the work done on the two springs will be

$$W_A = \frac{1}{2}k_A x^2 \quad W_B = \frac{1}{2}k_B x^2$$

$$\frac{W_A}{W_B} = \frac{k_A}{k_B}$$

Thus $W_A > W_B$, i.e. when two springs are stretched by the same amount, more work will be done on the stiffer spring.

(b) Let the two springs be stretched by equal force. Thus the work done

$$W_A = \frac{1}{2}k_A x^2 = \frac{1}{2}k_A\left(\frac{F}{k_A}\right)^2 = \frac{1}{2}\frac{F^2}{k_A}$$

$$W_B = \frac{1}{2}\frac{F^2}{k_B}$$

$$\therefore \quad \frac{W_A}{W_B} = \frac{k_B}{k_A}$$

Thus when two springs are stretched by the same force, less work will be done on the stiffer spring.

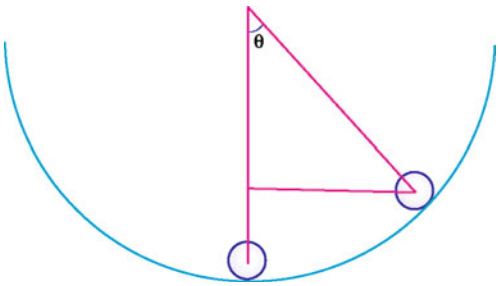


Fig. 6.19

6.30 $K_{\text{trans}} + K_{\text{rot}} + U = C = \text{constant}$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg(R-r)(1-\cos\theta) = C$$

Now $\quad I = \frac{1}{2}mr^2 \quad \omega = \frac{v}{r}$

$$\frac{3}{4}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + mg(R-r)\frac{\theta^2}{2} = C$$

Differentiating with respect to time

$$\frac{3}{2}m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\frac{\mathrm{d}x}{\mathrm{d}t} + mg(R-r)\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$$

Now $x = (R-r)\theta$

$$\therefore \quad \frac{3}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(R-r)\frac{\mathrm{d}\theta}{\mathrm{d}t} + gx\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$$

Cancelling $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ throughout

$$\frac{d^2x}{dt^2} + \frac{2}{3}\frac{gx}{(R-r)} = 0$$

which is the equation for SHM, with

$$\omega^2 = \frac{2}{3}\frac{g}{R-r}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$$

6.3.2 Physical Pendulums

6.31 If α is the angular acceleration, the torque τ is given by

$$\tau = I\alpha = I\frac{d^2\phi}{dt^2} \quad (1)$$

The restoring torque for an angular displacement ϕ is

$$\tau = -MgD\sin\phi \quad (2)$$

which arises due to the tangential component of the weight. Equating the two torques for small ϕ,

$$I\frac{d^2\phi}{dt^2} = -MgD\sin\phi = -MgD\,\phi$$

$$\text{or} \quad \frac{d^2\phi}{dt^2} + \frac{MgD}{I}\phi = 0 \quad (3)$$

which is the equation for SHM with

$$\omega^2 = \frac{MgD}{I} \quad (4)$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{MgD}}$$

6.32 Equation for the oscillatory motion is obtained by putting $I = \frac{1}{3}ML^2$ and $D = \frac{L}{2}$ in (3) of prob. (6.31).

$$\frac{d^2\theta}{dt^2} + \frac{MgD}{I}\theta = 0 \tag{3}$$

$$\frac{d^2\theta}{dt^2} + \frac{3}{2}\frac{g}{L}\theta = 0$$

$$\omega^2 = \frac{3}{2}\frac{g}{L}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2L}{3g}} \tag{4}$$

For a simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{5}$$

Comparing (4) and (5), the equivalent length of a simple pendulum is $l = \frac{2}{3}L$.

6.33 From the results of prob. (6.31) the time period of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{MgD}} \tag{1}$$

where I is the moment of inertia about the pivot A, Fig. 6.9.

Now $\quad I = I_C + MD^2$ and $I_C = Mk^2 \qquad (2)$

where k is the radius of gyration. Formula (1) then becomes

$$T = 2\pi\sqrt{\frac{k^2 + D^2}{gD}} \tag{3}$$

and the length of the simple equivalent pendulum is $D + \frac{k^2}{D}$.

If a point B be taken on AG such that AB $= D + \frac{k^2}{D}$, A and B are known as the centres of suspension and oscillation, respectively. Here G is the centre of mass (CM) of the physical pendulum.
Suppose now the body is suspended at B, then the time of oscillation is obtained by substituting $\frac{k^2}{D}$ for D in the expression

$$2\pi\sqrt{\frac{k^2+D^2}{gD}} \text{ and is therefore } 2\pi\sqrt{\frac{k^2+\frac{k^4}{D^2}}{g\frac{k^2}{D}}} \text{ i.e. } 2\pi\sqrt{\frac{D^2+k^2}{gD}}$$

Thus the centres of suspension and oscillation are convertible, for if the body be suspended from either it will make small vibrations in the same time as a simple pendulum whose length L is the distance between these centres.

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{or} \quad g = \frac{4\pi^2 L}{T^2}$$

6.34 $$\omega = \sqrt{\frac{mgd}{I}} \tag{1}$$

$$d = \frac{4R}{3\pi} \tag{2}$$

the distance of the point of suspension from the centre of mass

$$I = \frac{mR^2}{2} \tag{3}$$

Substituting (2) and (3) in (1) and simplifying

$$\omega = \sqrt{\frac{8g}{3\pi R}}$$

6.35 $$T = 2\pi\sqrt{\frac{I}{mgd}}$$

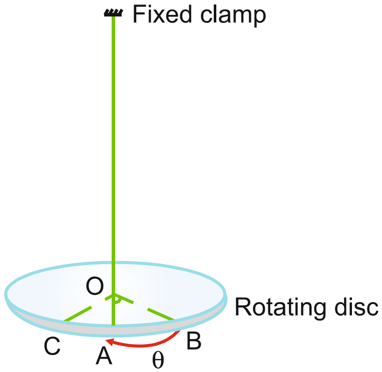
$$T_1 = 2\pi\sqrt{\frac{mr^2+mr^2}{mgr}} = 2\pi\sqrt{\frac{2r}{g}}$$

$$T_2 = 2\pi\sqrt{\frac{\frac{1}{2}mr^2+mr^2}{mgr}} = 2\pi\sqrt{\frac{3}{2}\frac{r}{g}}$$

$$\therefore \quad \frac{T_1}{T_2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

6.36 In Fig. 6.20 OA is the reference line or the disc in the equilibrium position. If the disc is rotated in the horizontal plane so that the reference line occupies the line OB, the wire would have twisted through an angle θ. The twisted wire will exert a restoring torque on the disc causing the reference line to move to

Fig. 6.20



its original position. For small twists the restoring torque will be proportional to the angular displacement in accordance with Hooke's law.

$$\tau = -C\theta \tag{1}$$

where C is known as torsional constant. If I is the moment of inertia of the disc about its axis, α the angular acceleration, the torque τ is given by

$$\tau = I\alpha = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{2}$$

Comparing (1) and (2)

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -C\theta$$

$$\text{or} \quad \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{C}{I}\theta = 0 \tag{3}$$

which is the equation for angular SHM with $\omega^2 = \frac{C}{I}$. Time period for small oscillations is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \tag{4}$$

6.37 Total kinetic energy of the system

$$K = K(\text{mass}) + K(\text{pulley}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

Replacing x by $r\theta$ and $\dot{x}$ by $r\dot{\theta}$

$$K = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(mr^2 + I)\dot{\theta}^2$$

Potential energy of the spring

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kr^2\theta^2$$

Total energy

$$E = K + U = \frac{1}{2}(mr^2 + I)\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2 = \text{constant}$$

Differentiating with respect to time

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (mr^2 + I)\dot{\theta} \cdot \ddot{\theta} + kr^2\theta \cdot \dot{\theta} = 0$$

Cancelling $\dot{\theta}$

$$\ddot{\theta} + \frac{kr^2\theta}{mr^2 + I} = 0$$

which is the equation for angular SHM with

$\omega^2 = \dfrac{kr^2}{mr^2 + I}$. Therefore

$$\omega = \sqrt{\frac{kr^2}{mr^2 + I}}$$

6.38 Let at any instant the centre of the cylinder be displaced by x towards right. Then the spring at C is compressed by x while the spring at P is elongated by $2x$. If $v = \dot{x}$ is the velocity of the centre of mass of the cylinder and $\omega = \dot{\theta}$ its angular velocity, the total energy in the displaced position will be

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_C\dot{\theta}^2 + \frac{1}{2}k_1x^2 + \frac{1}{2}k_2(2x)^2 \tag{1}$$

Substituting $x = r\theta$, $\dot{x} = r\dot{\theta}$, and $I_\mathrm{C} = \frac{1}{2}mr^2$, where r is the radius of the cylinder, (1) becomes

$$E = \frac{3}{4}mr^2\dot{\theta}^2 + \frac{1}{2}r^2(k_1 + 4k_2)\theta^2 = \text{constant}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{3}{2}mr^2\dot{\theta}\ddot{\theta} + r^2(k_1 + 4k_2)\theta\dot{\theta} = 0$$

$$\therefore \quad \ddot{\theta} + \frac{2}{3m}(k_1 + 4k_2)\theta = 0$$

which is the equation for angular SHM with $\omega^2 = \dfrac{2}{3m}(k_1 + 4k_2)$.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3m}{2(k_1 + 4k_2)}}$$

6.39 $U(x) = \dfrac{a}{x^2} - \dfrac{b}{x}$

Equilibrium position is obtained by minimizing the function $U(x)$.

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{2a}{x^3} + \frac{b}{x^2} = 0$$
$$x = x_0 = \frac{2a}{b}$$

Measuring distances from the equilibrium position and replacing x by $x + \dfrac{2a}{b}$

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{2a}{x^3} - \frac{b}{x^2}$$
$$F = \frac{2a}{(x + 2a/b)^3} - \frac{b}{(x + 2a/b)^2}$$
$$= \frac{2a}{(2a/b)^3}\left(1 + \frac{bx}{2a}\right)^{-3} - \frac{b}{(2a/b)^2}\left(1 + \frac{bx}{2a}\right)^{-2}$$

Since the quantity $bx/2a$ is assumed to be small, use binomial expansion retaining terms up to linear in x.

$$F = -\frac{b^4 x}{8a^3}$$

Acceleration $a = \dfrac{F}{m} = -\dfrac{b^4 x}{8a^3 m} = -\omega^2 x$

where $\omega = \sqrt{\dfrac{b^4}{8a^3 m}}$

$$T = \frac{2\pi}{\omega} = 4\pi\sqrt{\frac{2ma^2}{b^4}}$$

6.3.3 Coupled Systems of Masses and Springs

6.40 Let spring 1 undergo an extension x_1 due to force F. Then $x_1 = \dfrac{F}{k_1}$. Similarly, for spring 2, $x_2 = \dfrac{F}{k_2}$.
The force is the same in each spring, but the total displacement x is the sum of individual displacements:

$$x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2}$$

$$k_{\text{eq}} = \frac{F}{x} = \frac{F}{x_1 + x_2} = \frac{F}{\dfrac{F}{k_1} + \dfrac{F}{k_2}} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k_{\text{eq}}}} = 2\pi\sqrt{\frac{(k_1 + k_2)m}{k_1 k_2}}$$

6.41 The displacement is the same for both the springs and the total force is the sum of individual forces.

$$F_1 = k_1 x, \ \ F_2 = k_2 x$$
$$F = F_1 + F_2 = (k_1 + k_2)x$$
$$k_{\text{eq}} = \frac{F}{x} = k_1 + k_2$$
$$T = 2\pi\sqrt{\frac{m}{k_{\text{eq}}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

6.42 Let the centre of mass be displaced by x. Then the net force

$$F = -k_1 x - k_2 x = -(k_1 + k_2)x$$

$$\text{Acceleration } a = \frac{F}{m} = -(k_1 + k_2)\frac{x}{m} = -\omega^2 x$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

6.43 Spring constant of the wire is given by

$$k' = \frac{YA}{L} \tag{1}$$

Since the spring and the wire are in series, the effective spring constant k_{eff} is given by

$$k_{\mathrm{eff}} = \frac{k'k}{k + k} \tag{2}$$

The time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{m}{k_{\mathrm{eff}}}} \tag{3}$$

Combining (1), (2) and (3)

$$T = 2\pi\sqrt{\frac{m(YA + kL)}{YAk}}$$

6.44 In Fig. 6.15, C is the point of contact around which the masses M and m rotate. As it is the instantaneous centre of zero velocity, the equation of motion is of the form $\Sigma\tau_{\mathrm{c}} = I_{\mathrm{c}}\ddot{\theta}$, where I_{c} is the moment of inertia of masses M and m with respect to point C. Now

$$I_{\mathrm{c}} = \left(\frac{1}{2}MR^2 + MR^2\right) + md^2 \tag{1}$$

where $d^2 = L^2 + R^2 - 2RL\cos\theta$. (2)

For small oscillations, $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$ and

$$I_{\mathrm{c}} = \frac{3MR^2}{2} + m(L - d)^2 \tag{3}$$

Therefore the equation of motion become

$$\left[\frac{3MR^2}{2} + m(L - d)^2\right]\ddot{\theta} = -mgL\sin\theta = -mgL\theta$$

$$\text{or}\quad \ddot{\theta} + \frac{mgL}{3MR^2/2 + m(L - d)^2}\theta = 0$$

$$\therefore\quad \omega = \sqrt{\frac{mgL}{3MR^2/2 + m(L - d)^2}}\ \text{rad/s}$$

6.45 Figure 6.21 shows the semicircular disc tilted through an angle θ compared to the equilibrium position (b). G is the centre of mass such that $a = \mathrm{OG} = \dfrac{4r}{3\pi}$, where r is the radius.

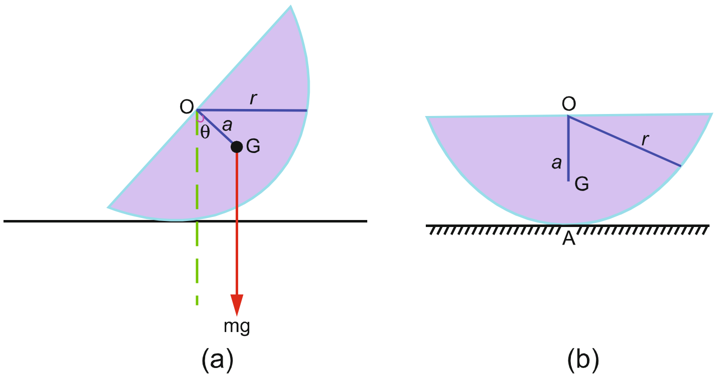


Fig. 6.21

We use the energy method.

$$K(\max) = \frac{1}{2} I_A \omega^2$$
$$= \frac{1}{2}(I_{\mathrm{G}} + \overline{\mathrm{GA}}^2)\omega^2$$
$$= \frac{1}{2}[I_0 - ma^2 + m(r-a)^2]\omega^2 = \frac{1}{2}\left[\frac{1}{2}mr^2 + mr(r-2a)\right]\omega^2$$
$$= mr\left(\frac{3}{4}r - a\right)\omega^2$$

$$K_{\max} = U_{\max}$$
$$mr\left(\frac{3}{4}r - a\right)\omega^2 = mga(1-\cos\theta)$$

But $a = \dfrac{4r}{3\pi}$

$$\omega = 4\sqrt{\frac{(1-\cos\theta)g}{(9\pi - 16)r}}$$

6.46 Referring to Fig. 6.16, take torques about the two hinged points P and Q.

$$mb^2\ddot{\theta}_1 = -mgb\theta_1 - kb^2(\theta_1 - \theta_2)$$

The left side gives the net torque which is the product of moment of inertia about P and the angular acceleration. The first term on the right side gives the torque of the force mg, which is force times the perpendicular distance from the vertical through P. The second term on the right side is the torque produced by the spring which is $k(x_1 - x_2)$ times the perpendicular distance from P, that is, $k(x_1 - x_2)b$ or $k(\theta_1 - \theta_2)b^2$. The second equation of motion can be similarly written. Thus, the two equations of motion are

$$mb\ddot{\theta}_1 + mg\theta_1 + kb(\theta_1 - \theta_2) = 0 \tag{1}$$

$$mb\ddot{\theta}_2 + mg\theta_2 + kb(\theta_2 - \theta_1) = 0 \tag{2}$$

The harmonic solutions are

$$\theta_1 = A \sin \omega t, \quad \theta_2 = B \sin \omega t \tag{3}$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -B\omega^2 \sin \omega t \tag{4}$$

Substituting (3) and (4) in (1) and (2) and simplifying

$$(mg + kb - mb\omega^2)A - kb\ B = 0 \tag{5}$$

$$-kbA + (mg + kb - mb\omega^2)B = 0 \tag{6}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B.

$$\begin{vmatrix} (mg + kb - mb\omega^2) & -kb \\ -kb & (mg + kb - mb\omega^2) \end{vmatrix} = 0$$

Expanding the determinant and solving for ω we obtain

$$\omega_1 = \sqrt{\frac{g}{b}}, \quad \omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$$

6.47 In prob. (6.46) equations of motion (1) and (2) can be re-written in terms of Cartesian coordinates x_1 and x_2 since $x_1 = b\theta_1$ and $x_2 = b\theta_2$.

$$m\ddot{x}_1 + \frac{mgx_1}{b} + k(x_1 - x_2) = 0 \tag{1}$$

$$m\ddot{x}_2 + \frac{mgx_2}{b} + k(x_2 - x_1) = 0 \tag{2}$$

It is possible to make linear combinations of x_1 and x_2 such that a combination involves but a single frequency. These new coordinates X_1 and X_2, called normal coordinates, vary harmonically with but a single frequency. No energy transfer occurs from one normal coordinate to another. They are completely independent.

$$x_1 = \frac{X_1 + X_2}{2}, \quad x_2 = \frac{X_1 - X_2}{2} \tag{3}$$

Substituting (3) in (1) and (2)

$$\frac{m}{2}(\ddot{X}_1 + \ddot{X}_2) + \frac{mg}{2b}(X_1 + X_2) + kX_2 = 0 \tag{4}$$
$$\frac{m}{2}(\ddot{X}_1 - \ddot{X}_2) + \frac{mg}{2b}(X_1 - X_2) - kX_2 = 0 \tag{5}$$

Adding (4) and (5)

$$m\ddot{X}_1 + \frac{mg}{b}X_1 = 0 \tag{6}$$

which is a linear equation in X_1 alone with constant coefficients.
Subtracting (5) from (4), we obtain

$$m\ddot{X}_2 + \left(\frac{mg}{b} + 2k\right) X_2 = 0 \tag{7}$$

This is again a linear equation in X_2 as the single dependent variable. Since the coefficients of X_1 and X_2 are positive, both (6) and (7) are differential equations of simple harmonic motion having frequencies $\omega_1 = \sqrt{\frac{g}{b}}$ and $\omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$. Thus when equations of motion are expressed in normal coordinates, the equations are linear with constant coefficients and each contains only one dependent variable.
We now calculate the energy in normal coordinates. The potential energy arises due to the energy stored in the spring and due to the position of the body.

$$V = \frac{1}{2}k(x_1 - x_2)^2 + mgb(1 - \cos\theta_1) + mgb(1 - \cos\theta_2) \tag{8}$$

Now $b(1 - \cos\theta_1) = b\frac{\theta_1^2}{2} = \frac{x_1^2}{2b}$

Similarly $b(1 - \cos\theta_2) = \frac{x_2^2}{2b}$

Hence $V = \frac{k}{2}(x_1 - x_2)^2 + \frac{mgx_1^2}{2b} + \frac{mgx_2^2}{2b}$ (9)

Kinetic energy $T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)$ (10)

Although there is no cross-product term in (10) for the kinetic energy, there is one in the potential energy of the spring in (9). The presence of the cross-product term means coupling between the components of the vibrating system. However, in normal coordinates the cross-product terms are avoided. Using (3) in (9) and (10)

$$V = \frac{mg}{4b}X_1^2 + \left(\frac{mg}{4b} + \frac{k}{2}\right)X_2^2 \tag{11}$$

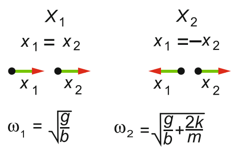
$$T = \frac{m}{4}(\dot{X}_1^2 + \dot{X}_2^2) \tag{12}$$

Thus the cross terms have now disappeared. The potential energy V is now expressed as a sum of squares of normal coordinates multiplied by constant coefficients and kinetic energy. T is expressed in the form of a sum of squares of the time derivatives of the normal coordination.

We can now describe the mode of oscillation associated with a given normal coordinate. Suppose $X_2 = 0$, then $0 = x_1 - x_2$, which implies $x_1 = x_2$. The mode X_1 is shown in Fig. 6.22, where the particles oscillate in phase with frequency $\omega_1 = \sqrt{g/b}$ which is identical for a simple pendulum of length b. Here the spring plays no role because it remains unstretched throughout the motion.

If we put $X_1 = 0$, then we get $x_1 = -x_2$. Here the pendulums are out of phase. The X_2 mode is also illustrated in Fig. 6.22, the associated frequency being $\omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$. Note that $\omega_2 > \omega_1$, because greater potential energy is now available due to the spring.

Fig. 6.22



6.48 $y = A\cos 6\pi t \sin 90\pi$

Now $\sin C + \sin D = 2\sin\frac{1}{2}(C + D)\cos\frac{1}{2}(C - D)$

Comparing the two equations we get

$$\frac{C + D}{2} = 90\pi \quad \frac{C - D}{2} = 6\pi$$

$$\therefore \quad C = 96\pi \text{ and } D = 84\pi$$

$$\omega_1 = 2\pi f_1 = 96\pi \quad \text{or} \quad f_1 = 48\,\text{Hz}$$

$$\omega_2 = 2\pi f_2 = 84\pi \quad \text{or} \quad f_2 = 42\,\text{Hz}$$

Thus the frequency of the component vibrations are 48 Hz and 42 Hz. The beat frequency is $f_1 - f_2 = 48 - 42 = 6$ beats/s.

6.49 The frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass given by

$$\mu = \frac{m_H m_{Cl}}{m_H + m_{Cl}} = \frac{10. \times 36.46}{1.0 + 36.46}$$
$$= 0.9733\,\text{amu} = 0.9733 \times 1.66 \times 10^{-27}\,\text{kg} = 1.6157 \times 10^{-27}\,\text{kg}$$
$$f = \frac{1}{2\pi}\sqrt{\frac{480}{1.6157 \times 10^{-27}}} = 8.68 \times 10^{13}\,\text{Hz}$$

6.50 Each vibration is plotted as a vector of magnitude which is proportional to the amplitude of the vibration and in a direction which is determined by the phase angle. Each phase angle is measured with respect to the x-axis. The vectors are placed in the head-to-tail fashion and the resultant is obtained by the vector joining the tail of the first vector with the head of the last vector, Fig. 6.23. $y_1 = \text{OA} = 1$ unit, parallel to x-axis in the positive direction, $y_2 = \text{AB} = \frac{1}{2}$ unit parallel to y-axis and $y_3 = \text{BC} = \frac{1}{3}$ unit parallel to the x-axis in the negative direction.

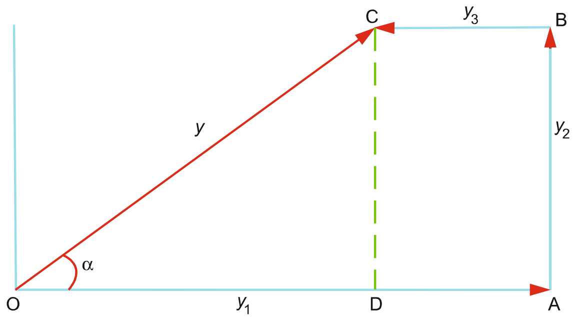


Fig. 6.23

The resultant is given by OC both in magnitude and in direction. From the geometry of the diagram

$$y = \text{OC} = \sqrt{\text{OD}^2 + \text{DC}^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = 5/6$$
$$\alpha = \tan^{-1}(\text{CD}/\text{OD}) = \tan^{-1}\left(\frac{1/2}{2/3}\right) = \tan^{-1}(3/4) = 37^\circ$$

6.3.4 Damped Vibrations

6.51 The logarithmic decrement Δ is given by

$$\Delta = bT' \tag{1}$$

where $T' = \dfrac{2\pi}{\omega'}$ is the time period for damped vibration and $b = \sqrt{\omega_0^2 - \omega'^2}$, where ω_0 and ω' are the angular frequencies for natural and damped vibrations, respectively.

$$\Delta = 2\pi\sqrt{\frac{\omega_0^2}{\omega'^2} - 1} = 2\pi\sqrt{\frac{f^2}{f'^2} - 1} = 2\pi\sqrt{\left(\frac{20}{16}\right)^2 - 1} = \frac{3\pi}{2}$$

6.52 The equation for damped oscillations is $4\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \dfrac{r\mathrm{d}x}{\mathrm{d}t} + 32x = 0$
Dividing the equation by 4

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{r}{4}\frac{\mathrm{d}x}{\mathrm{d}t} + 8x = 0$$

Comparing the equation with the standard equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{r}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0$$

$$m = 4, \quad \frac{k}{m} = 8 \rightarrow k = 32$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{8} = 2\sqrt{2}$$

The quantity $b = \dfrac{r}{2m}$ represents the decay rate of oscillation where r is the resistance constant.

(a) The motion will be underdamped if

$$b < \omega_0 \text{ or } \frac{r}{2m} < \sqrt{\frac{k}{m}} \text{ or } r < 2\sqrt{km}$$

$$\text{i.e.} \quad r < 2\sqrt{32 \times 4} \text{ or } r < 16\sqrt{2}$$

(b) The motion is overdamped if $r > 16\sqrt{2}$.
(c) The motion is critically damped if $r = 16\sqrt{2}$.

6.53 **(a)** $\omega_0 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{20}{4}} = 2.23\,\text{rad/s}$

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2.23} = 2.8\,\text{s}$$

(b) $\omega' = \dfrac{2\pi}{T'} = \dfrac{2\pi}{10} = 0.628\,\text{rad/s}$

$$b = \sqrt{\omega_0^2 - \omega'^2} = \sqrt{2.236^2 - 0.628^2} = 2.146$$

$$\frac{r}{2m} = b \text{ or } r = 2mb = 2 \times 4 \times 2.146 = 17.17\,\text{Ns/m}$$

(c) $\Delta = bT' = 2.146 \times 10 = 21.46$

6.54 $\dfrac{d^2x}{dt^2} + \dfrac{2dx}{dt} + 5x = 0$

Let $x = e^{\lambda t}$. The characteristic equation then becomes $\lambda^2 + 2\lambda + 5 = 0$ with the roots $\lambda = -1 \pm 2i$

$$x = Ae^{-(1-2i)t} + Be^{-(1+2i)t}$$

or $\quad x = e^{-t}[C\cos 2t + D\sin 2t]$

where A, B, C and D are constants.

C and D can be determined from initial conditions. At $t = 0$, $x = 5$. Therefore $C = 5$.

Also $\dfrac{dx}{dt} = -e^{-t}(C\cos 2t + D\sin 2t) + e^{-t}(-2C\sin 2t + 2D\cos 2t)$

At $t = 0$, $\dfrac{dx}{dt} = -3$

$\therefore \quad -3 = -C + 2D = -5 + 2D$

$\therefore \quad D = 1$

The complete solution is

$$x = e^{-t}(5\cos 2t + \sin 2t)$$

6.55 $F = mg = kx$

$$k = \frac{mg}{x} = \frac{(1.0)(9.8)}{0.2} = 49\,\text{N/m}$$

Equation of motion is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \tag{1}$$

Substituting $m = 1.0$, $r = 14$, $k = 49$, (1) becomes

$$\frac{d^2x}{dt^2} + 14\frac{dx}{dt} + 49x = 0 \tag{2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{49}{1}} = 7\,\text{rad/s}$$

$$b = \frac{r}{2m} = \frac{14}{2 \times 1} = 7$$

(a) Therefore the motion is critically damped.
(b) For critically damped motion, the equation is

$$x = x_0 e^{-bt}(1 + bt) \tag{3}$$

With $b = 7$ and $x_0 = 1.5$, (3) becomes

$$x = 1.5\,e^{-7t}(1 + 7t)$$

6.56 $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{150}{60}} = 5$

Damping force $f_r = r \cdot v$

or $r = \frac{f_r}{v} = \frac{80}{2} = 40$

$$b = \frac{r}{2m} = \frac{40}{2 \times 6} = 3.33\,\text{rad/s}$$

$$\omega(\text{res}) = \sqrt{\omega_0^2 - 2b^2} = \sqrt{5^2 - 2 \times (3.33)^2} = 1.66\,\text{rad/s}$$

$$f(\text{res}) = \frac{\omega(\text{res})}{2\pi} = 0.265\,\text{vib/s}$$

6.57 Equation of motion is

$$\frac{2d^2x}{dt^2} + 1.5\frac{dx}{dt} + 40x = 12\cos 4t$$

Dividing throughout by 2

$$\frac{d^2x}{dt^2} + 0.75\frac{dx}{dt} + 20x = 6\cos 4t$$

Comparing this with the standard equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = p\cos\omega t$$

$$b = 0.375;\ \omega_0 = \sqrt{20},\ \ p = 6,\ \omega = 4$$

$$Z_M = \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2} = \sqrt{(20-16)^2 + 4 \times 0.375^2 \times 4^2} = 5$$

(a) $A = \dfrac{p}{Z_m} = \dfrac{6}{5} = 1.2$

(b) $\tan\varepsilon = \dfrac{2b\omega}{\omega_0^2 - \omega^2} = \dfrac{2 \times 0.375 \times 4}{(20-16)} = 0.75 \to \varepsilon = 37°$

(c) $Q = \dfrac{\omega_0 m}{r} = \dfrac{\omega_0}{2b} = \dfrac{\sqrt{20}}{2 \times 0.375} = 5.96$

(d) $F = pm = 6 \times 2 = 12$

$$W = \frac{F^2}{2Z_m}\sin\varepsilon = \frac{12^2}{2 \times 5}\sin 37° = 8.64\,\text{W}$$

6.58 $Q = \dfrac{2\pi t_c}{T} = 2\pi t_c f = 2\pi \times 2 \times 100 = 1256$

6.59 **(a)** Energy is proportional to the square of amplitude

$$E = \text{const.}A^2$$

$$\frac{dE}{E} = \frac{2dA}{A} = \frac{2 \times 5}{100} = 10\%$$

(b) $E = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{E}{E_0} = \frac{A^2}{A_0^2} = e^{-t/t_c}$$

$$\therefore \quad \frac{A}{A_0} = \frac{95}{100} = e^{-t/2t_c}$$

$$\frac{t}{2t_c} = \ln\left(\frac{100}{95}\right) = 0.05126$$

$$t_c = \frac{3}{2 \times 0.05126} = 29.26\,\text{s}$$

(c) $Q = \dfrac{2\pi t_c}{T} = \dfrac{(2\pi)(29.26)}{3.0} = 61.25$

6.60 **(a)** $E = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{t}{t_c} = \ln\left(\frac{E_0}{E}\right) = \ln 2 = 0.693$$

Put $\quad t = nT$

$$\therefore \quad n = 0.693 \frac{t_c}{T}$$

But $\quad -\dfrac{\Delta E}{E} = \dfrac{3}{100} = \dfrac{T}{t_c}$

$$\therefore \quad n = 0.693 \times \frac{100}{3} = 23.1$$

(b) $Q = \dfrac{2\pi t_c}{T} = 2\pi \times \dfrac{100}{3} = 209.3$

6.61 $\omega' = \omega_0 \sqrt{1 - \dfrac{1}{4Q^2}} = \dfrac{9\omega_0}{10}$

$\therefore \quad Q = 1.147$

$$Q = \frac{2\pi t_c}{T}$$

or $\dfrac{T}{2t_c} = \dfrac{\pi}{Q} = \dfrac{3.14}{1.147} = 2.737$

$$\frac{A}{A_0} = e^{-T/2t_c} = e^{-2.737} = 0.065$$

6.62 $\omega'^2 = \omega_0^2 - b^2 \qquad (1)$

where $b = \dfrac{r}{2m} \qquad (2)$

$$\omega' = \omega_0 \left(1 - \frac{b^2}{\omega_0^2}\right)^{1/2} \approx \omega_0 \left(1 - \frac{b^2}{2\omega_0^2}\right) \qquad (3)$$

where we have expanded the radical binomially, assuming that $b/\omega_0 << 1$.

Now $\quad \omega_0^2 = \dfrac{k}{m} \qquad (4)$

$$\therefore \quad \frac{b^2}{2\omega_0^2} = \frac{r^2}{8mk} \qquad (5)$$

Substituting (5) in (3)

$$\omega' = \omega_0 \left(1 - \frac{r^2}{8mk}\right) \quad \text{(for small damping)}$$

6.63 The time elapsed between successive maximum displacements of a damped harmonic oscillator is represented by T', the period.

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}} = \frac{4\pi m}{\sqrt{4km - r^2}} = \text{constant}$$

6.64 Force $= mg = kx$

$$\therefore \quad \frac{k}{m} = \frac{g}{x} = \frac{980}{9.8} = 100$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{100} = 10\,\text{rad/s}$$

$$\Delta = bT' = \frac{2\pi b}{\sqrt{\omega_0^2 - b^2}} \tag{1}$$

Substituting $\Delta = 3.1$ and $\omega_0 = 10$ in (1), $b = 4.428$

$$T' = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\sqrt{10^2 - (4.428)^2}} = 0.7\,\text{s}$$

6.65 $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{2\mathrm{d}x}{\mathrm{d}t} + 8x = 16\cos 2t \tag{1}$$

This is the equation for the forced oscillations, the standard equation being

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + r\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F\cos\omega t \tag{2}$$

Comparing (1) and (2) we find

$$m = 1\,\text{kg},\ r = 2,\ k = 8,\ F = 16\,N,\ \omega = 2$$

(a) $\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1}} = 2\sqrt{2}$

$$\therefore \quad f_0 = \frac{2\sqrt{2}}{2\pi} = \frac{\sqrt{2}}{\pi}/\text{s}$$

(b) $\omega = 2\pi f = 2$

$$\therefore \quad f = \frac{2}{2\pi} = \frac{1}{\pi}/\text{s}$$

6.66 $E(t) = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{E(t_{1/2})}{E_0} = \frac{1}{2} = e^{-t_{1/2}/t_c}$$

or $t_{1/2} = t_c \ln 2$

6.67 $A(t) = A_0 e^{-t/2t_c}$

$$\frac{A(t)}{A_0} = \frac{1}{e}$$

If $t = 2t_c = 8T$

$$\therefore \quad t_c = 4T$$

$$Q = 2\pi \frac{t_c}{T} = 2\pi \times 4 = 25.1$$

Chapter 7
Lagrangian and Hamiltonian Mechanics

Abstract *Chapter 7* is devoted to problems solved by Lagrangian and Hamiltonian mechanics.

7.1 Basic Concepts and Formulae

Newtonian mechanics deals with force which is a vector quantity and therefore difficult to handle. On the other hand, Lagrangian mechanics deals with kinetic and potential energies which are scalar quantities while Hamilton's equations involve generalized momenta, both are easy to handle. While Lagrangian mechanics contains n differential equations corresponding to n generalized coordinates, Hamiltonian mechanics contains $2n$ equation, that is, double the number. However, the equations for Hamiltonian mechanics are linear.

The symbol q is a generalized coordinate used to represent an arbitrary coordinate x, θ, φ, etc.

If T is the kinetic energy, V the potential energy then the Lagrangian L is given by

$$L = T - V \tag{7.1}$$

Lagrangian Equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{q}_K}\right) - \frac{\partial L}{\partial q_K} = 0 \quad (K = 1, 2 \ldots) \tag{7.2}$$

where it is assumed that V is not a function of the velocities, i.e. $\dfrac{\partial v}{\partial \dot{q}_K} = 0$. Eqn (2) is applicable to all the conservative systems.

When n independent coordinates are required to specify the positions of the masses of a system, the system is of n degrees of freedom.

Hamilton $$H = \Sigma_{s=1}^{r} p_s \dot{q}_s - L \tag{7.3}$$

where p_s is the generalized momentum and $\dot{q}_K$ is the generalized velocity.

Hamiltonian's Canonical Equations

$$\frac{\partial H}{\partial p_r} = \dot{q}_r, \qquad \frac{\partial H}{\partial q_r} = -\dot{p}_r \tag{7.4}$$

7.2 Problems

7.1 Consider a particle of mass m moving in a plane under the attractive force $\mu m/r^2$ directed to the origin of polar coordinates r, θ. Determine the equations of motion.

7.2 **(a)** Write down the Lagrangian for a simple pendulum constrained to move in a single vertical plane. Find from it the equation of motion and show that for small displacements from equilibrium the pendulum performs simple harmonic motion.

(b) Consider a particle of mass m moving in one dimension under a force with the potential $U(x) = k(2x^3 - 5x^2 + 4x)$, where the constant $k > 0$. Show that the point $x = 1$ corresponds to a stable equilibrium position of the particle. Find the frequency of a small amplitude oscillation of the particle about this equilibrium position.

[University of Manchester 2007]

7.3 Determine the equations of motion of the masses of Atwood machine by the Lagrangian method.

7.4 Determine the equations of motion of Double Atwood machine which consists of one of the pulleys replaced by an Atwood machine. Neglect the masses of pulleys.

7.5 A particular mechanical system depending on two coordinates u and v has kinetic energy $T = v^2\dot{u}^2 + 2\dot{v}^2$, and potential energy $V = u^2 - v^2$. Write down the Lagrangian for the system and deduce its equations of motion (do not attempt to solve them).

[University of Manchester 2008]

7.6 Write down the Lagrangian for a simple harmonic oscillator and obtain the expression for the time period.

7.7 A particle of mass m slides on a smooth incline at an angle α. The incline is not permitted to move. Determine the acceleration of the block.

7.8 A block of mass m and negligible size slides on a frictionless inclined plane of mass M at an angle θ with the horizontal. The plane itself rests on a smooth horizontal table. Determine the acceleration of the block and the inclined plane.

7.9 A bead of mass m is free to slide on a smooth straight wire of negligible mass which is constrained to rotate in a vertical plane with constant angular speed ω about a fixed point. Determine the equation of motion and find the distance x from the fixed point at time t. Assume that at $t = 0$ the wire is horizontal.

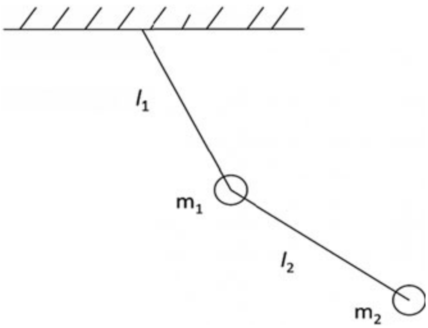
7.10 Consider a pendulum consisting of a small mass m attached to one end of an inextensible cord of length l rotating about the other end which is fixed. The pendulum moves on a spherical surface. Hence the name spherical pendulum. The inclination angle φ in the xy-plane can change independently.

(a) Obtain the equations of motion for the spherical pendulum.
(b) Discuss the conditions for which the motion of a spherical pendulum is converted into that of **(i)** simple pendulum and **(ii)** conical pendulum.

7.11 Two blocks of mass m and M connected by a massless spring of spring constant k are placed on a smooth horizontal table. Determine the equations of motion using Lagrangian mechanics.

7.12 A double pendulum consists of two simple pendulums of lengths l_1 and l_2 and masses m_1 and m_2, with the cord of one pendulum attached to the bob of another pendulum whose cord is fixed to a pivot, Fig. 7.1. Determine the equations of motion for small angle oscillations using Lagrange's equations.

Fig. 7.1



7.13 Use Hamilton's equations to obtain the equations of motion of a uniform heavy rod of mass M and length $2a$ turning about one end which is fixed.

7.14 A one-dimensional harmonic oscillator has Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. Write down Hamiltonian's equation and find the general solution.

7.15 Determine the equations for planetary motion using Hamilton's equations.

7.16 Two blocks of mass m_1 and m_2 coupled by a spring of force constant k are placed on a smooth horizontal surface, Fig. 7.2. Determine the natural frequencies of the system.

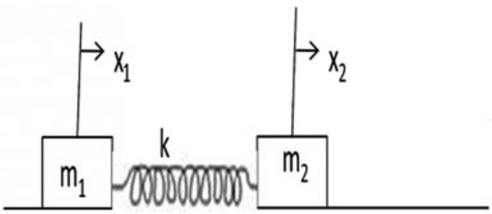
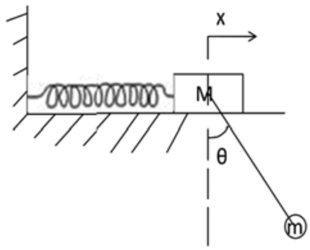


Fig. 7.2

7.17 A simple pendulum of length l and mass m is pivoted to the block of mass M which slides on a smooth horizontal plane, Fig. 7.3. Obtain the equations of motion of the system using Lagrange's equations.

Fig. 7.3



7.18 Determine the equations of motion of an insect of mass m crawling at a uniform speed v on a uniform heavy rod of mass M and length $2a$ which is turning about a fixed end. Assume that at $t = 0$ the insect is at the middle point of the rod and it is crawling downwards.

7.19 A uniform rod of mass M and length $2a$ is attached at one end by a cord of length l to a fixed point. Calculate the inclination of the string and the rod when the string plus rod system revolves about the vertical through the pivot with constant angular velocity ω.

7.20 A particle moves in a horizontal plane in a central force potential $U(r)$. Derive the Lagrangian in terms of the polar coordinates (r, θ). Find the corresponding momenta p_r and p_θ and the Hamiltonian. Hence show that the energy and angular momentum of the particle are conserved.

[University of Manchester 2007]

7.21 Consider the system consisting of two identical masses that can move horizontally, joined with springs as shown in Fig. 7.4. Let x, y be the horizontal displacements of the two masses from their equilibrium positions.

(a) Find the kinetic and potential energies of the system and deduce the Lagrangian.

(b) Show that Lagrange's equation gives the coupled linear differential equations

$$\begin{cases} m\ddot{x} = -4kx + 3ky \\ m\ddot{y} = 3kx - 4ky \end{cases}$$

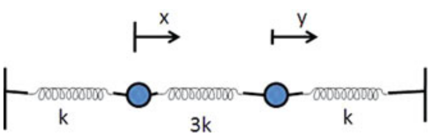
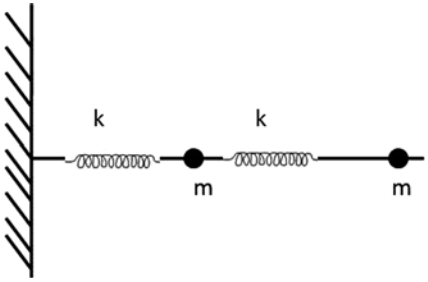


Fig. 7.4

(c) Find the normal modes of oscillation of this system and their period of oscillation.

7.22 Two identical beads of mass m each can move without friction along a horizontal wire and are connected to a fixed wall with two identical springs of spring constant k as shown in Fig. 7.5.

Fig. 7.5



(a) Find the Lagrangian for this system and derive from it the equations of motion.
(b) Find the eigenfrequencies of small amplitude oscillations.
(c) For each normal mode, sketch the system when it is at maximum displacement.

Note: Your sketch should indicate the relative sizes as well as the directions of the displacements.

[University of Manchester 2007]

7.23 Two beads of mass $2m$ and m can move without friction along a horizontal wire. They are connected to a fixed wall with two springs of spring constants $2k$ and k as shown in Fig. 7.6:

(a) Find the Lagrangian for this system and derive from it the equations of motion for the beads.
(b) Find the eigenfrequencies of small amplitude oscillations.
(c) For each normal mode, sketch the system when it is at the maximum displacement.

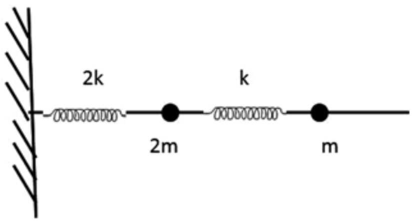


Fig. 7.6

7.24 Three identical particles of mass m, M and m with M in the middle are connected by two identical massless springs with a spring constant k. Find the normal modes of oscillation and the associated frequencies.

7.25 **(a)** A bead of mass m is constrained to move under gravity along a planar rigid wire that has a parabolic shape $y = x^2/l$, where x and y are, respectively, the horizontal and the vertical coordinates. Show that the Lagrangian for the system is

$$L = \frac{m(\dot{x})^2}{2}\left(1 + \frac{4x^2}{l^2}\right) - \frac{mgx^2}{l}$$

(b) Derive the Hamiltonian for a single particle of mass m moving in one dimension subject to a conservative force with a potential $U(x)$.

[University of Manchester 2006]

7.26 A pendulum of length l and mass m is mounted on a block of mass M. The block can move freely without friction on a horizontal surface as shown in Fig. 7.7.

(a) Show that the Lagrangian for the system is

$$L = \left(\frac{M+m}{2}\right)(\dot{x})^2 + ml\cos\theta\dot{x}\dot{\theta} + \frac{m}{2}l^2(\dot{\theta})^2 + mgl\cos\theta$$

(b) Show that the approximate form for this Lagrangian, which is applicable for a small amplitude swinging of the pendulum, is

$$L = \left(\frac{M+m}{2}\right)(\dot{x})^2 + ml\dot{x}\dot{\theta} + \frac{m}{2}l^2(\dot{\theta})^2 + mgl\left(1 - \frac{\theta^2}{2}\right)$$

(c) Find the equations of motion that follow from the simplified Lagrangian obtained in part (b),

(d) Find the frequency of a small amplitude oscillation of the system.

[University of Manchester 2006]

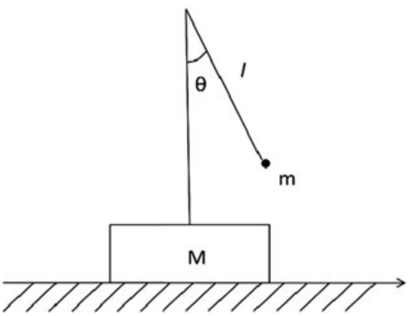


Fig. 7.7

7.27 **(a)** A particle of mass m slides down a smooth spherical bowl, as in Fig. 7.8. The particle remains in a vertical plane (the xz-plane). First, assume that the bowl does not move. Write down the Lagrangian, taking the angle ϑ with respect to the vertical direction as the generalized coordinate. Hence, derive the equation of motion for the particle.

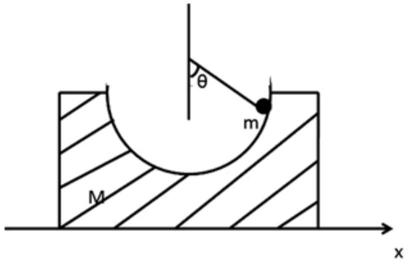


Fig. 7.8

(b) Assume now that the bowl rests on a smooth horizontal table and has a mass M, the bowl can slide freely along the x-direction.

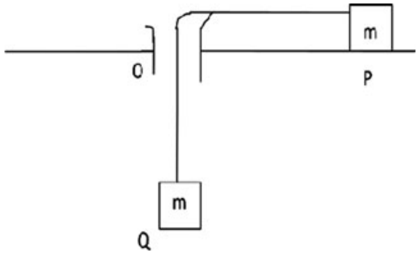
(i) Write down the Lagrangian in terms of the angle θ and the x-coordinate of the bowl, x.
(ii) Starting from the corresponding Lagrange's equations, obtain an equation giving $\ddot{x}$ in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$ and an equation giving $\ddot{\theta}$ in terms of $\ddot{x}$ and θ.
(iii) Hence, and assuming that $M >> m$, show that for small displacements about equilibrium the period of oscillation of the particle is smaller by a factor $[M/(M+m)]^{1/2}$ as compared to the case where the bowl is fixed. [You may neglect the terms in $\theta^2\ddot{\theta}$ or $\theta\dot{\theta}^2$ compared to terms in $\ddot{\theta}$ or θ.]

[University of Durham 2004]

7.28 A system is described by the single (generalized) coordinate q and the Lagrangian $L(q, \dot{q})$. Define the generalized momentum associated with q and the corresponding Hamiltonian, $H(q, p)$. Derive Hamilton's equations from Lagrange's equations of the system. For the remainder of the question, consider the system whose Lagrangian, $L(q, \dot{q})$. Find the corresponding Hamiltonian and write down Hamilton's equations.

7.29 Briefly explain what is the "generalized (or canonical) momentum conjugate to a generalized coordinate". What characteristic feature should the Lagrangian function have for a generalized momentum to be a constant of motion? A particle P can slide on a frictionless horizontal table with a small opening at O. It is attached, by a string of length l passing through the opening, to a particle Q hanging vertically under the table (see Fig. 7.9). The two particles have equal mass, m. Let τ denote the distance of P to the opening, θ the angle between OP and some fixed line through O and g the acceleration of gravity. Initially, $r = a$, Q does not move, and P is given an initial velocity of magnitude $(ag)^{1/2}$ at right angles to OP.

Fig. 7.9



(a) Write the Lagrangian in terms of the coordinates r and θ and derive the corresponding equations of motion.

(b) Using these equations of motion and the initial conditions, show that $\ddot{r} = a^3 g/r^3 - g$.

(c) Hence, (i) show that the trajectory of P is the circle $r = a$, (ii) show that P describes small oscillations about this circle if it is slightly displaced from it and (iii) calculate the period of these oscillations:

$[v_p^2 = \dot{r}^2 + r^2\dot{\theta}^2,$ where v_{p} is the speed of P]

7.30 A particle of mass m is constrained to move on an ellipse E in a vertical plane, parametrized by $x = a\cos\theta$, $y = b\sin\theta$, where $a, b > 0$ and $a \neq b$ and the positive y-direction is the upward vertical. The particle is connected to the origin by a spring, as shown in the diagram, and is subject to gravity. The potential energy in the spring is $\frac{1}{2}kr^2$ where r is the distance of the point mass from the origin (Fig. 7.10).

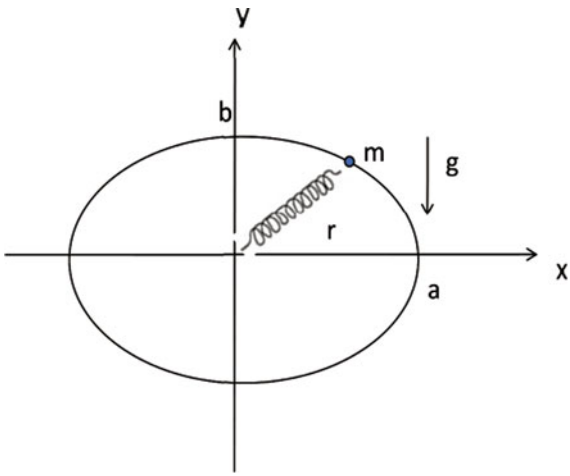


Fig. 7.10

(i) Using θ as a coordinate, find the kinetic and potential energies of the particle when moving on the ellipse. Write down the Lagrangian and show that Lagrange's equation becomes $m(a^2 \sin^2\theta + b^2\cos^2\theta)\ddot{\theta} = (a^2 - b^2)(k - m\dot{\theta}^2)\sin\theta\cos\theta - mgb\cos\theta$.

(ii) Show that $\theta = \pm\pi/2$ are two equilibrium points and find any other equilibrium points, giving carefully the conditions under which they exist. You may either use Lagrange's equation or proceed directly from the potential energy.

(iii) Determine the stabilities of each of the two equilibrium points $\theta = \pm\pi/2$ (it may help to consider the cases $a > b$ and $a < b$ separately).

(iv) When the equilibrium point at $\theta = -\pi/2$ is stable, determine the period of small oscillations.

[University of Manchester 2008]

7.31 In prob. (7.12) on double pendulum if $m_1 = m_2 = m$ and $l_1 = l_2 = l$, obtain the frequencies of oscillation.

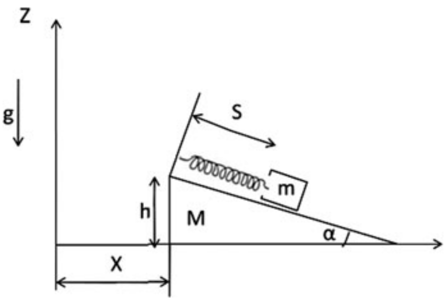
7.32 Use Lagrange's equations to obtain the natural frequencies of oscillation of a coupled pendulum described in prob. (6.46).

7.33 A bead of mass m slides freely on a smooth circular wire of radius r which rotates with constant angular velocity ω. On a horizontal plane about a point fixed on its circumference, show that the bead performs simple harmonic motion about the diameter passing through the fixed point as a pendulum of length $r = g/\omega^2$.

[with permission from Robert A. Becker, Introduction to theoretical mechanics, McGraw-Hill Book Co., Inc., 1954]

7.34 A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface as shown in Fig. 7.11.

Fig. 7.11



(a) Show that the Lagrangian of the system is

$$L = \frac{(M+m)}{2}\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\alpha - \frac{k}{2}(s-l)^2 - mg(h - s\sin\alpha),$$

where l is the natural length of the spring, x is the coordinate of the wedge and s is the length of the spring.

(b) By using the Lagrangian derived in (a), show that the equations of motion are as follows:

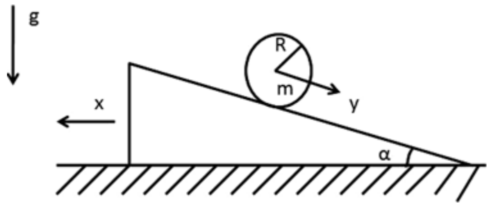
$$(m + M)\ddot{x} + m\ddot{s}\cos\alpha = 0,$$
$$m\ddot{x}\cos\alpha + m\ddot{s} + k(s - s_0) = 0,$$
where $s_0 = l + (mg\sin\alpha)/k$.

(c) By using the equations of motion in (b), derive the frequency for a small amplitude oscillation of this system.

[University of Manchester 2008]

7.35 A uniform spherical ball of mass m rolls without slipping down a wedge of mass M and angle α, which itself can slide without friction on a horizontal table. The system moves in the plane shown in Fig. 7.12. Here g denotes the gravitational acceleration.

Fig. 7.12



(a) Find the Lagrangian and the equations of motion for this system.

(b) For the special case of $M = m$ and $\alpha = \pi/4$ find

(i) the acceleration of the wedge and

(ii) the acceleration of the ball relative to the wedge.

[Useful information: Moment of inertia of a uniform sphere of mass m and radius R is $I = \dfrac{2}{5}mR^2$.]

[University of Manchester 2007]

7.3 Solutions

7.1 This is obviously a two degree of freedom dynamical system. The square of the particle velocity can be written as

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 \tag{1}$$

Formula (1) can be derived from Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

We thus obtain

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

The kinetic energy, the potential energy and the Lagrangian are as follows:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{2}$$

$$V = -\frac{\mu m}{r} \tag{3}$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r} \tag{4}$$

We take r, θ as the generalized coordinates q_1, q_2. Since the potential energy V is independent of $\dot{q}_i$, Lagrangian equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2) \tag{5}$$

Now $$\frac{\partial L}{\partial \dot{r}} = m\dot{r},\ \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\mu m}{r^2} \tag{6}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},\ \frac{\partial L}{\partial \theta} = 0 \tag{7}$$

Equation (5) can be explicitly written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{9}$$

Using (6) in (8) and (7) in (9), we get

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0 \tag{10}$$

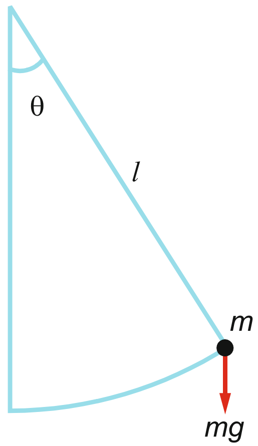
$$m\frac{\mathrm{d}(r^2\dot{\theta})}{\mathrm{d}t} = 0 \tag{11}$$

Equations (10) and (11) are identical with those obtained for Kepler's problem by Newtonian mechanics. In particular (11) signifies the constancy of areal velocity or equivalently angular momentum (Kepler's second law of planetary motion). The solution of (10) leads to the first law which asserts that the path of a planet describes an ellipse.

This example shows the simplicity and power of Lagrangian method which involves energy, a scalar quantity, rather than force, a vector quantity in Newton's mechanics.

7.2 The position of the pendulum is determined by a single coordinate θ and so we take $q = \theta$. Then (Fig. 7.13)

Fig. 7.13



$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 l^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{1}$$

$$V = mgl(1 - \cos\theta) \tag{2}$$

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \tag{3}$$

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial T}{\partial \theta} = 0 \tag{4}$$

$$\frac{\partial V}{\partial \dot{\theta}} = 0, \qquad \frac{\partial}{\partial \theta} = mgl\sin\theta \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{\theta}}(T - V)\right) - \frac{\partial}{\partial \theta}(T - V) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

or $l\ddot{\theta} + g\sin\theta = 0$ (equation of motion)

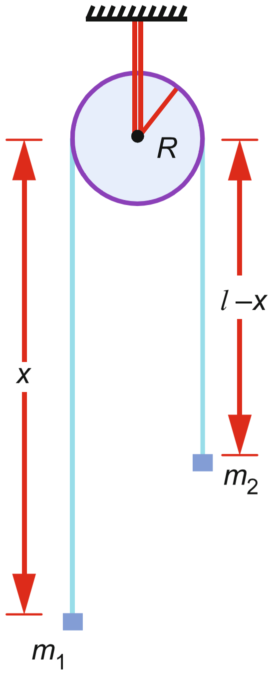
For small oscillation angles $\sin\theta \rightarrow \theta$

$\ddot{\theta} = -\dfrac{g\theta}{l}$ (equation for angular SHM)

$\therefore \quad \omega^2 = \dfrac{g}{l}$ or time period $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{g}}$

7.3 In this system there is only one degree of freedom. The instantaneous configuration is specified by $q = x$. Assuming that the cord does not slip, the angular velocity of the pulley is $\dot{x}/R$, Fig. 7.14.

Fig. 7.14



The kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{R^2} \tag{1}$$

The potential energy of the system is

$$V = -m_1 g x - m_2 g(l - x) \tag{2}$$

And the Lagrangian is

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2)gx + m_2 g l \tag{3}$$

The equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

yields

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} - g(m_1 - m_2) = 0$$

$$\text{or} \quad \ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}} \tag{5}$$

which is identical with the one obtained by Newton's mechanics.

7.4 By problem the masses of pulleys are negligible. The double machine is an Atwood machine in which one of the weights is replaced by a second Atwood machine, Fig. 7.15. The system now has two degrees of freedom, and its instantaneous configuration is specified by two coordinates x and x'. l and l' denote the length of the vertical parts of the two strings. Mass m_1 is at depth x below the centre of pulley A, m_2 at depth $l - x + x'$ and m_3 at depth $l + l' - x - x'$. The kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x} + \dot{x}')^2 + \frac{1}{2}m_3(-\dot{x} - \dot{x}')^2 \tag{1}$$

while the potential energy is given by

$$V = -m_1 g x - m_2 g(l - x + x') - m_3 g(l - x + l' - x') \tag{2}$$

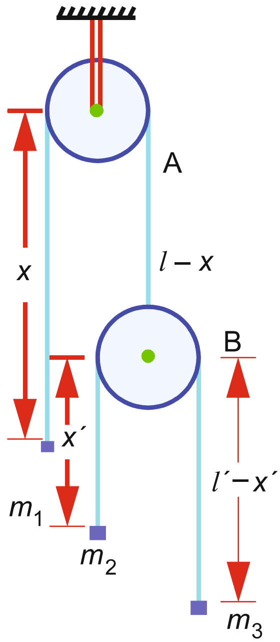


Fig. 7.15

The Lagrangian of the system takes the form

$$L = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x} + \dot{x}')^2 + \frac{1}{2}m_3(-\dot{x} - \dot{x}')^2 \\ + (m_1 - m_2 - m_3)gx + (m_2 - m_3)gx + m_2 gl + m_3 g(l + l') \tag{3}$$

The equations of motion are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0 \tag{5}$$

which yield

$$(m_1 + m_2 + m_3)\ddot{x} + (m_3 - m_2)\ddot{x}' = (m_1 - m_2 - m_3)g \tag{6}$$

$$(m_3 - m_2)\ddot{x} + (m_2 + m_3)\ddot{x}' = (m_2 - m_3)g \tag{7}$$

Solving (6) and (7) we obtain the equations of motion.

7.5

$$T = v^2\dot{u}^2 + 2\dot{v}^2 \tag{1}$$

$$V = u^2 - v^2 \tag{2}$$

$$L = T - V = v^2\dot{u}^2 + 2\dot{v}^2 - u^2 + v^2 \tag{3}$$

$$\frac{\partial L}{\partial \dot{u}} = 2v^2\dot{u},\ \ \frac{\partial L}{\partial u} = -2u \tag{4}$$

$$\frac{\partial L}{\partial \dot{v}} = 4\dot{v},\ \ \frac{\partial L}{\partial v} = 2v(\dot{u}^2 + 1) \tag{5}$$

The equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = 0 \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial v} = 0 \tag{7}$$

yield

$$2\frac{\mathrm{d}}{\mathrm{d}t}(v^2\dot{u}) + 2u = 0$$

$$\text{or} \quad v^2\ddot{u} + 2\dot{u}\dot{v} + 2u = 0 \tag{8}$$

$$2\ddot{v} + v(\dot{u}^2 + 1) = 0 \tag{9}$$

7.6 Here we need a single coordinate $q = x$:

$$T = \frac{1}{2}m\dot{x}^2, \quad V = \frac{1}{2}kx^2 \tag{1}$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{2}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -kx \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

$$m\ddot{x} + kx = 0 \tag{5}$$

$$\text{Let } x = A\sin\omega \mathrm{t} \tag{6}$$

$$\ddot{x} = -A\omega^2 \sin\omega \mathrm{t} \tag{7}$$

Inserting (6) and (7) and simplifying

$$-m\omega^2 + k = 0$$

$$\omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

where T_0 is the time period.

7.7 Only one coordinate $q = x$ (distance on the surface of the incline) is adequate to describe the motion:

$$T = \frac{1}{2}m\dot{x}^2, \quad V = -mgx\sin\alpha, \quad L = \frac{1}{2}m\dot{x}^2 + mgx\sin\alpha$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = mg\sin\alpha$$

Equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) - \frac{\partial}{\partial x}(mgx\sin\alpha) = 0$$

or $\quad \ddot{x} = g\sin\alpha$

7.8 This is a two degree of freedom system because both mass m and M are moving. The coordinate on the horizontal axis is described by x' for the inclined plane and x for the block of mass m on the incline. The origin of the coordinate system is fixed on the smooth table, Fig. 7.16. The x- and y-components of the velocity of the block are given by

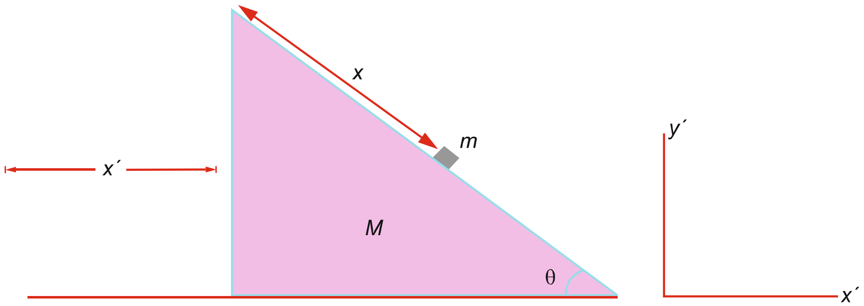


Fig. 7.16

$$v_x = \dot{x}' + \dot{x}\cos\theta \tag{1}$$
$$v_y = -\dot{x}\sin\theta \tag{2}$$
$$\therefore \quad v^2 = v_x^2 + v_y^2 = \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta + \dot{x}^2 \tag{3}$$

Hence, the kinetic energy of the system will be

$$T = \frac{1}{2}M\dot{x}'^2 + \frac{1}{2}m(\dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta + \dot{x}^2) \tag{4}$$

while the potential energy takes the form

$$V = -mgx\sin\theta \tag{5}$$

and the Lagrangian is given by

$$L = \frac{1}{2}M\dot{x}'^2 + \frac{1}{2}m(\dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta + \dot{x}^2) + mgx\sin\theta \tag{6}$$

The equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{7}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0 \tag{8}$$

yield

$$m(\ddot{x}'\cos\theta + \ddot{x}) - mg\sin\theta = 0 \tag{9}$$
$$M\ddot{x}' + m(\ddot{x}' + \ddot{x}\cos\theta) = 0 \tag{10}$$

Solving (9) and (10)

$$\ddot{x} = \frac{g\sin\theta}{1 - m\cos^2\theta/(M+m)} = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta} \tag{11}$$

$$\ddot{x}' = -\frac{g\sin\theta\cos\theta}{(M+m)/m - \cos^2\theta} = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \tag{12}$$

which are in agreement with the equations of prob. (2.14) derived from Newtonian mechanics.

7.9 $$T = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) \tag{1}$$

because the velocity of the bead on the wire is at right angle to the linear velocity of the wire:

$$V = mgx\sin\omega\mathrm{t} \tag{2}$$

because $\omega t = \theta$, where θ is the angle made by the wire with the horizontal at time t, and x sin θ is the height above the horizontal position:

$$L = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) - mgx\sin\omega t \tag{3}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = m\omega^2 x - mg\sin\omega t \tag{4}$$

Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \tag{5}$$

then becomes

$$\ddot{x} - \omega^2 x + g\sin\omega\mathrm{t} = 0 \quad \text{(equation of motion)} \tag{6}$$

which has the solution

$$x = A\mathrm{e}^{\omega t} + B\mathrm{e}^{-\omega t} + \frac{g}{2\omega^2}\sin\omega\mathrm{t} \tag{7}$$

where A and B are constants of integration which are determined from initial conditions.

$$\text{At } t = 0,\ x = 0 \text{ and } \dot{x} = 0 \tag{8}$$

$$\text{Further } \dot{x} = \omega(A\mathrm{e}^{\omega t} - B\mathrm{e}^{-\omega t}) + \frac{g}{2\omega}\cos\omega\mathrm{t} \tag{9}$$

Using (8) in (7) and (9) we obtain

$$0 = A + B \tag{10}$$

$$0 = \omega(A - B) + \frac{g}{2\omega} \tag{11}$$

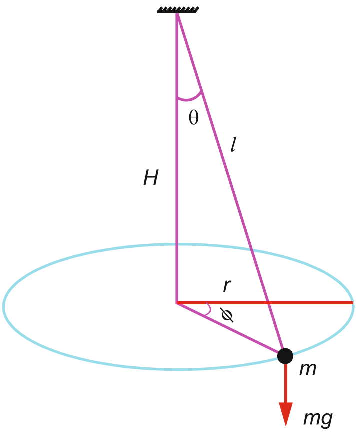
Solving (10) and (11) we get $A = -\frac{g}{4\omega^2}, \quad B = \frac{g}{4\omega^2}$ (12)

The complete solution for x is

$$x = \frac{g}{4\omega^2}(\mathrm{e}^{-\omega t} - \mathrm{e}^{\omega t} + 2\sin\omega \mathrm{t})$$

7.10 Let the length of the cord be l. The Cartesian coordinates can be expressed in terms of spherical polar coordinates (Fig. 7.17)

Fig. 7.17



$$x = l\sin\theta\cos\phi$$
$$y = l\sin\theta\sin\phi$$
$$z = -l\cos\theta$$
$$V = mgz = -mgl\cos\theta \tag{1}$$
$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$
$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{2}$$
$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mgl\cos\theta \tag{3}$$
$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial L}{\partial\theta} = ml^2\sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta \tag{4}$$
$$\frac{\partial L}{\partial\dot{\phi}} = ml^2\sin^2\theta\dot{\phi}, \quad \frac{\partial L}{\partial\phi} = 0 \tag{5}$$

The Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \tag{6}$$

give $\quad \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{g}{l}\sin\theta = 0 \quad (7)$

$$ml^2\frac{\mathrm{d}}{\mathrm{d}t}(\sin^2\theta\dot{\phi}) = 0 \tag{8}$$

Hence $\quad \sin^2\theta\dot{\phi} = \text{constant} = C \quad (9)$

and eliminating $\dot{\phi}$ in (7) with the use of (9) we get a differential equation in θ only.

$$\ddot{\theta} + \frac{g}{l}\sin\theta - C^2\frac{\cos\theta}{\sin^3\theta} = 0 \tag{10}$$

The quantity $P_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\sin^2\theta\dot{\phi} \quad (5)$

is a constant of motion and is recognized as the angular momentum of the system about the z-axis. It is conserved because torque is not produced either by gravity or the tension of the cord about the z-axis. Thus conservation of angular momentum is reflected in (5).

Two interesting cases arise. Suppose $\phi = \text{constant}$. Then $\dot{\phi} = 0$ and $C = 0$. In this case (10) reduces to

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{11}$$

which is appropriate for simple pendulum in which the bob oscillates in the vertical plane.

Suppose $\theta = \text{constant}$, then from (9) $\dot{\phi} = \text{constant}$. Putting $\ddot{\theta} = 0$ in (7) we get

$$\omega = \dot{\phi} = \sqrt{\frac{g}{l\cos\theta}} = \sqrt{\frac{g}{H}} \tag{12}$$

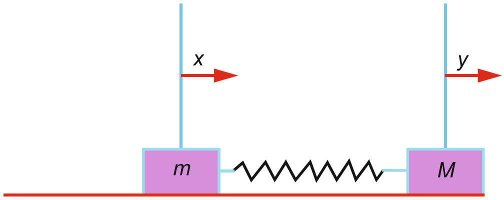
and time period

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{H}{g}} \tag{13}$$

appropriate for the conical pendulum in which the bob rotates on horizontal plane with uniform angular velocity with the cord inclined at constant angle θ with the vertical axis.

7.11 The two generalized coordinates x and y are indicated in Fig. 7.18. The kinetic energy of the system comes from the motion of the blocks and potential energy from the coupling spring:

Fig. 7.18



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 \tag{1}$$

$$V = \frac{1}{2}k(x-y)^2 \tag{2}$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - \frac{1}{2}k(x-y)^2 \tag{3}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -k(x-y) \tag{4}$$

$$\frac{\partial L}{\partial \dot{y}} = M\dot{y}, \quad \frac{\partial L}{\partial y} = k(x-y) \tag{5}$$

Lagrange's equations are written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{6}$$

Using (4) and (5) in (6) we obtain the equations of motion

$$m\ddot{x} + k(x-y) = 0 \tag{7}$$

$$m\ddot{y} + k(y-x) = 0 \tag{8}$$

7.12 This problem involves two degrees of freedom. The coordinates are θ_1 and θ_2 (Fig. 7.19)

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \tag{1}$$

$$v_1^2 = (l_1\dot{\theta}_1)^2 \tag{2}$$

$$v_2^2 = (l_1\dot{\theta}_1)^2 + (l_2\dot{\theta}_2)^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) \text{ (by parallelogram law)} \tag{3}$$

For small angles, $\cos(\theta_2 - \theta_1) \simeq 1$

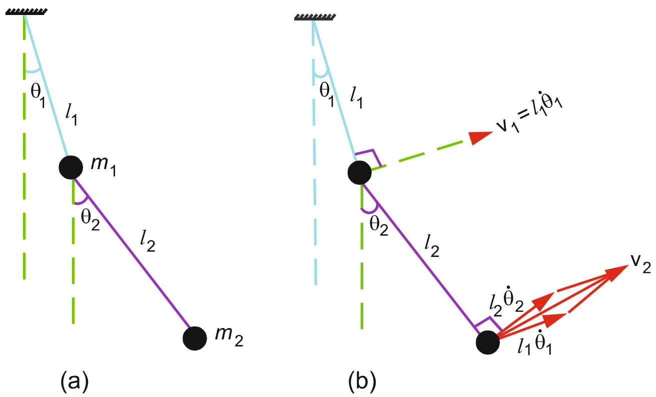


Fig. 7.19

$$T \simeq \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \right] \tag{4}$$

$$\begin{aligned} V &= m_1 g l_1 (1 - \cos\theta_1) + m_2 g l_1 (1 - \cos\theta_1) + m_2 g l_2 (1 - \cos\theta_2) \\ &\simeq m_1 g l_1 \frac{\theta_1^2}{2} + \frac{m_2 g}{2} \left[l_1 \theta_1^2 + l_2 \theta_2^2 \right] \end{aligned} \tag{5}$$

$$\begin{aligned} L =& \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \right] - m_1 g l_1 \frac{\theta_1^2}{2} \\ &- \frac{m_2 g}{2} \left(l_1 \theta_1^2 + l_2 \theta_2^2 \right) \end{aligned} \tag{6}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \tag{7}$$

$$\frac{\partial L}{\partial \theta_1} = -m_1 g l_1 \theta_1 - m_2 g l_1 \theta_1 = -(m_1 + m_2) g l_1 \theta_1 \tag{8}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \tag{9}$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 g l_2 \theta_2 \tag{10}$$

Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0, \tag{11}$$

using (7, (8), (9) and 10) in (11) we obtain the equations of motion

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 + (m_1 + m_2) g \theta_1 = 0 \tag{12}$$

$$l_2 \ddot{\theta}_2 + g \theta_2 + l_1 \ddot{\theta}_1 = 0 \tag{13}$$

7.13 Writing p_θ and p_ϕ for the generalized momenta, by (4) and (5) and V by (1) of prob. (7.10)

$$\frac{\partial L}{\partial \dot{\theta}} = P_\theta = ml^2\dot{\theta}, \quad \dot{\theta} = \frac{P_\theta}{ml^2} \tag{1}$$

$$\frac{\partial L}{\partial \dot{\phi}} = p_\phi = ml^2 \sin^2\theta\dot{\phi}, \quad \dot{\phi} = \frac{p_\phi}{ml^2\sin^2\theta} \tag{2}$$

$$H = \Sigma \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad \text{or} \quad H + L = 2T = \Sigma \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$2T = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{ml^2}\left(p_\theta^2 + \mathrm{cosec}^2\theta p_\phi^2\right) \tag{3}$$

$$\therefore \quad H = T + V = \frac{1}{2ml^2}\left(p_\theta^2 + \cos\mathrm{ec}^2\theta p_\phi^2\right) - mgl\cos\theta \tag{4}$$

The coordinate ϕ is ignorable, and therefore p_ϕ is a constant of motion determined by the initial conditions. We are then left with only two canonical equations to be solved. The canonical equations are

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \tag{5}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2}{ml^2}\frac{\cos\theta}{\sin^3\theta} - mgl\sin\theta \tag{6}$$

where p_ϕ is a constant of motion. By eliminating p_θ we can immediately obtain a second-order differential equation in θ as in prob. (7.10).

7.14 $$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \tag{1}$$

$$\frac{\partial H}{\partial p} = \dot{q} = p \tag{2}$$

$$\frac{\partial H}{\partial q} = -\dot{p} = \omega^2 q \tag{3}$$

Differentiating (2)

$$\ddot{q} = \dot{p} = -\omega^2 q \tag{4}$$

Let $q = x$, then (4) can be written as

$$\ddot{x} + \omega^2 x = 0 \tag{5}$$

This is the equation for one-dimensional harmonic oscillator. The general solution is

$$x = A\sin(\omega t + \varepsilon) + B\cos(\omega t + \varepsilon) \tag{6}$$

which can be verified by substituting (6) in (5). Here A, B and ε are constants to be determined from initial conditions.

7.15 Let r, θ be the instantaneous polar coordinates of a planet of mass m revolving around a parent body of mass M:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{1}$$

$$V = GMm\left(\frac{1}{2a} - \frac{1}{r}\right) \tag{2}$$

where G is the gravitational constant and $2a$ is the major axis of the ellipse:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \dot{r} = \frac{p_r}{m} \tag{3}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{mr^2} \tag{4}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + GMm\left(\frac{1}{2a} - \frac{1}{r}\right) \tag{5}$$

and the Hamiltonian equations are

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r}, \quad \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{GMm}{r^2} = -\dot{p}_r \tag{6}$$

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{\partial H}{\partial \theta} = 0 = -\dot{p}_\theta \tag{7}$$

Two equations in (7) show that

$$p_\theta = \text{constant} = mr^2\dot{\theta} \tag{8}$$

meaning the constancy of angular momentum or equivalently the constancy of areal velocity of the planet (Kepler's second law of planetary motion).

Two equations in (6) yield

$$\ddot{r} = \frac{\dot{p}_r}{m} = \frac{p_\theta^2}{m^2r^3} - \frac{GMm}{r^2} = r\dot{\theta}^2 - \frac{GMm}{r^2} \tag{9}$$

Equation (9) describes the orbit of the planet (Kepler's first law of planetary motion)

7.16 $$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{1}$$

$$V = \frac{1}{2}k(x_1 - x_2)^2 \tag{2}$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_1 - x_2)^2 \tag{3}$$

Equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{5}$$

Using (3) in (4) and (5)

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \tag{6}$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) = 0 \tag{7}$$

It is assumed that the motion is periodic and can be considered as superposition of harmonic components of various amplitudes and frequencies. Let one of these harmonics be represented by

$$x_1 = A\sin\omega t, \quad \ddot{x}_1 = -\omega^2 A\sin\omega t \tag{8}$$

$$x_2 = B\sin\omega t, \quad \ddot{x}_2 = -\omega^2 B\sin\omega t \tag{9}$$

Substituting (8) and (9) in (6) and (7) we obtain

$$(k - m_1\omega^2)\,A - kB = 0$$

$$-kA + (k - m_2\omega^2)B = 0$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0$$

Expansion of the determinant gives

$$m_1m_2\omega^4 - k(m_1 + m_2)\omega^2 = 0$$

or

$$\omega^2[m_1 m_2 \omega^2 - k(m_1 + m_2)] = 0$$

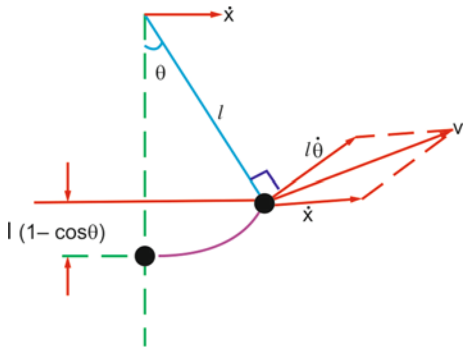
which yields the natural frequencies of the system:

$$\omega_1 = 0 \text{ and } \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} = \sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass. The frequency $\omega_1 = 0$ implies that there is no genuine oscillation of the block but mere translatory motion. The second frequency ω_2 is what one expects for a simple harmonic oscillator with a reduced mass μ.

7.17 Let $x(t)$ be the displacement of the block and $\theta(t)$ the angle through which the pendulum swings. The kinetic energy of the system comes from the motion of the block and the swing of the bob of the pendulum. The potential energy comes from the deformation of the spring and the position of the bob, Fig. 7.20.

Fig. 7.20



The velocity v of the bob is obtained by combining vectorially its linear velocity $(l\dot{\theta})$ with the velocity of the block $(\dot{x})$. The height through which the bob is raised from the equilibrium position is $l(1-\cos\theta)$, where l is the length of the pendulum:

$$v^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta \tag{1}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}) \tag{2}$$

$(\because \text{ for } \theta \to 0, \ \cos\theta \to 1)$

$$V = \frac{1}{2}kx^2 + mgl(1 - \cos\theta)$$

$$= \frac{1}{2}kx^2 + mgl\frac{\theta^2}{2} \tag{3}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}) - \frac{1}{2}kx^2 - mgl\frac{\theta^2}{2} \tag{4}$$

Applying Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{5}$$

we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} + kx = 0 \tag{6}$$

$$l\ddot{\theta} + \ddot{x} + g\theta = 0 \tag{7}$$

7.18 Considering that at $t = 0$ the insect was in the middle of the rod, the coordinates of the insect x, y, z at time t are given by

$$x = (a + vt)\sin\theta\cos\phi$$
$$y = (a + vt)\sin\theta\sin\phi$$
$$z = (a + vt)\cos\theta$$

and the square of its velocity is

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2 + (a + vt)^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$
$$\therefore \quad T = \frac{2}{3}Ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{m}{2}\{v^2 + (a + vt)^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)\}$$
$$V = -Mga\cos\theta - mg(a + vt)\cos\theta + \text{constant}$$
$$L = T - V$$

The application of the Lagrangian equations to the coordinates θ and ϕ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4}{3}Ma^2\dot{\theta} + m(a + vt)^2\dot{\theta}\right] - \left[\frac{4}{3}Ma^2 + m(a + vt)^2\right]\dot{\phi}^2\sin\theta\cos\theta$$
$$= -\{Ma + m(a + vt)\}g\sin\theta \tag{1}$$

and $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\{\frac{4}{5}Ma^2 + m(a + vt)^2\right\}\dot{\phi}\sin^2\theta\right] = 0 \tag{2}$$

Equation (2) can be integrated at once as it is free from ϕ:

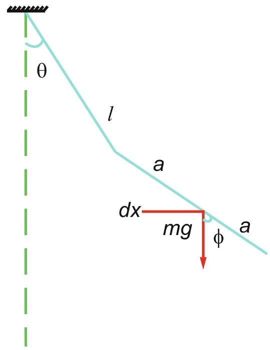
$$\left\{\frac{4}{5}Ma^2 + m(a + vt)^2\right\}\dot{\phi}\sin^2\theta = \text{constant} = C \tag{3}$$

Equation (3) is the equation for the constancy of angular momentum about the vertical axis.

When $\dot{\phi}$ in (2) is eliminated with the aid of (3) we obtain a second-order differential equation in θ.

7.19 Let ρ be the linear density of the rod, i.e. mass per unit length. Consider an infinitesimal element of length of the rod

Fig. 7.21



$$\mathrm{d}T = \rho\omega^2(l\sin\theta + x\sin\phi)^2\mathrm{d}x$$

$$T = \int \mathrm{d}T = \rho\omega^2\int_0^{2a}(l^2\sin^2\theta + 2lx\sin\theta\sin\phi + x^2\sin^2\phi)\mathrm{d}x$$

$$= \omega^2\left(Ml^2\sin^2\theta + 2Mla\sin\theta\sin\phi + \frac{4}{3}Ma^2\sin^2\phi\right) \quad (1)$$

where we have substituted $\rho = M/2a$:

$$V = -Mg(l\cos\theta + a\cos\phi) \quad (2)$$

$$L = \omega^2\left(Ml^2\sin^2\theta + 2Mla\sin\theta\sin\phi + \frac{4}{3}Ma^2\sin^2\phi\right) + Mg(l\cos\theta + a\cos\phi) \quad (3)$$

$$\frac{\partial L}{\partial\dot{\theta}} = 0,\quad \frac{\partial L}{\partial\theta} = \omega^2(2Ml^2\sin\theta\cos\theta + 2Mla\cos\theta\sin\phi) - Mgl\sin\theta \quad (4)$$

$$\frac{\partial L}{\partial\dot{\phi}} = 0,\quad \frac{\partial L}{\partial\phi} = \omega^2\left(2Mla\sin\theta\cos\phi + \frac{8}{3}Ma^2\sin\phi\cos\phi\right) - Mga\sin\phi \quad (5)$$

The Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \quad (6)$$

yield

$$2\omega^2(l\sin\theta + a\sin\phi) = g\tan\theta \tag{7}$$

$$2\omega^2\left(l\sin\theta + \frac{4}{3}a\sin\phi\right) = g\tan\phi \tag{8}$$

Equation (7) and (8) can be solved to obtain θ and ϕ.

7.20 Express the Cartesian coordinates in terms of plane polar coordinates (r, θ)

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{2}$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{3}$$

Square (2) and (3) and add

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{4}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{5}$$

$$V = U(r) \tag{6}$$

$$\therefore \quad L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \quad \text{(Lagrangian)} \tag{7}$$

Generalized momenta:

$$p_k = \frac{\partial L}{\partial \dot{q}_k}, \; p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \; p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{8}$$

Hamiltonian:

$$H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{9}$$

Conservation of Energy: In general H may contain an explicit time dependence as in some forced systems. We shall therefore write $H = H(q, p, t)$. Then H varies with time for two reasons: first, because of its explicit dependence on t, second because the variable q and p are themselves functions of time. Then the total time derivative of H is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\beta=1}^{n}\frac{\partial H}{\partial q_\beta}\dot{q}_\beta + \sum_{\beta=1}^{n}\frac{\partial H}{\partial p_\beta}\dot{p}_\beta \tag{10}$$

Now Hamilton's equations are

$$\frac{\partial H}{\partial p_\beta} = \dot{q}_\beta, \quad \frac{\partial H}{\partial q_\beta} = -\dot{p}_\beta \tag{11}$$

Using (11) in (10), we obtain

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\beta=1}^{n}\left[\frac{\partial H}{\partial q_\beta}\frac{\partial H}{\partial p_\beta} - \frac{\partial H}{\partial p_\beta}\frac{\partial H}{\partial q_\beta}\right] \tag{12}$$

whence

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} \tag{13}$$

Equation (13) asserts that H changes with time only by virtue of its explicit time dependence. The net change is induced by the fact that the variation of q and p with time is zero.

Now in a conservative system, neither T nor V contains any explicit dependence on time.

Hence $\frac{\partial H}{\partial t} = 0$. It follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{14}$$

which leads to the law of conservation of energy

$$H = T + V = E = \text{constant} \tag{15}$$

The Hamiltonian formalism is amenable for finding various conservation laws.
Conservation of angular momentum: The Hamiltonian can be written as

$$H = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta - L \tag{16}$$

Using the polar coordinates (r, θ)

$$H = p_r\dot{r} + p_\theta\dot{\theta} - \left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r)\right) \tag{17}$$

Using (7) and (8) in (17)

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r) \tag{18}$$

Now the second equation in (11) gives

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \qquad (\because \theta \text{ is absent in}(18)) \tag{19}$$

This leads to the conservation of angular momentum

$$p_\theta = J = \text{constant} \tag{20}$$

7.21 Let each mass be m.

(a) $$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{1}$$

$$V = \frac{1}{2}kx^2 + \frac{1}{2}3k(x-y)^2 + \frac{1}{2}ky^2 = k(2x^2 - 3xy + 2y^2) \tag{2}$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - k(2x^2 - 3xy + 2y^2) \tag{3}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

and $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{5}$$

yield

$$m\ddot{x} = -4kx + 3ky \tag{6}$$

$$m\ddot{y} = 3kx - 4ky \tag{7}$$

(c) Let $x = A \sin \omega t$ and $\quad y = B \sin \omega t$ (8)

$\ddot{x} = -A\omega^2 \sin \omega t$ and $\quad \ddot{y} = -B\omega^2 \sin \omega t$ (9)

Substituting (8) and (9) in (6) and (7) and simplifying we obtain

$$(4k - \omega^2 m)A - 3kB = 0 \tag{10}$$

$$-3kA + (4k - \omega^2 m)B = 0 \tag{11}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (4k - \omega^2 m) & -3k \\ -3k & (4k - \omega^2 m) \end{vmatrix} = 0 \tag{12}$$

Expanding the determinant

$$(4k - \omega^2 m)^2 - 9k^2 = 0 \tag{13}$$

This gives the frequencies

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{7k}{m}} \tag{14}$$

Periods of oscillations are

$$T_1 = \frac{2\pi}{\omega_1} = 2\pi\sqrt{\frac{m}{k}} \tag{15}$$

$$T_2 = \frac{2\pi}{\omega_2} = 2\pi\sqrt{\frac{m}{7k}} \tag{16}$$

If we put $\omega = \omega_1 = \sqrt{\frac{k}{m}}$ in (10) or (11) we get $A = B$ and if we put $\omega = \omega_2 = \sqrt{\frac{7k}{m}}$ in (10) or (11), we get $A = -B$. The first one corresponds to symmetric mode of oscillation and the second one to asymmetric one.

The normal coordinates q_1 and q_2 are formed by the linear combination of x and y:

$$q_1 = x - y, \quad q_2 = x + y \tag{17}$$

$$\therefore \quad x = \frac{q_1 + q_2}{2}, \quad y = \frac{q_2 - q_1}{2} \tag{18}$$

Substituting (18) in (6) and (7)

$$\frac{m}{2}(\ddot{q}_1 + \ddot{q}_2) = -2k(q_1 + q_2) + \frac{3k}{2}(q_2 - q_1) \tag{19}$$

$$\frac{m}{2}(\ddot{q}_2 - \ddot{q}_1) = \frac{3k}{2}(q_1 + q_2) - 2k(q_2 - q_1) \tag{20}$$

Adding (19) and (20), $m\ddot{q}_2 = -kq_2$ (21)

Subtracting (20) from (19), $m\ddot{q}_1 = -7kq_1$ (22)

Equation (21) is a linear equation in q_2 alone, with constant coefficients. Similarly (22) is a linear equation in q_1 with constant coefficients. Since the coefficients on the right sides are positive quantities, we note that both (21) and (22) are differential equations of simple harmonic motion having the frequencies given in (14). It is the characteristic of normal coordinates that when the equations of motion are expressed in terms of normal coordinates they are linear with constant coefficients, and each contains but one dependent variable.

Another feature of normal coordinates is that both kinetic energy and potential energy will have quadratic terms and the cross-products will be absent. Thus in this example, when (18) is used in (1) and (2) we get the expressions

$$T = \frac{m}{4}(\dot{q}_1^2 + \dot{q}_2^2),\ \ V = \frac{k}{4}(7q_1^2 + q_2^2)$$

The two normal modes are depicted in Fig. 7.22.

(a) Symmetrical with $\omega_1 = \sqrt{\dfrac{k}{m}}$ and (b) asymmetrical with $\omega_2 = \sqrt{\dfrac{7k}{m}}$

Fig. 7.22

7.22 **(a)** See Fig. 7.23 : $T = \dfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ (1)

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2$$

$$= k\left(x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \quad (2)$$

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k\left(x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \quad (3)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \quad (4)$$

$$m\ddot{x}_1 + k(2x_1 - x_2) = 0 \quad (5)$$

$$m\ddot{x}_2 + k(x_1 - x_1) = 0 \quad (6)$$

(5) and (6) are equations of motion

(b) Let the harmonic solutions be $x_1 = A\sin\omega\mathrm{t}, \quad x_2 = B\sin\omega\mathrm{t}$ (7)

Then $\ddot{x}_1 = -A\omega^2\sin\omega\mathrm{t}, \quad \ddot{x}_2 = -B\omega^2\sin\omega\mathrm{t},$ (8)

Using (7) and (8) in (5) and (6) we get

$$(2k - m\omega^2)A - kB = 0 \quad (9)$$

$$-kA + (k - m\omega^2)B = 0 \quad (10)$$

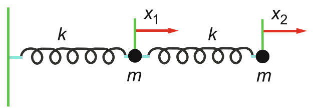


Fig. 7.23

The eigenfrequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (2k - m\omega^2) & -k \\ -k & (k - m\omega^2) \end{vmatrix} = 0 \tag{11}$$

Expanding the determinant, we obtain

$$m^2\omega^4 - 3mk\omega^2 + k^2 = 0$$

$$\omega^2 = \left(\frac{3 \pm \sqrt{5}}{2}\right)\frac{k}{m} \tag{12}$$

$$\therefore \quad \omega_1 = 1.618\sqrt{\frac{k}{m}}, \quad \omega_2 = 0.618\sqrt{\frac{k}{m}} \tag{13}$$

(c) Inserting $\omega = \omega_1$ in (10) we find $B = 1.618\,A$. This corresponds to a symmetric mode as both the amplitudes have the same sign.
Inserting $\omega = \omega_2$ in (10), we find $B = -0.618\,A$. This corresponds to asymmetric mode. These two modes of oscillation are depicted in Fig. 7.24 with relative sizes and directions of displacement.

(a) Symmetric mode $\omega_1 = 1.618\sqrt{\frac{k}{m}}$

(b) Asymmetric mode $\omega_2 = 0.618\sqrt{\frac{k}{m}}$

Fig. 7.24

7.23 **(a)** Let x_1 and x_2 be the displacements of the beads of mass $2m$ and m, respectively.

$$T = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2}(m)\dot{x}_2^2 \tag{1}$$

$$V = \frac{1}{2} \cdot 2kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 \tag{2}$$

$$L = m\left(\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2\right) - k\left(\frac{3}{2}x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \tag{3}$$

Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{4}$$

which yield equations of motion

$$2m\ddot{x}_1 + k(3x_1 - x_2) = 0 \tag{5}$$

$$m\ddot{x}_2 - k(x_1 - x_2) = 0 \tag{6}$$

(b) Let the harmonic solutions be

$$x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t \tag{7}$$

$$\ddot{x}_1 = -A\omega^2 \sin \omega t, \quad \ddot{x}_2 = -B\omega^2 \sin \omega t \tag{8}$$

Substituting (7) and (8) in (5) and (6) we obtain

$$(3k - 2m\omega^2)A - kB = 0 \tag{9}$$

$$kA + (m\omega^2 - k)B = 0 \tag{10}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (3k - 2m\omega^2) & -k \\ k & m\omega^2 - k \end{vmatrix} = 0$$

Expanding the determinant

$$2m^2\omega^4 - 5km\,\omega^2 + 2k^2 = 0$$

$$\omega_1 = \sqrt{\frac{2k}{m}}, \quad \omega_2 = \sqrt{\frac{k}{2m}}$$

(c) Put $\omega = \omega_1 = \sqrt{\frac{2k}{m}}$ in (9) or (10). We find $B = -A$.

Put $\omega = \omega_2 = \sqrt{\frac{k}{2m}}$ in (9) or (10). We find $B = +2A$.

The two normal modes are sketched in Fig. 7.25.

Fig. 7.25

(a) Asymmetric mode

$$\omega_1 = \sqrt{\frac{2k}{m}} \quad B = -A$$

(b) Symmetric mode $\omega_2 = \sqrt{\dfrac{k}{2m}} \quad B = +2A$

7.24 There are three coordinates x_1, x_2 and x_3, Fig. 7.26:

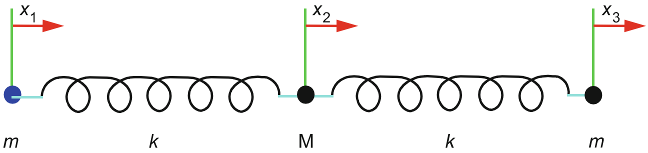


Fig. 7.26

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}Mx_2^2 + \frac{1}{2}m\dot{x}_3^2 \tag{1}$$

$$\begin{aligned} V &= \frac{1}{2}k\left[(x_2 - x_1)^2 + (x_3 - x_2)^2\right] \\ &= \frac{1}{2}k\left(x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2\right) \end{aligned} \tag{2}$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}k\left(x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2\right) \tag{3}$$

Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0, \ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} = 0 \tag{4}$$

yield

$$m\ddot{x}_1 + k(x_1 - x_2) = 0 \tag{5}$$

$$M\ddot{x}_2 + k(-x_1 + 2x_2 - x_3) = 0 \tag{6}$$

$$m\ddot{x}_3 + k(-x_2 + x_3) = 0 \tag{7}$$

Let the harmonic solutions be

$$x_1 = A\sin\omega\mathrm{t}, \quad x_2 = B\sin\omega\mathrm{t}, \quad x_3 = C\sin\omega\mathrm{t} \tag{8}$$

$$\therefore \quad \ddot{x}_1 = -A\omega^2\sin\omega\mathrm{t}, \ \ddot{x}_2 = -B\omega^2\sin\omega\mathrm{t}, \ \ddot{x}_3 = -C\omega^2\sin\omega\mathrm{t} \tag{9}$$

Substituting (8) and (9) in (5), (6) and (7)

$$(k - m\omega^2)A - kB = 0 \tag{10}$$

$$-kA + (2k - M\omega^2)B - kC = 0 \tag{11}$$

$$-kB + (k - m\omega^2)C = 0 \tag{12}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A, B and C

$$\begin{vmatrix} (k - m\omega^2) & -k & 0 \\ -k & (2k - M\omega^2) & -k \\ 0 & -k & (k - m\omega^2) \end{vmatrix} = 0$$

Expanding the determinant we obtain

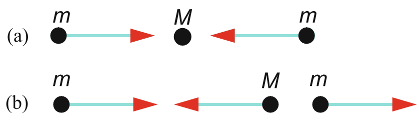
$$\omega^2 (k - m\omega^2)\,(\omega^2 Mm - 2km - Mk) = 0 \tag{13}$$

The frequencies are

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{k(2m + M)}{Mm}} \tag{14}$$

The frequency $\omega_1 = 0$ simply means a translation of all the three particles without vibration. Ratios of amplitudes of the three particles can be found out by substituting ω_2 and ω_3 in (10), (11) and (12). Thus when $\omega = \omega_2 \sqrt{\frac{k}{m}}$ is substituted in (10), we find the amplitude for the central atom $B = 0$. When $B = 0$ is used in (11) we obtain $C = -A$. This mode of oscillation is depicted in Fig. 7.27a.

Fig. 7.27



Substituting $\omega = \omega_3 = \sqrt{\frac{k(2m + M)}{Mm}}$ in (10) and (12) yields

$$B = -\left(\frac{2m}{M}\right) A = -\left(\frac{2m}{M}\right) C$$

Thus in this mode particles of mass m oscillate in phase with equal amplitude but out of phase with the central particle.

This problem has a bearing on the vibrations of linear molecules such as CO_2. The middle particle represents the C atom and the particles on either side represent O atoms. Here too there will be three modes of oscillations. One will have a zero frequency, $\omega_1 = 0$, and will correspond to a simple translation of the centre of mass. In Fig. 7.27a the mode with $\omega_1 = \omega_2$ is such that the carbon atom is stationary, the oxygen atoms oscillating back and forth in opposite phase with equal amplitude. In the third mode which has frequency ω_3, the carbon atom undergoes motion with respect to the centre of mass and is in opposite phase from that of the two oxygen atoms. Of these two modes only ω_3 is observed optically. The frequency ω_2 is not observed because in this mode, the electrical centre of the system is always coincident with the centre of mass, and so there is no oscillating dipole moment Σer available. Hence dipole radiation is not emitted for this mode. On the other hand in the third mode characterized by ω_3 such a moment is present and radiation is emitted.

7.25 **(a)**
$$y = \frac{x^2}{l} \tag{1}$$
$$\dot{y} = \frac{2x \cdot \dot{x}}{l} \tag{2}$$
$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2\left(1 + \frac{4x^2}{l^2}\right)$$
$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2\left(1 + \frac{4x^2}{l^2}\right)$$
$$V = mgy = \frac{mgx^2}{l}$$
$$L = T - V = \frac{1}{2}m\dot{x}^2\left(1 + \frac{4x^2}{l^2}\right) - \frac{mgx^2}{l}$$

(b)
$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$
$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \dot{r} = \frac{p_r}{m}$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$
$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r)$$

7.26 **(a)** At any instant the velocity of the block is $\dot{x}$ on the plane surface. The linear velocity of the pendulum with respect to the block is $l\dot{\theta}$, Fig. 7.28. The velocity $l\dot{\theta}$ must be combined vectorially with $\dot{x}$ to find the velocity v or the pendulum with reference to the plane:

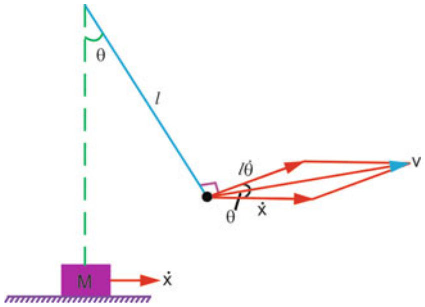
$$v^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\,l\dot{\theta}\,\cos\theta \tag{1}$$

The total kinetic energy of the system

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\,l\dot{\theta}\,\cos\theta) \tag{2}$$

Taking the zero level of the potential energy at the pivot of the pendulum, the potential energy of the system which comes only from the pendulum is

Fig. 7.28



$$V = -mgl\cos\theta \tag{3}$$

$$\therefore \quad L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + ml\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \tag{4}$$

where we have used (2) and (3).

(b) For small angles $\cos\theta \simeq 1 - \frac{\theta^2}{2}$, in the first approximation, and $\cos\theta \simeq 1$, in the second approximation. Thus in this approximation (4) becomes

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1 - \frac{\theta^2}{2}\right) \tag{5}$$

(c) The Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{6}$$

lead to the equations of motion

$$\ddot{x} + l\ddot{\theta} + g\theta = 0 \tag{7}$$

$$(M+m)\ddot{x} + ml\ddot{\theta} = 0 \tag{8}$$

(d) Eliminating $\ddot{x}$ between (7) and (8) and simplifying

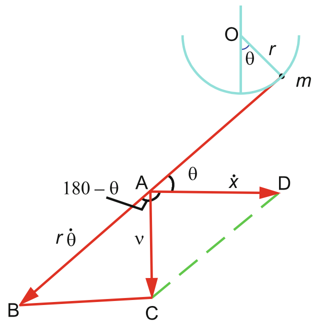
$$\ddot{\theta} + \frac{(M+m)}{M}\frac{g}{l}\theta = 0 \tag{9}$$

This is the equation for angular simple harmonic motion whose frequency is given by

$$\omega = \sqrt{\frac{(M+m)g}{Ml}} \tag{10}$$

7.27 **(a)** First, we assume that the bowl does not move. Both kinetic energy and potential energy arise from the particle alone. Taking the origin at O, the centre of the bowl, Fig. 7.29, the linear velocity of the particle is $v = r\dot{\theta}$. There is only one degree of freedom:

Fig. 7.29



$$T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\dot{\theta}^2 \tag{1}$$

$$V = -mgr\cos\theta \tag{2}$$

$$L = \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta \tag{3}$$

Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{4}$$

becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{5}$$

which yields the equation of motion

$$mr^2\ddot{\theta} + mgr\sin\theta = 0$$

$$\text{or} \quad \ddot{\theta} + \frac{g}{r}\sin\theta = 0 \quad \text{(equation of motion)} \tag{6}$$

For small angles, $\sin\theta \simeq \theta$. Then (6) becomes

$$\ddot{\theta} + \frac{g}{r}\theta = 0 \tag{7}$$

which is the equation for simple harmonic motion of frequency $\omega = \sqrt{\dfrac{g}{r}}$ or time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g}} \tag{8}$$

(b) **(i)** The bowl can now slide freely along the x-direction with velocity $\dot{x}$. The velocity of the particle with reference to the table is obtained by adding $l\dot{\theta}$ to $\dot{x}$ vectorially, Fig. 7.29. The total kinetic energy then comes from the motion of both the particle and the bowl. The potential energy, however, is the same as in (a):

$$v^2 = r^2\dot{\theta}^2 + x^2 - 2r\,\dot{\theta}\dot{x}\cos(180 - \theta) \tag{9}$$

from the diagonal AC of the parallelogram ABCD

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\,(r^2\dot{\theta}^2 + \dot{x}^2 - 2r\dot{x}\dot{\theta}\cos\theta) \tag{10}$$

$$V = -mgr\cos\theta \tag{11}$$

$$\begin{aligned} L &= T - V \qquad \text{(Lagrangian)} \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{x}^2 - 2r\dot{x}\dot{\theta}\cos\theta) + mgr\cos\theta \end{aligned} \tag{12}$$

(ii) and (iii).
In the small angle approximation the $\cos\theta$ in the kinetic energy can be neglected as $\cos\theta \to 1$ but can be retained in the potential energy in order to avoid higher order terms.
Equation (12) then becomes

$$L = \frac{1}{2}(M+m)\dot{x}^2 - mr\dot{x}\dot{\theta} + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta \tag{13}$$

We have now two degrees of freedom, x and θ, and the corresponding Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{14}$$

which yield the equations of motion

$$(M + m)\ddot{x} - mr\,\ddot{\theta} = 0 \tag{15}$$

$$\ddot{x} - r\,\ddot{\theta} - g\,\theta = 0 \tag{16}$$

Equations (15) and (16) constitute the equations of motion. Eliminating $\ddot{x}$ we obtain

$$\ddot{\theta} + \left(\frac{M + m}{M}\right)\frac{g}{r}\theta = 0 \tag{17}$$

which is the equation for simple harmonic motion with frequency $\omega = \sqrt{\frac{(M + m)}{M}\frac{g}{r}}$ and time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{(M + m)}\frac{r}{g}} \tag{18}$$

On comparing (18) with (8) it is observed that the period of oscillation is smaller by a factor $[M/(M + m)^{1/2}]$ as compared to the case where the bowl is fixed.

7.28 Take the differential of the Lagrangian

$$L(q_1, \ldots, q_n,\ \dot{q}_1, \ldots, \dot{q}_n,\ t)$$

$$\mathrm{d}L = \sum_{r=1}^{n}\left(\frac{\partial L}{\partial q_r}\mathrm{d}q_r + \frac{\partial L}{\partial \dot{q}_r}\mathrm{d}\dot{q}_r\right) + \frac{\partial L}{\partial t}\mathrm{d}t \tag{1}$$

Now the Lagrangian equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0 \tag{2}$$

and the generalized momenta are defined by

$$\frac{\partial L}{\partial \dot{q}_r} = p_r \tag{3}$$

Using (3) in (2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = \dot{p}_r \tag{4}$$

Using (2), (3) and (4) in (1)

$$\mathrm{d}L - \sum_{r=1}^{n} (\dot{p}_r \mathrm{d}q_r + p_r \mathrm{d}\dot{q}_r) + \frac{\partial L}{\partial t}\mathrm{d}t \tag{5}$$

Equation (5) can be rearranged in the form

$$\mathrm{d}\left(\sum_{r=1}^{n} p_r \dot{q}_r - L\right) = \sum_{r=1}^{n} (\dot{q}_r \mathrm{d}p_r - \dot{p}_r \mathrm{d}q_r) - \frac{\partial L}{\partial t}\mathrm{d}t \tag{6}$$

The Hamiltonian function H is defined by

$$H = \sum_{r=1}^{n} p_r \dot{q}_r - L(q_1, \ldots, q_n,\ \dot{q}_1, \ldots, \dot{q}_n,\ t) \tag{7}$$

Equation (6) therefore my be written as

$$\mathrm{d}H = \sum_{r=1}^{n} (\dot{q}_r dp_r - \dot{p}_r dq_r) - \frac{\partial L}{\partial t}\mathrm{d}t \tag{8}$$

While the Lagrangian function L is an explicit function of $q_1, \ldots, q_n$, $\dot{q}_1, \ldots, \dot{q}_n$ and t, it is usually possible to express H as an explicit function only of $q_1, \ldots, q_n,\ p_1, \ldots, p_n,\ t$, that is, to eliminate the n generalized velocities from (7). The n equation of type (3) are employed for this purpose. Each provides one of the p's in terms of the $\dot{q}'s$. Assuming that the elimination of the generalized velocities is possible, we may write

$$H = H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \tag{9}$$

H now depends explicitly on the generalized coordinates and generalized momenta together with the time. Therefore, taking the differential dH, we obtain

$$\mathrm{d}H = \sum_{r=1}^{n} \left(\frac{\partial H}{\partial q_r}\mathrm{d}q_r + \frac{\partial H}{\partial p_r}\mathrm{d}p_r\right) + \frac{\partial H}{\partial t}\mathrm{d}t \tag{10}$$

Comparing (8) and (10), we have the relations

$$\frac{\partial H}{\partial p_r} = \dot{q}_r, \quad \frac{\partial H}{\partial q_r} = -\dot{p}_r \tag{11}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{12}$$

Equations (11) are called Hamilton's canonical equations. These are $2n$ is number. For a system with n degrees of freedom the n Lagrangian equations (2) of the second order are replaced by $2n$ Hamiltonian equations of the first order. We note from the second equation of (11) that if any coordinate q_l is not contained explicitly in the Hamiltonian function H, the conjugate momentum p_l is a constant of motion. Such coordinates are called ignorable coordinates.

7.29 The generalized momentum p_r conjugate to the generalized coordinate q_r is defined as $\frac{\partial L}{\partial \dot{q}_r} = p_r$. If the Lagrangian of a dynamical system does not contain a certain coordinate, say q_s, explicitly then p_s is a constant of motion.

(a) The kinetic energy arises only from the motion of the particle P on the table as the particle Q is stationary. The potential energy arises from the particle Q alone.
When P is at distance r from the opening, Q will be at a depth l–x from the opening:

$$T = \frac{1}{2}mv_p^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{1}$$

$$V = -mg(l - r) \tag{2}$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mg(l - r) \tag{3}$$

For the two coordinates r and θ, Lagrange's equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{4}$$

Equations (4) yield

$$\ddot{r} = r\dot{\theta}^2 - g \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0$$

$$\therefore \quad r^2\dot{\theta} = C = \text{constant} \tag{6}$$

Equations (5) and (6) constitute the equations of motion.

(b) Initial conditions: At $r = a$, $r\dot{\theta} = \sqrt{ag}$

$$\therefore \quad \dot{\theta} = \sqrt{\frac{g}{a}} \tag{7}$$

Using (7) in (6) with $r = a$, we obtain

$$C^2 = a^3 g \tag{8}$$

Using (6) and (8) in (5)

$$\ddot{r} = \frac{rc^2}{r^4} - g = \frac{a^3 g}{r^3} - g \tag{9}$$

(c) **(i)** Since Q does not move, P must be at constant distance $r = a$ from the opening. Therefore P describes a circle of constant radius a.

(ii) Let P be displaced by a small distance x from the stable circular orbit of radius a, that is

$$r = a + x \tag{10}$$

$$\therefore \quad \ddot{r} = \ddot{x} \tag{11}$$

Using (10) and (11) in (9)

$$\ddot{x} = g\left[\frac{a^3}{(a+x)^3} - 1\right] = g\left[\left(1 + \frac{x}{a}\right)^{-3} - 1\right]$$

$$\text{or} \quad \ddot{x} \simeq -\frac{3gx}{a}$$

$$\text{or} \quad \ddot{x} + \frac{3gx}{a} = 0 \tag{12}$$

which is the equation for simple harmonic motion. Thus the particle P when slightly displaced from the stable orbit of radius a executes oscillations around $r = a$.

This aspect of oscillations has a bearing on the so-called betatron oscillations of ions in circular machines which accelerate charged particles to high energies. If the amplitudes of the betatron oscillations are large then they may hit the wall of the doughnut and be lost, resulting in the loss of intensity of the accelerated particles.

7.30 **(i)** $x = a\cos\theta,\ y = b\sin\theta, \quad r^2 = a^2\cos^2\theta + b^2\sin^2\theta$ (1)

$$\dot{x} = -a\dot{\theta}\sin\theta,\ \dot{y} = b\dot{\theta}\cos\theta \tag{2}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(a^2\sin^2\theta + b^2\cos^2\theta)\dot{\theta}^2 \tag{3}$$

$$V = mgy + \frac{1}{2}kr^2 = mgb\sin\theta + \frac{1}{2}k(a^2\cos^2\theta + b^2\sin^2\theta) \tag{4}$$

$$L = \frac{1}{2}m(a^2\sin^2\theta + b^2\cos^2\theta)\dot{\theta}^2 - mgb\sin\theta$$
$$- \frac{1}{2}k(a^2\cos^2\theta + b^2\sin^2\theta) \tag{5}$$

Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

yields

$$m(a^2 \sin^2\theta + b^2\cos^2\theta)\ddot{\theta} + m(a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta)\dot{\theta}^2 + mgb\cos\theta + k(-a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta) = 0$$

or $$m(a^2\sin^2\theta + b^2\cos^2\theta)\ddot{\theta} - (a^2 - b^2)(k - m\dot{\theta}^2)\sin\theta\cos\theta + mgb\cos\theta = 0 \tag{6}$$

(ii) Equilibrium point is located where the force is zero, or $\partial V/\partial\theta = 0$. Differentiating (4)

$$\frac{\partial V}{\partial \theta} = mgb\cos\theta + k(b^2 - a^2)\sin\theta\cos\theta \tag{7}$$

Clearly the right-hand side of (7) is zero for $\theta = \pm\dfrac{\pi}{2}$
Writing (7) as

$$[mgb + k(b^2 - a^2)\sin\theta]\cos\theta \tag{8}$$

Another equilibrium point is obtained when

$$\sin\theta = \frac{mgb}{k(a^2 - b^2)} \tag{9}$$

provided $a > b$.

(iii) An equilibrium point will be stable if $\dfrac{\partial^2 V}{\partial\theta^2} > 0$ and will be unstable if $\dfrac{\partial^2 V}{\partial\theta^2} < 0$. Differentiating (8) again we have

$$\frac{\partial^2 V}{\partial\theta^2} = k(b^2 - a^2)(\cos^2\theta - \sin^2\theta) - mgb\sin\theta \tag{10}$$

For $\theta = \dfrac{\pi}{2}$, (10) reduces to

$$k(a^2 - b^2) - mgb \tag{11}$$

Expression (11) will be positive if $a^2 > b^2 + \dfrac{mgb}{k}$, and $\theta = \dfrac{\pi}{2}$ will be a stable point.

For $\theta = -\dfrac{\pi}{2}$, (10) reduces to

$$k(a^2 - b^2) + mgb \tag{12}$$

Expression (12) will be positive if $a^2 > b^2 - \dfrac{mgb}{k}$, and $\theta = -\dfrac{\pi}{2}$ will be a stable point.

(iv) $T = 2\pi\sqrt{\dfrac{A(\theta)}{V''(\theta)}}$

$V''\left(-\dfrac{\pi}{2}\right) = k(a^2 - b^2) + mgb$, by (12)

$A(\theta)$ is the coefficient of $\dfrac{1}{2}\dot{\theta}^2$ in (3)

$\therefore \quad A(\theta) = m(a^2 \sin^2\theta + b^2\cos^2\theta)$

$\therefore \quad A\left(-\dfrac{\pi}{2}\right) = ma^2$

$$\therefore \quad T = 2\pi\sqrt{\frac{ma^2}{k(a^2 - b^2) + mgb}}$$

7.31 In prob. (7.12) the following equations were obtained:

$$(m_1 + m_2)\, l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2 + (m_1 + m_2)g\theta_1 = 0 \tag{1}$$

$$l_2\ddot{\theta}_2 + g\theta_2 + l_1\ddot{\theta}_1 = 0 \tag{2}$$

For $l_1 = l_2 = l$ and $m_1 = m_2 = m$, (1) and (2) become

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2 + 2g\theta_1 = 0 \tag{3}$$

$$l\ddot{\theta}_2 + l\ddot{\theta}_1 + g\theta_2 = 0 \tag{4}$$

The harmonic solutions of (3) and (4) are written as

$$\theta_1 = A\sin\omega t, \quad \theta_2 = B\sin\omega t \tag{5}$$

$$\ddot{\theta}_1 = -A\omega^2\sin\omega t, \quad \ddot{\theta}_2 = -B\omega^2\sin\omega t \tag{6}$$

Substituting (5) and (6) in (3) and (4) and simplifying

$$2(l\omega^2 - g)A + l\omega^2 B = 0 \tag{7}$$

$$l\omega^2 A + (l\omega^2 - g)B = 0 \tag{8}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} 2(l\omega^2 - g) & l\omega^2 \\ l\omega^2 & (l\omega^2 - g) \end{vmatrix} = 0$$

Expanding the determinant

$$l^2\omega^4 - 4\,\text{lg}\,\omega^2 + 2g^2 = 0$$
$$\omega^2 = \left(2 \pm \sqrt{2}\right)\frac{g}{l}$$
$$\therefore \quad \omega = \sqrt{\left(2 \pm \sqrt{2}\right)\frac{g}{l}}$$
$$\therefore \quad \omega_1 = 0.76\sqrt{\frac{g}{l}},\ \omega_2 = 1.85\sqrt{\frac{g}{l}}$$

7.32 While the method employed in prob. (6.46) was based on forces or torques, that is, Newton's method, the Lagrangian method is based on energy:

$$T = \frac{1}{2}m\,(\dot{x}_1^2 + \dot{x}_2^2) \tag{1}$$

For small angles $\dot{y}_1$ and $\dot{y}_2$ are negligibly small

$$V = \frac{1}{2}k(x_1 - x_2)^2 + mgb(1 - \cos\theta_1) + mgb(1 - \cos\theta_2)$$

For small angles $1 - \cos\theta_1 = \dfrac{\theta_1^2}{2} = \dfrac{x_1^2}{2b^2}$.

Similarly, $1 - \cos\theta_2 = \dfrac{x_2^2}{2b^2}$

$$\therefore \quad V = \frac{1}{2}\,k(x_1 - x_2)^2 + \frac{mg}{2b}(x_1^2 + x_2^2) \tag{2}$$
$$\therefore \quad L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 - 2x_1x_2 + x_2^2) - \frac{mg}{2b}(x_1^2 - x_2^2) \tag{3}$$

The Lagrange's equations for the coordinates x_1 and x_2 are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{4}$$

Using (3) in (4) we obtain

$$m\ddot{x}_1 + \left(k + \frac{mg}{b}\right)x_1 - kx_2 = 0 \tag{5}$$
$$m\ddot{x}_2 - kx_1 + \left(k + \frac{mg}{b}\right)x_2 = 0 \tag{6}$$

Assuming that x_1 and $x_2$are periodic with the same frequency but different amplitudes, let

$$x_1 = A \sin \omega t, \quad \ddot{x}_1 = -A\omega^2 \sin \omega t \tag{7}$$

$$x_2 = B \sin \omega t, \quad \ddot{x}_2 = -B\omega^2 \sin \omega t \tag{8}$$

Substituting (7) and (8) in (5) and (6) and simplifying

$$\left(k + \frac{mg}{b} - m\omega^2\right) A - kB = 0 \tag{9}$$

$$-kA + \left(k + \frac{mg}{b} - m\omega^2\right) B = 0 \tag{10}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} \left(k + \frac{mg}{b} - m\omega^2\right) & -k \\ -k & \left(k + \frac{mg}{b} - m\omega^2\right) \end{vmatrix} = 0$$

Expanding the determinant and solving gives

$$\omega_1 = \sqrt{\frac{g}{b}} \text{ and } \omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}},$$

In agreement with the results of prob. (6.46).

7.33 Let the origin be at the fixed point O and OB be the diameter passing through C the centre of the circular wire, Fig. 7.30. The position of m is indicated by the angle θ subtended by the radius CP with the diameter OB. Only one general coordinate $q = \theta$ is sufficient for this problem. Let $\phi = \omega t$ be the angle which the diameter OB makes with the fixed x-axis. From the geometry of the diagram (Fig. 7.30) the coordinates of m are expressed as

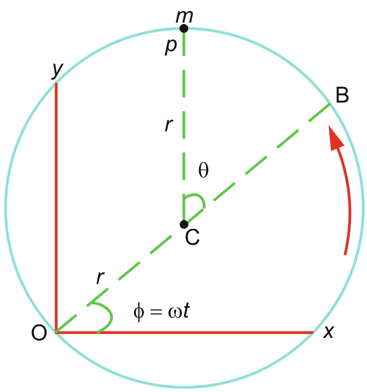


Fig. 7.30

$$x = r\cos\omega t + r\cos(\theta + \omega t) \tag{1}$$

$$y = r\sin\omega t + r\sin(\theta + \omega t) \tag{2}$$

The velocity components are found as

$$\dot{x} = -r\,\omega\sin\omega t - r(\dot{\theta} + \omega)\sin(\theta + \omega t) \tag{3}$$

$$\dot{y} = r\,\omega\cos\omega t - r(\dot{\theta} + \omega)\cos(\theta + \omega t) \tag{4}$$

Squaring and adding and simplifying we obtain

$$\dot{x}^2 + \dot{y}^2 = r^2\omega^2 + r^2(\dot{\theta} + \omega)^2 + 2r^2\omega(\dot{\theta} + \omega)\cos\theta \tag{5}$$

$$\therefore \quad T = \frac{1}{2}mr^2[\omega^2 + (\dot{\theta} + \omega)^2 + 2\omega(\dot{\theta} + \omega)\cos\theta] \tag{6}$$

Here $V = 0$, and so $L = T$. The Lagrange's equation then simply reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = 0 \tag{7}$$

Cancelling the common factor mr^2 (7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\theta} + \omega + \omega\cos\theta) + \omega(\dot{\theta} + \omega)\sin\theta = 0 \tag{8}$$

which reduces to

$$\ddot{\theta} + \omega^2\sin\theta = 0 \tag{9}$$

which is the equation for simple pendulum. Thus the bead oscillates about the rotating line OB as a pendulum of length $r = a/\omega^2$.

7.34 **(a)** The velocity v of mass m relative to the horizontal surface is given by combining $\dot{s}$ with $\dot{x}$. The components of the velocity v are

$$v_x = \dot{x} + \dot{s}\cos\alpha \tag{1}$$

$$v_y = -\dot{s}\sin\alpha \tag{2}$$

$$\therefore \quad v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{s}^2 + 2\dot{x}\,\dot{s}\cos\alpha \tag{3}$$

Kinetic energy of the system

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\,(\dot{s}^2 + \dot{x}^2 + 2\dot{s}\dot{x}\cos\alpha) \tag{4}$$

Potential energy comes exclusively from the mass m (spring energy + gravitational energy)

$$V = \frac{k}{2}(s-l)^2 + mg(h - s\ \sin\alpha) \tag{5}$$

$$L = T - V = \frac{(M+m)}{2}\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\alpha - \frac{k}{2}(s-l)^2 - mg(h - s\sin\alpha) \tag{6}$$

(b) The generalized coordinates are $q_1 = x$ and $q_2 = s$. The Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \tag{7}$$

Using (6) in (7), equations of motion become

$$(M+m)\ddot{x} + m\ddot{s}\cos\alpha = 0 \tag{8}$$

$$m\ddot{x}\cos\alpha + m\ddot{s} + k(s - s_0) = 0 \tag{9}$$

where $s_0 = l + (mg\sin\alpha)/k$.

Let $x = A\sin\omega\mathrm{t}$ and $s - s_0 = B\sin\omega\mathrm{t}$ (10)

$$\ddot{x} = -\omega^2 A\sin\omega\mathrm{t}, \quad \ddot{s} = -B\omega^2\sin\omega\mathrm{t} \tag{11}$$

Substituting (10) and (11) in (8) and (9) we obtain

$$A(M+m) + B\cos\alpha = 0 \tag{12}$$

$$Am\omega^2\cos\alpha + B(m\omega^2 - k) = 0 \tag{13}$$

Eliminating A and B, we find

$$\omega = \sqrt{\frac{k(M+m)}{m(M + m\ \sin^2\alpha)}} \tag{14}$$

Components of the velocity of the ball as observed on the table are

7.35

$$v_x = \dot{x} + \dot{y}\cos\alpha \tag{1}$$

$$v_y = \dot{y}\sin\alpha \tag{2}$$

$$v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha \tag{3}$$

$$T(\text{ball}) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\times\frac{2}{5}mr^2\omega^2 = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2 \tag{4}$$

$$T(\text{wedge}) = \frac{1}{2}(M+m)\dot{x}^2 \tag{5}$$

$$\therefore \quad T(\text{system}) = \frac{7}{10}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha) + \frac{1}{2}(M+m)\dot{x}^2 \tag{6}$$

$$V(\text{system}) = V(\text{ball}) = -mgy\sin\alpha \tag{7}$$

$$L = \frac{7}{10}m(\dot{x}^2+\dot{y}^2+2\dot{x}\dot{y}\cos\alpha)+\frac{1}{2}(M+m)\dot{x}^2 + mgy\sin\alpha \tag{8}$$

Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{9}$$

become

$$\frac{7m}{5}\ddot{x} + \frac{7}{5}m\ddot{y}\cos\alpha + (M+m)\ddot{x} = 0 \tag{10}$$

$$\frac{7m}{5}\ddot{y} + \frac{7}{5}m\ddot{x}\cos\alpha - mg\sin\alpha = 0 \tag{11}$$

Solving (10) and (11) and simplifying

$$\ddot{x} = -\frac{5mg\sin\alpha\cos\alpha}{5M + (5+7\sin^2\alpha)m} \quad \text{(for the wedge)}$$

$$\ddot{y} = \frac{5(5M+12m)g\sin\alpha}{7(5M+(5+7\sin^2\alpha)m)} \quad \text{(for the ball)}$$

For $M = m$ and $\alpha = \pi/4$

$$\ddot{x} = \frac{5g}{27}$$

$$\ddot{y} = \frac{85\sqrt{2}}{189}$$

Chapter 8
Waves

Abstract *Chapter 8* deals with waves. The topics covered are wave equation, progressive and stationary waves, vibration of strings, wave velocity in solids, liquids and gases, capillary waves and gravity waves, the Doppler effect, shock wave, reverberation in buildings, stationary waves in pipes and intensity level.

8.1 Basic Concepts and Formulae

The travelling wave: The simple harmonic progressive wave travelling in the positive x-direction can be variously written as

$$\begin{aligned} y &= A \sin \frac{2\pi}{\lambda}(vt - x) \\ &= A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \\ &= A \sin(\omega t - kx) \\ &= A \sin 2\pi f \left(t - \frac{x}{v}\right) \end{aligned} \tag{8.1}$$

where A is the amplitude, f the frequency and v the wave velocity, λ the wavelength, $\omega = 2\pi f$ the angular frequency and $k = 2\pi/\lambda$, the wave number.

Similarly, the wave in the negative x-direction can be written as

$$y = A \sin \frac{2\pi}{\lambda}(vt + x) \tag{8.2}$$

and so on.

The superposition principle states that when two or more waves traverse the same region independently, the displacement of any particle at a given time is given by the vector addition of the displacement due to the individual waves.

Interference of waves: Interference is the physical effect caused by the superposition of two or more wave trains crossing the same region simultaneously. The wave trains must have a constant phase difference.

Vibrating strings: Stationary waves are formed by the superposition of two similar progressive waves travelling in the opposite direction over a taut string clamped by rigid supports.

Wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{F}{\mu}\frac{\partial^2 y}{\partial x^2} \tag{8.3}$$

Wave velocity:

$$v = \sqrt{F/\mu} \tag{8.4}$$

where F is the tension in the string and μ is the linear density, i.e. mass per unit length.

The general solution of (8.3) is

$$y = f_1(vt - x) + f_2(vt + x) \tag{8.5}$$

Harmonic solution:

$$y = 2A \sin kx \cos \omega t \tag{8.6}$$

When the displacement in the y-direction is maximum (antinode) the amplitude is $2A$, the antinodes are located at $x = \lambda/4, 3\lambda/4, 5\lambda/4 \ldots$ and are spaced half a wavelength apart. The amplitude has a minimum value of zero (nodes). The nodes are located at $x = 0, \lambda/2, \lambda \ldots$ and are also spaced half a wavelength apart. Ends of the strings are always nodes. Neighbouring nodes and antinodes are spaced one-quarter wavelength apart.

The frequency of vibration is given by

$$f = \frac{v}{\lambda} = \frac{N}{2L}\sqrt{\frac{F}{\mu}} = \frac{N}{2L}\sqrt{\frac{F}{\rho A}} \tag{8.7}$$

where ρ is the density and A the cross-sectional area of the string and $N = 1, 2, 3, \ldots$. Vibration with $N = 1$ is called the fundamental or the first harmonic, $N = 2$ is called the first overtone or the second harmonic, etc.

Power: The energy per unit length of the string is given by

$$E = \tfrac{1}{2}\mu V_0^2 \tag{8.8}$$

where V_0 is the velocity amplitude of any particle on the string. Since the wave is travelling with velocity v, the power (P) is given by

$$P_{\mathrm{av}} = Ev = \frac{1}{2}\mu V_0^2 v = \frac{1}{2}V_0^2\sqrt{F\mu} \tag{8.9}$$

Waves in Solids

In solids, compressional and shearing forces are readily transmitted.

(i) Transverse waves in wires/strings in which the elastic properties of the material are disregarded:

$$v = \sqrt{F/\mu} \tag{8.4}$$

(ii) Transverse waves in bars/wires

$$V \propto \frac{1}{\lambda}\sqrt{Y/\rho} \tag{8.10}$$

where Y is Young's modulus of elasticity.

(iii) Longitudinal waves in wires and bars

$$V = \sqrt{Y/\rho} \tag{8.11}$$

(iv) Torsional vibrations in wires/bars

$$V = \sqrt{\eta/\rho} \tag{8.12}$$

where η is the shear modulus of elasticity.

In all these cases the material of restricted dimension is considered.

Waves in Liquids

The wave motion through liquids is influenced by the gravity and the characteristics of the medium such as the depth and surface tension.

Canal Waves

If the wavelength is large compared with the wave amplitude, surface tension effect is small. The controlling factors are then basically gravity (g) and the boundary conditions. Furthermore, if the surface is sufficiently extensive so that the wall effects are negligible then the depth (h) alone is the main boundary condition. The velocity (v) of the canal waves is given by

$$V = \sqrt{gh} \tag{8.13}$$

Surface Waves

These are the waves found on relatively deep water. The velocity of deep water waves is given by

$$V = \sqrt{g\lambda/2\pi} \tag{8.14}$$

For long waves in shallow water

$$V = \sqrt{gh} \tag{8.15}$$

Capillary Waves

Surface waves are modified by surface tension S. If h is large compared with λ

$$V^2 = \frac{2\pi s}{\rho\lambda} + \frac{g\lambda}{2\pi} \tag{8.16}$$

The minimum value of λ is given by minimizing (8.16)

$$\lambda_{\min} = 2\pi\sqrt{\frac{s}{g\rho}} \tag{8.17}$$

If λ is sufficiently large the second term dominates and the controlling factor being mainly gravity. Thus, the velocity of the gravity waves is given by

$$V = \sqrt{g\lambda/2\pi} \tag{8.18}$$

If λ is very small, the first term in (8.16) dominates and the motion is mainly controlled by capillarity and

$$V = \sqrt{\frac{2\pi s}{\rho\lambda}} \tag{8.19}$$

Acoustic Waves

$$\frac{\partial^2 \xi}{\partial t^2} = V^2 \frac{\partial^2 \xi}{\partial x^2} \quad \text{(plane wave equation for displacement)} \tag{8.20}$$

$$\frac{\partial^2 P}{\partial t^2} = V^2 \frac{\partial^2 P}{\partial x^2} \quad \text{(plane wave equation for pressure)} \tag{8.21}$$

where

$$V = \sqrt{B/\rho_0} \tag{8.22}$$

B being the bulk modulus of elasticity.

Sound Velocity in a Gas

$$V = \sqrt{\frac{\gamma P}{\rho}} \quad \text{(Laplace formula)} \tag{8.23}$$

Sound Velocity in a Liquid

$$V = \sqrt{\frac{\gamma B_\tau}{\rho}} \tag{8.24}$$

where B_T is the isothermal bulk modulus.

The energy in length λ is given by

$$E_\lambda = \frac{1}{2}\rho_0 \omega^2 A^2 \lambda \tag{8.25}$$

The energy density

$$E = E_\lambda/\lambda = \frac{1}{2}\rho_0 \omega^2 A^2 \tag{8.26}$$

The intensity, i.e. the time rate of flow of energy per unit area of the wave front

$$I = \frac{1}{2}\rho V A^2 \omega^2 \tag{8.27}$$

Intensity Level (IL):Decibel Scale

$$\mathrm{IL} = 10 \log(I/I_0) \tag{8.28}$$

where log is logarithmic to base 10, I_0 is the reference intensity (the zero of the scale) and IL is expressed in decibels.

Stationary Waves in Pipes

(i) *Closed pipe* (pipe closed at one end and opened at the other)

$$f_1 = V/4L,\ f_2 = 3V/4L,\ f_3 = 5V/4L \ldots$$

(ii) *Open pipe*_(pipe opened at both ends)

$$f_1 = V/2L,\ f_2 = V/L,\ f_3 = 3V/2L \ldots$$

Doppler effect is the apparent change in frequency of a wave motion when there is relative motion between the source and the observer.

(a) Moving Source but Stationary Observer

If the source of waves of frequency f moves with velocity v and if v_s is the sound velocity in still air then the apparent frequency f would be

$$f^/ = \frac{fv}{v \pm v_s} \tag{8.29}$$

where the minus sign is for approach and plus sign for separation.

(b) Source is At Rest, Observer in Motion

Let the observer be moving with speed v_0. Then

$$f' = f\frac{(v \pm v_0)}{v} \tag{8.30}$$

where the plus sign is for motion towards the source and the minus sign for motion away from the source.

(c) Both Source and Observer in Motion

$$f' = f\frac{(v \pm v_0)}{(v \mp v_s)} \tag{8.31}$$

(d) If the medium moves with velocity W relative to the ground along the line joining source and observer,

$$f' = f\frac{(v + W \pm v_0)}{(v + W \mp v_s)} \tag{8.32}$$

Shock waves are emitted when the observer's velocity or the source velocity exceeds the sound velocity and Doppler's formulae break down. The wave front assumes the shape of a cone with the moving body at the apex. The surface of the cone makes an angle with the line of flight of the source such that

$$\sin\theta = v/v_s \tag{8.33}$$

The ratio v_s/v is called Mach number. An example of shock waves is the wave resulting from a bow boat speeding on water, a second example is a jet-plane or missile moving at the supersonic speed, a third example is the emission of Cerenkov radiation when a charged particle moves through a transparent medium with a speed exceeding that of the phase velocity of light in that medium.

Echo is defined as direct reflection of short duration sound from the surface of a large area. If d is the distance of the reflector, V the speed of sound then the time interval between the direct and reflected waves is

$$T = 2d/v \tag{8.34}$$

Reverberation: A sound once produced in a room will get reflected repeatedly from the walls and become so feeble that it will not be heard. The time t taken for the steady intensity level to reach the inaudible level is called the time of reverberation:

$$T_R = 0.16V/KS \qquad \text{(Sabine law)} \tag{8.35}$$

where V is in cubic metres and S in square metre for the volume and surface area of the room, respectively, and K is the absorption coefficient of the material of the floor, ceiling, walls, etc. summed over these components.

Beats: When two wave trains of slightly different frequencies travel through the same region, a regular swelling and fading of the sound is heard, a phenomenon called beats.

At a given point let the displacements produced by the two waves be

$$y = A \sin \omega_1 t \tag{8.36}$$

$$y = A \sin \omega_2 t \tag{8.37}$$

By the superposition principle, the resultant displacement is given by

$$y = y_1 + y_2 = [2A \cos 2\pi (f_1 - f_2)t/2] \sin 2\pi (f_1 + f_2)t/2 \tag{8.38}$$

The resulting vibration has a frequency

$$f = (f_1 + f_2)/2 \tag{8.39}$$

and an amplitude given by the expression in the square bracket of (8.38). The beat frequency is given by $f_1 \sim f_2$.

8.2 Problems

8.2.1 Vibrating Strings

8.1 Show that the one-dimensional wave equation is satisfied by the function $y = A\sqrt{(x + vt)}$.

8.2 Show that the equation $y = 2A \sin(n\pi x/L) \cos 2\pi f t$ for a standing wave is a solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F} \frac{\partial^2 y}{\partial t^2}$$

where F is the tension and μ the mass/unit length.

8.3 A cord of length L fixed at both ends is set in vibration by raising its centre a distance h and let go. Obtain an expression for the displacement y at any point x and time t as a series expansion assuming that initially the velocity is zero. Also show that even harmonics are absent.

8.4 Show that the superposition of the waves $y_1 = A \sin(kx - \omega t)$ and $y_2 = 3A \sin(kx + \omega t)$ is a pure standing wave plus a travelling wave in the negative direction along the x-axis. Find the amplitude of **(a)** the standing wave and **(b)** the travelling wave.

8.5 A sinusoidal wave on a string travelling in the $+x$ direction at 8 m/s has a wavelength 2 m. (a) Find its wave number, frequency and angular frequency. (b) If the amplitude is 0.2 m, and the point $x = 0$ on the string is at its equilibrium position ($y = 0$) at time $t = 0$, find the equation for the wave.

8.6 A sinusoidal wave on a string travelling in the $+x$ direction has wave number $3/m$ and angular frequency 20 rad/s. If the amplitude is 0.2 m and the point at $x = 0$ is at its maximum displacement and $t = 0$, find the equation of the wave.

8.7 Show that when a standing wave is formed, each point on the string is undergoing SHM transverse to the string.

8.8 The length of the longest string in a piano is 2.0 m and the wave velocity of the string is 120 m/s. Find the frequencies of the first three harmonics.

8.9 Two strings are tuned to fundamentals of $f_1 = 4800$ Hz and $f_1' = 32$ Hz. Their lengths are 0.05 and 2.0 m, respectively. If the tension in these two strings is the same, find the ratio of the masses per unit length of the two strings.

8.10 The equation of a transverse wave travelling on a rope is given by $y = 5 \sin \pi(0.02x - 4.00t)$, where y and x are expressed in centimetres and t is in seconds. Find the amplitude, frequency, velocity and wavelength of the wave.

8.11 A string vibrates according to the equation $y = 4 \sin \frac{1}{2}\pi x \cos 20\pi t$, where x and y are in centimetres and t is in seconds. **(a)** What are the amplitudes and velocity of the component waves whose superposition can give rise to this vibration? **(b)** What is the distance between the nodes? **(c)** What is the velocity of the particle in the transverse direction at $x = 1.0$ cm and when $t = 9/4$ s?

8.12 A wave of frequency 250 cycles/s has a phase velocity 375 m/s. **(a)** How far apart are two points 60° out of phase? **(b)** What is the phase difference between two displacements at a certain point at time 10^{-3} s apart?

8.13 Two sinusoidal waves having the same frequency and travelling in the same direction are combined. If their amplitudes are 6.0 and 8.0 cm and have a phase difference of $\pi/2$ rad, determine the amplitude of the resultant motion.

8.14 Show that the one-dimensional wave equation is satisfied by the following functions:

(a) $y = A \ln(x + vt)$ and **(b)** $y = A \cos(x + vt)$.

8.15 **(a)** A cord of length L is rigidly attached at both ends and is plucked to a height h at a point 1/3 from one end and let it go. Show that the displacement y at any distance x along the string at time t in the subsequent motion is given by

$$y = \frac{3^{5/2}}{2\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} + \frac{1}{4}\sin\frac{2\pi x}{L}\cos\frac{2\pi vt}{L} - \frac{1}{16}\sin\frac{4\pi x}{L}\cos\frac{4\pi vt}{L}\ldots\right]$$

(b) and that the third, sixth and ninth harmonics are absent.

8.16 Given the amplitude $A = 0.01$ m, frequency $f = 170$ vibrations/s, the wave velocity $v = 340\,\text{m/s}$, write down the equation of the wave in the negative x-direction.

8.17 **(a)** Show that the superposition of the waves $y_1 = A\sin(kx - \omega t)$ and $y_2 = +A\sin(kx + \omega t)$ is a standing wave. **(b)** Where are its nodes and antinodes?

8.18 The wave function for a harmonic wave travelling in the positive x-direction with amplitude A, angular frequency ω and wave number k is $y_1 = A\sin(kx - \omega t)$.
The wave interferes with another harmonic wave travelling in the same direction with the same amplitude, frequency and wave number, but with a phase difference δ. By using the principle of superposition, obtain an expression for the wave function of the resultant wave and show its amplitude is $|2A\cos\frac{1}{2}\delta|$.
If each wave has an amplitude of 6 cm and they differ in phase by $\pi/2$, what is the amplitude of the resultant wave?
For what phase differences would the resultant amplitude be equal to 6 cm?
Describe the effects that would be heard if the two waves were sound waves, but with slightly different frequencies. How could you determine the difference between the frequencies of the two harmonic sound sources? [you may use the result $\sin\theta_1 + \sin\theta_2 = 2\cos\frac{1}{2}(\theta_1 - \theta_2)\sin\frac{1}{2}(\theta_1 + \theta_2)$].

[University of Durham]

8.19 Show that the average rate of energy transmission $\bar{P}$, of a travelling sine of velocity v, angular frequency ω, amplitude A, along a stretched string of mass per unit length, μ, is $\bar{P} = \frac{1}{2}\mu v\omega^2 A^2$.

8.20 A fork and a monochord string of length 100 cm give 4 beats/s. The string is made shorter, without any change of tension, until it is in unison with the fork. If its new length is 99 cm, what is the frequency of the fork?

8.21 $y(x,t) = \frac{0.10}{4+(2x-t)^2}$ represents a moving pulse, where x and y are in metres and t in seconds. Find out the velocity of the pulse (magnitude and direction) and point out whether it is symmetric or not.

[adapted from Hyderabad Central University 1995]

8.22 **(a)** A piano string of length 0.6 m is under a tension of 300 N and vibrates with a fundamental frequency of 660 Hz. What is the mass density of the string?
(b) What are the frequencies of the first two harmonics?
(c) A flute organ pipe (opened at both ends) also plays a note of 660 Hz. What is the length of the pipe? (you may take the speed of sound as $V = 340\,\text{m/s}$).

[University of Manchester 2006]

8.23 **(a)** Sketch the first and second harmonic standing waves on a stretched string of length L. Deduce an expression for the frequencies of the family of standing waves that can be excited on the string.

(b) The wave function of a standing wave on a string that is fixed at both ends is given in SI units by $y(x, t) = (0.024)\sin(62.8x)\cos(471t)$.
Find the speed of the waves on the string, and the distance between nodes for the standing wave.
[hint: You may need to use $\sin\theta_1 + \sin\theta_2 = 2\cos\frac{1}{2}(\theta_2 - \theta_1)\sin\frac{1}{2}(\theta_1 + \theta_2)$].

8.24 A progressive wave travelling along a string has maximum amplitude $A = 0.0821\,\mathrm{m}$, angular frequency $\omega = 100\,\mathrm{rad/s}$ and wave number $k = 22.0\,\mathrm{rad/m}$. If the wave has zero amplitude at $t = 0$ and $x = 0$ for its starting conditions

(i) State the wave function that represents the progressive wave motion for this wave travelling in the negative x-direction.
(ii) State the wave function for this wave travelling in the positive x-direction.
(iii) Find the wavelength (λ), period (T) and the speed (v) of this wave.
(iv) Find its amplitude at a time $t = 2.5\,\mathrm{s}$ at a distance $x = 3.2\,\mathrm{m}$ from its origin, for this wave travelling in the negative x-direction.

[University of Wales 2008]

8.25 The speed of a wave on a string is given by $v = \sqrt{\frac{F}{\mu}}$. Show that the right-hand side of this equation has the units of speed.

8.26 For a sinusoidal wave travelling along a string show that at any time t the slope $\frac{\partial y}{\partial x}$ at any point x is equal to the negative of the instantaneous transverse velocity $\frac{\partial y}{\partial t}$ of the string at x divided by the wave velocity v.

8.27 **(a)** Consider a small segment of a string upon which a wave pulse is travelling.
Using this diagram, or otherwise, show that the wave equation for transverse waves on a stretched string is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

where μ is the mass per unit length and F is the tension.
(b) Show that the wave function representing a wave travelling in the positive x-direction, $y(x - vt)$, is a solution of the wave equation. Obtain an expression for the velocity, v, of the wave (Fig. 8.1).

8.28 Two wires of different densities are joined as in Fig. 8.2. An incident wave $y_1 = A_1\sin(\omega t - k_1 x)$ travelling in the positive x-direction along the wire at the boundary is partly transmitted. (a) Find the reflected and transmitted amplitudes in terms of the incident amplitude. (b) When will the amplitude of the reflected wave be negative?

Fig. 8.1

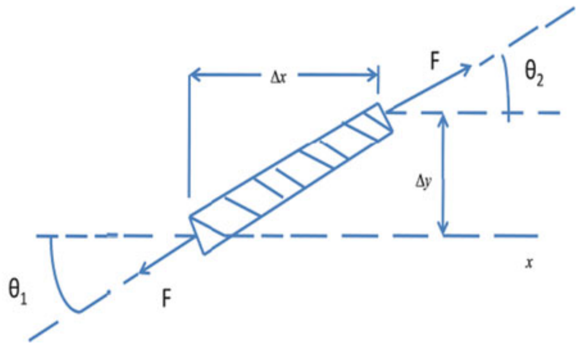
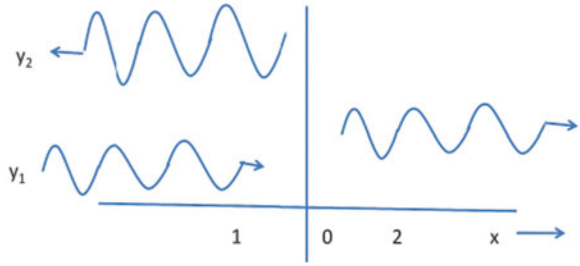


Fig. 8.2



8.29 For the wave shown in Fig. 8.3 find its frequency and wavelength if its speed is 24 m/s. Write the equation for this wave as it travels along the $+x$-axis if its position at $t = 0$ is as shown in Fig. 8.3.

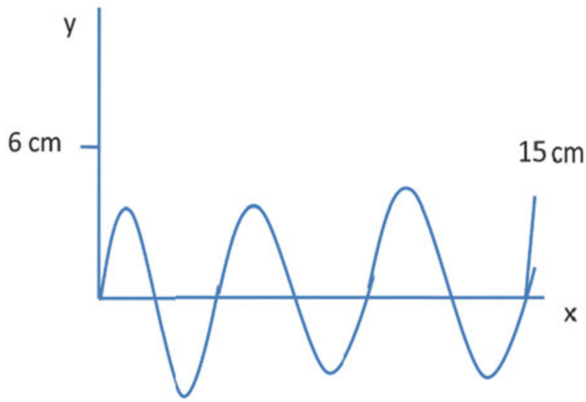


Fig. 8.3

8.30 In prob. (8.29) if the linear density of the string is 0.25 g/m, how much energy is sent down the string per second?

8.31 **(a)** Show that when a string of length L plucked at the centre through height h, the energy in the nth mode is given by $E_n = \frac{16Mh^2v^2}{n^2\pi^2L^2}$, where v is the wave velocity and M is the total mass of the string.

(b) Compare the energies in the first and the third harmonics of a string plucked at the centre.

8.2.2 Waves in Solids

8.32 **(a)** A steel bar of density $7860\,\text{kg/m}^3$ and Young's modulus $2 \times 10^{11}\,\text{N/m}^2$ and of length 0.25 m is rigidly clamped at one end and free to move at the other end. Determine the fundamental frequency of the bar for longitudinal harmonic vibrations.

(b) How do the frequencies compare with **(i)** rod free at both ends; **(ii)** bar clamped at the midpoint; and **(iii)** bar clamped at both the ends.

8.33 A 2 kg mass is hung on a steel wire of $1 \times 10^{-5}\,\text{m}^2$ cross-sectional area and 1.0 m length. **(a)** Calculate the fundamental frequency of vertical oscillations of the mass by considering it to be a simple oscillator and **(b)** calculate the fundamental frequency of vertical oscillations of the mass by regarding it as a system of longitudinally vibrating bar fixed at one end and mass-loaded at the other. Assume $Y = 210^{11}\,\text{N/m}^2$ and $\rho = 7800\,\text{kg/m}^2$ for steel.

8.34 Show that for $kl < 0.2$, the frequency equation derived for the mass loaded system for the bar of length l clamped at one end and loaded at the other reduces to that of a simple harmonic oscillator (you may assume that the frequency condition for this system is, $kl \tan kl = M/m$).

8.2.3 Waves in Liquids

8.35 **(a)** Find the velocity of long waves for a liquid whose depth is $\lambda/4$ and compare it with **(b)** the velocity for a similar wavelength λ in a deep liquid and **(c)** that for canal waves.

8.36 Find the maximum depth of liquid for which the formula $v^2 = gh$ represents the velocity of waves of length λ within 1%. You may assume that the velocity of surface waves is given by $v = \sqrt{\frac{g\tanh(kh)}{k}}$ which is valid for relatively deep waters.

8.37 In an experiment to measure the surface tension of water by the ripple method, the waves were created by a tuning fork of frequency 100 Hz and the wavelength was 3.66 mm. Calculate the surface tension of water.

8.38 Compare the minimum velocities of surface waves at 10°C for mercury and water if the surface tensions are 544 and 74 dyne/cm, respectively, and the specific gravity of mercury is 13.56.

8.39 It is only when a string is perfectly flexible that the phase velocity of a wave on a string is given by $\sqrt{T/\mu}$. The dispersion relations for the real piano wire can be written as

$$\frac{\omega^2}{k^2} = \frac{T}{\mu} + ak^2$$

where α is a small positive quantity which depends on the stiffness of the string. For perfectly flexible string, $\alpha = 0$. Obtain expressions for phase velocity (v_p) and group velocity v_g and show that v_p increases as wavelength decreases.

8.40 The dispersion relation for water waves of very short wavelength in deep water is $\omega^2 = \frac{S}{\rho}k^3$, where S is the surface tension and ρ is the density.

(a) What is the phase velocity of these waves?
(b) What is the group velocity?
(c) Is the group velocity greater or less than the phase velocity?

8.41 The general dispersion relation for water waves can be written as

$$\omega^2 = \left(gk + \frac{S}{\rho}k^3\right)\tanh kh$$

where g is acceleration due to gravity, ρ is the density of water, S is the surface tension and h is the water depth. Use the properties of tanh x function viz. for $x >> 1$, $\tanh x = 1$ and for $x << 1$, $\tanh x = x$.
Show that **(a)** in shallow water the group velocity and the phase velocity are both equal to $\sqrt{gh}$ if the wavelength is long enough to ensure that $Sk^2/v = 4\pi^2 S/\lambda^2\rho << g$. **(b)** Show that for deep water the phase velocity is given by $v_p = \sqrt{\frac{g}{k} + Sk/\rho}$ and find the group velocity.

8.42 For water $\rho = 10^3\,\text{kg/m}^3$ and $S = 0.075\,\text{N/m}$. Calculate v_p and v_g in deep water for small ripples with $\lambda = 1$ cm and for large waves with $\lambda = 1$ m.

8.43 The relation for total energy E and momentum p for a relativistic particle is $E^2 = c^2p^2 + m^2c^4$, where m is the rest mass and c is the velocity of light. Using the relations, $E = \hbar\omega$ and $p = \hbar k$, where ω is the angular frequency and k is the wave number and $\hbar = h/2\pi$, h being Planck's constant. Show that the product of group velocity v_g and the phase velocity v_p, $v_p v_g = c^2$.

8.44 Taking the surface tension of water as 0.075 N/m its density as $1000\,\text{kg/m}^3$, find the wavelength of surface waves on water with a velocity of 0.3 m/s.

Which one of these would be preferable to use in determining the surface tension by means of ripples?

8.45 Waves in deep water travel with phase velocity given by $v_{\mathrm{p}}{}^2 = g/k$, where g is the acceleration due to gravity and k is the wave number, $2\pi/\lambda$. Obtain an expression for the group velocity and show that it is equal to $v_{\mathrm{p}}/2$.
[University of Manchester 2006]

8.46 The dispersion relation for sound waves in air is $\omega = \sqrt{\frac{\gamma RT}{M}}k$. Find the phase velocity and the group velocity.

8.47 The phase velocity v_{p} for deep water waves is given by $v_{\mathrm{p}}{}^2 = (g/k + Sk/\rho)$. Show that the phase velocity is minimum at $\lambda = 2\pi\sqrt{\frac{S}{\rho g}}$.

8.2.4 Sound Waves

8.48 Let both displacement and pressure of a plane wave vary harmonically. Obtain a relation between pressure amplitude and displacement amplitude. Also show that the displacement is 90° out of phase with the pressure wave.

8.49 Assuming $\rho = 1.29\,\mathrm{kg/m^3}$ for the density of air and $v = 331\,\mathrm{m/s}$ for the speed of sound, find the pressure amplitude corresponding to the threshold of hearing intensity of $10^{-12}\,\mathrm{W/m^2}$.

8.50 For ordinary conversation, the intensity level is given as 60 dB. What is the intensity of the wave?

8.51 A small source of sound radiates energy uniformly at a rate of 4 W. Calculate the intensity and the intensity level at a point 25 cm from the source if there is no absorption.

8.52 The maximum pressure variation that the ear can tolerate is about $29\,\mathrm{N/m^2}$. Find the corresponding maximum displacement for a sound wave in air having a frequency of 2000 Hz. Assume the density of air as $1.22\,\mathrm{kg/m^3}$ and the speed of sound as 331 m/s.

8.53 If two sound waves, one in air and the other in water, have equal pressure amplitudes, what is the ratio of the intensities of the waves? Assume that the density of air is $1.293\,\mathrm{kg/m^3}$, and the speed of sound in air and water is 330 and 1450 m/s, respectively.

8.54 The pressure in a progressive sound wave is given by the equation $P = 2.4\sin\pi(x - 330t)$, where x is in metres, t in seconds and P in $\mathrm{N/m^2}$. Find **(a)** the pressure amplitude, **(b)** frequency, **(c)** wavelength and **(d)** speed of the wave.

8.55 A note of frequency 1200 vibrations/s has an intensity of $2.0\,\mu\,\mathrm{W/m^2}$. What is the amplitude of the air vibrations caused by this sound?

8.56 Show that a plane wave having an effective acoustic pressure of a microbar in air has an intensity level of approximately 74 dB. Assume that the density and speed of air is $1.293\,\mathrm{kg/m^3}$ and sound velocity is 330 m/s.

8.57 Calculate the energy density and effective pressure of a plane wave in air of 70 dB intensity level. Assume the velocity of sound in air to be 331 m/s and the air density $1.293\,\mathrm{kg/m^3}$.

8.58 Find the pressure amplitude for an intensity of $1\,\mathrm{W/m^2}$ at the pain threshold. Assume that sound velocity is 331 m/s and gas density is $1.293\,\mathrm{kg/m^3}$.

8.59 Find the theoretical speed of sound in hydrogen at 0°C. For a diatomic gas $\gamma = 1.4$ and for hydrogen $M = 2.016\,\mathrm{g/mol}$; the universal gas constant $R = 8.317\,\mathrm{J/mol/K}$.

8.60 The density of oxygen is 16 times that of hydrogen. For both $\gamma = 1.4$. If the speed of sound is 317 m/s in oxygen at 0°C what is the speed in hydrogen at the same pressure?

8.61 Two sound waves have intensities 0.4 and $10\,\mathrm{W/m^2}$, respectively. How many decibels is one louder than the other?

8.62 If one sound is 6.0 dB higher than another, what is the ratio of their intensities?

8.63 A small source radiates uniformly in all directions at a rate of 0.009 W. If there is no absorption, how far from the source is the sound audible?

8.64 For the faintest sound that can be heard at 1000 Hz the pressure amplitude is about $2 \times 10^5\,\mathrm{N/m^2}$. Find the corresponding displacement amplitude. Assume that the velocity of sound is 331 m/s and the air density is $1.22\,\mathrm{kg/m^3}$.

8.65 Two sound waves of equal pressure amplitudes and frequencies traverse two liquids for which the velocities of propagation are in the ratio 3:2 and the densities of the liquids are in the ratio 3:4. Compare the **(a)** displacement amplitudes, **(b)** intensities and **(c)** energy densities.

8.66 One sound wave travels in air and the other in water, their intensities and frequencies being equal. Calculate the ratio of their **(a)** wavelength, **(b)** pressure amplitudes and **(c)** amplitudes of vibration of particles in air and water. Assume that the density of air is $1.293\,\mathrm{kg/m^3}$, and sound velocity in air and water is 331 and 1450 m/s, respectively.

8.67 Show that the characteristic impedance ρv of a gas is inversely proportional to the square root of its absolute temperature T. What is the characteristic impedance at **(a)** 0°C and **(b)** 80°C?

8.68 A beam of plane waves in water contains 50 W of acoustic power distributed uniformly over a circular cross-section of 50 cm diameter. The frequency of the waves is 25 kc/s. Determine **(a)** the intensity of the beam, **(b)** the sound pressure amplitude, **(c)** the acoustic particle velocity amplitude, **(d)** the acoustic particle displacement amplitude and **(e)** the condensation amplitude. Assume that the velocity of sound in water is 1450 m/s.

8.69 Derive Laplace formula for the sound velocity in a gas.

8.70 An empirical formula giving the velocity of sound in distilled water as a function of temperature at a pressure of one atmosphere in the range $0 - 60°C$ is $v = 1403 + 5t - 0.06t^2 + 0.0003t^3$ where t is the temperature of water in °C and v is in m/s. **(a)** Determine the velocity of sound in distilled water at 20°C and **(b)** find the change of velocity of sound per degree Celsius at this temperature.

8.2.5 Doppler Effect

8.71 A railway engine whistles as it approaches a tunnel, and the sound is reflected back by the wall of the rock at the opening. If the train is proceeding at a speed of 72 km/h and if the effect of the wind be neglected, find the ratio of the relative frequencies of the reflected and direct sounds as heard by the driver of the engine.

[University of Aberystwyth, Wales]

8.72 Two trains move towards each other at a speed of 90 km/h relative to the earth's surface. One gives a 520 Hz signal. Find the frequency heard by the observer on the other train.

8.73 Two trains move away from each other at a speed of 25 m/s relative to the earth's surface. One gives a 520 Hz signal. Find the frequency heard by the observer on the other train (sound velocity = 330 m/s).

8.74 A whistle of frequency 540 Hz rotates in a circle of radius 2 m at an angular speed of 15.0 rad/s. What are the maximum and minimum frequencies heard by a listener, standing at a long distance away at rest from the centre of the circle (sound velocity = 330 m/s).

8.75 In the Kundt's tube experiment, the length of the steel rod which is stroked is 120 cm long and the distance between heaps of cork dust is 8 cm when the rod is caused to vibrate longitudinally in air. If the ends of the tube are sealed and the air replaced by a gas and the experiment repeated, the distance between heaps is observed to be 10 cm. **(a)** What is the velocity of sound in the gas if the velocity in air is 340 m/s. **(b)** What is the velocity of sound in the rod?

8.76 A sound source from a motionless train emits a sinusoidal wave with a source frequency of $f_s = 514$ Hz. Given that the speed of sound in air is 340 m/s and that you are a stationary observer. Find the wavelength of the wave you observe

(i) When the train is at rest
(ii) When the train is moving towards you at 15 m/s

(iii) When the train is moving away from you at 15 m/s>

[University of Aberystwyth, Wales 2007]

8.2.6 Shock Wave

8.77 **(a)** What is a shock wave?

(b) What is the Mach number when a plane travels with a speed twice the speed of the sound?

(c) Calculate the angle of Mach cone in (b).

8.2.7 Reverberation

8.78 Calculate the reverberation time of a room, 10 m wide by 18 m long by 4 m high. The ceiling is acoustic, the walls are plastered, the floor is made of concrete and there are 50 persons in the room. Sound absorption coefficients are acoustic ceiling 0.60, plaster 0.03, concrete 0.02, the absorbing power per person is 0.5.

8.2.8 Echo

8.79 A man standing in front of mountain at a certain distance beats a drum at regular intervals. The drumming rate is gradually increased and he finds the echo is not heard distinctly when the rate becomes 40/min. He then moves nearer to the mountain by 90 m and finds that the echo is again not heard when the drumming rate becomes 60/min. Calculate **(a)** the distance between the mountain and the initial position of the man and the mountain and **(b)** the velocity of sound.

[Indian Institute of Technology 1974]

8.80 A rifle shot is fired in a valley formed between two parallel mountains. The echo from one mountain is heard 2 s after the first one.

(a) What is the width of the valley?

(b) Is it possible to hear the subsequent echoes from the two mountains simultaneously, at the same point? If so, after what time, given sound velocity $= 360\,\text{m/s}$.

[Indian Institute of Technology 1973]

8.2.9 Beat Frequency

8.81 Two whistles are sounded with frequencies of 548 and 552 cycles/s, respectively. A man directly in the line between them walks towards the lower pitched whistle at 1.5 m/s. Find the beat frequency that he hears. Assume the sound velocity of 330 m/s.

8.82 A tuning fork of frequency 300 c/s gives 2 beats/s with another fork of unknown frequency. On loading the unknown fork the beats increase to 5/s, while transferring the load to the fork of known frequency increases the number of beats per second to 9. Calculate the frequency of the unknown fork (unloaded) assuming the load produces the same frequency change in each fork.

[University of Newcastle]

8.2.10 Waves in Pipes

8.83 An open organ pipe sounding its fundamental note is in tune with a fork of frequency 439 cycles/s. How much must the pipe be shortened or lengthened in order that 2 beats/s shall be heard when it sounded with the fork? Assume the speed of sound is 342 m/s.

[University of Durham]

8.84 A light pointer fixed to one prong of a tuning fork touches a vertical plate. The fork is set vibrating and the plate is allowed to fall freely. Eight complete oscillations are counted when the plate falls through 10 cm. What is the frequency of the tuning fork?

[Indian Institute of Technology 1997]

8.85 Air in a tube closed at one end vibrates in resonance with tuning fork whose frequencies are 210 and 350 vibrations/s, when the temperature is 20°C. Explain how this is possible and find the effective length of the tube. Assume that the velocity in air at 0°C is 33, 150 cm/s.

[University of London]

8.86 An open organ pipe is suddenly closed with the result that the second overtone of the closed pipe is found to be higher in frequency by 100 vibrations/s than the first overtone of the original pipe. What is the fundamental frequency of the open pipe?

[University of Bristol]

8.3 Solutions

8.3.1 Vibrating Strings

8.1 The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad (1)$$

Given function is $y = A\sqrt{x + vt}$

$$\frac{\partial y}{\partial x} = \frac{A}{2\sqrt{x + vt}}$$
$$\frac{\partial^2 y}{\partial x^2} = -\frac{A}{4(x + vt)^{3/2}} \tag{2}$$
$$\frac{\partial y}{\partial t} = \frac{Av}{2\sqrt{(x + vt)}}$$
$$\frac{\partial^2 y}{\partial t^2} = -\frac{A}{4}\frac{v^2}{(x + vt)^{3/2}} \tag{3}$$

Equation (1) is satisfied with the use of (2) and (3).

8.2 The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{1}$$
$$y = 2A \sin\left(\frac{n\pi x}{L}\right)\cos(2\pi ft) \quad \text{(standing wave)}$$
$$\frac{\partial y}{\partial x} = \frac{2An\pi}{L}\cos\left(\frac{n\pi x}{L}\right)\cos(2\pi ft)$$
$$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{2An^2\pi^2}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos(2\pi ft) = -\frac{n^2\pi^2 y}{L^2} \tag{2}$$
$$\frac{\partial y}{\partial t} = -4\pi fA \sin\left(\frac{n\pi x}{L}\right)\sin(2\pi ft)$$
$$\frac{\partial^2 y}{\partial t^2} = -8\pi^2 f^2 A \sin\left(\frac{n\pi x}{L}\right)\cos(2\pi ft) = -4\pi^2 f^2 y$$
$$\text{but} \quad \sqrt{\frac{F}{\mu}} = v = f\lambda$$
$$\therefore \quad \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} = -\frac{4\pi^2 f^2 y}{v^2} = -\frac{4\pi^2 y}{\lambda^2} = -\frac{n^2\pi^2 y}{L^2} \tag{3}$$
$$\because \quad L = \frac{n\lambda}{2}$$

Thus

$$\frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

8.3 Let the string AB of length L be plucked at the point C, distant d from the end A and be raised through height h, Fig. 8.4.

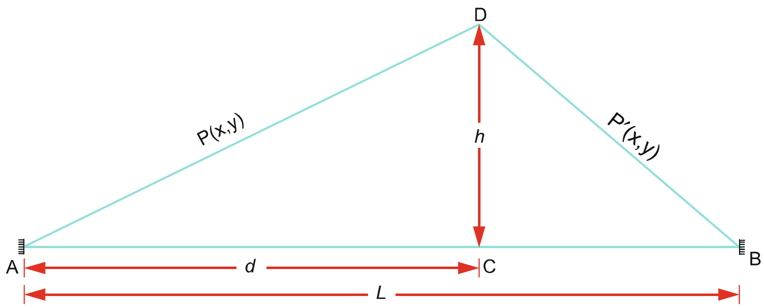


Fig. 8.4

The general form of the displacement at any point x and time t is given by the Fourier expansion

$$y = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) \tag{1}$$

The coefficient a_n is obtained from

$$a_\mathrm{n} = \frac{2}{L}\int_0^L y_0 \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \tag{2}$$

where $y_0 = y(x, 0)$.
We break the integral into two parts, one from 0 to d and the other from d to L. In the interval from 0 to d the equation of the initial configuration of the string for a typical point $p(x, y)$ is

$$\frac{y}{x} = \frac{h}{d} \quad \text{or} \quad y = \frac{hx}{d} \quad \text{for} \quad o < x < d$$

and in the interval d to L, the equation for $P'(x, y)$ is

$$\frac{y}{L-x} = \frac{h}{L-d} \quad \text{or} \quad y = \frac{h(L-x)}{L-d} \text{ for } d < x < L$$

so that by substituting (1) into (2) with $t = 0$

$$a_\mathrm{n} = \frac{2}{L}\left[\int_0^d \frac{hx}{d}\sin\left(\frac{\pi nx}{L}\right)\mathrm{d}x + \int_d^L \frac{h(L-x)}{L-d}\sin\frac{\pi nx}{L}\mathrm{d}x\right] \tag{3}$$

Integrating by parts

$$a_{\mathrm{n}} = \frac{2hL^2}{n^2\pi^2 d(L-d)} \sin\left(\frac{n\pi d}{L}\right) \tag{4}$$

Here $d = \frac{1}{2}L$, so that (4) becomes

$$a_{\mathrm{n}} = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

If n is an even integer then the corresponding a_{n} is zero. If n is an odd integer, then the sine term alternates in sign as $\sin\frac{\pi}{2} = 1$, $\sin\frac{3\pi}{2} = -1$, $\sin\frac{5\pi}{2} = 1\ldots$, so that we may write

$$a_{\mathrm{n}} = \frac{8h}{\pi^2 n^2}(-1)^{(n-1)/2} \tag{5}$$

Using (5) in (1)

$$y = \frac{8h}{\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi v\mathrm{t}}{L} - \frac{1}{9}\sin\frac{3\pi x}{L}\cos\frac{3\pi v\mathrm{t}}{L} + \frac{1}{25}\sin\frac{5\pi x}{L}\cos\frac{5\pi v\mathrm{t}}{L} - \frac{1}{49}\sin\frac{7\pi x}{L}\cos\frac{7\pi v\mathrm{t}}{L} + \cdots\right]$$

Note that the even harmonics are absent. Since the intensity of a wave is proportional to the square of its amplitude, then for the sound emitted by the string, the fundamental would have an intensity of 81 times the third harmonic and 625 times the fifth harmonic, etc.

Formula (4) shows that the nth harmonic will be absent if $\sin\left(\frac{n\pi d}{L}\right) = 0$. $a_{\mathrm{n}} = 0$ if $d = L/n$, $2L/n$, $3L/n$, i.e. nd/L is any integer or whenever there is any node of the nth harmonic situated at D, Fig. 8.4. If the string is divided into n equal parts and is plucked at any dividing point, the nth harmonic will disappear from the resultant vibration. In particular, any force applied at the midpoint of the string cannot produce even harmonics. Further after the application of force at the midpoint of the string, if this point be lightly touched the string ceases to vibrate. This is because odd harmonics cannot be sustained with a node at the midpoint, and the even harmonics are already absent for reasons discussed above.

8.4
$$\begin{aligned} y &= y_1 + y_2 \\ &= A\sin(kx - \omega\mathrm{t}) + 3A\sin(kx + \omega\mathrm{t}) \\ &= [A\sin(kx - \omega\mathrm{t}) + A\sin(kx + \omega\mathrm{t})] + 2A\sin(kx + \omega\mathrm{t}) \\ &= 2A\sin kx\cos\omega\mathrm{t} + 2A\sin(kx + \omega\mathrm{t}) \end{aligned}$$

where we have used the identity

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Thus the resultant wave = standing wave + travelling wave in the negative direction.
The amplitudes are (a) $2A$ (b) $2A$

8.5 **(a)** The wave number $k = \frac{2\pi}{\lambda} = \frac{2\pi}{2} = \pi/\text{m}$

Frequency $f = \frac{v}{\lambda} = \frac{8}{2} = 4\,\text{Hz}$

Angular frequency $\omega = 2\pi f = (2\pi)(4) = 8\pi$ rad/s

(b) $y = A\sin(kx - \omega \text{t}) = A\sin\pi(x - 8\text{t})$

8.6 Let $y = A\sin(kx - \omega \text{t} + \phi)$

At $x = 0$, $t = 0$, the wave has the maximum displacement and $y = A$:

$$A = A\sin(0 - 0 + \phi)$$

$$\text{or} \quad \sin\phi = 1 \rightarrow \phi = \frac{\pi}{2}$$

$$\therefore \quad y = A\sin\left(kx - \omega t + \frac{1}{2}\pi\right) = A\cos(kx - \omega \text{t})$$

$$\therefore \quad y = 0.2\cos(3x - 20t)$$

8.7 $y = 2A\sin kx\cos\omega \text{t}$ (standing wave)

$$\frac{\partial y}{\partial t} = -2A\omega\sin kx\sin\omega \text{t}$$

Acceleration, $a = \frac{\partial^2 y}{\partial t^2} = -\omega^2 2A\sin kx\cos\omega t = -\omega^2 y$.

This is the defining equation for the SHM.

8.8 $f_N = \dfrac{Nv}{2L}$

$$f_1 = \frac{1 \times 120}{2 \times 2} = 30\text{ Hz}$$

$$f_2 = \frac{2 \times 120}{2 \times 2} = 60\text{ Hz}$$

$$f_3 = \frac{3 \times 120}{2 \times 2} = 90\text{ Hz}$$

$$f_4 = \frac{4 \times 120}{2 \times 2} = 120\text{ Hz}$$

8.9 $f_1 = \dfrac{1}{2L_1}\sqrt{\dfrac{F}{\mu_1}}$

$$f_2 = \frac{1}{2L_2}\sqrt{\frac{F}{\mu_2}}$$

$$\therefore \frac{\mu_2}{\mu_1} = \frac{(L_1 f_1)^2}{(L_2 f_2)^2} = \frac{(0.05 \times 4800)^2}{(2.0 \times 32)^2} \simeq 14$$

8.10 $y = 5 \sin \pi(0.02x - 4.00t) = 5 \sin 2\pi(0.01x - 2.00t)$ (given equation) (1)

$y = A \sin 2\pi \left(\dfrac{x}{\lambda} - ft\right)$ (standard equation) (2)

Comparing (1) and (2)

$$A = 5\,\text{cm},\ f = 2\,\text{Hz}\ \frac{1}{\lambda} = 0.01 \quad \text{or} \quad \lambda = 100\,\text{cm}$$

$$v = f\lambda = 2 \times 100 = 200\,\text{cm/s}$$

8.11 $y = 4 \sin \dfrac{1}{2}\pi x \cos 20\pi t$ (standing wave) (1)

$y = 2A \sin kx \cos \omega t$ (standard equation) (2)

Comparing (1) and (2)

(a) $2A = 4$ or $A = 2\,\text{cm},\ k = \dfrac{\pi}{2}, \quad \omega = 20\pi$

$$v = \frac{\omega}{k} = \frac{20\pi}{\pi/2} = 40\,\text{cm/s}$$

(b) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{\pi/2} = 4\,\text{cm}$

Distance between nodes $= \dfrac{\lambda}{2} = \dfrac{4}{2} = 2\,\text{cm}$

(c) $\dfrac{\partial y}{\partial t} = -(4)(20\pi) \sin \dfrac{1}{2}\pi x \sin 20\pi \text{t}$

$$\frac{\partial y}{\partial t}\Big|_{x=1.0,\ t=9/4} = -80\pi \sin \frac{\pi}{2} \sin 45\pi = 0$$

8.12 The wave is of the form

$$y = A \sin(kx - \omega t + \phi)$$

(a) $\omega = 2\pi f = (2\pi)(250) = 500\pi$ rad/s

$$k = \frac{\omega}{v} = \frac{500\pi}{375} = \frac{4\pi}{3}\text{m}^{-1}$$

$$\phi = 60^\circ = \frac{\pi}{3}\text{ rad}$$
$$x = \frac{\phi}{k} = \frac{\pi/3}{4\pi/3} = 0.25\text{ m}$$

(b) $\phi = \omega t = (500\pi)(10^{-3}) = \frac{\pi}{2}\text{ rad} = 90^\circ$

8.13 $y_1 = A_1 \sin(kx - \omega t)$

$$y_2 = A_2 \sin\left(kx - \omega t + \frac{\pi}{2}\right) = A_2 \cos(kx - \omega t)$$
$$y = y_1 + y_2$$
$$= A_1 \sin(kx - \omega t) + A_2 \cos(kx - \omega t)$$
$$= \sqrt{A_1^2 + A_2^2}\left[\frac{A_1}{\sqrt{A_1^2 + A_2^2}} \sin(kx - \omega t) + \frac{A_2}{\sqrt{A_1^2 + A_2^2}} \cos(kx - \omega t)\right]$$

Put $\frac{A_1}{\sqrt{A_1^2 + A_2^2}} = \cos\alpha$. Then $\frac{A_2}{\sqrt{A_1^2 + A_2^2}} = \sin\alpha$

$$\therefore \quad y = \sqrt{A_1^2 + A_2^2}[\sin(kx - \omega t)\cos\alpha + \cos(kx - \omega t)\sin\alpha]$$
$$= \sqrt{A_1^2 + A_2^2}\sin(kx - \omega t + \alpha)$$

which has the amplitude $A = \sqrt{A_1^2 + A_2^2} = \sqrt{6^2 + 8^2} = 10\,\text{cm}$.

Graphical Method

This method was outlined in prob. (6.50). The waves are represented as vectors, the magnitudes being proportional to the amplitudes, the orientation according to the phase difference. Here the vectors OA and AB are laid in the head-to-tail fashion, Fig. 8.5. The amplitude of the resultant wave is given by OB which is found to be 10 cm from the right angle triangle OAB

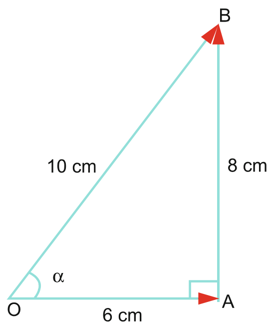


Fig. 8.5

8.14 **(a)** $y = A\ln(x + vt)$

$$\frac{\partial y}{\partial x} = \frac{A}{x + vt}, \quad \frac{\partial^2 y}{\partial x^2} = -\frac{A}{(x + vt)^2}$$
$$\frac{\partial y}{\partial t} = \frac{Av}{x + vt}, \quad \frac{\partial^2 y}{\partial t^2} = -\frac{Av^2}{(x + vt)^2}$$
$$\therefore \quad \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = -\frac{A}{(x + vt)^2} = \frac{\partial^2 y}{\partial x^2}$$

Thus the wave equation is satisfied.

(b) $y = A\cos(x + vt)$

$$\frac{\partial y}{\partial x} = -A\sin(x + vt) \qquad \frac{\partial^2 y}{\partial x^2} = -A\cos(x + vt)$$
$$\frac{\partial y}{\partial t} = -vA\sin(x + vt)$$
$$\frac{\partial^2 y}{\partial t^2} = -v^2 A\cos(x + vt)$$
$$\therefore \quad \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = -A\cos(x + vt) = \frac{\partial^2 y}{\partial x^2}$$

Thus the wave equation is satisfied.

8.15 **(a)** By prob. (8.3)

$$y = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) \tag{1}$$
$$a_n = \frac{2hL^2}{n^2\pi^2 d(L - d)}\sin\left(\frac{n\pi d}{L}\right) \tag{2}$$

Here $d = \dfrac{L}{3}$ and (2) becomes

$$a_n = \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3} \tag{3}$$

Inserting (3) in (1)

$$\therefore \quad y = \frac{3^{5/2}h}{2\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi vt}{L}\right) + \frac{1}{4}\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi vt}{L}\right) - \frac{1}{16}\sin\left(\frac{4\pi x}{L}\right)\cos\left(\frac{4vt}{L}\right)\ldots\right] \tag{4}$$

(b) For $n = 3$, 6 or 9, the sine term in (3) becomes zero. Therefore, the third, sixth and ninth harmonics will be absent.

8.16 General equation for a progressive wave in the negative x-direction is

$$y = A\sin(kx + \omega t)$$
$$\omega = 2\pi f = 2\pi \times 170 = 340\pi \text{ rad/s}$$
$$k = \frac{\omega}{v} = \frac{340\pi}{340} = \pi/\text{m}$$
$$\therefore \quad y = 0.01 \sin\pi(x + 340t)$$

8.17 **(a)** $y_1 = A\sin(kx - \omega t)$

$y_2 = A\sin(kx + \omega t)$

$y = y_1 + y_2 = 2A\sin kx \cos\omega t$

where we have used the identity stated in prob. (8.4).

(b) The nodes are formed when $kx = n\pi$ or $\frac{2\pi}{\lambda}x = n\pi$

or $\quad x = \frac{n\lambda}{2}$

$x = 0,\ \frac{\lambda}{2}, \lambda, \ldots$

The antinodes are formed when $kx = \frac{n\pi}{2}$ or $x = \frac{n\lambda}{4}$

$x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4} \ldots$

8.18 $y_1 = A\sin(kx - \omega t)$

$$y_2 = A\sin(kx - \omega t + \delta)$$
$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx - \omega t + \delta)]$$
$$= 2A\cos\frac{1}{2}\delta \sin\left(kx - \omega t + \frac{\delta}{2}\right)$$

Thus the amplitude of the resultant wave is $2A\cos\frac{1}{2}\delta$.

For $A = 6$ cm and $\delta = \frac{\pi}{2}$, the amplitude of the resultant wave will be $2 \times 6\cos\frac{\pi}{4}$ or $6\sqrt{2}$ cm.

For $2A\cos\frac{1}{2}\delta = 6$

$$\cos\frac{1}{2}\delta = \frac{6}{2A} = \frac{6}{2\times 6} = \frac{1}{2} = \cos\frac{\pi}{3}$$
$$\therefore \quad \frac{1}{2}\delta = \frac{\pi}{3} \quad \text{or} \quad \delta = \frac{2\pi}{3}$$

If two sound waves with slightly different frequencies are produced then beats are heard. These consist of regular swelling and fading of the sound. In one set of waves compressions and rarefactions will be spaced further apart, in another they will be close enough. At some instant, two compressions arrive together at the ear of the listener and the sound is loud. At a later time, the compression of one wave arrives with the rarefaction of the other and the sound will be faint. Beats are thus caused due to interference of sound waves of neighbouring frequencies in time. The beat frequency is equal to the difference $f_1 \sim f_2$ for the two component waves. Beats between two tones can be detected by the ear up to a frequency of about 7/s.

8.19 Consider an infinitesimal element of length $\mathrm{d}x$ of the string of linear mass density μ. The mass element $\mu \mathrm{d}x$ will execute SHM with amplitude A. The maximum kinetic energy will be $\frac{1}{2}(\mu \mathrm{d}x)\omega^2 A^2$.
Energy transmitted across the string per second, i.e. power

$$P = \frac{1}{2}\left(\mu \frac{\mathrm{d}x}{\mathrm{d}t}\right)\omega^2 A^2 = \frac{1}{2}\mu v \omega^2 A^2$$

8.20 Let the fork of frequency f be in unison with 99 cm of the string. Then

$$f = \frac{1}{2 \times 99}\sqrt{\frac{F}{\mu}} \tag{1}$$

When the length of the string was 100 cm the frequency must have been less by 4 beats. Thus

$$f - 4 = \frac{1}{2 \times 100}\sqrt{\frac{F}{\mu}} \tag{2}$$

Dividing (1) by (2) and solving

$$\frac{f}{f-4} = \frac{100}{99}$$

We get $f = 400/\mathrm{s}$.

8.21
$$y(x,t) = \frac{0.10}{(2x-t)^2+4}$$

$$\therefore \quad y(0,0) = \frac{0.10}{4} = 0.025$$

$$\text{Let} \quad y(x,t) = 0.025 = \frac{0.10}{4+(2x-t)^2}$$

Solving we find

$$v = \frac{x}{t} = 0.5\,\text{m/s along the} + x - \text{direction.}$$

$$\text{Now,}\ \ y(-x, t) = \frac{0.10}{4 + (2x + t)^2} \neq y(x, t)$$

Therefore, the pulse is not symmetric.

8.22 **(a)** $f = \frac{N}{2l}\sqrt{\frac{F}{\mu}} \quad (N = 1)$

$$\mu = \frac{F}{4f^2L^2} = \frac{300}{(4)(660)^2(0.6)^2} = 4.78 \times 10^{-4}\ \text{kg/m}$$

(b) The frequencies of the first two harmonics are $f_2 = 2f = 1320\,\text{Hz}$ and $f_3 = 3f = 1980\,\text{Hz}$.

(c) For open pipe length is

$$L = \frac{\lambda}{2} = \frac{v}{2f} = \frac{340}{2 \times 660} = 0.2576\,\text{m}$$

8.23 **(a)** First harmonic – second harmonic (Fig. 8.6)

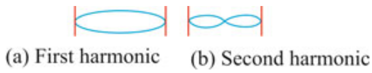


Fig. 8.6

$$v = \sqrt{\frac{F}{\mu}}, \quad \lambda = \frac{2L}{N}$$

$$f_N = \frac{v}{\lambda_N}$$

$$f_N = \frac{N}{2L}\sqrt{\frac{F}{\mu}}, \quad N = 1, 2, 3, \ldots$$

(b) The standard equation for the standing wave is

$$y(x, t) = 2A \sin kx \cos \omega t \tag{1}$$

Given equation is

$$y(x, t) = 0.024 \sin(62.8x) \cos(471t) \tag{2}$$

Comparison shows that

$$k = 62.8 \quad \text{and} \quad \omega = 471$$

$$\text{Wave velocity } v = \frac{\omega}{k} = \frac{471}{62.8} = 7.5 \text{ m/s}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{62.8} = 0.1\,\text{m}$$

$$\text{Distance between nodes} = \frac{\lambda}{2} = \frac{0.1}{2} = 0.05\,\text{m}$$

8.24 $y = A\sin(kx + \omega t)$

(i) $y = 8.2 \times 10^{-2}\sin(22x + 100t)$ (negative x-direction)
(ii) $y = 8.2 \times 10^{-2}\sin(100t - 22x)$ (positive x-direction)
(iii) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{22} = 0.2856\,\text{m}$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{100} = 0.0628\,\text{m}$$
$$v = \frac{\omega}{k} = \frac{100}{22} = 4.545\,\text{m/s}$$

(iv) $y = 8.2 \times 10^{-2} \times \sin(22 \times 3.2 + 100 \times 2.5)$
$= 8.2 \times 10^{-2} \times \sin(51 \times 2\pi) = 0$

8.25 $\left[\frac{F}{\mu}\right]^{1/2} = \left[\frac{MLT^{-2}}{ML^{-1}}\right]^{1/2} = \left[LT^{-1}\right] = [v]$

8.26 Let the travelling wave be represented by

$$y = A\sin(kx - \omega t)$$

$$\text{Then } \frac{\partial y}{\partial x} = kA\cos(kx - \omega t) \tag{1}$$

$$\frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$$

$$= -vkA\cos(kx - \omega t) = -v\frac{\partial y}{\partial x} \tag{2}$$

Combining (1) and (2), $\frac{\partial y}{\partial x} = -\frac{\partial y}{\partial t}/v$.

8.27 **(a)** Let a long string of linear density μ be stretched by a force F. Assume that the damping is negligible. Take the x-axis in the direction of the undisplaced string and y-axis in the direction perpendicular to it. If θ is the angle between the tangent to the string and the x-axis, the tension in the horizontal direction (x-axis) would be $T\cos\theta$ and in the vertical direction (y-axis) it would be $T\sin\theta$. Assuming that θ is very small, $\cos\theta \simeq 1$ and consequently the x-component of the tension remains constant. We are therefore concerned only with the y-component of the tension.

Referring to Fig. 8.1 note that the forces across Δx the element of length of the string make angles θ_1 and θ_2 with the x-axis. Let $\theta_2 = \theta$ and $\theta_1 = \theta + \mathrm{d}\theta$. To find the equation of motion of this element subject to these forces, the difference in tension acting across Δx in the y-direction is

$$\begin{aligned}
\mathrm{d}F_y &= F\{\sin(\theta + \mathrm{d}\theta) - \sin\theta\} \\
&= F\{\sin\theta\cos(\mathrm{d}\theta) + \cos\theta\sin(\mathrm{d}\theta) - \sin\theta\}
\end{aligned}$$

$$\text{but } \cos(\mathrm{d}\theta) \simeq 1 \quad \text{and} \quad \sin(\mathrm{d}\theta) \simeq \mathrm{d}\theta, \text{ since } \mathrm{d}\theta \text{ is small :}$$

$$\therefore \quad \mathrm{d}F_y = F\cos\theta\,\mathrm{d}\theta = F\mathrm{d}(\sin\theta)$$

In the small angle approximation

$$\sin\theta \simeq \tan\theta = \frac{\partial y}{\partial x}$$

this last quantity being the gradient of the curve

$$\therefore \quad \mathrm{d}F_y = F\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathrm{d}x = F\left(\frac{\partial^2 y}{\partial x^2}\right)\mathrm{d}x \tag{1}$$

The mass of the element Δx is $\mu\,\mathrm{d}x$, and its acceleration in the y-direction is $\mathrm{d}^2y/\mathrm{d}t^2$. Hence by Newton's second law of motion

$$\mu\mathrm{d}x\frac{\partial^2 y}{\partial t^2} = F\left(\frac{\partial^2 y}{\partial x^2}\right)\mathrm{d}x$$

$$\text{or} \quad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{2}$$

(b) Let $y(x - vt)$ be a solution of (2)

$$\frac{\partial y}{\partial \mathrm{t}}(x - vt) = y'(x - vt)\frac{\partial}{\partial \mathrm{t}}(x - vt) = -vy'(x - vt)$$

where y' is another function of $(x - vt)$ defined by $y'(x - vt) = \dfrac{\mathrm{d}y(x - vt)}{\mathrm{d}(x - vt)}$.

The second derivative with respect to time gives

$$\frac{\partial^2 y(x - vt)}{\partial t^2} = v^2 y''(x - vt) \tag{3}$$

where $y''(x - vt)$ is yet another function of $(x - vt)$ defined by

$$y''(x - vt) = \frac{\mathrm{d}y'(x - vt)}{\mathrm{d}(x - vt)} = \frac{\mathrm{d}^2 y'(x - vt)}{\mathrm{d}(x - vt)^2}$$

proceeding along similar lines, differentiation of the function $y(x - vt)$ with respect to x yields

$$\frac{\partial y(x-vt)}{\partial x} = y'(x-vt)\frac{\partial}{\partial x}(x-vt) = y''(x-vt)$$
$$\frac{\partial^2 y(x-vt)}{\partial x^2} = f''(x-vt) \tag{4}$$

where $y(x - vt)$ and $y''(x - vt)$ are the same functions of $(x - vt)$ as in (3). Substitution of (3) and (4) into (2) shows that $y(x - vt)$ is a solution, provided we set

$$v^2 = \frac{F}{\mu}$$

8.28 **(a)** The incident wave has the form

$$y_1 = A_1 \sin(\omega t - k_1 x) \tag{1}$$

The reflected wave has the form

$$y_2 = A_2 \sin(\omega t + k_1 x) \tag{2}$$

The transmitted wave has the form

$$y_3 = A_3 \sin(\omega t - k_2 x) \tag{3}$$

The boundary conditions at the boundary ($x = 0$) are that the displacement and its first derivative be single valued:

$$y_1|_{x=0} + y_2|_{x=0} = y_3|_{x=0}$$
$$A_1 + A_2 = A_3 \tag{4}$$
$$\frac{\partial y_1}{\partial x}\Big|_{x=0} + \frac{\partial y_2}{\partial x}\Big|_{x=0} = \frac{\partial y_3}{\partial x}\Big|_{x=0}$$
$$-k_1 A_1 + k_1 A_2 = -k_2 A_3 \tag{5}$$

Solving (4) and (5)

$$A_2 = \frac{(k_1 - k_2)A_1}{k_1 + k_2}; \quad A_3 = \frac{2k_1 A_1}{k_1 + k_2} \tag{6}$$

(b) A_2 is negative when $k_2 > k_1$ or $\frac{\omega}{v_2} > \frac{\omega}{v_1}$ or $\mu_2 > \mu_1$, i.e. wire 2 has greater linear density than wire 1.

8.29 $\frac{5}{2}\lambda = 15\,\text{cm}$

$\therefore \quad \lambda = 6\,\text{cm}$

$f = \frac{v}{\lambda} = \frac{2400}{6} = 400\,\text{Hz}$

$\omega = 2\pi f = 2512\,\text{rad/s}$

$k = \frac{2\pi}{\lambda} = 1.047/\text{cm}$

$A = 6\,\text{cm}$

$y = A\sin(\omega t - kx) = 6\sin(2512t - 1.047x)\,\text{cm}$

8.30 $P = \frac{1}{2}\omega^2 A^2 \mu v = 2\pi^2 f^2 A^2 \mu v$

$= 2\pi^2 \times (400)^2(0.06)^2(2.5 \times 10^{-4})(24) = 68.2\,\text{W}$

8.31 **(a)** The amplitude of any point of the plucked string at time t may be written as

$$y = \sum_{n=1}^{\infty} a_n \cos\omega_n t\ \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\omega_n t\ \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

The kinetic energy of vibration of an element of length of string dx in the nth mode is given by

$$\mathrm{d}K_n = \frac{1}{2}(\mu \mathrm{d}x)(\dot{y})^2$$
$$= \frac{1}{2}\mu\omega_n^2(-a_n \sin\omega_n t + b_n \cos\omega_n t)^2 \sin^2(k_n x)\mathrm{d}x \tag{2}$$

where we have used the value of velocity $\dot{y}$ by differentiating (1) for the nth mode with respect to t.
The potential energy of an element of string of length dx is

$$\mathrm{d}U_n = \frac{1}{2}ky^2\mathrm{d}x$$
$$= \frac{1}{2}\mu\omega_n^2(a_n \cos\omega_n t + b_n \sin\omega_n t)^2 \sin^2 k_n x\,\mathrm{d}x \tag{3}$$

where we have used (1).
Adding (2) and (3), the total energy

$$\mathrm{d}E_n = \mathrm{d}K_n + \mathrm{d}U_n = \frac{1}{2}\mu\omega_n^2(a_n^2 + b_n^2)\sin^2 k_n x\,\mathrm{d}x \tag{4}$$

The total energy of the entire string is obtained by integrating from 0 to L

$$E_n = \int \mathrm{d}E_n = \frac{1}{2}\mu\omega_n^2(a_n^2 + b_n^2)\int_0^L \sin^2(k_n x)\mathrm{d}x$$

$$\text{Now} \quad \int_0^L \sin^2(k_n x)\mathrm{d}x = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{L}{2}$$

$$\therefore \quad E_n = \frac{1}{4}\mu L\omega_n^2(a_n^2 + b_n^2) = \frac{1}{4}M\omega_n^2(a_n^2 + b_n^2)$$

where M is the total mass of the string.

(b) For the string plucked at the centre $a_n = \dfrac{8h}{n^2\pi^2}$ (see prob. 8.3). Further $\omega_n = k_n v = \dfrac{n\pi v}{L}$ and $b_n = 0$. Thus the energy of vibration

$$E_n = \frac{M}{4}\left[\frac{n\pi v}{L}\right]^2\left[\frac{8h}{n^2\pi^2}\right]^2 = \frac{16Mh^2v^2}{n^2\pi^2L^2}$$

$$\therefore \quad \frac{E_1}{E_3} = \frac{9}{1}$$

8.3.2 *Waves in Solids*

8.32 **(a)** For the rod clamped at one end and free at the other (fixed–free)

$$f_n = \frac{n}{4L}\sqrt{\frac{\gamma}{\rho}} \quad (n = 1, 3, 5, \ldots)$$

$$f_1 = \frac{1}{4L}\sqrt{\frac{\gamma}{\rho}} = \frac{1}{4 \times 0.25}\sqrt{\frac{2 \times 10^{11}}{7860}} = 5044\,\mathrm{Hz}$$

(b) **(i)** For the rod free at both ends (free–free)

$$f_n = \frac{n}{2L}\sqrt{\frac{Y}{\rho}} \quad (n = 1, 2, 3, \ldots)$$

(ii) For the rod clamped at the midpoint

$$f_n = \frac{n}{2L}\sqrt{\frac{Y}{\rho}} \quad (n = 1, 3, 5, \ldots)$$

(iii) For the bar clamped at both ends (fixed–fixed)

$$f_n = \frac{n}{2L}\sqrt{\frac{Y}{\rho}} \quad (n = 1, 2, 3, \ldots)$$

For the case (a) (bar clamped at one end only) the frequency of the fundamental is half that of a similar free–free (case (b) (i)) or fixed–fixed (case (b) (iii)) bar and only the odd-numbered harmonic overtones are present. This is to be expected since the effect of clamping a free–free bar at its centre is to suppress all its even harmonics.

For case (b) (ii) rod clamped at the midpoint only odd partials are present similar to case (a) (fixed–free) and differs from case (b) (i) (free–free) where all the partials are present. However the fundamental has the same frequency in cases (b) (ii) and (b) (i).

8.33 **(a)** $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Now $k = \frac{F}{x}$ and $Y = \frac{F/A}{x/L} = \frac{FL}{xA} = \frac{kL}{A}$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\frac{AY}{mL}} = \frac{1}{2\pi}\sqrt{\frac{1 \times 10^{-5} \times 2 \times 10^{11}}{2 \times 1.0}} = 159\,\text{c/s}$$

(b) For the given system

$$kL\tan(kL) = \frac{M}{m}$$

Mass of the bar, $M = \rho AL = 7800 \times 1 \times 10^{-5} \times 1.0 = 0.078\,\text{kg}$. The frequency condition becomes

$$kL\tan(kL) = \frac{0.078}{2.0} = 0.039$$

The solution to the above equation is

$$kL = 0.196$$

$$\therefore \quad \frac{\omega L}{v} = 0.196$$

$$\text{or} \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{0.196}{L}\sqrt{\frac{Y}{\rho}} = \frac{0.196}{2\pi \times 1.0}\sqrt{\frac{2 \times 10^{11}}{7800}} = 158\,\text{c/s}$$

Observe that the results of (b) are nearly the same as those for (a), showing thereby for small values of kL, the mass loaded system approximates that of a simple harmonic oscillator with the mass fixed at the end.

8.34 The frequency condition for this system is

$$kL\tan(kL) = \frac{M}{m}$$

Expanding tan (kL) by series

$$kL\left[kL + \frac{(kL)^3}{3} + 2\frac{(kL)^5}{15} + \cdots\right] = \frac{M}{m}$$

If $kL < 0.2$, we may retain only the first term within the brackets:

$$k^2L^2 = \frac{M}{m}$$

$$\frac{\omega^2 L^2}{v^2} = \frac{M}{m}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{v}{L}\sqrt{\frac{M}{m}} = \frac{1}{2\pi L}\sqrt{\frac{Y}{\rho}\frac{M}{m}}$$

But $Y = \dfrac{kL}{A}$ and $M = AL\rho$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

8.3.3 Waves in Liquids

8.35 **(a)** $v^2 = \dfrac{g}{k}\tanh(kh) = \dfrac{g\lambda}{2\pi}\tanh\left(\dfrac{2\pi}{\lambda}\dfrac{\lambda}{4}\right) = \dfrac{g\lambda}{2\pi}\tan h\left(\dfrac{\pi}{2}\right)$

$$= \frac{9.8}{2\pi} \times 0.917\,\lambda$$

$$v = 1.2\sqrt{\lambda}\ \text{m/s}$$

(b) $v = \sqrt{\dfrac{g}{k}} = \sqrt{\dfrac{g\lambda}{2\pi}} = 1.25\sqrt{\lambda}\ \text{m/s}$

(c) $v = \sqrt{gh} = \sqrt{\dfrac{g\lambda}{4}} = \sqrt{\dfrac{9.8\lambda}{4}} = 1.56\sqrt{\lambda}\ \text{m/s}$

8.36 The fractional error introduced by the use of the formula $v = \sqrt{gh}$ is

$$\frac{\sqrt{gh} - \sqrt{\dfrac{g}{k}\tanh(kh)}}{\sqrt{\dfrac{g}{k}\tanh(kh)}} = 0.01$$

Put $kh = x$, then $\dfrac{\tanh x}{x} = 0.96$

This gives the solution $x = 0.25$ or $h = \dfrac{x}{k} = \dfrac{0.25\lambda}{2\pi} = 0.04\lambda$.

8.37 Surface conditions are modified by the surface tension S. For the capillary waves

$$v^2 = \left(\frac{2\pi S}{\rho\lambda} + \frac{g\lambda}{2\pi}\right)\tanh\left(\frac{2\pi h}{\lambda}\right) \tag{1}$$

If $h >> \lambda$, $\tanh\left(\dfrac{2\pi\, h}{\lambda}\right) \to 1$, and

$$v^2 = \frac{2\pi S}{\rho\lambda} + \frac{g\lambda}{2\pi} \tag{2}$$

Substituting $\lambda = 0.366\,\text{cm}$, $\rho = 1.0\,\text{g/cm}^3$, $g = 980\,\text{cm/s}^2$ and $v = f\lambda = 100 \times 0.366 = 36.6\,\text{cm/s}$ in (2) we find $S = 74.7$ dynes/cm.

8.38 For capillary waves when $h >> \lambda$

$$v^2 = \frac{2\pi S}{\rho\lambda} + \frac{g\lambda}{2\pi} \tag{1}$$

The minimum value of the wavelength λ_m can be found out by minimizing (1):

$$\frac{\partial(v^2)}{\partial\lambda} = -\frac{2\pi \mathrm{S}}{\rho\lambda_m^2} + \frac{g}{2\pi} = 0$$

$$\lambda_m = 2\pi\sqrt{\frac{S}{g\rho}} \tag{2}$$

Ignoring the second term in the right-hand side of (1) and using (2)

$$v = \left(\frac{gs}{\rho}\right)^{1/4}$$

For mercury and water

$$v_1 : v_2 = \left(\frac{S_1}{\rho_1}\right)^{1/4} : \left(\frac{S_2}{\rho_2}\right)^{1/4}$$

$$= \left(\frac{544}{13.56}\right)^{1/4} : \left(\frac{74}{1}\right)^{1/4} = 0.858 : 1$$

8.39 $$\frac{\omega^2}{k^2} = \frac{F}{\mu} + \alpha k^2 \tag{1}$$

The phase velocity $v_\mathrm{p} = \dfrac{\omega}{k} = \sqrt{\dfrac{F}{\mu} + \alpha k^2} = \sqrt{\dfrac{F}{\mu}}\left[1 + \dfrac{\alpha \mu k^2}{F}\right]^{1/2}$

$$= \sqrt{\frac{F}{\mu}}\left[1 + \alpha\frac{k^2\mu}{2F} + \cdots\right] \text{ (for small } \alpha) \tag{2}$$

Since $k = 2\pi/\lambda$, v_p increases as λ decreases.
The group velocity is given by

$$v_\mathrm{g} = v_\mathrm{p} + \frac{k\,\mathrm{d}v_\mathrm{p}}{\mathrm{d}k} = v_\mathrm{p} + \alpha k^2\sqrt{\frac{\mu}{F}}$$

8.40 **(a)** $\omega = \dfrac{S}{\rho}k^{3/2}$

$$v_\mathrm{p} = \frac{\omega}{k} = \frac{S}{\rho}\sqrt{k}$$

(b) $v_\mathrm{g} = v_\mathrm{p} + \dfrac{k\,\mathrm{d}v_\mathrm{p}}{\mathrm{d}k}$

$$= \frac{S}{\rho}\sqrt{k} + \frac{kS}{2\rho\sqrt{k}} = \frac{3}{2}\frac{S}{\rho}\sqrt{k}$$

(c) From (a) and (b) $v_\mathrm{g} > v_\mathrm{p}$

8.41 **(a)** $$\omega^2 = \left(gk + \frac{S}{\rho}k^3\right)\tanh(kh) \tag{1}$$

If $kh << 1$, then $\tanh(kh) = kh$ and (1) becomes

$$\omega^2 = \left(gk + \frac{S}{\rho}k^3\right)kh \tag{2}$$

If the second term in the brackets is smaller than the first one $\left(\dfrac{S}{\rho}k^2 << g\right)$

$$\omega^2 = ghk^2$$

$$\therefore \quad \omega = k\sqrt{gh} \tag{3}$$

$$v_\mathrm{p} = \frac{\omega}{k} = \sqrt{gh} \tag{4}$$

$$v_g = \frac{d\omega}{dk} = \sqrt{gh} \tag{5}$$

$$\therefore \quad v_g = v_p$$

(b) $kh >> 1$, $\tanh(kh) = 1$ and (1) becomes

$$\omega^2 = gk + \frac{S}{\rho}k^3$$

$$\frac{\omega^2}{k^2} = \frac{g}{k} + \frac{S}{\rho}k$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \frac{Sk}{\rho}} \tag{6}$$

$$v_g = v_p + k\frac{dv_p}{dk}$$

$$= v_p + \frac{k}{2}\left(\frac{S}{\rho} - \frac{g}{k^2}\right)\left(\frac{g}{k} + \frac{Sk}{\rho}\right)^{-1/2}$$

Using (6)

$$\frac{v_g}{v_p} = \frac{\frac{g}{2k} + \frac{3}{2}\frac{ks}{\rho}}{\frac{g}{k} + \frac{ks}{\rho}} \tag{7}$$

For short wavelengths k is larger, the first term in both the numerator and denominator will be smaller and $v_g = \frac{3}{2}v_p$, while for long wavelengths, k is smaller and $v_g = \frac{1}{2}v_p$.

8.42 With reference to prob. (8.41) for small ripples, λ is small and k is large so that the second term in (6) dominates over the first term in the radical.

$$v_p = \sqrt{\frac{Sk}{\rho}} = \sqrt{\frac{2\pi\ S}{\lambda\rho}} = \sqrt{\frac{2\pi \times 0.075}{0.01 \times 1000}} = 0.217\,\text{m/s}$$

$$v_g = \frac{3}{2}v_p = 0.325\,\text{m/s}$$

For large waves, first term in (6) is important:

$$v_{\mathrm{p}} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8 \times 1.0}{2\pi}} = 1.25\,\mathrm{m/s}$$

$$v_{\mathrm{g}} = \frac{1}{2}v_{\mathrm{p}} = 0.625\,\mathrm{m/s}$$

8.43 $E^2 = c^2 p^2 + m^2 c^4$

$$\hbar^2\omega^2 = c^2\hbar^2 k^2 + m^2 c^4$$

$$\therefore \quad \omega = \sqrt{c^2 k^2 + \frac{m^2 c^4}{\hbar^2}}$$

$$v_{\mathrm{p}} = \frac{\omega}{k} = \sqrt{c^2 + \frac{m^2 c^4}{\hbar^2 k^2}}$$

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c^2 k}{\sqrt{c^2 k^2 + \frac{m^2 c^4}{\hbar^2}}} = \frac{c^2}{\sqrt{c^2 + \frac{m^2 c^4}{\hbar^2 k^2}}}$$

$$\therefore \quad v_{\mathrm{p}} v_{\mathrm{g}} = c^2$$

8.44 $$v_{\mathrm{p}} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi S}{\rho\lambda}} \tag{1}$$

Substituting $v_{\mathrm{p}} = 30\,\mathrm{cm/s}$, $g = 980\,\mathrm{cm/s^2}$, $S = 75$ dynes/cm and $\rho = 1\,\mathrm{g/cm^3}$, on simplification (1) reduces to the quadratic equation in λ:

$$\lambda^2 - 5.767\lambda + 1.153 = 0$$

The two roots are $\lambda_1 = 5.56\,\mathrm{cm}$ and $\lambda_2 = 0.207\,\mathrm{cm}$.
In determining surface tension it is preferable to use the shorter wavelength because the surface effect will dominate over gravity:

8.45 $$v_{\mathrm{p}} = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$$

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

$$\therefore \quad v_{\mathrm{g}} = \frac{1}{2}v_{\mathrm{p}}$$

8.46 $\omega = \sqrt{\dfrac{\gamma RT}{M}}k$

$$v_{\mathrm{p}} = \frac{\omega}{k} = \sqrt{\frac{\gamma RT}{M}}$$

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \sqrt{\frac{\gamma RT}{M}}$$

$$\therefore \quad v_{\mathrm{g}} = v_{\mathrm{p}}$$

8.47 $v_{\mathrm{p}}^2 = \dfrac{g}{k} + \dfrac{Sk}{\rho}$ (1)

Maximize (1)

$$\frac{\mathrm{d}(v_{\mathrm{p}}^2)}{\mathrm{d}k} = -\frac{g}{k^2} + \frac{S}{\rho} = 0$$

$$\therefore \quad k = \frac{2\pi}{\lambda} = \sqrt{\frac{g\rho}{S}}$$

$$\therefore \quad \lambda_{\min} = 2\pi\sqrt{\frac{S}{g\rho}}$$

8.3.4 Sound Waves

8.48 Let the displacement be represented by

$$y = A\cos(kx - \omega t) \tag{1}$$

$$\frac{\partial y}{\partial x} = -kA\sin(kx - \omega t)$$

but $P = -B\dfrac{\partial y}{\partial x} = BkA\sin(kx - \omega t)$

where B is the bulk modulus:

$$B = v^2\rho_0$$

$$P = [k\rho_0 v^2 A]\sin(kx - \omega t) \tag{2}$$

P represents the change from standard pressure P_0. The term in square bracket represents maximum change in pressure and is called the pressure amplitude $P_{\max}$. Then

$$P = P_{max} \sin(kx - \omega t) \tag{3}$$

$$\text{where } P_{max} = k\rho_0 v^2 A \tag{4}$$

If the displacement wave is represented by the cosine function, (1), then the pressure wave is represented by the sine function, (3). Here the displacement wave is 90° out of phase with the pressure wave.

8.49 $I = \frac{1}{2} P_{max}^2 / \rho_0 v$

$$\therefore \quad P_{max} = \sqrt{2I\rho_0 v} = \sqrt{2 \times 10^{-12} \times 1.29 \times 331} = 2.92 \times 10^{-5}\ \text{N/m}^2$$

8.50 $IL = 10 \ \log \frac{I}{I_0}$

$$60 = 10 \ \log \frac{I}{10^{-12}}$$

$$\log I + \log 10^{12} = 6 \quad \log I = -6$$

$$\therefore \quad I = 10^{-6}\ \text{W/m}^2 = 1\ \mu\ \text{W/m}^2$$

8.51 $I = \frac{\text{Power}}{4\pi r^2} = \frac{4}{4\pi \times 25^2} = 5.093 \times 10^{-4}\ \text{W/m}^2$

$$IL = 10 \ \log \frac{I}{I_0} = 10 \ \log \frac{5.093 \times 10^{-4}}{10^{-12}} = 10 \ \log(5.093 \times 10^8)$$

$$= 10[\log 5.093 + 8] = 87\ \text{dB}$$

8.52 $A = \frac{P_{max}}{k\rho_0 v^2} = \frac{P_{max}}{2\pi\rho_0 f v}$

where we have substituted $k = \frac{2\pi}{\lambda}$ and $v = f\lambda$:

$$\therefore \quad A = \frac{29}{2\pi \times 1.22 \times 2000 \times 331} = 5.7 \times 10^{-6}\ \text{m}$$

8.53 $I = \frac{P_{max}^2}{2\rho_0 v}$

By problem, $P_{max}(\text{air}) = P_{max}(\text{water})$

$$\therefore \quad \frac{I_{\text{Water}}}{I_{\text{Air}}} = \frac{\rho_A v_A}{\rho_W v_W} = \frac{1.293 \times 330}{1000 \times 1450} = 2.94 \times 10^{-4}$$

8.54 $P = 2.4 \sin \pi(x - 330t) = 2.4 \sin 2\pi \left(\frac{1}{2}x - 165t\right)$

$P = P_{\max} \sin 2\pi \left(\frac{x}{\lambda} - ft\right)$ (standard expression)

On comparing the two expressions we find

(a) $2.4\,\mathrm{N/m^2}$, (b) 165 Hz, (c) 2.0 m, (d) $v = f\lambda = 165 \times 2 = 330\,\mathrm{m/s}$

8.55 $I = 2\pi^2 \rho_0 A^2 f^2 v$

$$\therefore \quad A = \frac{1}{\pi f}\sqrt{\frac{I}{2\rho_0 v}} = \frac{1}{1200\pi}\sqrt{\frac{2 \times 10^{-6}}{2 \times 1.293 \times 330}} = 1.28 \times 10^{-4}\,\mathrm{m}$$

8.56 $P_\mathrm{e} = 1$ microbar $= 10^{-6}$ bar $= 0.1\,\mathrm{N/m^2}$

$$I = \frac{P_\mathrm{e}^2}{\rho_0 v} = \frac{(0.1)^2}{1.293 \times 330} = 2.34 \times 10^{-5}\,\mathrm{W/m^2}$$

$$IL = 10 \log \frac{I}{I_0} = 10 \log \left(\frac{2.34 \times 10^{-5}}{10^{-12}}\right)$$

$$= 10(7 + \log 2.34) = 73.7\,\mathrm{dB} \simeq 74\,\mathrm{dB}$$

8.57 $\mathrm{IL} = 10 \log \frac{I}{I_0}$

$$70 = 10 \log \frac{I}{10^{-12}} = 10\,[\log I + 12]$$

$\log I = -5 \quad I = 10^{-5}\mathrm{W/m^2}$

Energy density

$$E = \frac{I}{v} = \frac{10^{-5}}{331} = 3 \times 10^{-8}\,\mathrm{J/m^3}$$

Effective pressure

$$P_\mathrm{e} = \sqrt{I\rho_0 v} = \sqrt{10^{-5} \times 1.293 \times 331} = 0.0654\,\mathrm{N/m^2}$$

8.58 $P_{\max} = \sqrt{2I\rho_0 v} = \sqrt{2 \times 1 \times 1.293 \times 331} = 29.26\,\mathrm{N/m^2}$

8.59 $v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{1.4 \times 8.317 \times 273}{2.016 \times 10^{-3}}} = 1256\,\mathrm{m/s}$

8.60 $v = \sqrt{\dfrac{\gamma P}{\rho}}$ (Laplace formula)

$$\therefore \quad \frac{v_H}{v_O} = \sqrt{\frac{\rho_O}{\rho_H}} = \sqrt{16} = 4$$

$$\therefore \quad v_H = 4v_O = 4 \times 317 = 1268\,\text{m/s}$$

8.61 $\text{IL} = 10\,\log\left(\dfrac{I_2}{I_1}\right) = 10\,\log\left(\dfrac{10}{0.4}\right) = 14\,\text{dB}$

8.62 $\text{IL} = 10\,\log\dfrac{I_2}{I_1} = 6$

$$\log\frac{I_2}{I_1} = 0.6$$

$$\therefore \quad \frac{I_2}{I_1} = 3.98 \text{ or } 4$$

8.63 The threshold of hearing intensity is taken as $10^{-12}\,\text{W/m}^2$. Let r be the distance from the source at which the sound can be audible

$$I = \frac{\text{power}}{4\pi r^2} = 10^{-12}$$

$$\therefore \quad r = \sqrt{\frac{\text{power}}{4\pi \times 10^{-12}}} = \sqrt{\frac{0.009}{4\pi \times 10^{-12}}} = 2.677 \times 10^4\,\text{m} = 26.8\,\text{km}$$

8.64 $P_{\max} = k\rho_0 v^2 A = \dfrac{2\pi}{\lambda}\rho_0 v(f\lambda)A = 2\pi\rho_0 v f A$

$$\therefore \quad A = \frac{P_{\max}}{2\pi\rho_0 v f} = \frac{2 \times 10^{-5}}{2\pi \times 1.22 \times 331 \times 1000} = 7.9 \times 10^{-12}\,\text{m}$$

8.65 **(a)** $A = \dfrac{P_{\max}}{2\pi\rho_0 f v}$

$$\frac{A_1}{A_2} = \frac{\rho_2}{\rho_1}\frac{v_2}{v_1} = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$$

(b) $I = \dfrac{1}{2}\dfrac{p_{\max}^2}{\rho_0 v}$

$$\frac{I_1}{I_2} = \frac{\rho_2 v_2}{\rho_1 v_1} = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$$

(c) $E = \dfrac{I}{v}$

$$\frac{E_1}{E_2} = \frac{I_1}{I_2}\frac{v_2}{v_1} = \frac{8}{9} \times \frac{2}{3} = \frac{16}{27}$$

8.66 **(a)** $\lambda = \dfrac{v}{f}$

$$\therefore \quad \frac{\lambda_A}{\lambda_W} = \frac{v_A}{v_W} = \frac{331}{1450} = 0.228 \quad (\because f_A = f_W)$$

(b) $P_{max} = \sqrt{2I\rho_0 v}$

$$\therefore \quad \frac{P_A}{P_W} = \sqrt{\frac{\rho_A}{\rho_W}\frac{v_A}{v_W}} = \sqrt{\frac{1.293}{1000} \times \frac{331}{1450}} = 0.0172 \quad (\because I_A = I_W)$$

(c) $A = \sqrt{\dfrac{2I}{\rho v \omega^2}}$

$$\frac{A_A}{A_W} = \sqrt{\frac{\rho_W}{\rho_A}\frac{v_W}{v_A}} = \sqrt{\frac{1000}{1.293} \times \frac{1450}{331}} = 33.88 \quad (\because I_A = I_W \text{ and } f_A = f_w)$$

8.67 Characteristic impedance of a gas

$$Z = \rho_0 v \tag{1}$$

Now $v = \sqrt{\dfrac{B}{\rho_0}}$ or $\rho_0 = \dfrac{B}{v^2}$

$$\therefore \quad Z = \frac{B}{v} = B\sqrt{\frac{M}{\gamma RT}} \tag{2}$$

Thus $\quad Z \propto \dfrac{1}{\sqrt{T}}$

(a) At 0°C, $v = 331\,\text{m/s}$, $\rho_0 = 1.293\,\text{kg/m}^3$

$$Z = \rho_0 v = 1.293 \times 331 = 428\,\text{rayl}$$

(b) $Z \propto \dfrac{1}{\sqrt{T}}$

$$\therefore \quad Z(80°\text{C}) = Z(0°\text{C}) \times \sqrt{\frac{273}{273 + 80}} = 428 \times 0.879 = 376\,\text{rayl}$$

8.68 **(a)** $I = \dfrac{\text{Power}}{\text{area}} = \dfrac{50}{\pi(0.25)^2} = 255\,\text{W/m}^2$

(b) $P_{\max} = \sqrt{2I\rho_0 v} = \sqrt{2 \times 255 \times 10^3 \times 1450} = 2.72 \times 10^4\,\text{N/m}^2$

(c) $A = \sqrt{\dfrac{2I}{\rho_0 v 4\pi^2 f^2}} = \sqrt{\dfrac{2 \times 255}{1000 \times 1450 \times 4\pi^2 \times (25 \times 10^3)^2}}$
$= 1.19 \times 10^{-7}\,\text{m}$

(d) $U_{\max} = A\omega = 1.19 \times 10^{-7} \times 2\pi \times 25 \times 10^3 = 0.019\,\text{m}$

(e) $S_{\max} = \dfrac{2\pi A}{\lambda} = \dfrac{2\pi A f}{v} = \dfrac{2\pi \times 1.19 \times 10^{-7} \times 25 \times 10^3}{1450}$
$= 1.29 \times 10^{-5}$

8.69 Consider sound waves of finite amplitude. Now, the bulk modulus is constant only for infinitesimal volume changes:

$$B = -V\frac{\mathrm{d}p}{\mathrm{d}V} \tag{1}$$

where the acoustic pressure p has been replaced by the pressure change dp. Now, $V\rho$ = mass = constant

$$\therefore \quad V\frac{\mathrm{d}\rho}{\mathrm{d}P} + \rho\frac{\mathrm{d}V}{\mathrm{d}P} = 0$$

$$\text{or} \quad -V\frac{\mathrm{d}P}{\mathrm{d}V} = \rho\frac{\mathrm{d}P}{\mathrm{d}\rho} = B$$

where we have used (1)

$$\therefore \quad v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\mathrm{d}P}{\mathrm{d}\rho}} \tag{2}$$

When a second wave passes through a gas the changes in volume are assumed to be adiabatic so that

$$\frac{P}{\rho^\gamma} = C = \text{constant} \tag{3}$$

where γ is the ratio of specific heats of the gas at constant pressure to that at constant volume. Differentiation of (3) gives

$$-\gamma P\rho^{\gamma-1}\mathrm{d}\rho + \rho^\gamma \mathrm{d}P = 0$$

$$\text{or} \quad \frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{\gamma P}{\rho} \tag{4}$$

Using (4) in (2), we get

$$v = \sqrt{\frac{\gamma P_0}{\rho_0}} \qquad \text{(Laplace formula)} \tag{5}$$

where P_0 and ρ_0 refer to equilibrium conditions of pressure and density. The velocity v_0 at 0°C can be found out by substituting $\gamma = 1.4$, $P_0 = 1.013 \times 10^5\,\mathrm{N/m^2}$ and $\rho_0 = 1.293\,\mathrm{kg/m^3}$. We find $v_0 = 331.2\,\mathrm{m/s}$, in good agreement with the experiment.

With the assumption of isothermal changes we would have obtained the formula $v = \sqrt{P_0/\rho_0}$, (Newton's treatment) which gives a value of 20% lower.

8.70 **(a)** $v|_{t=20} = 1403 + 5 \times 20 - 0.06 \times (20)^2 + 0.0003 \times (20)^3 = 1481.4\,\mathrm{m/s}$

(b) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 5 - 0.12t + 0.0009t^2$

$$\frac{\mathrm{d}v}{\mathrm{d}t}\Big|_{t=20} = 5 - 0.12 \times 20 + 0.0009 \times (20)^2 = 2.62\,\mathrm{m/s/°C}$$

8.3.5 Doppler Effect

8.71 $v_s = 72\,\mathrm{km/h} = 72 \times \dfrac{5}{18}\,\mathrm{m/s} = 20\,\mathrm{m/s}$

$$f_0 = \frac{vf_s}{v + v_s} \quad \text{(direct)}$$

$$f_0' = \frac{vf_s}{v - v_s} \quad \text{(reflected from the wall of the rock)}$$

$$\frac{f_0'}{f_0} = \frac{v + v_s}{v - v_s} = \frac{340 + 20}{340 - 20} = \frac{9}{8}$$

8.72 $v_0 = v_s = 90\,\mathrm{km/h} = 90 \times \dfrac{5}{18} = 25\,\mathrm{m/s}$

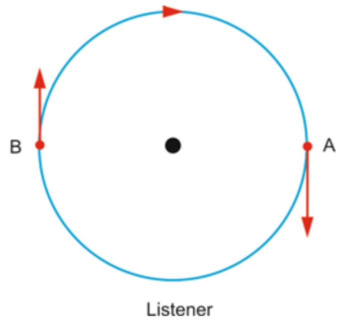
$$v = 350\,\mathrm{m/s}$$

$$f' = \frac{f(v + v_0)}{v - v_s} = \frac{520(350 + 25)}{350 - 25} = 600\,\mathrm{Hz}.$$

8.73 $f' = \dfrac{f(v - v_0)}{v + v_s} = 520\dfrac{(330 - 25)}{330 + 25} = 446.8\,\mathrm{Hz}.$

8.74 Let the whistle rotate clockwise, Fig. 8.7. At point A the linear velocity of the whistle will be towards the distant listener and at B away from the listener.

Fig. 8.7



Maximum frequency will be heard when the whistle will be at A and minimum when it is at B:

$$v_s = \omega r = 15 \times 2 = 30\,\text{m/s}$$
$$f_{\max} = \frac{v f_s}{v - v_s} = \frac{330 \times 540}{330 - 30} = 594\,\text{Hz}$$
$$f_{\min} = \frac{v f_s}{v - v_s} = \frac{330 \times 540}{330 + 30} = 495\,\text{Hz}$$

8.75 **(a)** The frequency of the rod is fixed, and so also for the air and the gas. The distance between successive heaps of cork dust is equal to the distance between two neighbouring nodes which is $\frac{1}{2}\lambda$, Fig. 8.8:

Fig. 8.8

$$v_{\text{air}} = f\lambda_{\text{air}}$$
$$v_{\text{gas}} = f\lambda_{\text{gas}}$$
$$\therefore \quad v_{\text{gas}} = v_{\text{air}}\frac{\lambda_{\text{gas}}}{\lambda_{\text{air}}} = 330 \times \frac{10}{8} = 425\,\text{m/s}$$

(b) $$v_{\text{rod}} = f\lambda_{\text{rod}} = \frac{v_{\text{air}}}{\lambda_{\text{air}}} \cdot \lambda_{\text{rod}} = \left(\frac{340}{2 \times 0.08}\right)(2L_{\text{rod}})$$
$$= \left(\frac{340}{0.16}\right)(2 \times 1.2) = 5100\,\text{m/s}$$

8.76 **(i)** $v = f\lambda$

$$\therefore \quad \lambda = \frac{v}{f} = \frac{340}{514} = 0.66\,\text{m}$$

(ii) $\lambda' = \dfrac{v - v_\mathrm{s}}{f} = \dfrac{340 - 15}{514} = 0.63\,\mathrm{m}$

(iii) $\lambda' = \dfrac{v + v_\mathrm{s}}{f} = \dfrac{340 + 15}{514} = 0.69\,\mathrm{m}$

8.3.6 Shock Wave

8.77 **(a)** If an object flies with a supersonic speed (speed greater than that of sound) a shock wave is emitted, a booming sound. In the two-dimensional drawing, Fig. 8.9, the wave fronts CB and DB represent the V-shaped wave. In three dimensions the bunching of the wave fronts actually forms a cone called the Mach cone. The shock wave lies on the surface of the cone.

(b) The half-angle θ of the cone called the Mach cone is given by

$$\sin\theta = \frac{vt}{v_\mathrm{s}t} = \frac{v}{v_\mathrm{s}} = \frac{1}{2}$$

$$\text{The Mach number} = \frac{v_\mathrm{s}}{v_\mathrm{p}} = 2$$

(c) The Mach angle $\theta = 30°$

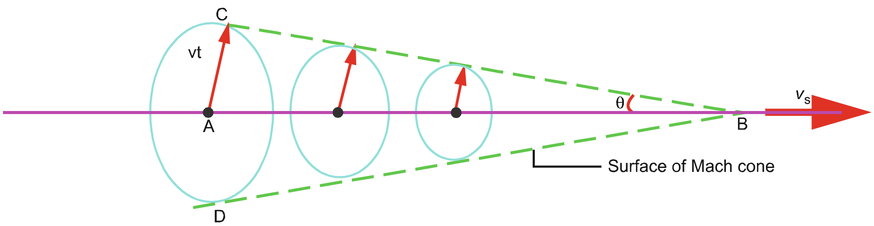


Fig. 8.9 Shock wave

8.3.7 Reverberation

8.78 If V is the volume, S the surface area and K the absorption coefficient then the reverberation time t_R is given by

$$t_\mathrm{R} = \frac{0.16V}{\sum_i k_i S_i} \quad \text{(Sabine Law)}$$

$$V = 10 \times 18 \times 4 = 720\,\mathrm{m}^3$$

$$\begin{aligned}\Sigma K_i S_i &= 10 \times 18 \times 0.6 \\ &\quad + 10 \times 18 \times 0.02 + (18 \times 4 + 10 \times 4) \times 2 \times 0.03 + 50 \times 0.5 \\ &= 143.3\end{aligned}$$

$$t_R = \frac{0.16 \times 720}{143.3} = 0.8\,\text{s}$$

8.3.8 Echo

8.79 The drum rate, that is, frequency when the echo is inaudible, is 40/min or 2/3 per second. Therefore, the time period of drum beats $t_1 = \frac{3}{2}$ s. Time for the echo, $t_1 = \frac{2x}{v}$, where x is the initial distance from the mountain and v is the sound velocity.

$$\text{Thus} \quad \frac{2x}{v} = \frac{3}{2} \tag{1}$$

On moving 90 m towards the mountain ht is $x - 90$ m from the mountain, the drum rate is 60/min or 1/s and again the echo is not heard.
Thus

$$\frac{2(x-90)}{v} = 1 \tag{2}$$

Solving (1) and (2) we get

$$x = 270\,\text{m} \quad \text{and} \quad v = 360\,\text{m/s}.$$

8.80 **(a)** If the width of the valley is d m and the rifle shot is fired at a distance x from one of the mountains the echoes will be heard in time t_1 and t_2 s:

$$t_1 = \frac{2x}{v} \tag{1}$$

$$t_2 = 2\frac{(d-x)}{v} \tag{2}$$

Adding (1) and (2)

$$t_1 + t_2 = 2 + 4 = \frac{2d}{v} = \frac{2d}{360}$$

Therefore $d = 1080$ m.

(b) Solving (1) with $t_1 = 2$ s and $v = 360$ m we find $x = 360$ m and therefore $d - x = 720$ m. Subsequent echoes will be heard after 6, 8, 10,... s.

8.3.9 Beat Frequency

8.81 When the man moves towards the source

$$f_0 = \frac{(v + v_0) f_s}{v}$$

When the man moves away from the source

$$f_0' = \frac{(v - v_0) f_s'}{v}$$

$$\therefore \quad f_0 - f_0' = \frac{v_0}{v}(f_s + f_s') + f_s - f_s'$$

$$= \frac{1.5}{330}(548 + 552) + 548 - 552 = 1\ \text{s}$$

$\therefore$ Beat frequency $= 1/\text{s}$.

8.82 Suppose the frequency of the unknown fork (unloaded) is n. Then $n = 300 \pm 2$
Case (*i*) Suppose $n = 300 - 2 = 298$
Let the frequency of the loaded unknown fork be n_1 and the loaded known fork be n_2:

$$300 - n_1 = 5 \tag{1}$$

$$298 - n_2 = 9 \tag{2}$$

Also frequency changes in both the forks are the same

$$\therefore \quad 300 - n_2 = 298 - n_1$$

$$\text{or} \quad n_2 - n_1 = 2 \tag{3}$$

Subtracting (2) from (1)

$$n_2 - n_1 = -6 \tag{4}$$

Obviously (3) and (4) are inconsistent.
Case (*ii*) Suppose $n = 300 + 2 = 302$

$$300 - n_1 = 5 \tag{5}$$

$$302 - n_2 = 9 \tag{6}$$

$$\text{also } 300 - n_2 = 302 - n_1$$

$$\text{or} \quad n_1 - n_2 = 2 \tag{7}$$

Subtracting (5) from (6)

$$n_1 - n_2 = 2 \tag{8}$$

Thus (7) and (8) are consistent. Therefore, correct solution is $n = 302$.

8.3.10 Waves in Pipes

8.83 $L = \dfrac{v}{2f} = \dfrac{342}{2 \times 439} = 0.3895\,\text{m}$

The new frequency with the changed length L_1 is

$$f_1 = 439 + 2 = 441$$

$$f_2 = 439 - 2 = 437$$

$$L_1 = \frac{v}{2f_1} = \frac{342}{2 \times 441} = 0.3877\,\text{m}$$

$$L_2 = \frac{v}{2f_2} = \frac{342}{2 \times 437} = 0.3913\,\text{m}$$

The pipe must be shortened by $0.3895 - 0.3877 = 0.0018\,\text{m}$ or 1.8 mm or lengthened by $0.3913 - 0.3895 = 0.0018\,\text{m} = 1.8\,\text{mm}$, so that 2 beats/s may be heard when it is sounded with the fork.

8.84 Time taken for the plate to fall is

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 10}{980}} = \frac{1}{7}\,\text{s}$$

Time period

$$T = \frac{t}{8} = \frac{1}{7 \times 8} = \frac{1}{56}\,\text{s}$$

Frequency, $f = \dfrac{1}{T} = 56\,\text{Hz}$

8.85 $v = v_\text{t} = v_0\sqrt{\dfrac{t + 273}{273}} = 33150\sqrt{\dfrac{20 + 273}{273}} = 34343\,\text{cm/s}$

$$L = (2N + 1)\frac{\lambda}{4} = \frac{(2N + 1)v}{4f_1} = \frac{(2N + 1) \times 34,343}{4 \times 210} \ (N = 0, 1, 2)\ldots \quad (1)$$

$$\text{Also } L = (2M + 1)\frac{v}{4f_2} = \frac{(2M + 1) \times 34343}{4 \times 350} \ (M = 0, 1, 2\ldots) \quad (2)$$

Equating right-hand sides of (1) and (2) and simplifying

$$\frac{(2N + 1)}{(2M + 1)} = \frac{3}{5} \quad (3)$$

The choice of $N = 1$ and $M = 2$ satisfies (3). Using $N = 1$ in (1) or M in (2) gives $L = 122.65\,\text{cm}$.

Resonance with tuning forks of different frequencies is possible because resonance for $f = 210$ occurs with the first overtone of the tube and for $f = 350$ it occurs with the second overtone.

8.86 $f = \dfrac{v}{2L}$ (open pipe, fundamental)

$f_1 = \dfrac{v}{L}$ (open pipe, first overtone)

$f_2 = \dfrac{5v}{4L}$ (closed pipe, second overtone)

$f_2 - f_1 = 100$ (by problem)

$$\frac{5v}{4L} - \frac{v}{L} = \frac{v}{4L} = 100$$

$\therefore$ Fundamental frequency of open pipe, $f = \dfrac{v}{2L} = 200\,\text{Hz}$.

Chapter 9
Fluid Dynamics

Abstract *Chapter 9* is concerned with fluid dynamics comprising equation of continuity, Bernoulli's equation, Torricelli's theorem, Reynolds number, viscosity and terminal velocity.

9.1 Basic Concepts and Formulae

Steady flow (laminar flow): In this type of flow, the velocity of the fluid (liquid or gas) at a point is always the same although the velocity of the fluid may be different at different points along the line of flow.

Irrotational flow: In this type of flow the element of fluid at each point has no net angular velocity about that point. It implies the absence of eddies and vortices.

Incompressible fluid flow: A liquid is incompressible if its density is constant.

Stream lines are imaginary curves drawn through a fluid to indicate the direction of motion of the flow of the fluid.

Tube of flow is a bundle of stream lines which define the boundary of the fluid.

Turbulent motion and Reynold's number: For any liquid through a pipe there exists a critical velocity at which the laminar flow suddenly changes into turbulent type of flow. The stability of fluid flow is described by a dimensionless quantity called Reynold's number (R).

$$R = \rho v d/\eta \tag{9.1}$$

where η and ρ are the viscosity and density of the fluid, respectively, d is the diameter of the pipe and v is the velocity. If

$R < 2200$, the flow is steady
$R = 2200$, the flow is unstable
$R > 2200$, the flow is usually turbulent

The Equation of Continuity

The principle of conservation of mass leads to the equation of continuity. For steady flow, the mass of fluid passing all sections in a stream of fluid per unit time is constant. At two points 1 and 2,

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 = \text{constant (one-dimensional flow)} \tag{9.2}$$

For incompressible fluids, $\rho_1 = \rho_2$, and (9.2) is reduced to

$$Q = A_1 v_1 = A_2 v_2 = \text{constant} \tag{9.3}$$

$$\nabla.\rho v + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(equation of continuity in three dimensions)} \tag{9.4}$$

For steady incompressible flow, ρ = constant and (9.4) reduces to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{9.5}$$

Bernoulli's equation for steady, non-viscous, incompressible flow

$$\frac{p}{\rho} + gh + \frac{1}{2}v^2 = \text{constant} \tag{9.6}$$

$$\text{or } P + \rho gh + \frac{1}{2}\rho v^2 = \text{constant} \tag{9.7}$$

In (9.6) the first term P/ρ is the pressure head or the potential energy per unit mass of the liquid due to the pressure, the second term hg is the elevation head or the potential energy per unit mass of the liquid due to gravity and the third term $v^2/2$ is the kinetic energy per unit mass of the liquid. Thus, Bernoulli's equation results from the conservation of energy.

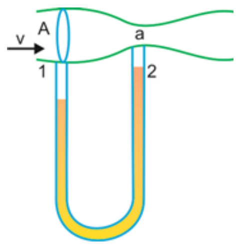
Torricelli's theorem: A tank is filled with a liquid. An orifice is located at the side of the tank, at a depth h below the surface of the liquid. Then the velocity of emergence of the liquid from the orifice is given by

$$v = \sqrt{2gh} \tag{9.8}$$

The venturi meter: It works as a gauge to measure the flow speed of a liquid. Let a liquid of density ρ flow through a pipe of cross-sectional area A with velocity v. At the constriction, called throat, the area is reduced to a (Fig. 9.1). A manometer containing a suitable liquid of density serves to register the pressure difference between points 1 and 2. In Bernoulli's equation the gravitational energy term will be absent as the centre of the cross-sectional areas A and a is at the same horizontal level. Finally, we obtain

$$v = a\sqrt{\frac{2(\rho' - \rho)gh}{\rho(A^2 - a^2)}} \tag{9.9}$$

Fig. 9.1



The pitot tube is a device to measure the flow speed of a gas. If ρ' is the density of the liquid in the manometer, ρ the density of the flowing gas and h the difference in height in the limbs of the manometer, then

$$v = \sqrt{\frac{2gh\rho'}{\rho}} \tag{9.10}$$

Viscosity: The coefficient of viscosity of a liquid is the tangential force per unit area per unit velocity gradient. The backward tangential force acting on any liquid layer is

$$F = -\eta A \frac{\mathrm{d}v}{\mathrm{d}y} \tag{9.11}$$

where $\mathrm{d}v/\mathrm{d}y$ is the velocity gradient which is identical with v/y for constant gradient. The negative sign shows that the viscous drag acts opposite to the velocity of the liquid.

Poiseuille's Method for Viscosity Determination

Volume V flowing per second through a tube of radius a and length L under pressure head P is given by

$$V = \frac{\pi P a^4}{8\eta L} \tag{9.12}$$

Terminal velocity

$$\text{Drag force } F = 6\pi\eta r v \quad \text{(Stokes law)} \tag{9.13}$$

When a sphere of radius r and of density ρ_0 is dropped in an extensive fluid of density ρ its speed increases linearly as for a free fall ($v = gt$). However, due to viscous drag v approaches asymptotically to a constant value v_T given by

$$v_\mathrm{T} = \frac{2gr^2}{9\eta}(\rho_0 - \rho) \tag{9.14}$$

9.2 Problems

9.2.1 Bernoulli's Equation

9.1 The radius of a water pipe decreases from 10 to 5 cm. If the average velocity in the wider portion is 4 m/s, find the average velocity in the narrower region.

9.2 Verify if the continuity equation for steady incompressible flow is satisfied for the following velocity components:

$$v_x = 3x^2 - xy + 2z^2,\ v_y = 2x^2 - 6xy + y^2,\ v_z = -2xy - yz + 2y^2$$

9.3 Air streams horizontally across an aeroplane wing of area 4 m^2, weighing 300 kg. The air speed is 70 and 55 m/s over the top surface and the bottom surface, respectively. Find **(a)** the lift on the wing; **(b)** the net force on it.

9.4 A venturi meter has a pipe diameter of 20 cm and a throat diameter of 10 cm. If the water pressure in the pipe is 60,000 Pa and in the throat is 45,000 Pa, calculate the rate of flow of water in m^3/s.

9.5 A pitot tube which is used to determine the speed of an aircraft relative to air is mounted on the wing of a plane. The tube contains alcohol of density 810 kg/m^3 and registers a level difference of 15.0 cm. Assuming that the density of air at NTP is 1.293 kg/m^3, find the plane's speed in km/h relative to the air.

9.6 A garden sprinkler has 80 small holes each 2.5 mm^2 in area. If water is supplied at the rate of 2×10^{-3} m^3/s, find the average velocity of the spray.

9.7 For steady, incompressible flow which of the following values of velocity components are possible?

(a) $v_x = 3xy + y^2,\ v_y = 5xy + 2x$

(b) $v_x = 3x^2 + y^2,\ v_y = -6xy$

9.8 If the speed of flow past the lower surface of the wing of an aeroplane is 100 m/s, what speed of flow over the upper surface would give a pressure difference of 1000 Pa? Assume an air density of 1.293 kg/m^3.

9.9 A venturi meter has a pipe diameter of 4 cm and a throat diameter of 2 cm. The velocity of water in the pipe section is 10 cm/s. Find **(a)** the pressure drop; **(b)** the velocity in the throat.

9.10 Water is observed to flow through a capillary of diameter 1.0 mm with a speed of 3 m/s. Viscosity of water in CGS units is

(a) 0.018 at 0°C

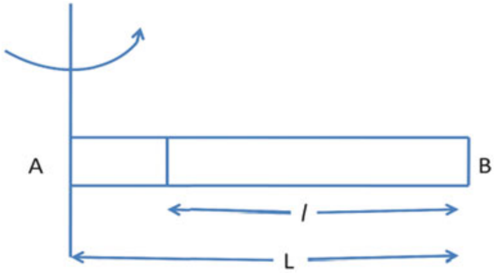
(b) 0.008 at 30°C

(c) 0.004 at 70°C

Calculate the Reynold's number and test at which of these three temperatures is the flow likely to be streamlined. Assume that for Reynold's number $R <$ 2200 flow is steady.

9.11 A horizontal tube AB of length L, open at A and closed at B, is filled with an ideal fluid. The end B has a small orifice. The tube is set in rotation in the horizontal plane with angular velocity ω about a vertical axis passing through A, Fig. 9.2. Show that the efflux velocity of the fluid is given by $v = \omega l\sqrt{\frac{2L}{l} - 1}$ where l is the length of the fluid.

Fig. 9.2



9.12 A pitot tube, Fig. 9.3, is mounted along the axis of a gas pipeline of cross-sectional area A. Calculate the rate of flow of the gas across the section of the pipe if h is the difference in the liquid column and ρ_L and ρ_g are the densities of the liquid and the gas, respectively.

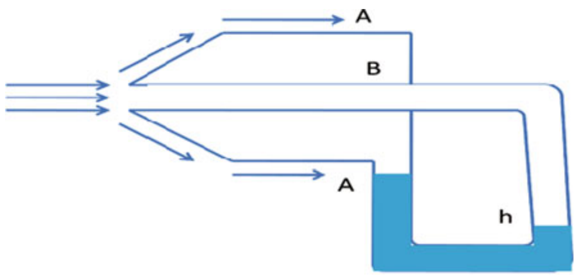
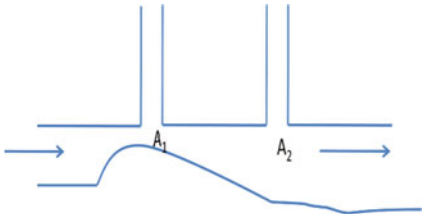


Fig. 9.3 Pitot tube

9.13 Water flows in a horizontal pipe of varying cross-section. Two manometer tubes fixed on the pipe, Fig. 9.4, at sections A_1 and A_2 indicate a difference Δh in the water columns. Calculate the rate of flow of water in the pipe.

9.14 A cylinder filled with water of volume V is fitted with a piston and is placed horizontally. There is a small hole of cross-sectional area s at the other end of

Fig. 9.4

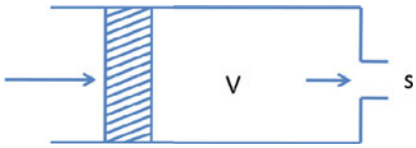


the cylinder, s being much smaller than the cross-sectional area of the piston (Fig. 9.5). Show that the work to be done by a constant force acting on the piston to squeeze all water from the cylinder in time t is given by

$$W = \frac{1}{2}\frac{\rho V^3}{s^2 t^2}$$

where ρ is the density of water. Neglect friction and viscosity.

Fig. 9.5



9.15 A cylindrical vessel with water is rotated about its vertical axis with a constant angular velocity ω. Show that

(a) the water pressure distribution along its radius is given by $P = P_0 + \frac{1}{2}\rho\omega^2 r^2$, where ρ is the density of water and P_0 is the pressure at the central point.

(b) Show that the figure of revolution of water is a paraboloid.

9.16 A manometer is fixed to a water tap. When the valve is closed the manometer shows the reading of 3.5×10^5 Pa. When the valve is open the reading becomes 3.1×10^5 Pa. Find the speed of water.

9.2.2 Torricelli's Theorem

9.17 A water container is filled up to a height H. A small hole is punched at the side wall at a depth h below the water surface. Show **(a)** that the distance from the foot of the wall at which the stream strikes the floor is $2\sqrt{h(H-h)}$; **(b)** the second hole through which the second stream has the same range must be punched at a depth $H - h$.

9.18 In prob. (9.17) show that the hole must be punched at a depth $h = H/2$ for maximum range and that this maximum distance is H.

9.19 A large tank is filled with water at the rate of $70\,\text{cm}^3/\text{s}$. A hole of cross-section $0.25\,\text{cm}^2$ is punched at the bottom of the tank. Find the maximum height to which the tank can be filled.

9.20 A tank of cross-sectional area A is filled with water up to a height h_1. Water leaks out from a small hole of area 'a' at the bottom. Find the time taken for the water level to decrease from h_1 to h_2.

9.21 A large tank is filled with water. The total pressure at the bottom is 3.0 atm. If a small hole is punched at the bottom what is the velocity of efflux?

9.22 Two tanks with a large opening are filled with a liquid. A hole of cross-sectional area A_1 is punched in tank 1 and another of cross-sectional area A_2 in tank 2 at depths h_1 and h_2, respectively. If $A_1 = 2A_2$ and the volume flux is identical, then what should be the ratio h_1/h_2?

9.23 A wide container with a small orifice in the bottom is filled with water and kerosene. If the water column measures 60 cm and kerosene column 40 cm, calculate the efflux velocity of water. Take the specific gravity of water as 1.0 and kerosene as 0.8 and neglect viscosity.

9.24 A wide vessel filled with water is punched with two holes on the opposite side each with cross-sectional area of $1.0\,\text{cm}^2$. If the difference in height of the holes is 51 cm, calculate the resultant force of reaction of the water flowing out of the vessel.

9.2.3 Viscosity

9.25 Water is conveyed through a tube 8 cm in diameter and 4 km in length at the rate of 120 l/min. Calculate the pressure required to maintain the flow. Coefficient of viscosity of water, $\eta = 0.001$ SI units. 1 atm $= 1.013 \times 10^5$ Pa.

9.26 Two capillary tubes AB and BC are joined end to end at B. AB is 16 cm long and of diameter 0.4 cm. BC is 4 cm long and of diameter 0.2 cm. The composite tube is held horizontally as in Poiseuille's experiment, with A connected to a vessel of water giving a constant head of 3 cm and C open to air. Calculate the pressure difference between B and C.

9.27 Two raindrops fall through air with terminal velocity of v_{T} cm/s. If the drops coalesce what will be the new terminal velocity?

9.28 $Q\,\text{cm}^3$ of water flows per second through a horizontal tube of uniform bore of radius r and of length l. Another tube of half the length but radius $2r$ is connected in parallel to the same pressure head. What will be the total quantity of water flowing per second through these two tubes?

9.29 In prob. (9.28) if the tubes are connected in series then what quantity will flow through the composite tube?

9.3 Solutions

9.3.1 Bernoulli's Equation

9.1 From continuity equation

$$A_1 v_1 = A_2 v_2$$
$$\therefore \quad v_2 = \frac{A_1 v_1}{A_2} = \frac{\pi r_1^2 v_1}{\pi r_2^2} = \frac{10^2 \times 4}{5^2} = 16\,\text{m/s}$$

9.2 $v_x = 3x^2 - xy + 2z^2$

$$v_y = 2x^2 - 6xy + y^2$$
$$v_z = -2xy - yz + 2y^2$$
$$\therefore \quad \frac{\partial v_x}{\partial x} = 6x - y; \ \frac{\partial v_y}{\partial y} = -6x + 2y; \ \frac{\partial v_z}{\partial z} = -y$$
$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = (6x - y) + (-6x + 2y) - y = 0$$

Thus the continuity equation for steady incompressible flow is satisfied.

9.3 Pressure difference across the wing

$$\Delta p = \frac{1}{2}\rho(v_1^2 - v_2^2)$$
$$= \frac{1}{2} \times 1.293 \times (70^2 - 55^2) = 1212\,\text{Pa}$$

(a) Lift = (pressure difference) (area)
= 1212 × 4 = 4848 N

(b) Net force = Lift − Weight of plane

= 4848 − (300 × 9.8)
= 1908 N in the upward direction

9.4 $\Delta P = \frac{1}{2}\rho v^2 \left(\frac{A^2}{a^2} - 1\right)$

$$\frac{A}{a} = \frac{\pi R^2}{\pi r^2} = \left(\frac{10}{5}\right)^2 = 4$$

$$60{,}000 - 45{,}000 = \frac{1}{2} \times 1000 \times 15v^2$$

or $\quad v = 1.414\,\mathrm{m\,/\,s}$ (throat)

Rate of flow of water

$$Q = vA = (1.414)(\pi \times 0.01^2) = 0.0444\,\mathrm{m^3/s}$$

9.5 $v = \sqrt{\dfrac{2gh\rho'}{\rho}} = \sqrt{\dfrac{2 \times 9.8 \times 0.15 \times 810}{1.293}} = 42.9\,\mathrm{m/s} = 154.5\,\mathrm{km/h}$

9.6 Total area of the holes

$$A = 80 \times 2.5 \times 10^{-6}\,\mathrm{m^2} = 2 \times 10^{-4}\,\mathrm{m^2}$$

$$Q = Av$$

$$v = \frac{Q}{A} = \frac{2 \times 10^{-3}}{2 \times 10^{-4}} = 10\,\mathrm{m/s}$$

9.7 **(a)** $v_x = 3xy + y^2 \quad v_y = 5xy + 2x$

$$\frac{\partial v_x}{\partial x} = 3y;\ \frac{\partial v_y}{\partial y} = 5x$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 3y + 5x \neq 0$$

Therefore, steady incompressible flow is not possible.

(b) $v_x = 3x^2 + y^2 \quad v_y = -6xy$

$$\frac{\partial v_x}{\partial x} = 6x;\ \frac{\partial v_y}{\partial y} = -6x$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 6x - 6x = 0$$

Thus, steady incompressible flow is possible.

9.8 $\Delta P = \dfrac{1}{2}\rho(v_1^2 - v_2^2)$

$$v_1 = \sqrt{\frac{2\Delta P}{\rho} + v_2^2} = \sqrt{\frac{2 \times 1000}{1.293} + 100^2} = 107.45\,\mathrm{m/s}$$

9.9 **(a)** $Q = v_{\text{throat}} a = v_{\text{pipe}} A$

$$\therefore \quad v_{\text{throat}} = v_{\text{pipe}} \frac{A}{a} = v_{\text{pipe}} \frac{D^2}{d^2} = 10 \times \frac{4^2}{2^2} = 40\,\text{m/s}$$

(b) $\Delta p = \frac{1}{2}\rho v^2 \left(\frac{A^2}{a^2} - 1\right)$

$$= \frac{1}{2} \times 1000 \times 40^2 \left(\frac{4^4}{2^4} - 1\right) = 12 \times 10^6\,\text{Pa}$$

9.10 Reynold's number $R = \frac{\rho D v}{\eta}$, where ρ is density, D diameter, v velocity and η coefficient of viscosity.

(a) $R = \frac{1 \times 0.1 \times 300}{0.018} = 1667$

Flow is steady because $R < 2200$

(b) $R = \frac{1 \times 0.1 \times 300}{0.008} = 3750$

Flow is turbulent because $R > 2200$

(c) $R = \frac{1 \times 0.1 \times 300}{0.004} = 7500$

Flow is turbulent because $R > 2200$

9.11 Consider a mass element dm of the fluid at distance x from the vertical axis. The centrifugal force on dm is

$$\mathrm{d}F = \mathrm{d}m\,\omega^2 x = \mathrm{d}m \frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{d}m \frac{\mathrm{d}v}{\mathrm{d}x} v$$

$$v\mathrm{d}v = \omega^2 x\,\mathrm{d}x$$

$$\int v\,\mathrm{d}v = \omega^2 \int x\,\mathrm{d}x$$

$$\frac{v^2}{2} = \frac{\omega^2}{2} x^2 \Big|_{L-l}^{L}$$

$$\therefore \quad v = \omega l \sqrt{\frac{2L}{l} - 1}$$

9.12 Applying Bernoulli's equation to points A and B,

$$p_{\text{A}} + \frac{1}{2}\rho_{\text{g}} v^2 = P_{\text{B}} \tag{1}$$

$$p_{\text{A}} + \rho_{\text{L}} g h = P_{\text{B}} \tag{2}$$

Comparing (1) and (2)

$$v = \sqrt{\frac{2gh\rho_{\mathrm{L}}}{\rho_{\mathrm{g}}}}$$

9.13 Apply Bernoulli's equation at the sections A_1 and A_2:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \tag{1}$$

$$\therefore \quad p_2 - p_1 = \Delta p = \Delta h \rho g = \frac{1}{2}\rho(v_1^2 - v_2^2)$$

$$\therefore \quad 2g\Delta h = v_1^2 - v_2^2 \tag{2}$$

$$Q = v_1 A_1 = v_2 A_2 \tag{3}$$

$$v_2 = \frac{v_1 A_1}{A_2} \tag{4}$$

Using (4) in (2)

$$2g\Delta h = v_1^2 \frac{(A_2^2 - A_1^2)}{A_2^2}$$

$$v_1 = A_2 \sqrt{\frac{2g\Delta h}{A_2^2 - A_1^2}}$$

$$Q = A_1 v_1 = A_1 A_2 \sqrt{\frac{2g\Delta h}{A_2^2 - A_1^2}}$$

9.14 Volume of water flowing out per second

$$Q = sv \tag{1}$$

where v is the speed and s is the cross-sectional area.
Volume flowing out

$$V = Qt = svt \tag{2}$$

$$\frac{1}{2}\rho v^2 = P = \frac{F}{A} = \frac{FL}{AL} = \frac{W}{V} \tag{3}$$

where L is the length of the cylinder and W is the work done.

$$\therefore \quad W = \frac{1}{2}\frac{\rho V^3}{s^2 t^2} \tag{4}$$

where we have used (2).

9.15 **(a)** The components of $m\omega^2 r$ parallel to the x-axis and z-axis are $m\omega^2 x$ and $m\omega^2 z$, respectively. Taking y in the upward direction

$$\mathrm{d}p = \rho(\omega^2 x \mathrm{d}x + \omega^2 z \mathrm{d}z - g\mathrm{d}y) \tag{1}$$

In the $x - z$-plane, $y =$ constant. Hence $\mathrm{d}y = 0$.
Integrating (1)

$$p = \frac{\rho\omega^2 x^2}{2} + \frac{\rho\omega^2 z^2}{2} + C$$

where C is the constant of integration.

$$\begin{aligned} p &= \frac{\rho\omega^2}{2}(x^2 + z^2) + C \\ &= \frac{1}{2}\omega^2 r^2 + C \end{aligned}$$

$p = p_0$ at $r = 0$, then $C = p_0$

$$\therefore \quad p = p_0 + \frac{1}{2}\rho\omega^2 r^2$$

(b) Particle at P is in equilibrium under centrifugal force and gravity, Fig. 9.6. Let PM be tangent at $P(r, y)$ making an angle θ with the r-axis. PN is normal at P. If N is the normal reaction

$$N \cos\theta = mg$$

$$N \sin\theta = m\omega^2 r$$

$$\therefore \quad \tan\theta = \frac{\omega^2 r}{g}$$

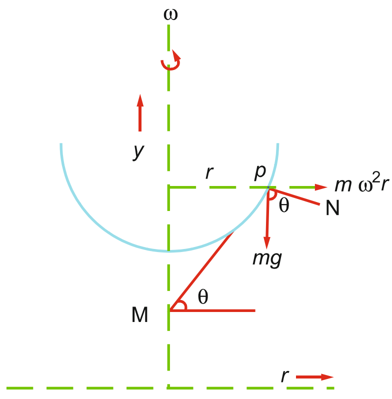


Fig. 9.6

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}r} = \frac{\omega^2 r}{g}$$

$$y = \int \mathrm{d}y = \frac{\omega^2}{g} \int r\mathrm{d}r + c$$

$$y = \frac{\omega^2 r^2}{2g} + c$$

$$y = 0, r = 0, c = 0$$

$$y = \frac{1}{2} \frac{\omega^2 r^2}{g}$$

Figure of revolution of the curve is a paraboloid.

9.3.2 Torricelli's Theorem

9.16 Using Bernoulli's equation

$$P_2 + \frac{1}{2}\rho v_2^2 = P_1 + \frac{1}{2}\rho v_1^2$$
$$3.1 \times 10^5 + \frac{1}{2} \times 1000 v_2^2 = 3.5 \times 10^5 + 0$$
$$v_2 = 8.94\,\mathrm{m/s}$$

9.17 **(a)** Use Bernoulli's equation at two points A and B at height h_A and h_B, respectively, Fig. 9.7.

$$P + \rho g h_A = P + \rho g h_B + \frac{\rho v^2}{2} \tag{1}$$

where P is the atmospheric pressure, ρ is the density of water and v is the efflux velocity.

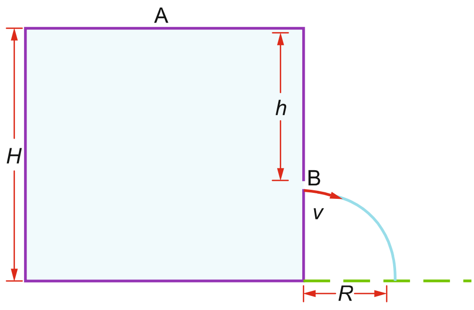


Fig. 9.7

Calling $h_A - h_B = h$ (2)

$$v = \sqrt{2gh} \tag{3}$$

Using simple kinematics, the range

$$R = vt = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}}$$

$$R = 2\sqrt{h(H-h)} \tag{4}$$

(b) In (4) R is unchanged if we replace h by $H - h$. Therefore, the second hole must be punched at a depth $H - h$ to get the same range.

9.18 From prob. (9.17)

$$R = 2\sqrt{h(H-h)} \tag{1}$$

Maximum range is obtained by setting $\mathrm{d}R/\mathrm{d}h = 0$ and holding H as constant. This gives $h = H/2$ and substituting this value in (1), we get $R_{\max} = H$.

9.19 For the water level to remain stationary volume efflux = rate of filling = x

$$vA = \left(\sqrt{2gh}\right) A = x = 70\,\mathrm{cm}^3/\mathrm{s}$$

$$h = \frac{x^2}{2gA^2} = \frac{(70)^2}{2 \times 980 \times (0.25)^2} = 40\,\mathrm{cm}$$

9.20 Let the water level be at a height x at any instant. The efflux velocity will be $v = \sqrt{2gx}$. As the water flows out, the level of water comes down, Fig. 9.8.

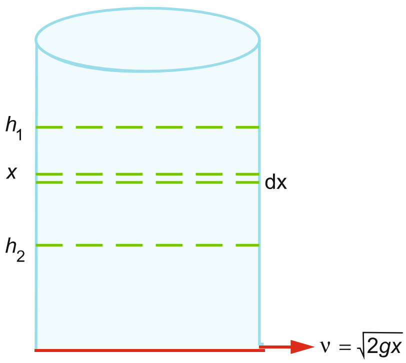


Fig. 9.8

Volume flux, $Q = av = a\sqrt{2gx}$

Volume flux is also equal to $Q = A\dfrac{\mathrm{d}x}{\mathrm{d}t}$

We then have $a\sqrt{2gx} = A\dfrac{\mathrm{d}x}{\mathrm{d}t}$

$$t = \int \mathrm{d}t = \frac{A}{a\sqrt{2g}} \int_{h_2}^{h_1} \frac{\mathrm{d}x}{\sqrt{x}} = \frac{A}{a}\sqrt{\frac{2}{g}}\left[\sqrt{h_1} - \sqrt{h_2}\right]$$

9.21 Pressure at the bottom due to water column = (3 − 1) atm = 2 atm = 2 × 10^5 Pa.

$$P = h\rho g$$

$$\therefore \quad h = \frac{P}{\rho g} = \frac{2 \times 10^5}{1000g} = \frac{200}{g}$$

$$v = \sqrt{2gh} = \sqrt{2g\frac{200}{g}} = 20\,\mathrm{m/s}$$

Second method

Apply Bernoulli's equation

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

where the left side refers to the point inside the tank and right side to a point outside the tank.

$$3 \times 10^5 + 0 = 1 \times 10^5 + \frac{1}{2} \times 1000 v_2^2$$

$$\therefore \quad v_2 = 20\,\mathrm{m/s}$$

9.22 $Q = v_1 A_1 = v_2 A_2$

$$\left(\sqrt{2gh_1}\right)(2A_2) = \left(\sqrt{2gh_2}\right)A_2$$

$$\therefore \quad \frac{h_1}{h_2} = \frac{1}{4}$$

9.23 Apply Bernoulli's equation to a point just outside the hole and a point at the top of the kerosene surface. If P is the atmospheric pressure, h_1 and h_2 the heights of water and kerosene columns, respectively, ρ_1 and ρ_2 the respective densities,

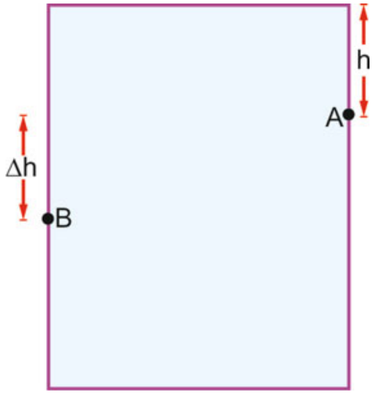
$$P + \frac{1}{2}\rho_1 v_1^2 = P + h_1\rho_1 g + h_2\rho_2 g$$

$$\therefore \quad v_1 = \sqrt{2g\left(h_1 + \frac{h_2\rho_2}{\rho_1}\right)}$$

Substituting $h_1 = 60\,\text{cm}$, $h_2 = 40\,\text{cm}$, $\rho_1 = 1$, $\rho_2 = 0.8$ and $g = 980$, we find $v_1 = 425\,\text{cm/s}$ or $4.25\,\text{m/s}$.

9.24 Volume efflux at A and B, Fig. 9.9

Fig. 9.9



$$Q_A = v_A S$$
$$Q_B = v_B S$$
$$(\text{Mass efflux})_A = \rho v_A S$$
$$(\text{Mass efflux})_B = \rho v_B S$$

Force $F_A = (\text{rate of change of momentum})_A$

$$= \rho v_A S v_A = \rho S v_A^2$$
$$= \rho S(2gh) = 2\rho S g h$$
$$F_B = 2\rho S g(h + \Delta h)$$
$$F_B - F_A = 2\,\rho\,S\,g\,\Delta h$$

(because the vector force is in the opposite direction)

$$= 2 \times 1000 \times 1.0 \times 10^{-4} \times 9.8 \times 0.51 = 1.0\,\text{N}$$

9.3.3 Viscosity

9.25 Volume of liquid flowing per second

$$V = \frac{\pi r^4 P}{8\eta l}$$

$$P = \frac{8\eta l V}{\pi r^4} = \frac{8 \times 0.001 \times 4000 \times 0.002}{3.14 \times (0.04)^4} = 0.0796 \times 10^5\,\text{Pa}$$

$$\text{Pressure head } h = \frac{P}{\rho g} = \frac{0.0796 \times 10^5}{1000 \times 9.8} = 0.8\,\text{m}$$

9.26 $$P_A - P_B = \frac{8\eta l_1 Q}{\pi r^4} = \frac{8\eta Q}{\pi}\frac{(0.16)}{(2 \times 10^{-3})^4} = \frac{8\eta Q\,(0.01)}{\pi \times 10^{-12}} \quad (1)$$

$$P_B - P_0 = \frac{8\eta Q \times (0.04)}{\pi \times 10^{-12}} \quad (2)$$

Adding (1) and (2)

$$P_A - P_0 = \frac{8\eta Q \times 0.05}{\pi \times 10^{-12}} \quad (3)$$

Dividing (2) by (3)

$$\frac{P_B - P_0}{P_A - P_0} = 0.8$$

$$\therefore \quad P_B - P_0 = 0.8 \times (P_A - P_0) = 0.8 \times 3 = 2.4\,\text{cm of water.}$$

9.27 The terminal velocity v_T is given by

$$v_T = \frac{2}{9} r^2 g \frac{(\rho_1 - \rho_2)}{\eta} \quad (1)$$

where r is the radius of the drop, ρ_1 and ρ_2 are the densities of the drop and air, respectively, g is the gravity and η is the coefficient of viscosity. If the new radius is r' and the new terminal velocity v'_T, then

$$\frac{v'_T}{v_T} = \frac{r'^2}{r^2} \quad (2)$$

Under the assumption that the drops are incompressible, the volume remains constant:

$$\frac{4\pi}{3}(r')^3 = 2 \times \frac{4\pi}{3} r^3$$

$$\therefore \quad r' = 2^{1/3} r \quad (3)$$

Using (3) in (2)

$$v'_T = 2^{2/3} v_T = 4^{1/3} v_T$$

9.28 For the first tube $Q_1 = \dfrac{\pi P r^4}{8\eta l} = Q$

For the second tube $Q_2 = \dfrac{\pi P (2r)^4}{8\eta l/2} = \dfrac{32\pi P r^4}{8\eta l} = 32Q$

Total quantity of water flowing is

$$Q_1 + Q_2 = Q + 32Q = 33Q$$

9.29 Let the pressure at the beginning of the first tube be P_1 and at the end P_2. Since the water flow must be continuous, the rate of flow in the two tubes must be identical, that is, $Q_1 = Q_2$. Let the atmospheric pressure be P_0.

$$P_1 - P_2 = \frac{8\eta l Q_1}{\pi r^4} \quad \text{(for the first tube)} \tag{1}$$

$$P_2 - P_0 = \frac{8\eta (l/2) Q_2}{\pi (2r)^4} = \frac{8\eta l Q_1}{32\pi r^4} \quad \text{(for the second tube)} \tag{2}$$

Adding (1) and (2)

$$P_1 - P_0 = \frac{8\eta l}{\pi r^4}\frac{33Q_1}{32} \tag{3}$$

But $P_1 - P_0 = \dfrac{8\eta l Q}{\pi r^4}$ (for single tube of length l and radius r) (4)

Comparing (3) and (4), we get $Q_1 = \dfrac{32Q}{33}$.

Chapter 10
Heat and Matter

Abstract *Chapter 10* is devoted to kinetic theory of gases, collision cross-section, mean free path, van der Waal's equation, thermal expansion of solids, liquids and gases; gas equation, heat conduction in composite slabs and sphere; Newton's law of cooling, radiation problems covering Boltzmann law and Wien's law, specific heat and latent heat, thermodynamics, indicator diagrams, the Carnot, Otto and Sterling's cycles, thermodynamic relations, elasticity and surface tension.

10.1 Basic Concepts and Formulae

The mean free path λ of a gas molecule is the average distance travelled by the molecule between successive collisions

$$\lambda = \sum x_i / N \tag{10.1}$$

The average time of collision T is related to λ and $<v>$ the mean speed by

$$\lambda = <v>T = <v>/f \tag{10.2}$$

$$\text{where}\, f = 1/T \tag{10.3}$$

is the collision frequency.

$$N(v)\,\mathrm{d}v = 4\pi N[m/2\pi kT]^{3/2} v^2 \exp[-mv^2/2kT]\mathrm{d}v \text{ (Maxwell's law)} \tag{10.4}$$

Assuming Maxwell's law of velocity distribution,

$$\text{Most probable speed } v_\mathrm{p} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}} \tag{10.5}$$

$$\text{Average speed } < v >= \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}} \tag{10.6}$$

$$\text{Root-mean-square speed } \sqrt{< v^2 >} = \sqrt{\frac{8kT}{m}} = \sqrt{\frac{8RT}{M}} \tag{10.7}$$

where k is Boltzmann constant, T the absolute temperature, m the particle mass, R the gas constant and M the molecular weight.

$$\lambda = \frac{v}{4\pi\sqrt{2}\,r^2\,N} \tag{10.8}$$

where $N = nV$, n is the number of molecules per unit volume and V is the volume of the gas.

Gas Laws

$$PV = nRT \qquad \text{(gas equation)} \tag{10.9}$$

$$\frac{\rho_1 T_1}{P_1} = \frac{\rho_2 T_2}{P_2} \tag{10.10}$$

Thermal Expansion

$$L = L_0[1 + \alpha\Delta T] \qquad \text{(linear expansion)} \tag{10.11}$$

$$\beta = 2\alpha, \gamma = 3\alpha \tag{10.12}$$

where α is the coefficient of linear expansion, β is the coefficient of areal expansion and γ that of volume expansion.

Thermal Expansion and Elasticity

$$\text{Force } F = Ya\alpha\Delta T \tag{10.13}$$

where Y is Young's modulus and a the cross-sectional area.

The coefficient A of apparent expansion of a liquid

$$A = \gamma - g = \gamma - 3\alpha \tag{10.14}$$

where γ is the absolute volume coefficient of expansion of liquid and g that of the container.

$$\text{Apparent expansion of liquid} = \frac{\text{mass expelled}}{(\text{mass left})(\text{temperature rise})}$$

Heat Transfer

Heat conduction through a slab:

$$-\frac{\mathrm{d}Q}{\mathrm{d}r} = k_1 A\left(\frac{T_1 - T_2}{\mathrm{d}}\right) \tag{10.15}$$

Rate of heat flow is directly proportional to the temperature gradient $\mathrm{d}T/\mathrm{d}x$ and the cross-sectional area and inversely proportional to the thickness. The constant of proportionality k is known as thermal conductivity.

Heat conduction is through a composite slab made of two slabs of thickness d_1 and d_2 and thermal conductivity k_1 and k_2, respectively, in series, the end temperatures being T_1 and T_2. The equivalent conductivity is given by

$$k_{\mathrm{eq}} = \frac{d_1 + d_2}{\left(\frac{d_1}{k_1}\right) + \left(\frac{d_2}{k_2}\right)} \tag{10.16}$$

The temperature of the interface is given by

$$T = \frac{\left(\frac{k_1 T_1}{d_1}\right) + \left(\frac{k_2 T_2}{d_2}\right)}{\left(\frac{k_1}{d_1}\right) + \left(\frac{k_2}{d_2}\right)} \tag{10.17}$$

The rate of flow of heat in a composite slab made of n slabs in parallel:

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{(T_1 - T_2)}{\mathrm{d}} \sum\nolimits_{i=1}^{n} k_i A_i \tag{10.18}$$

$$k_{\mathrm{eq}} = \frac{\sum k_i A_i}{\sum A_i} \tag{10.19}$$

Convection

Rate of cooling:

$$-\frac{d\theta}{dt} = C(\theta - \theta_0) \qquad \text{(Newton's law of cooling)} \tag{10.20}$$

where θ is the mean temperature, θ_0 the room temperature and C a constant.

Radiation

Rate of energy loss:

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma A \left(T_1^4 - T_2^4\right) \qquad \text{(Stefan–Boltzmann formula)} \tag{10.21}$$

where A is the area of the radiator, T_1 is its absolute temperature, T_2 is the absolute temperature of the surroundings and σ is known as Stefan–Boltzmann constant.

Wien's Law

$$\lambda_{\mathrm{m}} \cdot T = b = 3 \times 10^{-3}\,\mathrm{m} - \mathrm{K} \tag{10.22}$$

The wavelength λ_m for the maximum intensity of black body spectrum is inversely proportional to the absolute temperature of the body.

Thermodynamics

Process	Quantity that remains constant
(i) Isobaric	Pressure
(ii) Isochoric	Volume
(iii) Isothermal	Temperature
(iv) Adiabatic	Heat

First law of Thermodynamics

$$\Delta Q = \Delta U + W \tag{10.23}$$

ΔQ is positive if heat is absorbed by the system and negative if heat is evolved. Internal energy U of a system tends to increase if energy is added as heat and tends to decrease if energy is lost as work done by the system. Both Q and W are dependent while U is path independent.
Work done by the system:

Isobaric process: $W = \int P\,\mathrm{d}V = P(V_1 - V_2)$
Isochoric process: $W = 0$, $\Delta Q = \Delta U$
Isothermal process: $W = -nRT \ln\left(\frac{V_2}{V_1}\right)$, $\mathrm{d}U = 0$, $\mathrm{d}Q = \mathrm{d}W$
Adiabatic process: $W = \frac{1}{\gamma - 1}(P_2V_2 - P_1V_1)$, $\mathrm{dQ} = 0$, $\mathrm{d}U = -\mathrm{d}W$
where γ is the ratio of two specific heats (c_p/c_v).

The change in entropy

$$\Delta S = \int \frac{\mathrm{d}Q}{T} \tag{10.24}$$

Enthalpy (H) is the total heat and is defined by

$$H = U + PV \tag{10.25}$$

Gibb's function (G) is defined by

$$G = U + PV - TS \tag{10.26}$$

Number of degrees of freedom

$$f = \frac{2}{\gamma - 1} \tag{10.27}$$

Heat Engines Thermodynamic efficiency

$$e = \frac{\text{Work done by the gas}}{\text{Heat put into the system}} = \frac{W}{Q_{\text{in}}} \tag{10.28}$$

The Carnot cycle consists of two isothermal processes and two adiabatic processes.

The Sterling cycle consists of two isothermal processes and two isochoric processes.

The Otto cycle consists of two adiabatic processes and two isochoric processes.

$$e = (Q_{\mathrm{H}} - Q_{\mathrm{C}})/Q_{\mathrm{H}} \tag{10.29}$$

$$e = (T_{\mathrm{H}} - T_{\mathrm{C}})T_{\mathrm{H}} \tag{10.30}$$

where the symbols H and C are for hot and cold reservoirs.

Elasticity

Stress = force/area = F/A
Strain = elongation/original length = $\Delta L/L$
Young's modulus(Y) = stress/strain = $FL/A\Delta L$
Shear modulus (η) = shear stress/shear strain = $Fy/A\Delta x$
Bulk modulus (K) = pressure increment/volume strain = $\dfrac{\Delta P}{-\Delta V/V}$

Poisson's ratio (σ) = lateral contraction per unit length/longitudinal elongation per unit length.

Relations for the elastic moduli:

$$Y = \frac{9\eta K}{3K + \eta} = 2n(1 + \sigma) = 3K(1 - 2\sigma) \tag{10.31}$$

$$\sigma = \frac{3K - 2\eta}{6K + 2\eta} \tag{10.32}$$

Surface Tension

Excess pressure in a drop

$$P = 2S/r \tag{10.33}$$

Excess pressure in a bubble

$$P = 4S/r \tag{10.34}$$

Pressure in a bubble due to electric charges

$$P = \sigma^2/2\varepsilon_0 \tag{10.35}$$

where σ is the charge density and ε_0 is the permittivity.

Energy released in coalescing n droplets each of radius r into a large drop of radius R

$$\Delta W = 4\pi r^2(n - n^{2/3})S \tag{10.36}$$

$$\Delta W = 4\pi R^2(n^{\frac{1}{3}} - 1)S \tag{10.37}$$

Capillary rise:

$$2S\cos\theta = (h + r/3)r\rho g \tag{10.38}$$

where θ is the angle of contact, r is the radius of the bore and h is the height of the liquid column.

10.2 Problems

10.2.1 Kinetic Theory of Gases

10.1 Define the mean free path of a gas both mathematically and in words. Calculate the mean free path of a molecule in a gas if the number of collisions is 2×10^{10}/s and the mean molecular velocity is 1000 m/s.

[University of Manchester 2008]

10.2 **(a)** Consider a gas that has a molecular weight of 28 and a temperature of 27°C. What is the rms speed of molecule of the gas if it has a Maxwellian velocity distribution? The ideal gas constant is 8.31 J/mol K.

(b) What is the mean free path of a molecule if the pressure is 2 atm (1 atm = 101.3 kPa), the temperature is 27°C, and the cross-section is 0.43 nm^2? Using the average velocity from part **(a)** calculate the collision frequency for the molecule. Boltzmann's constant is 1.38×10^{-23} J/K.

10.3 **(a)** By considering a volume, V, of ideal gas, containing N spherical molecules with radius r, show that the mean free path of the molecules can be defined by the equation

$$\lambda = \frac{V}{4\pi\sqrt{2}r^2N}$$

(b) Hence, or otherwise, calculate the mean free path of air at 100°C and 1.01×10^5 Pa. Assume that the molecules of air are spheres of radius $r = 2.0 \times 10^{-10}$ m.
($k = 1.38 \times 10^{-23}$ J/K)

[University of Aberystwyth, Wales]

10.4 Figure 10.1 shows the Maxwell–Boltzmann velocity distribution functions of a gas for two different temperatures, of which first (curve f_1) is for $T_1 = 300$ K.

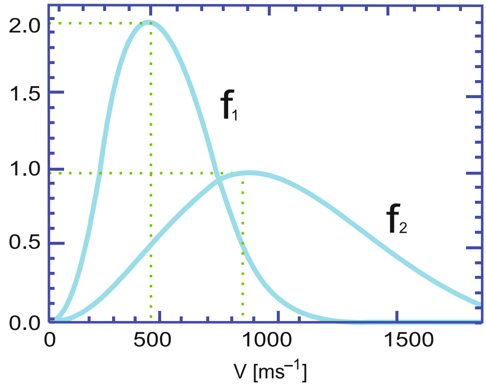


Fig. 10.1 Maxwell–Boltzmann velocity distributions

(a) Read the approximate value for the most probable speed of the molecules from the diagram for each of the two cases.
(b) What is the temperature, T_2, when the velocity distribution is given by f_2?
(c) Indicate the average speeds in the diagram for each of the two temperatures and give the ratio between the two average speeds.
(d) The gas consists of 5 mol of molecules. If the molecular velocity distribution is given by f_2, estimate the number of those molecules in the gas which have a speed between $v_A = 800\,\text{m/s}$ and $v_B = 900\,\text{m/s}$.

10.5 If the Maxwell–Boltzmann distribution of speeds is given by

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{\frac{-mv^2}{2kT}}$$

show that the most probable speed is defined by the equation

$$v_{mp} = \left(\frac{2kT}{m}\right)^{1/2}$$

10.6 For carbon dioxide gas (CO_2, molar mass = 44.0 g/mol) at $T = 300\,\text{K}$, calculate

(i) the mean kinetic energy of one molecule
(ii) the root mean square speed, v_{rms}
(iii) the most probable speed, v_{mp}
(iv) the average speed, v_{av}

($R = 8.31$ J/K/mol, $k = 1.38 \times 10^{-23}$ J/K, $N_A = 6.02 \times 10^{23}$/mol)

10.7 **(a)** Give three assumptions that are made when deriving the properties of an ideal gas using a molecular model.

(b) A weather balloon is loosely filled with $2\,\text{m}^3$ of helium at 1 atm. and 27°C. The balloon is then released, and by the time it has reached an elevation of 7000 m the pressure has dropped to 0.5 atm. and the balloon has expanded. If the temperature at this elevation is −48°C, what is the new volume of the balloon?

10.8 van der Waal's equation can be written in terms of moles per volume as

$$\frac{n}{V} = \left(\frac{p + a\dfrac{n^2}{V^2}}{RT} \right) \left(1 - b\frac{n}{V}\right)$$

The van der Waal's parameters for hydrogen sulphide gas (H_2S) are $a = 0.448\,\text{J}\,\text{m}^3/\text{mol}^2$ and $b = 4.29 \times 10^{-5}\,\text{m}^3/\text{mol}$. Determine an estimate of the number of moles per volume of H_2S gas at 127°C and a pressure of 9.80×10^5 Pa as follows:

(a) Calculate as a first approximation using the ideal gas equation, $\dfrac{n}{V} = \dfrac{p}{RT}$.

(b) Substitute this first approximation into the right-hand side of the equation derived in part **(a)** to find a new approximation of $\dfrac{n}{V}$ (on the left-hand side) that takes into account real gas effects.

10.2.2 Thermal Expansion

10.9 Two parallel bars of different material with linear coefficient of expansion α_1 and α_2, respectively, are riveted together at a distance d apart. An increase in temperature ΔT will cause them to bend into circular arcs with a common centre subtending an angle θ at the centre (Fig. 10.2). Find the mean radius of curvature.

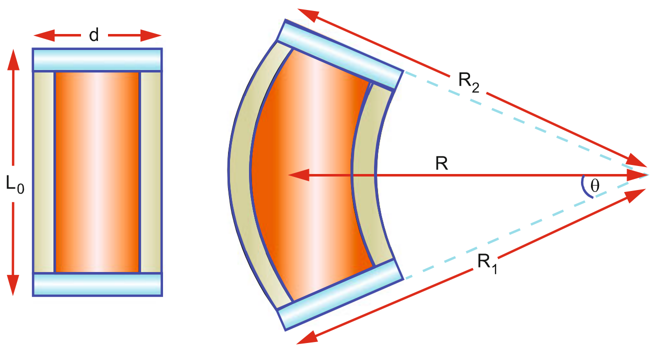


Fig. 10.2 Expansion of a bimetal strip of bars

10.10 A 20 m long steel rail is firmly attached to the road bed only at its ends. The sun raises the temperature of the rail by 30°C, causing the rail to buckle. Assuming that the buckled rail consists of two straight parts meeting in the centre, calculate how much the centre of the rail rises? For steel $\alpha = 12 \times 10^{-6}/°C$.

10.11 What should be the lengths of a steel and copper rod if the steel rod is 4 cm longer than the copper rod at any temperature. α(steel) $= 1.1 \times 10^{-5}/°C$; α (copper) $= 1.7 \times 10^{-5}/°C$.

10.12 A 1 l glass flask contains some mercury. It is found that at different temperatures the volume of air inside that flask remains the same. What is the volume of mercury in this flask? Coefficient of linear expansion of glass $= 9 \times 10^{-6}/°C$; coefficient of volume expansion of mercury $= 1.8 \times 10^{-4}/°C$.

[Indian Institute of Technology 1973]

10.13 A steel wire of cross-sectional area 0.5 mm^2 is held between two fixed supports. If the tension in the wire is negligible and it is just taut at a temperature of 20°C, determine the tension when the temperature falls to 0°C (assume that the distance between the supports remains the same). Young's modulus of steel $= 2.1 \times 10^{11}$ dynes/cm^2; $\alpha = 12 \times 10^{-6}/°C$.

[Indian Institute of Technology 1973]

10.14 A glass vessel just holds 50 g of toluene at 0°C. What mass of toluene will it hold at 80°C if between 0 and 80°C the expansion coefficients are constant. The coefficient of linear expansion of glass is $8 \times 10^{-6}/°C$ and the absolute expansion of toluene is 11×10^{-4}°C.

[University of Dublin]

Gas Laws

10.15 Determine the constant in the gas equation given that a gram molecule of a gas occupies a volume of 22.4 l at NTP. [University of Durham]

10.16 A bubble of gas rises from the bottom of a lake 30 m deep. At what depth will the volume be thrice as great as it was originally (atmospheric pressure = 0.76 m of mercury; specific gravity of mercury = 13.6)?

10.17 A balloon will carry a total load of 175 kg when the temperature and pressure are normal. What load will the balloon carry on rising to a height at which the barometric pressure is 50 cm of mercury and the temperature is −10°C, assuming the envelope maintains a constant volume?

[University of London]

10.18 Two glass bulbs of volume 500 and 100 cc are connected by a narrow tube whose volume is negligible. When the apparatus is sealed off, the pressure of the air inside is 70 cm of Hg and the temperature 20°C. What does the pressure become if the 100 cc bulb is kept at 20°C and the other is heated to 100°C?

[University of Durham]

10.2.3 Heat Transfer

10.19 Two slabs of cross-sectional area A and of thickness d_1 and d_2 and thermal conductivities k_1 and k_2 are arranged in contact face to face. The outer face of the first slab is maintained at temperature T_1°C, that of the second one at T_2°C and the interface at T°C. Calculate

(a) Rate of flow of heat through the composite slab
(b) The interface temperature
(c) The equivalent conductivity

10.20 n slabs of the same thickness, the cross-sectional area $A_1, A_2, \ldots, A_n$ and thermal conductivities $k_1, k_2, \ldots, k_n$ are placed in contact in parallel and maintained at temperatures T_1 and T_2. Calculate

(a) the rate of flow of heat through the composite slab
(b) the equivalent conductivity

10.21 A bar of copper and a bar of iron of equal length are welded together end to end and are lagged. Determine the temperature of the interface when the free end of the copper bar is at 100°C and the free end of the iron is at 0°C and the conditions are steady. Thermal conductivities: copper = 92, iron = 16 cal/m/s/°C.

[University of Durham]

10.22 A block of ice is kept pressed against one end of a circular copper bar of diameter 2 cm, length 20 cm, thermal conductivity 90 SI units and the other end is kept at 100°C by means of a steam chamber. How long will it take to melt 50 g of ice assuming heat is only supplied to the ice along the bar, $L = 8 \times 10^4$ cal/kg.

[University of Dublin]

10.23 At low temperatures, say below 50 K, the thermal conductivity of a metal is proportional to the absolute temperature, that is, $k = aT$, where a is a constant with a numerical value that depends on the particular metal. Show that the rate of heat flow through a rod of length L and cross-sectional area A and whose ends are at temperatures T_1 and T_2 is given by $Q = \frac{aA}{2L}(T_1^2 - T_2^2)$.

10.24 Find the radial flow of heat in a material of thermal conductivity placed between two concentric spheres of radii r_1 and r_2 ($r_1 < r_2$) which are maintained at temperatures T_1 and T_2 ($T_1 > T_2$).

10.25 Find the radial rate of flow of heat in a material of thermal conductivity k placed between a co-axial cylinder of length L and radii r_1 and r_2, respectively ($r_1 < r_2$), maintained at temperatures T_1 and T_2, respectively ($T_1 > T_2$).

10.26 A small pond has a layer of ice on the surface that is 1 cm thick. If the air temperature is -10°C, find the rate (in m/h) at which ice is added to the bottom of the layer. The density of ice is 917 kg/m^3, the thermal conductivity of ice is 0.59 W/m/K, and the latent heat of fusion is 333 kJ/kg. Assume that the underlying water is at 0°C.

10.27 An object is cooled from 85 to 75°C in 2 min in a room at 30°C. What time will be taken for the object to cool from 55 to 45°C.

10.28 A calorimeter containing first 40 g and then 100 g of water is heated and suspended in the same constant temperature enclosure. It is found that the time to cool from 50 to 40°C in the two cases was 15 and 33 min, respectively. Calculate the water equivalent of the calorimeter.

10.29 Two steel balls of identical material and surface quality have their radii in the ratio 1:2. When heated to 100°C and left to cool, they lose their heat by radiation. Find the rate of cooling $d\theta/dt$ for the balls.

10.30 A resistance thermometer gives readings of 24.9 Ω at the ice point, 29.6 Ω at the steam point and 26.3 Ω at some unknown temperature. What is the unknown temperature on the Celsius scale?

[The University of Wales, Aberystwyth 2004]

10.31 Solar constant (S) is defined as the average power received from the sun's radiation per square metre of earth's surface. Calculate S assuming sun's radius (R) as 6.95×10^8 m, the mean earth–sun distance (r) as 1.49×10^{11} m, sun's surface temperature $T = 5740$ K and Boltzmann's constant $\sigma = 5.67 \times 10^{-8}$ W/m^2/K^4.

10.32 Calculate the temperature of the solar surface if the radiant intensity at the sun's surface is 63 MW/m^2. Stefan–Boltzmann constant $\sigma = 5.67 \times 10^{-8}$ W/m^2/K^4.

10.33 Calculate the amount of heat lost per second by radiation by a sphere 10 cm diameter at a temperature of 227°C when placed in an enclosure at 27°C ($\sigma = 5.67 \times 10^{-8}$ W/m^2/K^4)

[Nagarjuna University 2002]

10.34 A body emits most intense radiation at $\lambda_m = 480$ nm. If the temperature of the body is lowered so that total radiation is now 1/16 of the previous value, what is the wavelength of the most intense radiation under new conditions? Wien's constant $b = 3 \times 10^{-3}$ m K.

10.2.4 Specific Heat and Latent Heat

10.35 The latest heat of fusion of a material is 6 kJ/mol and the heat capacity (C_p) in solid and liquid phases of the material is a linear function of temperature $C_\mathrm{p} = 30.6 + 0.0103\,T$, with units J/mol/K. How much heat is required to increase the temperature of 1 mol of the material from 20 to 200°C if the fusion phase transition occurs at 80°C?

[University of Manchester 2007]

10.36 The variation of the specific heat of a substance is given by the expression $C = A + BT^2$, where A and B are constants and T is Celsius temperature. Show that the difference between the mean specific heat and the specific heat at midpoint $T/2$ is $BT^2/12$.

10.37 The temperature of equal masses of three different liquids A, B and C is 12, 18 and 28°C, respectively. When A and B are mixed the temperature is 16°C. When B and C are mixed, it is 23°C. What would be the temperature when A and C are mixed?

[Indian Institute of Technology 1976]

10.38 A 3.0 g bullet moving at 120 m/s on striking a 50 g block of wood is arrested within the block. Calculate the rise of temperature of the bullet if (a) the block is fixed; (b) the block is free to move. The specific heat of lead is 0.031 cal/g°C.

10.39 Calculate the difference in temperature between the water at the top and bottom of a 25 m high waterfall assuming that 15% of the energy of fall is spent in heating the water ($J = 4.18$ J/Cal).

[University of Durham]

10.40 A piece of lead falls from a height of 100 m on to a fixed non-conducting slab which brings it to rest. Show that its temperature immediately after the collision is raised by approximately 7.1 K (the specific heat of lead is 30.6 cal/kg 0°C between 0 and 100°C).

10.2.5 Thermodynamics

10.41 **(a)** Define

(i) an isobaric process
(ii) an isochoric process

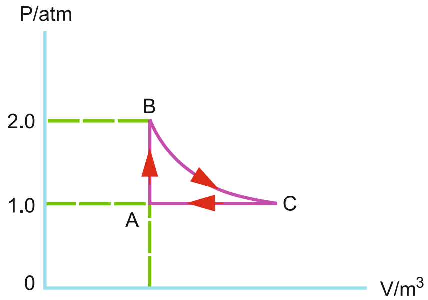
(iii) an adiabatic process
(iv) an isothermal process

(b) Show that the work done on a gas during an adiabatic compression from initial conditions (P_1, V_1) to final conditions (P_2, V_2) is given by the equation

$$W = \frac{1}{\gamma - 1}(p_2 V_2 - p_1 V_1)$$

10.42 A sample containing 2 k mol of monatomic ideal gas is put through the cycle of operations as in Fig. 10.3. Find the values of T_A, T_B and V_C.

Fig. 10.3 A thermodynamic cycle



10.43 Show that for a monatomic ideal gas undergoing an adiabatic process, $PV^{5/3} = \text{constant}$.

10.44 **(a)** State the first law of thermodynamics, expressing the law in its infinitesimal form. Explain carefully each term used and note whether or not each term is path dependent.

(b) Show that the work done on a gas during an isothermal compression from an initial volume V_1 to a final volume V_2 is given by the equation

$$W = -nRT \ln\left(\frac{V_2}{V_1}\right)$$

(c) An ideal gas system, with an initial volume of 1.0 m^3 at standard temperature and pressure, undergoes the following three-stage cycle:
Stage 1 – an isothermal expansion to twice its original volume.
Stage 2 – a process by which its volume remains constant, its pressure returns to its original value and 10^4 J of heat is added to the system.
Stage 3 – an isobaric compression to its original volume, with 3×10^4 J of heat being removed from the system.

(i) How many moles of gas are present in the system?
(ii) Calculate the work done on the system during each of the three stages.

(iii) What is the resultant change in the internal energy over the whole three-stage cycle?
(At STP, temperature = 0°C = 273.15 K and pressure = 1 atm = 1.01×10^5 Pa, $R = 8.31$/J/K/mol.)

10.45 The initial values for the volume and pressure of a certain amount of nitrogen gas are $V_1 = 0.06\,\mathrm{m}^3$ and $p_1 = 10^5\,\mathrm{N/m}^2$, respectively.

First, the gas undergoes an isochoric process (process 1–2), which triples the pressure; then it is followed by an isobaric process (process 2–3), which reduces the volume by a factor of three; finally, the volume of the gas is tripled by an isothermal process (process 3–4).

(a) Give the initial and final temperatures, T_1 and T_4, of the nitrogen gas if the temperature after the first (isochoric) process is $T_2 = 1083$ K.
(b) Find the volume, V_4, and pressure, p_4, at the final state of the gas, then sketch the three processes in a p–V diagram.
(c) How much heat is gained by the nitrogen gas during the first (isochoric) process and how much heat is given away by the nitrogen gas during the second (isobaric process)? The amount of heat required to raise the temperature of 1 mol of nitrogen by 1 K while the gas pressure is kept constant is $c_\mathrm{p} = 29.12$ J/(mol K).
(d) Find the change in the internal energy of the nitrogen gas by the end of the final process compared to the initial value.

[University of Aberystwyth, Wales]

10.46 When a gas expands adiabatically, its volume is doubled while its absolute temperature is decreased by a factor 1.32. Compute the number of degrees of freedom for the gas molecules.

10.47 A heat engine absorbs heat of 10^5 k cal from a source, which is at 127°C and rejects a part of heat to sink at 27°C. Calculate the efficiency of the engine and the work done by it.

[Osmania University 2004]

10.48 A reversible engine has an efficiency of 1/6. When the temperature of the sink is reduced by 62°C its efficiency gets doubled. Find the temperatures of the source and the sink.

10.49 Assuming that air temperature remains constant at all altitudes and that the variation of g with altitude is negligible

(a) show that the pressure P at an altitude h above sea level is given by $p = p_0 \exp(-Mgh/RT)$, where M is the molecular weight of the gas.
(b) show that $n = n_0 \exp(-Mgh/RT)$ where n is the number of molecules per unit volume.
(c) taking the average molecular weight of air to be 29 g, calculate the height at which the air pressure would be half the value at sea level.

10.50 **(a)** Write down the efficiency for a Carnot cycle as a function of

(i) the heat flows to and from the reservoirs and
(ii) the temperatures of the two reservoirs.

(b) Describe the working of an Otto engine and efficiency for the air standard Otto cycle as a function of temperature as well as volume. Start by sketching this cycle in a standard P–V diagram. Explain the four steps of this cycle in terms of associated temperature and volume changes as well as the heat exchanged with external reservoirs.
(c) Compare the Carnot and the Sterling cycle using P–V diagram.

10.51 **(a)** $1 \times 10^{-3}\,\mathrm{m}^3$ of He at normal conditions ($p_0 = 1\,\mathrm{bar}$, $T_0 = 0°\mathrm{C}$) is heated to a final temperature of 500 K. What is the entropy change for

(i) an isobaric and
(ii) an isochoric process?
Use ${C_\mathrm{P}}^{\mathrm{He}} = 21\,\mathrm{J/(mol\,K)}$ and ${C_\mathrm{V}}^{\mathrm{He}} = 12.7/(\mathrm{mol\,K})$.

(b) Calculate the change in entropy ΔS_1 for 1 kg of water being heated from 0 to 50°C. Compare this change in entropy ΔS_2 for 0.5 kg of water at 0°C being mixed with 0.5 kg of water at a temperature of 100°C. Use ${C_\mathrm{V}}^{\mathrm{H_2O}} = 4.13 \times 10^3\,\mathrm{J/(kg\,K)}$.

10.52 Consider a reversible isothermal expansion of an ideal gas in contact with the reservoir at temperature T, from an initial volume V_1 to a final volume V_2.

(a) What is the change in the internal energy of the system?
(b) Calculate the work done by the system.
(c) What is the amount of heat absorbed by the system?
(d) Find the change of entropy of the system.
(e) Find the change of the entropy of the system plus the reservoir.

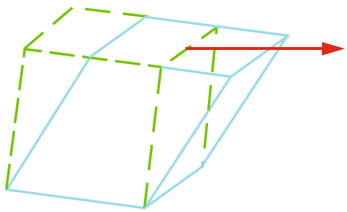
10.53 Internal energy, heat, enthalpy, work, and the Gibbs free energy (Gibbs function) are all measured in units of joules.

(a) What is the difference between these forms of energy? Write down the equations relating these forms of energy.
(b) Which of the above are state variables? What properties distinguish a state variable from other variables?

10.2.6 Elasticity

10.54 **(a)** A 100 MPa force is applied to the surface of a material (surface area, 1 m^2) that exerts a shear across the material (Fig. 10.4). The sample has a thickness of 10 cm and causes the surface to be displaced by 0.1 cm. What is the shear modulus of the material?

Fig. 10.4 Shear deformation



(b) What is the bulk modulus of a material if a 100 MPa increase in pressure causes a 1% reduction in its volume?

[University of Manchester 2008]

10.55 A wire has a length of 10 m and a cross-sectional area of 20 mm^2. When a 20 kg block of lead is attached to it, it stretches by 2.5 cm. Find

(i) the stress
(ii) the strain
(iii) Young's modulus for the wire

10.56 Show that the isothermal elasticity $K_T = P$ and adiabatic elasticity $K_H = \gamma P$.

10.57 For a given material, the Young's modulus is 2.5 times the rigidity modulus. Find its Poisson's ratio.

10.58 A 1.2 m long metal wire is fixed securely at both ends to two solid supports so that the wire is initially horizontal. When a 29 g mass is attached from the midpoint of the wire, the midpoint is observed to move down by 20 mm. If the diameter of the wire is 0.1 mm, estimate the Young's modulus for the wire material.

[The University of Wales, Aberystwyth 2004]

10.59 The rubber cord of a catapult has a cross-sectional area of 2 mm and an initial length of 0.2 m and is stretched to 0.25 m to fire a small object of mass 15 g. If the Young's modulus is $Y = 6 \times 10^8\ \text{N/m}^2$, what is the initial velocity of the object that is released?

10.60 A 10 kg object is whirled in a horizontal circle on the end of a wire. The wire is 0.3 m long and has a cross-section $10^{-6}\ \text{m}^2$ and has the breaking stress $4.8 \times 10^7\ \text{N/m}^2$. What is the maximum angular speed the object can have?

10.61 A steel wire is fixed at one end and hangs freely. The breaking stress for steel is equal to $7.8 \times 10^8\ \text{N/m}^2$ and its density is 7800 kg/m^2. Find the maximum length of the wire so that it does not break under its own weight.

10.2.7 Surface Tension

10.62 If the surface tension of the liquid–gas interface is 0.072 N/m, the density is 1 kg/L and the radius of the capillary is 1 mm, to what height will the liquid rise up the capillary?

[University of Manchester 2007]

10.63 A mole of gaseous molecules in a bubble obeys the ideal gas law. What is the volume of the bubble at a 100 m depth of water if the temperature is 293 K, the atmospheric pressure is 101 kPa, density of water is $1000\,\mathrm{kg/m^3}$ and the ideal gas constant is 8.314 J/mol/K.

[University of Manchester 2008]

10.64 Let n droplets each of radius r coalesce to form a large drop of radius R. Assuming that the droplets are incompressible and S is the surface tension calculate the rise in temperature if c is the specific heat and ρ is the density.

10.65 A soap bubble of surface tension 0.03 N/m is blown from 1 cm radius to 5 cm radius. Find the work done.

10.66 A small hollow vessel which has a small hole in it is immersed in water to a depth of 45 cm before any water penetrates into the vessel. If the surface tension of water is 0.073 N/m, what should be the radius of the hole?

10.67 What will be the depth of water at which an air bubble of radius 0.3×10^{-3} m may remain in equilibrium (surface tension of water = 0.072 N/m and $g = 9.8\,\mathrm{m/s^2}$)?

10.68 A capillary tube of radius 0.2 mm and of length 6 cm is barely dipped in water. Will the water overflow through the capillary? If not what happens to the meniscus (surface tension of water = 0.073 N/m and angle of contact = 0°)?

10.69 A soap bubble of radius 2.0 cm is charged so that the excess of pressure due to surface tension is neutralized. If the surface tension is 0.03 N/m, what is the charge on the bubble?

10.70 Two soap bubbles with radii r_1 and r_2 coalesce to form a bigger bubble of radius r. Show that $r = \sqrt{r_1^2 + r_2^2}$.

10.3 Solutions

10.3.1 Kinetic Theory of Gases

10.1 The mean free path λ of a gas molecule is the average distance travelled by the molecule between successive collisions.

$$\lambda = x/N \tag{1}$$

where x is the total distance travelled and N is the number of collisions.
In terms of frequency f, average time of collision T and the mean molecular velocity v,

$$\lambda = vT = \frac{v}{f} = \frac{1000}{2 \times 10^{10}} = 5 \times 10^{-8}\,\mathrm{m}$$

10.2 **(a)** $\sqrt{\langle v^2 \rangle} = \sqrt{\dfrac{3RT}{M}} = \sqrt{\dfrac{3 \times 8.31 \times 300}{28 \times 10^{-3}}} = 516.8\,\mathrm{m/s}$

(b) $R = kn_0 = 1.38 \times 10^{-23} \times 6.02 \times 10^{23} = 8.3\,\mathrm{J/mol\,K}$

$$N = \frac{PV}{RT} = \frac{2 \times 1.013 \times 10^5 \times 1}{8.3 \times 300} = 81.365\,\mathrm{mol/m^3}$$

$$n = \text{number/m}^3 = \frac{6.02 \times 10^{23} \times 81.365}{28 \times 10^{-3}} = 2.15 \times 10^{25}/\mathrm{m}^3$$

$$\lambda = \frac{1}{\sigma n} = \frac{1}{0.43 \times 10^{-18} \times 2.15 \times 10^{25}} = 1.082 \times 10^{-7}\,\mathrm{m}$$

$$f = \frac{v}{\lambda} = \frac{516.8}{1.082 \times 10^{-7}} = 4.78 \times 10^9/\mathrm{s}$$

10.3 **(a)** Collision cross-section between two molecules each of radius r is equivalent to collision of one molecule of radius $2r$ with another point size molecule. Therefore the cross -section will be

$$\sigma = \pi(2r)^2 = 4\pi r^2 \tag{1}$$

Consider a rectangular box of face area $1\,\mathrm{m}^2$ and length λ metres. Then the volume of the box $V = \lambda\,\mathrm{m}^3$. If n is the number of molecules per unit volume then λ is such that the total projected area arising from n molecules will just fill up an area of $1\,\mathrm{m}^2$.

$$\therefore \quad n\lambda\sigma = 1$$

$$\text{or} \quad \lambda = \frac{1}{n\sigma} = \frac{V}{4\pi r^2 N} \tag{2}$$

where we have used (1) and set $n = N/V$.
Equation (2) in based on the assumption that the target molecules are stationary. In practice, the molecule hits moving targets. This leads to an increase of collision frequency by a factor of $\sqrt{2}$ and therefore a decrease in the cross-section by a factor $\sqrt{2}$. The corrected expression for mean free path is then

$$\lambda = \frac{V}{4\pi\sqrt{2}r^2 N} \tag{3}$$

(b) $R = N_0 k = 6.02 \times 10^{23} \times 1.38 \times 10^{-23} = 8.3\,\mathrm{J/mol\,K}$ (1)

$PV = NRT$ (2)

$1.01 \times 10^5\, V = 8.13 \times 373\, N$ (3)

whence $N = 33.3\,\mathrm{mol/m^3}$

$n = \mathrm{number/m^3} = 33.3 \times 6.02 \times 10^{23} = 2 \times 10^{25}/\mathrm{m^3}$

$$\lambda = \frac{1}{n\sigma\sqrt{2}} = \frac{1}{2 \times 10^{25} \times 4\pi \left(10^{-10}\right)^2 \sqrt{2}} = 10^{-7}\,\mathrm{m}.$$

10.4 **(a)** For curve f_1, $v_{mp} = 425\,\mathrm{m/s}$; for curve f_2, $v_{mp} = 850\,\mathrm{m/s}$ (Fig. 10.5).

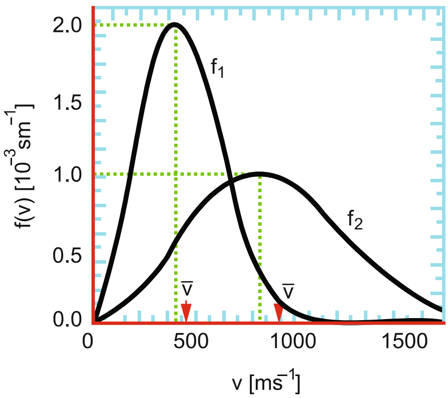


Fig. 10.5 Maxwell–Boltzman velocity distribution

(b) $v_{mp} \propto \sqrt{T}$

$$\therefore \quad T_2 = \frac{T_1 v_{mp}^2(1)}{v_{mp}^2(2)} = 300\left(\frac{850}{425}\right)^2 = 1200\,\mathrm{K}$$

(c) $\bar{v} = \sqrt{4/\pi}\, v_{mp}$

$\bar{v}_1 = 1.1287 \times 425 = 480\,\mathrm{m/s}$

$\bar{v}_2 = 1.1287 \times 850 = 959\,\mathrm{m/s}$

$\bar{v}_1/\bar{v}_2 = 1/2$

(d) and **(e)**

$$v_{mp} = \sqrt{\frac{2kT}{m}}$$

$$\therefore \quad m = \frac{2kT}{v_{\mathrm{mp}}^2} = \frac{2 \times 1.38 \times 10^{-23} \times 1200}{(850)^2} = 4.584 \times 10^{-26}\,\mathrm{kg}$$

$$= \frac{4.584 \times 10^{-26}}{1.66 \times 10^{-27}}\,\mathrm{amu} = 27.6\,\mathrm{amu}$$

Therefore the gas is N_2.
In 5 mol of gas total number (N) of gas molecules will be

$$N = \frac{5}{28} \times 6.02 \times 10^{23} = 1.075 \times 10^{23}$$

Number of molecules $N(v)\,\mathrm{d}v$ in the interval v and $v + \mathrm{d}v$ will be

$$N(v)\mathrm{d}v = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left[-\frac{mv^2}{2kT}\right] \mathrm{d}v \text{ (Maxwellian distribution)} \tag{2}$$

The mean value of the interval is $v = \dfrac{800 + 900}{2} = 850\,\mathrm{m/s}$
which happens to be identical with v_{mp} found in (a). In this case (2) is reduced to a simpler form

$$N(v)\mathrm{d}v = \frac{4N\,\mathrm{d}v}{\sqrt{\pi}\,e\,v_{\mathrm{mp}}}$$

Number of N_2 molecules in 5 mol will be

$$N = \frac{5}{28} \times 6.02 \times 10^{23} = 1.075 \times 10^{23}$$

The speed interval

$$\mathrm{d}v = 900 - 800 = 100\,\mathrm{m/s}$$

Thus the required number of molecules is

$$N(v)\mathrm{d}v = \frac{4 \times 1.075 \times 10^{23} \times 100}{\sqrt{\pi} \times 2.718 \times 850} = 1.05 \times 10^{22}$$

10.5 $f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \mathrm{e}^{-mv^2/2kT}$

$$\frac{\mathrm{d}f(v)}{\mathrm{d}v} = \text{const.}\frac{\mathrm{d}}{\mathrm{d}v}\left[v^2 \mathrm{e}^{-mv^2/2kT}\right] = 0$$

$$\therefore \qquad \mathrm{e}^{-mv^2/2kT}\left[2v - \frac{m}{kT}v^3\right] = 0$$

whence $\quad v = 0, \infty, \sqrt{\frac{2kT}{m}}$

The most probable speed is $v_{\text{mp}} = \sqrt{\frac{2kT}{m}}$

10.6 (i) $\bar{E} = \frac{3}{2}kT = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21}\,\text{J}$

(ii) $\sqrt{\overline{v^2}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 8.31 \times 300}{44 \times 10^{-3}}} = 412.3\,\text{m/s}$

(iii) $v_{\text{mp}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2 \times 8.31 \times 300}{44 \times 10^{-3}}} = 336.6\,\text{m/s}$

(iv) $v_{\text{av}} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8 \times 8.31 \times 300}{44 \times 10^{-3}\pi}} = 380.0\,\text{m/s}$

10.7 **(a) Assumptions:**

(i) The molecules of a gas behave like hard, smooth spheres and of negligible size compared to that of the container.
(ii) The molecules are in random motion undergoing collisions with one another and with the walls of the container for negligible duration.
(iii) Newton's laws of motion are applicable and the number of molecules is large so that statistics may be applied.

(b) $\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$ (from the gas equation)

$$\therefore \quad V_2 = \frac{P_1}{P_2} \cdot \frac{T_2}{T_1} V_1 = \frac{1}{0.5} \times \frac{(273 - 40)}{(273 + 27)} \times 2 = 3.1\,\text{m}^3$$

10.8 **(a)** $\frac{n}{V} = \frac{P}{RT} = \frac{9.8 \times 10^5}{8.31 \times 400} = 294.8\,\text{mol/m}^3$

(b) $\frac{n}{V} = \left[\frac{P + a\frac{n^2}{V^2}}{RT}\right]\left(1 - \frac{bn}{V}\right)$

$$= \left[\frac{9.8 \times 10^5 + 0.448 \times (294.8)^2}{8.31 \times 400}\right](1 - 4.29 \times 10^{-5} \times 294.8)$$

$$= 302.66\,\text{mol/m}^3$$

10.3.2 Thermal Expansion

10.9 Let the length of the bars be L_0, each at 0°C when they are straight. With the rise of temperature ΔT, their lengths will be (Fig. 10.2)

$$L_1 = \theta R_1 = L_0(1 + \alpha_1 \Delta T) \quad (1)$$
$$L_2 = \theta R_2 = L_0(1 + \alpha_2 \Delta T) \quad (2)$$

Subtracting (2) from (1)

$$\theta(R_1 - R_2) = \theta d = (\alpha_1 - \alpha_2)L_0 \Delta T \quad (3)$$
$$\because \quad R_1 - R_2 = d$$

Adding (1) and (2)

$$\theta(R_1 + R_2) = 2L_0 + (\alpha_1 + \alpha_2)L_0 \Delta T \simeq 2L_0 \quad (4)$$
$$\because \quad (\alpha_1 + \alpha_2)\Delta T << 2$$
$$R = \frac{R_1 + R_2}{2} = \frac{L_0}{\theta} = \frac{d}{(\alpha_1 - \alpha_2)\Delta T} \quad (5)$$

where we have used (3).

10.10 Let the initial length of the rod be $2x$ and the final total length be $2(x + \Delta x)$. Let the centre of the buckled rod be raised by y, then

$$\Delta x = \alpha x \Delta T$$

From the geometry of Fig. 10.6,

$$\begin{aligned} y &= [(x + \Delta x)^2 - x^2]^{1/2} = [2x\,\Delta x + (\Delta x)^2]^{1/2} \\ &\simeq \sqrt{2x\Delta x} \quad (\because\ \Delta x << 2x) \\ &= \sqrt{2\alpha x^2 \Delta T} \\ &= \sqrt{2 \times 12 \times 10^{-6} \times 20^2 \times 30} = 0.5367\,\text{m} \\ &= 53.67\,\text{cm} \end{aligned}$$

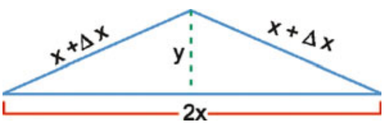


Fig. 10.6 Buckling of rail

10.11 Let L_0(S) and L_0(Cu) be the lengths of steel and copper rod at 0°C, respectively. Let the respective lengths be L(S) and L(Cu) at temperature T°C. Then

$$L(\mathrm{S}) = L_0(\mathrm{S})(1 + \alpha_\mathrm{s} T) \tag{1}$$

$$L(\mathrm{Cu}) = L_0(\mathrm{Cu})(1 + \alpha_\mathrm{Cu} T) \tag{2}$$

Subtracting (2) from (1)

$$L(\mathrm{S}) - L(\mathrm{Cu}) = L_0(\mathrm{S}) - L_0(\mathrm{Cu}) + [L_0(\mathrm{S})\alpha_\mathrm{s} - L_0(\mathrm{Cu})\alpha_\mathrm{Cu}]T \tag{3}$$

Now in the RHS, $L_0(\mathrm{S}) - L_0(\mathrm{Cu})$ is constant. If $L(\mathrm{S}) - L(\mathrm{Cu})$ is to remain constant, then $[L_0(\mathrm{S})\alpha_\mathrm{s} - L_0(\mathrm{Cu})\alpha_\mathrm{Cu}] = 0$, at any temperature T. This gives us

$$\frac{L_0(\mathrm{S})}{L_0(\mathrm{Cu})} = \frac{\alpha_\mathrm{cu}}{\alpha_\mathrm{s}} = \frac{1.7 \times 10^{-5}}{1.1 \times 10^{-5}} = \frac{17}{11} \tag{4}$$

Furthermore, $\quad L_0(\mathrm{S}) - L_0(\mathrm{Cu}) = 4\,\mathrm{cm}.$ (5)

Solving (4) and (5) we obtain

$$L_0(\mathrm{S}) = 11.33\,\mathrm{cm}; \quad L_0(\mathrm{Cu}) = 7.33\,\mathrm{cm}$$

10.12 Let the volume of mercury in the flask be $V_0\,\mathrm{cm}^3$ and that of glass flask $1000\,\mathrm{cm}^3$ at initial temperature T_1. At a higher temperature T_2 the volume of glass will be

$$V_\mathrm{g} = 1000(1 + \gamma_\mathrm{g}\Delta T) \tag{1}$$

where $\Delta T = T_2 - T_1$. The volume of mercury will be

$$V = V_0(1 + \gamma\Delta T) \tag{2}$$

The volume of air inside the flask at temperature T_2 will be

$$\begin{aligned} V_\mathrm{g} - V &= 1000(1 + \gamma_g\Delta T) - V_0(1 + \gamma\Delta T) \\ &= 1000 - V_0 + (1000\gamma_\mathrm{g} - V_0\gamma)\Delta T \end{aligned} \tag{3}$$

The RHS will be constant if

$$1000\gamma_g - V_0\gamma = 0 \tag{4}$$

for any value of ΔT. Therefore,

$$V_0 = 1000\frac{\gamma_\mathrm{g}}{\gamma} = \frac{1000 \times 27 \times 10^{-6}}{1.8 \times 10^{-4}} = 150\,\mathrm{cm}^3$$

where we have used $\gamma_\mathrm{g} = 3\alpha_\mathrm{g}$.

10.13 $$Y = \frac{F/A}{\Delta L/L} \quad (1)$$

$$\text{or} \quad F = \frac{YA\Delta L}{L} = \frac{YA\Delta L}{L_0(1+\alpha\Delta T)} \simeq \frac{YA\Delta L}{L_0} \quad (2)$$

$(\because \ \alpha\Delta T << 1)$

$$\text{But} \quad \Delta L = \alpha L_0 \Delta T$$

$$\therefore \quad F = YA\alpha\Delta T = 2.1 \times 10^{10} \times 0.5 \times 10^{-6} \times 12 \times 10^{-6} \times 20 = 2.52\,\text{N}$$

where we have used SI units.

10.14 The coefficient of apparent expansion of a liquid

$$A = \gamma - g = \gamma - 3\alpha = 11 \times 10^{-4} - 3 \times 8 \times 10^{-6} = 1.076 \times 10^{-3}$$

$$\text{Apparent expansion} = \frac{\text{mass expelled}}{\text{(mass left) (temperature rise)}}$$

$$A = \frac{W}{W_0\Delta T} = \frac{50 - W_0}{W_0\Delta T}$$

$$\therefore \quad W_0 = \frac{50}{1 + A\Delta T} = \frac{50}{1 + 1.076 \times 10^{-3} \times 80} = 46\,\text{g}$$

Gas Laws

10.15 $PV = n\,RT$ (gas equation)

$$\therefore \quad R = \frac{PV}{nT} = \frac{(1.0129 \times 10^5)\,(22.4 \times 10^{-3})}{(1.0)\,(273)}$$
$$= 8.31\,\text{J/mol/K}$$

10.16 $P_0 = 0.76 \times 13,600 \times 9.8 = 1.0129 \times 10^5$ Pa
Pressure at depth 30 m, $P = 30 \times 1000 \times 9.8 = 2.94 \times 10^5$ Pa.
$\therefore$ Total pressure inside the bubble, $P_1 = P_0 + P$

$$= (1.0129 + 2.94) \times 10^5 = 3.9529 \times 10^5\,\text{Pa}$$

$$P_2 V_2 = P_1 V_1 \qquad \text{(Boyle's law)}$$

$$\therefore \quad P_2(3V_1) = P_1 V_1$$

$$\therefore \quad P_2 = P_1/3 = 1.376 \times 10^5\,\text{Pa}$$

This corresponds to a water depth equivalent of $1.3176 \times 10^5 - 1.0129 \times 10^5 = 0.3047 \times 10^5$ Pa.

$$\text{Therefore water depth} = \frac{30 \times 0.3047 \times 10^5}{2.94 \times 10^5} = 3.11\,\text{m}$$

10.17 $\dfrac{\rho_1 T_1}{P_1} = \dfrac{\rho_2 T_2}{P_2}$

$$M_1 = V\rho_1, \qquad M_2 = V\,\rho_2$$

$$\therefore \quad \frac{M_1}{M_2} = \frac{\rho_1}{\rho_2} = \frac{P_1 T_2}{P_2 T_1}$$

$$\therefore \quad M_2 = \frac{P_2}{P_1}\cdot\frac{T_1}{T_2}M_1 = \frac{50}{76}\times\frac{273}{263}\times 175$$

$$= 119.5\,\text{kg}$$

10.18 Let n moles be total mass of air in the two bulbs.
Initially, $T = 273 + 20 = 293\,\text{K}$, $P = 76\,\text{cm}$ of Hg, $V = V_1 + V_2 = 100 + 500 = 600\,\text{cc}$.

$$n = \frac{PV}{RT} = \frac{70\times 600}{293R}$$

Finally, let n_1 and n_2 moles be the mass of air in the small and large bulb, respectively. Under new conditions

$$n_1 = \frac{P_1 V_1}{RT_1} = \frac{100P_1}{293R}$$

$$n_2 = \frac{P_2 V_2}{RT_2} = \frac{500P_2}{293R} \quad (\because \quad P_2 = P_1)$$

$$\text{But} \;\; n = n_1 + n_2 = \frac{100P_1}{293R} + \frac{500P_1}{373R} = \frac{70\times 600}{293R}$$

Cancelling off R, we find $P_1 = 85.23\,\text{cm}$ of Hg.

10.3.3 Heat Transfer

10.19 **(a)** In the first slab, heat flow is given by

$$-\frac{\mathrm{d}Q_1}{\mathrm{d}t} = \frac{k_1 A\,(T_1 - T)}{d_1} \tag{1}$$

In the second slab, heat flow is given by

$$-\frac{\mathrm{d}Q_2}{\mathrm{d}t} = \frac{k_2 A(T - T_2)}{d_2} \tag{2}$$

Now the continuity of heat flow requires that heat flow must be the same in both the slabs (Fig. 10.7). Thus

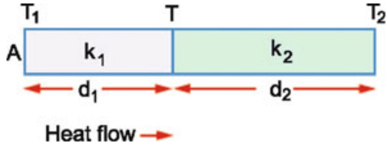


Fig. 10.7 Heat flow in the composite slab made of two slabs in series

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = \frac{\mathrm{d}Q_2}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{3}$$

Using (3) in (1) and (2)

$$T_1 - T = \frac{-d_1}{k_1 A}\frac{\mathrm{d}Q}{\mathrm{d}t} \tag{4}$$

$$T - T_2 = \frac{-d_2}{k_2 A}\frac{\mathrm{d}Q}{\mathrm{d}t} \tag{5}$$

Adding (4) and (5)

$$T_1 - T_2 = \frac{-1}{A}\left[\frac{d_1}{k_1} + \frac{d_2}{k_2}\right]\frac{\mathrm{d}Q}{\mathrm{d}t} \tag{6}$$

$$\text{or} \qquad -\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{A(T_1 - T_2)}{\frac{d_1}{k_1} + \frac{d_2}{k_2}} \tag{7}$$

(b) Rewriting (7)

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{A(T_1 - T_2)}{d}\frac{d}{\frac{d_1}{k_1} + \frac{d_2}{k_2}} = \frac{A(T_1 - T_2)k}{d} \tag{8}$$

with $d = d_1 + d_2$, and the equivalent conductivity

$$k_{\mathrm{eq}} = \frac{d_1 + d_2}{\frac{d_1}{k_1} + \frac{d_2}{k_2}} \tag{9}$$

Formula (9) can be generalized to any number of slabs in series.

$$k = \frac{\sum d_i}{\sum \frac{d_i}{k_i}} \tag{10}$$

(c) Eliminating $\mathrm{d}Q/\mathrm{d}t$ between (4) and (5)

$$T = \frac{(k_1 T_1/d_1) + (k_2 T_2/d_2)}{(k_1/d_1) + (k_2/d_2)} \tag{11}$$

10.20 **(a)** The rate of flow of heat through the composite slab (Fig. 10.8) is given by

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{(T_1 - T_2)}{d} \sum_{l}^{n} k_i A_i \tag{1}$$

(b) Rewriting (1)

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{(T_1 - T_2)}{d} \left(\sum_{1}^{n} A_i\right) \left(\frac{\sum k_i A_i}{\sum A_i}\right) \tag{2}$$

The equivalent conductivity of the system is

$$k_{\mathrm{eq}} = \frac{\sum k_i A_i}{\sum A_i} \tag{3}$$

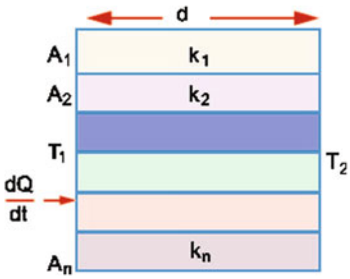


Fig. 10.8 Heat flow in a composite slab made of n slabs in parallel

10.21 $T = \dfrac{(k_1 T_1/d_1) + (k_2 T_2/d_2)}{(k_1/d_1) + (k_2/d_2)}$

as $d_1 = d_2$

$$T = \frac{k_1 T_1 + k_2 T_2}{k_1 + k_2} = \frac{92 \times 100 + 16 \times 0}{92 + 16} = 85.18\,°\mathrm{C}$$

10.22 Heat transferred/second

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = kA\frac{(\theta_2 - \theta_1)}{d} = k\pi r^2 \frac{(\theta_2 - \theta_1)}{d}$$
$$= 90\pi (0.01)^2 \frac{(100 - 0)}{0.2} = 14.137\,\mathrm{J/s}$$

Heat required to melt 0.05 kg of ice

$= 0.05 \times 8 \times 10^4 = 4000\,\mathrm{cal} = 16720\,\mathrm{J}$

Time required $= 16720/14.137 = 1183$ s.

10.23 Heat flow

$$\mathrm{d}Q = A\frac{\mathrm{d}T}{\mathrm{d}x}k = A\,aT\frac{\mathrm{d}T}{\mathrm{d}x}$$

or $\quad A\,aT\,\mathrm{d}T = \mathrm{d}Q\,\mathrm{d}x$

Integrating $\quad A\,a\int_{T_2}^{T_1} T\,\mathrm{d}T = \int_0^Q \mathrm{d}Q\int_0^L \mathrm{d}x$

$$\therefore \quad \frac{Aa}{2}\left(T_1^2 - T_2^2\right) = QL$$

or $\quad Q = \frac{aA}{2L}\left(T_1^2 - T_2^2\right)$

10.24 $\dfrac{\mathrm{d}Q}{\mathrm{d}t} = -kA\dfrac{\mathrm{d}T}{\mathrm{d}r}$

$$\therefore \quad \mathrm{d}T = -\frac{1}{k}\frac{\mathrm{d}Q}{\mathrm{d}t}\frac{\mathrm{d}r}{4\pi r^2}$$

When steady state is reached, $\mathrm{d}Q/\mathrm{d}t$ will be independent of r and is constant (Fig. 10.9). Integrating

$$\int_{T_1}^{T_2}\mathrm{d}T = -\frac{1}{4\pi k}\frac{\mathrm{d}Q}{\mathrm{d}t}\int_{r_1}^{r_2}\frac{\mathrm{d}r}{r^2} = -\frac{1}{4\pi k}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\frac{\mathrm{d}Q}{\mathrm{d}t}$$

$$\therefore \quad T_2 - T_1 = \frac{1}{4\pi k}\frac{(r_1 - r_2)}{r_1 r_2}\frac{\mathrm{d}Q}{\mathrm{d}t}$$

or $\quad T_1 - T_2 = \dfrac{1}{4\pi k}\dfrac{(r_2 - r_1)}{r_1 r_2}\dfrac{\mathrm{d}Q}{\mathrm{d}t}$

$$\therefore \quad \frac{\mathrm{d}Q}{\mathrm{d}t} = 4\pi k\frac{r_1 r_2}{r_2 - r_1}(T_1 - T_2)$$

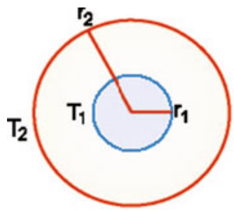


Fig. 10.9 Radial flow of heat through two concentric spheres

10.25 Rate of flow of heat

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kA\frac{\mathrm{d}T}{\mathrm{d}r} \tag{1}$$

Neglecting the area of the faces, area of the cylinder $A = 2\pi r L$. For steady state, $\mathrm{d}Q/\mathrm{d}t =$ constant. We can then write (1) as

$$\frac{\mathrm{d}r}{r} = -\frac{2\pi Lk}{\mathrm{d}Q/\mathrm{d}t}\,\mathrm{d}T$$

Integrating

$$\int_{r_1}^{r_2} \frac{\mathrm{d}r}{r} = \ln(r_2/r_1) = -\frac{2\pi Lk}{\mathrm{d}Q/\mathrm{d}t}(T_2 - T_1)$$

or $$\mathrm{d}Q/\mathrm{d}t = \frac{2\pi Lk}{\ln(r_2/r_1)}(T_1 - T_2)$$

10.26 $\mathrm{d}Q/\mathrm{d}t = kA(T_1 - T_2)/d$

$$= (0.59A) \times \frac{10}{0.01} = 590\,\mathrm{A\,J/s} \quad (1)$$

Let x m/s ice be added at the bottom of the layer.
Mass of ice formed per second

$$M = \rho \times A \quad (2)$$

The required energy per second

$$E = \rho \times AL \quad (3)$$

Equating (1) and (3), $\rho \times AL = 590\,\mathrm{A}$

$$\therefore \quad x = \frac{590A}{\rho AL} = \frac{590}{917 \times 333 \times 10^3} = 1.932 \times 10^{-6}\,\mathrm{m/s} = 0.00695\,\mathrm{m/h}$$

10.27 $$-\frac{\mathrm{d}\theta}{\mathrm{d}t} = C(\theta - \theta_0) \quad \text{(Newton's law of cooling)}$$

$$\frac{\theta_1 - \theta_2}{t} = C\left[\frac{\theta_1 + \theta_2}{2} - \theta_0\right] \quad (C = \text{constant})$$

$$\frac{85 - 75}{2} = C\left[\frac{85 + 75}{2} - 30\right]$$

$$\therefore \quad C = 0.1$$

$$\frac{55 - 45}{t} = 0.1\left[\frac{55 + 45}{2} - 30\right]$$

$$\therefore \quad t = 5 \text{ min}$$

10.28 $$-\frac{d\theta}{dt} = \frac{K}{ms}(\theta - \theta_0) \qquad (K = \text{constant}) \tag{1}$$

Let x be the water equivalent of the calorimeter.

$$\frac{50-40}{15} = \frac{K}{(x + 40 \times 1)}\left[\frac{50+40}{2} - \theta_0\right] \tag{2}$$

$$\frac{50-40}{33} = \frac{K}{(x + 100 \times 1)}\left[\frac{50+40}{2} - \theta_0\right] \tag{3}$$

Dividing (2) by (3)

$$\frac{x+100}{x+40} = \frac{11}{5}$$

or $\qquad x = 10\,\text{g}$

10.29 $$-\frac{d\theta}{dt} = \frac{K}{ms}(\theta - \theta_0)$$

Now $K \propto$ surface area or $\propto r^2$ and $m \propto r^3$.

$$\therefore \qquad d\theta/dt \propto 1/r$$

$$\therefore \qquad \frac{(d\theta/dt)_1}{(d\theta/dt)_2} = \frac{r_2}{r_1} = \frac{2}{1}$$

10.30 Assuming a linear variation of resistance with temperature

$$R_T = R_0(1 + \alpha T) \tag{1}$$

$$29.6 = 24.9(1 + 100\alpha) \tag{2}$$

$$\text{whence} \qquad \alpha = 1.8875 \times 10^{-3}\,\text{W}/^\circ\text{C} \tag{3}$$

$$26.3 = 24.9(1 + 1.8875 \times 10^{-3}\,T) \tag{4}$$

whence $T = 29.79^\circ$C

10.31 If R is the radius of the sun, r the mean distance of the earth from the sun, E the energy emitted from 1 m^2 of the sun's surface per second, T the absolute temperature of the sun's surface and σ the Boltzmann–Stefan constant, then

$$S = \frac{4\pi R^2 E}{4\pi r^2} = \frac{R^2 \sigma T^4}{r^2}$$

$$= \frac{(6.95 \times 10^8)^2(5.67 \times 10^{-8})(5740)^4}{(1.49 \times 10^{11})^2} = 1339\,\text{W/m}^2$$

10.32 Radiant intensity at the sun's surface is the power emitted by 1 m^2 of sun's surface.

$$\sigma T^4 = 63 \times 10^6$$

$$T = \left(\frac{63 \times 10^6}{5.67 \times 10^{-8}}\right)^{1/4} = 5773\,\mathrm{K}$$

10.33 $$-\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma A\left(T_1^4 - T_2^4\right) \qquad \text{(Stefan–Boltzmann formula)}$$

$$= \sigma\,\pi\,r^2\left(T_1^4 - T_2^4\right)$$

$$= \pi(5.67 \times 10^{-8})(0.05)^2(500^2 - 300^2)$$

$$= 7.12 \times 10^{-5}\,\mathrm{W}$$

10.34 $$\lambda_\mathrm{m} T = b = 3 \times 10^{-3}\,\mathrm{m\,K} \qquad \text{(Wien's law)} \qquad (1)$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma T^4 \qquad \text{(Boltzmann law)} \qquad (2)$$

If $\mathrm{d}E/\mathrm{d}t$ goes down to 1/16 of its original value then by (2) the temperature $T \to T/2$. Therefore $\lambda_\mathrm{m} \to 2\lambda_\mathrm{m}$ by (1). Thus the wavelength under new conditions $\lambda'_\mathrm{m} = 2\lambda_\mathrm{m} = 2 \times 480 = 960\,\mathrm{nm}$.

10.3.4 Specific Heat and Latent Heat

10.35 $$Q = \int_{20}^{80} mC_\mathrm{p}\mathrm{d}T + mL + \int_{80}^{200} mC_\mathrm{p}\mathrm{d}T$$

$$= \int_{20}^{200} mC_\mathrm{p}\mathrm{d}T + mL \qquad (\because\ C_\mathrm{p} \text{ relation for solid and liquid phase is identical})$$

$$= \int_{20}^{200} 1 \times (30.6 + 0.0103\,T)\mathrm{d}T + 1 \times 6000$$

$$= 11710\,\mathrm{J}$$

10.36 $$\overline{C} = \frac{\int C\mathrm{d}T}{\int \mathrm{d}T} = \frac{\int_0^T (A + BT^2)\mathrm{d}T}{T} = \frac{1}{T}\left(AT + \frac{BT^3}{3}\right) = A + \frac{BT^2}{3}$$

$$C\,(\text{midpoint}) = A + B(T/2)^2 = A + \frac{BT^2}{4}$$

$$\therefore \qquad \overline{C} - C\,(\text{midpoint}) = A + \frac{BT^2}{3} - \left(A + \frac{BT^2}{4}\right) = \frac{BT^2}{12}$$

10.37 Let the specific heats of liquids A, B and C be, respectively, C_A, C_B and C_C. When A and B are mixed, equilibrium of the mixture requires that

$$MC_A(16-12) = MC_B(18-16)$$
$$\text{or} \qquad C_B = 2C_A$$

When B and C are mixed

$$MC_B(23-18) = MC_C(28-23)$$
$$\text{or} \qquad C_C = C_B = 2C_A$$

When A and C are mixed, let the equilibrium temperature be T.

$$MC_A(T-12) = MC_C(28-T) = M2C_C(28-T)$$
$$\therefore \qquad T = 22.67\,°\text{C}$$

10.38 **(a)** The block is fixed. The kinetic energy of the bullet is entirely converted into heat energy. Let m be the mass and v the velocity of the bullet.

$$Q = \frac{1}{2}mv^2 = \frac{1}{2}(3 \times 10^{-3})(120)^2 = 21.6\,\text{J} = 5.167\,\text{cal}$$

Rise in temperature

$$\Delta T = \frac{Q}{mc} = \frac{5.167}{3 \times 0.031} = 55.56\,°\text{C}$$

(b) The block is free to move. In this case, after the collision some kinetic energy will go into the block + bullet system.

$$\frac{1}{2}mv^2 = \frac{1}{2}(M+m)v_1^2 + Q \qquad \text{(energy conservation)} \qquad (1)$$
$$mv = (M+m)v_1 \qquad \text{(momentum conservation)} \qquad (2)$$

where M is the mass of the block and v_1 the final velocity of the block + bullet system. Eliminating v_1 and simplifying

$$Q = \frac{1}{2}mv^2\left(\frac{M}{M+m}\right) = \frac{1}{2} \times 3 \times 10^{-3} \times (120)^2\left(\frac{50}{50+3}\right)$$
$$= 20.38\,\text{J} = 4.875\,\text{cal}$$
$$\Delta T = \frac{Q}{mc} = \frac{4.875}{3 \times 0.031} = 52.42\,°\text{C}$$

10.39 Potential energy available from m kg of water through a fall of h metres is mgh J. $15/100\,mgh$. Mechanical energy is converted into heat.

$$\therefore \quad \frac{15}{100} mgh = mc\Delta T \times 4.18 = m \times 1000\Delta T \times 4.18$$

$$\therefore \quad \Delta T = \frac{15 \times 9.8 \times 25}{100 \times 1000 \times 4.18} = 0.0088°\text{C}$$

10.40 Mechanical energy available, $W = mgh$
Heat absorbed, $H = mc\Delta T$ J.
Loss of mechanical energy = gain of heat energy

$$\therefore \quad mgh = mc\Delta T \text{ J}$$

$$\therefore \quad \Delta T = \frac{gh}{cJ} = \frac{9.8 \times 100}{30.6 \times 4.18} = 7.66°\text{C} = 7.66\,\text{K}$$

10.3.5 Thermodynamics

10.41 **(a)** **(i)** In an isobaric process pressure remains constant.
(ii) In an isochoric process volume remains constant.
(iii) In an adiabatic process heat is neither absorbed nor evolved by the system.
(iv) In an isothermal process temperature remains constant.

(b) For adiabatic process use the relation

$$PV^{\gamma} = P_1 V_1^{\gamma} = \text{const.} \tag{1}$$

$$\text{or} \quad P = \frac{P_1 V_1^{\gamma}}{V^{\gamma}} \tag{2}$$

Work done on the gas

$$W = -\int_{v_1}^{v_2} P\mathrm{d}V = -\int_{v_1}^{v_2} P_1 V_1^{\gamma} \frac{\mathrm{d}V}{V^{\gamma}} = -P_1 V_1^{\gamma} \int_{v_1}^{v_2} \frac{\mathrm{d}V}{V^{\gamma}}$$

$$\frac{P_1 V_1^{\gamma}}{\gamma - 1}\left(V_2^{1-\gamma} - V_1^{1-\gamma}\right) = \frac{1}{\gamma - 1}\left(P_1 V_1^{\gamma} V_2^{1-\gamma} - P_1 V_1\right)$$

$$\therefore \quad W = \frac{1}{\gamma - 1}(P_2 V_2 - P_1 V_1) \tag{3}$$

where we have used the relation $P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$.
W is positive if $V_1 > V_2$ (compression) and negative if $V_1 < V_2$ (expansion).

10.42 Applying the gas equation

$$T_A = \frac{P_A V_A}{nR} = \frac{(1.013 \times 10^5)(44.8)}{(2000)(8.31)} = 273\,\text{K}$$

As the process AB is isometric (isochoric), $V_B = V_A$.

$$T_B = \frac{T_A P_B}{P_A} = \frac{273 \times 2}{1} = 546\,\text{K}$$

As the process CA is isobaric, Charles' first law applies. Thus

$$\frac{V_C}{T_C} = \frac{V_A}{T_A} \rightarrow V_C = \frac{V_A T_C}{T_A}$$

As BC is an isothermal process, $T_C = T_B = 546\,\text{K}$.

$$\therefore \quad V_C = \frac{V_A T_B}{T_A} = \frac{(44.8)(546)}{273} = 89.6\,\text{m}^3$$

10.43 For adiabatic process, $\text{d}Q = 0$ so that $\text{d}U = -\text{d}W$. The energy of 1 mol of monatomic gas is given by

$$U = \frac{3}{2}RT$$

$$\therefore \quad \text{d}U = \frac{3}{2}R\,\text{d}T$$

$$\text{d}W = P\text{d}V = \frac{RT}{V}\,\text{d}V$$

$$\therefore \quad \frac{3}{2}R\text{d}T = -\frac{RT}{V}\,\text{d}V$$

$$\therefore \quad \frac{\text{d}T}{T} + \frac{2}{3}\frac{\text{d}V}{V} = 0$$

Integrating

$$\ln\, T + \frac{2}{3}\ln\, V = \text{constant}$$

$$\text{or} \quad TV^{2/3} = \text{constant}$$

Eliminating T from the gas equation

$$PV^{5/3} = \text{constant}$$

10.44 **(a)** $U_f - U_i = \Delta U = Q - W$ (First law of thermodynamics)

Let a system change from an initial equilibrium state i to a final equilibrium state f in a definite way, the heat absorbed by the system being Q and the work done by the system being W. The quantity $Q - W$ represents the change in internal energy of the system.
Both Q and W are path dependent while ΔU is path independent.

(b) $$W = \int \mathrm{d}W = -\int P\,\mathrm{d}V \tag{1}$$

$PV = nRT_1 = \text{constant}$

or $$P = \frac{nRT_1}{V} \quad (T = \text{constant for isothermal process}) \tag{2}$$

Substituting (2) in (1)

$$W = -nRT\int_{V_1}^{V_2}\frac{\mathrm{d}V}{V} = -nRT\ \ln\left(\frac{V_2}{V_1}\right)$$

(c) At the beginning of the cycle, $P_1 = 1.01 \times 10^5$ Pa, $V_1 = 1\,\text{m}^3$, $T_1 = 273.15$ K.
At the end of stage 1, $P_2 = 5.05 \times 10^4$ Pa, $V_2 = 2\,\text{m}^3$, $T_2 = 273.15$ K.
At the end of stage 2, $P_3 = 1.01 \times 10^5$ Pa, $V_3 = 2\,\text{m}^3$, $T_3 = T_2\,P_3/P_2 = 546.3$ K.

(i) $$n = \frac{P_1V_1}{RT_1} = \frac{1.01 \times 10^5 \times 1.0}{8.31 \times 273.15} = 44.5\,\text{mol}$$

(ii) Stage1, $$W_1 = -nRT\ \ln\ (V_2/V_1)$$
$$= -44.5 \times 8.31 \times 273.15\ \ln(2/1) = -70046\,\text{J}$$

Stage2, $$W_2 = P_2\Delta V = P_3(V_3 - V_2) = P(V_2 - V_2) = 0$$

Stage3, $$W_3 = P_3(V_1 - V_3) = 1.01 \times 10^5(1 - 2)$$
$$= -1.015 \times 10^5\,\text{J}$$

(iii) $\Delta U = 0$, overall.

10.45 **(a)** and **(b)** $$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

$$\therefore \quad T_1 = \frac{P_1}{P_2}\frac{V_1}{V_2}T_2 = \frac{P_1}{3P_1}\frac{V_1}{V_1}1083 = 361\,\text{K}$$

$$n = \frac{P_1V_1}{RT_1} = \frac{(10^5)(0.06)}{(8.31)(361)} = 2\,\text{mol}$$

$$\frac{P_3V_3}{T_3} = \frac{P_2V_2}{T_2}$$

$$\therefore \quad T_3 = \frac{P_3}{P_2}\frac{V_3}{V_2}T_2 = \frac{P_2}{P_2}\frac{(V_2/3)}{V_2}1083 = 361\,\text{K}$$

$T_4 = T_3 = 361$ K $\quad$ ($\because$ process $3 \rightarrow 4$ isothermal process)

$$\frac{P_4V_4}{T_4} = \frac{P_3V_3}{T_3}$$

$$\therefore \qquad P_4 = \frac{V_3}{V_4}\frac{T_4}{T_3}P_3 = \frac{V_3}{3V_3}\frac{T_3}{T_3}P_2 = \frac{P_2}{3} = \frac{3P_1}{3} = P_1 = 10^5\,\mathrm{N/m^2}$$

$$V_4 = \frac{nRT_4}{P_4} = \frac{2 \times 8.31 \times 361}{10^5} = 0.06\,\mathrm{m^3}$$

Thus the P, V, T coordinates of the initial and final points on the indicator diagram are identical.

(c) Heat gained by N_2 in the first (isochoric) process (Fig. 10.10):

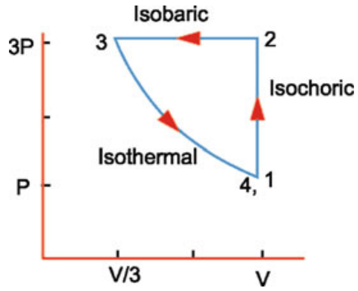


Fig. 10.10 The P–V diagram

$$C_V = C_P/\gamma = 29.12/1.4 = 20.8\,\mathrm{J/(mol\,K)}$$
$$Q_1 = nC_v\Delta T = 2 \times 20.8 \times (1083 - 311) = 32115\,\mathrm{J}$$
$$W_{12} = 0 \qquad (\because \text{ process } 1 \to 2 \text{ is isochoric})$$

$$\therefore \qquad \Delta U_1 = Q_1 = 32115\,\mathrm{J}$$
$$Q_2 = nC_p\Delta T = 2 \times 29.12 \times (361 - 1083) = -42049\,\mathrm{J}$$
$$W_{23} = P\Delta V = P_2(V_3 - V_2) = -3 \times 10^5 \times \frac{2}{3} \times 0.06 = -12{,}000\,\mathrm{J}$$
$$\therefore \qquad \Delta U_2 = Q_3 - W_{23} = -42049 + 12000 = -30049\,\mathrm{J}$$
$$\Delta U_3 = 0 \qquad (\because \text{ process } 3 \to 4 \text{ is isothermal})$$

(d) Net change in energy

$$\Delta U = \Delta U_1 + \Delta U_2 + \Delta U_3 = 32115 - 30049 + 0 = 2066\,\mathrm{J}$$

The expected value is zero.

10.46 Number of degrees of freedom, $f = \dfrac{2}{\gamma - 1}$ we can find γ from the relation

$$T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1}$$

$$\therefore \qquad \left(\frac{V_2}{V_1}\right)^{\gamma-1} = \frac{T_1}{T_2} = 1.32$$

$$\therefore \qquad 2^{\gamma-1} = 1.32$$

whence $\gamma = 1 + \dfrac{\log 1.32}{\log 2.0} = 1.4$

$$\therefore \qquad f = \frac{2}{1.4 - 1} = 5$$

10.47 The efficiency of the engine is

$$e = 1 - \frac{T_2}{T_1} = 1 - \frac{300}{400} = 0.25 \quad \text{or} \quad 25\%$$

Work done by the engine

$$\begin{aligned} W &= e \times Q_1 = 0.25 \times 10^8\,\text{cal} \\ &= 0.25 \times 10^8 \times 4.18\,\text{J} \\ &= 1.05 \times 10^8\,\text{J} \end{aligned}$$

10.48 $e = 1 - \dfrac{T_2}{T_1} = \dfrac{1}{6}$ (1)

$\therefore \qquad 5T_1 - 6T_2 = 0$ (2)

When the temperature of the sink is reduced by 62°C, the efficiency becomes

$e' = 2e$ (3)

$e' = 2e = 1 - \dfrac{(T_2 - 62)}{T_1} = \dfrac{1}{3}$ (4)

$\therefore \quad 2T_1 - 3T_2 - 186 = 0$ (5)

Solving (2) and (5), $T_1 = 372\,\text{K} = 99°\text{C}$, $T_2 = 310\,\text{K} = 37°\text{C}$.

10.49 **(a)** $PV = P_0V_0$ (isothermal conditions) (1)

Dividing by m, the mass of air

$$\frac{PV}{m} = \frac{P_0V_0}{m}$$

or $\dfrac{P}{\rho} = \dfrac{P_0}{\rho_0}$

$\therefore \qquad \rho = \dfrac{P\rho_0}{P_0}$ (2)

where the density of air on earth's surface is ρ_0 and pressure is P_0, the corresponding quantities at height y being ρ and P.

Now, for a small increase in height $\mathrm{d}y$, the pressure decreases by $\mathrm{d}p$ and is given by

$$\mathrm{d}p = -\rho g\,\mathrm{d}y = -\frac{P}{P_0}\rho_0 \mathrm{g}\,\mathrm{dy} \tag{3}$$

where we have used (2). The negative sign shows that the pressure decreases as the height increases.
Integrating

$$\int_0^h \mathrm{d}y = -\frac{P_0}{\rho_0 g}\int_{P_0}^{P}\frac{\mathrm{d}p}{P}$$

$$\therefore \quad h = -\frac{\mathrm{p}_0}{\mathrm{\rho}_0 g}\ln\left(\frac{p}{P_0}\right)$$

$$\therefore \quad p = p_0 \exp\left(-\frac{\rho_0 g h}{P_0}\right) \tag{4}$$

$$\text{Now,} \quad P_0 V_0 = \mu RT \tag{5}$$

since the temperature is assumed to be constant and μ is in moles. Furthermore

$$\rho_0 = \frac{\mu M}{V_0} \tag{6}$$

where M is the molecular weight. Combining (5) and (6)

$$\frac{\rho_0}{P_0} = \frac{M}{RT} \tag{7}$$

Substituting (7) in (4)

$$p = p_0 \exp\left(-\frac{Mgh}{RT}\right) \tag{8}$$

(b) If n is the number density, that is, the number of molecules per unit volume and m_0 the mass of each molecule then

$$\rho = m_0 n$$

$$\text{and} \quad \rho_0 = m_0 n_0$$

$$\therefore \quad \frac{\rho}{\rho_0} = \frac{n}{n_0} \tag{9}$$

Combining (9) with (2)

$$\frac{p}{P_0} = \frac{n}{n_0} \tag{10}$$

Using (10) in (8)

$$n = n_0 \exp\left(-\frac{Mgh}{RT}\right) \tag{11}$$

(c) $\frac{P}{P_0} = \frac{1}{2} = \exp\left(-\frac{Mgh}{RT}\right)$

$$\therefore \quad h = \frac{RT}{Mg} \ln 2 = \frac{8.31 \times 273 \times 0.693}{0.029 \times 9.8} = 54224\,\text{m} \simeq 54\,\text{km}$$

10.50 **(a)** For a Carnot cycle

(i) $e = \frac{Q_H - Q_C}{Q_H}$

(ii) $e = \frac{T_H - T_C}{T_H}$

The symbols H and C are for hot and cold reservoirs.

(b) The Otto cycle, Fig. 10.11, consists of two reversible adiabatic processes (paths AB and CD) and two reversible isochoric processes (path DA and BC).

Suppose we start at the point C. The temperature T_C at C is low, slightly above atmospheric temperature. The cylinder is filled with air charged with the combustible gas or vapour. The air is compressed adiabatically to the point D. At D a spark causes combustion, heating the air at constant volume to the point A. The heated air expands adiabatically along the path AB. At B, a valve is opened and the pressure drops to that of the atmosphere. The point C is reached at constant volume. The cycle is complete.

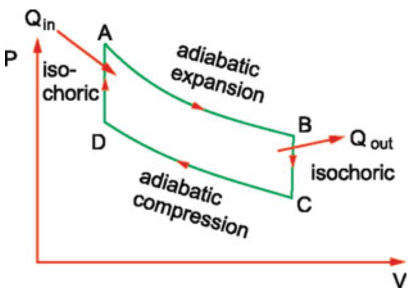


Fig. 10.11 The Otto cycle

During the isochoric heating and cooling no work can be done by or on the gas:

$$\Delta Q = \Delta U = \int_1^2 \mathrm{d}U = C_v \int_{T_1}^{T_2} \mathrm{d}T = C_v(T_2 - T_1) \tag{1}$$

so that

$$Q_{\text{out}} = C_{\text{v}}(T_{\text{B}} - T_{\text{C}}),\ \ Q_{\text{in}} = C_v(T_A - T_{\text{D}}) \tag{2}$$

where the heats (positive) are those which are given out and put into the system, respectively. The thermodynamic efficiency

$$e = \frac{\text{work done by the gas}}{\text{heat put into the system}} = \frac{W}{Q_{\text{in}}} \tag{3}$$

Since the internal energy does not change over the entire cycle, by first law of thermodynamics, net heat added to the system equals the work done by the system, so that

$$W = Q_{\text{in}} - Q_{\text{out}} \tag{4}$$

$$e = \frac{W}{Q_{\text{in}}} = \frac{Q_{\text{in}} - Q_{\text{out}}}{Q_{\text{in}}} = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}} = \left(1 - \frac{T_{\text{B}} - T_{\text{C}}}{T_{\text{A}} - T_{\text{D}}}\right) \tag{5}$$

In an adiabatic expansion or compression

$$TV^{\gamma-1} = \text{constant} \tag{6}$$

$$\therefore \quad T_{\text{A}} V_{\text{A}}^{\gamma-1} = T_{\text{B}} V_{\text{B}}^{\gamma-1}, T_{\text{D}} V_{\text{D}}^{\gamma-1} = T_{\text{C}} V_{\text{C}}^{\gamma-1} \tag{7}$$

$$\text{or} \quad T_{\text{A}} = T_{\text{B}} \left(\frac{V_{\text{B}}}{V_{\text{A}}}\right)^{\gamma-1}, \quad T_{\text{D}} = T_{\text{C}} \left(\frac{V_{\text{C}}}{V_{\text{D}}}\right)^{\gamma-1} \tag{8}$$

From Fig. 10.11 we note that

$$\frac{V_{\text{B}}}{V_{\text{A}}} = \frac{V_{\text{C}}}{V_{\text{D}}} = r \qquad \text{(compression ratio)} \tag{9}$$

Using (8) and (9) in (5)

$$e = (1 - r^{1-\gamma}) \tag{10}$$

Thus the higher the compression ratio the greater is the efficiency.

(c)

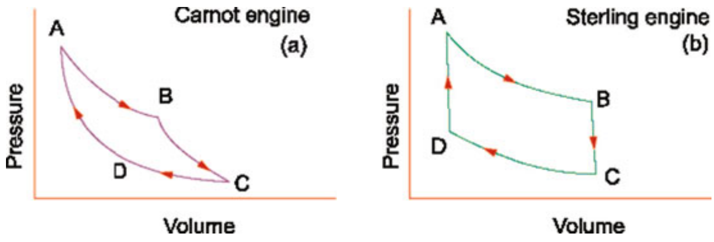


Fig. 10.12 **(a)** Carnot cycle consisting of two isothermic processes AB and CD; two adiabatic processes BC and DA. **(b)** Sterling cycle consisting of two isothermic process AB and CD, two isochoric processes BC and DA

10.51 **(a)** $n = \dfrac{PV}{RT} = \dfrac{1.013 \times 10^5 \times 1 \times 10^{-3}}{8.31 \times 273} = 0.04465\,\text{mol}$

(i) $\mathrm{d}Q = nC_\mathrm{P}\mathrm{d}T$

$$\Delta S = \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{\mathrm{d}Q}{T} = nc_\mathrm{p} \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{\mathrm{d}T}{T} = nc_\mathrm{p} \ln\left(\frac{T_\mathrm{f}}{T_\mathrm{i}}\right)$$
$$= 0.04465 \times 21 \times \ln(500/273) = 0.567\,\text{J/K}$$

(ii) $\mathrm{d}Q = nC_\mathrm{v}\mathrm{d}T$

$$\Delta S = \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{\mathrm{d}Q}{T} = nc_\mathrm{v} \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{\mathrm{d}T}{T} = nc_\mathrm{v} \ln\left(\frac{T_\mathrm{f}}{T_\mathrm{i}}\right)$$
$$= 0.04465 \times 12.7 \times \ln(500/273) = 0.343\,\text{J/K}$$

(b) $\mathrm{d}Q_1 = mC_\mathrm{v}\mathrm{d}T$

$$\Delta S_1 = m\,C_\mathrm{v} \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{\mathrm{d}T}{T} = mC_\mathrm{v} \ln\left(\frac{T_\mathrm{f}}{T_\mathrm{i}}\right)$$
$$= 1.0 \times 4.13 \times 10^3 \times \ln(313/273) = 695\,\text{J/K}$$

When 0.5 kg water at 0°C is mixed with 0.5 kg water at 100°C, the final temperature would be 50°C.

$$\begin{aligned}\Delta S_2 &= 0.5 \times 4.13 \times 10^3 \times \ln(313/273) \\ &\quad + 0.5 \times 4.13 \times 10^3 \times \ln(313/373) \\ &= 282.35 - 362.15 = -79.8\,\text{J/K}\end{aligned}$$

10.52 **(a)** The internal energy of an ideal gas is given by $U = nC_V T$. In isothermal expansion where the temperature and the amount of gas remain constant, the internal energy does not change. Thus $\Delta U = 0$.

(b) The work done is

$$W = -\int_{V_1}^{V_2} P\,\mathrm{d}V = -\int_{V_1}^{V_2} \left(\frac{nRT}{V}\right)\mathrm{d}V$$

$$= -nRT\int_{V_1}^{V_2} \frac{\mathrm{d}V}{V} = -nRT\ln(V_2/V_1).$$

(c) Using the first law of thermodynamics, $\Delta Q = \Delta U + W$, and putting $\Delta U = 0$, we have

$$\Delta Q = W = nRT\ln(V_2/V_1)$$

(d) The change in entropy is

$$\Delta S = \int \mathrm{d}s = \int \frac{\mathrm{d}Q}{T} = \frac{1}{T}\int \mathrm{d}Q$$

because the temperature T does not change. Thus

$$\Delta S = nR\ln(V_2/V_1)$$

(e) By assumption the temperature of the reservoir does not change and because it loses heat ΔQ to the gas, the entropy change of the reservoir will be

$$\Delta S_{\text{res}} = -\Delta S = -nR\ln(V_2/V_1)$$

Therefore the entropy change of the system plus the reservoir equals zero, which is the definition of a reversible process.

10.53 **(a)** The internal energy U of a system tends to increase if energy is added as heat Q and tends to decrease if energy is lost as work W done by the system.
Heat is energy that is transferred from one body to another due to difference in temperature of the bodies.
Enthalpy (H) is the total heat and is defined by

$$H = U + PV$$

Work (W) is energy that is transferred from one body to another body due to a force that acts between them.
The function, $G = U + PV - TS$, is known as Gibb's function or Gibb's energy.

Relations:

(i) *Enthalpy*

$$H = U + PV \tag{1}$$

$$\mathrm{d}H = \mathrm{d}U + P\mathrm{d}V + V\mathrm{d}P = T\mathrm{d}s + V\mathrm{d}p \tag{2}$$

$$\because \quad T\mathrm{d}s = \mathrm{d}U + P\mathrm{d}V \tag{3}$$

It follows that enthalpy is a function of entropy (S) and pressure (P).

$$\therefore \quad H = f(S, P)$$

$$\mathrm{d}H = \left(\frac{\partial H}{\partial S}\right)_{\mathrm{P}} \mathrm{d}S + \left(\frac{\partial H}{\partial P}\right)_{\mathrm{S}} \mathrm{d}P \tag{4}$$

Comparing (4) with (2)

$$\left(\frac{\partial H}{\partial S}\right)_{\mathrm{P}} = T \text{ and } \left(\frac{\partial H}{\partial P}\right)_{\mathrm{S}} = V \tag{5}$$

(ii) *Gibb's function:*

$$G = U + PV - TS \tag{6}$$

$$\mathrm{d}G = \mathrm{d}U + P\mathrm{d}V + V\mathrm{d}P - T\mathrm{d}S - S\mathrm{d}T$$

$$\text{But} \quad T\mathrm{d}S = \mathrm{d}U + P\mathrm{d}V$$

$$\therefore \quad \mathrm{d}G = V\mathrm{d}P - S\mathrm{d}T \tag{7}$$

Thus G is a function of two independent variables P and T.

$$G = f(P, T)$$

$$\mathrm{d}G = \left(\frac{\partial G}{\partial P}\right)_{\mathrm{T}} \mathrm{d}P + \left(\frac{\partial G}{\partial T}\right)_{\mathrm{P}} \mathrm{d}T \tag{8}$$

Comparing (8) with (7)

$$\left(\frac{\partial G}{\partial P}\right)_{\mathrm{T}} = V \tag{9}$$

$$\left(\frac{\partial G}{\partial P}\right)_{\mathrm{P}} = -S \tag{10}$$

(iii) *Internal energy*:

$$\mathrm{d}Q = T\mathrm{d}S \tag{11}$$
$$\mathrm{d}U = \mathrm{d}Q - \mathrm{d}W \tag{12}$$
$$\mathrm{d}W = P\mathrm{d}V \qquad \text{(isobaric process)} \tag{13}$$
$$\therefore \quad \mathrm{d}U = \mathrm{d}Q - P\mathrm{d}V \tag{14}$$
$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V \tag{15}$$
$$\therefore \quad \left(\frac{\partial U}{\partial S}\right)_{\mathrm{V}} = T \tag{16}$$
$$\therefore \quad \left(\frac{\partial U}{\partial V}\right)_{\mathrm{S}} = -P \tag{17}$$

(b) The quantities U, T, S, P and V are functions of the condition or state of the body only, in other words, all the differentials are perfect differentials and are state variables. Since the differentials which occur in (15) are perfect differentials, they are valid for all changes whatever their nature. On the other hand, dQ is not a perfect differential, but represents only an infinitesimal quantity of heat, and for a cycle $\int \mathrm{d}Q$ is not zero, but is equal to the work done. Similarly, dW is also not a perfect differential.

Note that the internal energy, the entropy and the volume are all proportional to the mass of the substance under consideration, while the temperature and the pressure are independent of it.

The condition of a given mass of a body (say 1 mol) can be defined by U, T, S, P, V or combinations of them, of which only two are independent. It follows that enthalpy and Gibb's function are also acceptable as state functions, apart from the internal energy but not the heat or work.

10.3.6 Elasticity

10.54 **(a)** $\eta = \dfrac{\text{shear stress}}{\text{shear strain}} = \dfrac{F/A}{\Delta x/y} = \dfrac{100 \times 10^6/1^2}{0.1/10} = 10^{10}\,\text{Pa}$

(b) $K = \dfrac{\Delta P}{(-\Delta V/V)} = \dfrac{100 \times 10^6}{1/100} = 10^{10}\,\text{Pa}$

10.55 **(i)** $\text{Stress} = \dfrac{\text{force}}{\text{area}} = \dfrac{mg}{A} = \dfrac{20 \times 9.8}{20 \times 10^{-6}} = 9.8 \times 10^6\,\text{Pa}$

(ii) $\text{Strain} = \dfrac{\text{elongation}}{\text{original length}} = \dfrac{2.5 \times 10^{-2}}{10} = 2.5 \times 10^{-3}$

(iii) $\text{Young's modulus} = \dfrac{\text{stress}}{\text{strain}} = \dfrac{9.8 \times 10^6}{2.5 \times 10^{-3}} = 3.92 \times 10^9\,\text{Pa}$

10.56 For a perfect gas of 1 mol

$$PV = RT \tag{1}$$

Under isothermal conditions T = constant. Differentiating (1)

$$P\mathrm{d}V + V\mathrm{d}P = 0 \tag{2}$$

The bulk modulus for the isothermal process

$$K_\mathrm{T} = -V\left(\frac{\mathrm{d}P}{\mathrm{d}V}\right)_\mathrm{T} = P \tag{3}$$

For adiabatic compression in which heat of compression remains in the gas

$$PV^\gamma = \text{constant} \tag{4}$$

where $\gamma = C_\mathrm{p}/C_\mathrm{v}$ is the ratio of specific heats at constant pressure and constant volume. Differentiating (4)

$$\gamma PV^{\gamma-1}\,\mathrm{d}V + V^\gamma \mathrm{d}P = 0 \tag{5}$$

Thus adiabatic elasticity K_H is given by

$$K_\mathrm{H} = -V\left(\frac{\mathrm{d}P}{\mathrm{d}V}\right)_\mathrm{H} = \gamma P \tag{6}$$

It follows that

$$K_\mathrm{H} = \gamma K_\mathrm{T} \tag{7}$$

The adiabatic elasticity is greater than the isothermal elasticity by a factor γ which is always greater than unity.

10.57 $Y = 2\eta(1+\sigma)$

$$\therefore \qquad \sigma = \frac{Y}{2\eta} - 1 = \frac{1}{2} \times 2.5 - 1 = 0.25$$

10.58 From Fig. 10.13 the new length $L' = 2\mathrm{AD} = 2\sqrt{\mathrm{AC}^2 + \mathrm{CD}^2}$

$$= 2\sqrt{(0.6)^2 + (0.02)^2} = 1.200666\,\mathrm{m}$$

Elongation of the wire, $\Delta L = L' - L = 1.200666 - 1.20 = 0.000666\,\mathrm{m}$.

Strain $= \Delta L/L = 0.000666/1.2 = 5.55 \times 10^{-4}$

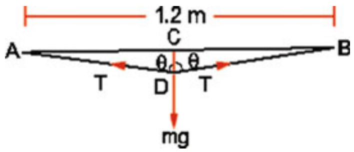


Fig. 10.13 Load fixed to the midpoint of a horizontal wire

For equilibrium

$$2T\cos\theta = \text{mg}$$

$$F = T = \frac{mg}{2\cos\theta} = \frac{29\times 10^{-3}\times 9.8}{2\times(0.02/60)} = 426.3\,\text{N}$$

$$\text{Stress} = \frac{F}{A} = \frac{426.3}{\pi(0.05\times 10^{-3})^2} = 5.43\times 10^{10}\,\text{Pa}$$

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{5.43\times 10^{10}}{5.55\times 10^{-4}} = 9.78\times 10^{13}\,\text{Pa}$$

10.59 Elastic energy $E = \frac{1}{2}Y\ (\text{strain})^2\ (\text{volume})$

$$E = \frac{1}{2}\times 6\times 10^8\left(\frac{0.05}{0.20}\right)^2(2\times 10^{-6}\times 0.25) = 9.375\,\text{J}$$

The elastic energy is converted into kinetic energy.

$$E = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2\times 9.375}{15\times 10^{-3}}} = 35.3\,\text{m/s}$$

10.60 $F = m\omega^2 r$

$$\text{Breaking stress} = \frac{F}{A} = \frac{m\omega^2 r}{A} = 4.8\times 10^7$$

$$\omega = \sqrt{\frac{4.8\times 10^7 A}{mr}} = \sqrt{\frac{4.8\times 10^7\times 10^{-6}}{10\times 0.3}} = 4\,\text{rad/s}$$

10.61 Stretching force = weight of the wire = (volume) (density) $\times g$

$$F = LA\rho g$$

where L is the length of wire, ρ the density, A the area of cross-section and g the acceleration due to gravity.

Breaking stress = maximum stretching force/area

$$= \frac{LA\rho g}{A} = L\rho g$$

$$7.8 \times 10^8 = L \times 7800 \times 9.8$$

$$\therefore \quad L = 1.021 \times 10^4\,\text{m} = 10.2\,\text{km}$$

Note that the result is independent of cross-sectional area of the wire.

10.3.7 Surface Tension

10.62 $S = \dfrac{\left(h + \frac{r}{3}\right) r\rho g}{2\cos\theta}$

Assuming that the contact angle $\theta = 0$

$$h = \frac{2s}{r\,\rho g} - \frac{r}{3} = \frac{2 \times 0.072}{10^{-3} \times 10^3 \times 9.8} - \frac{10^{-3}}{3}$$

$$= 0.01436\,\text{m}$$

$$= 1.436\,\text{cm}$$

10.63 Pressure due to water column of depth h is

$$P = hg\rho$$

Total pressure of the bubble, ignoring surface tension,

$$P' = P + P_0 = hg\rho + P_0$$

$$= 100 \times 9.8 \times 1000 + 1.01 \times 10^5$$

$$= 10.81 \times 10^5$$

$$P'V = nRT$$

$$\therefore \quad V = \frac{nRT}{P'} = \frac{1 \times 8.314 \times 293}{10.81 \times 10^5} = 2.25 \times 10^{-3}\,\text{m}^3$$

10.64 As the drops are incompressible, the volume is constant.

$$n\,\frac{4}{3}\pi r^3 = \frac{4\pi}{3}R^3$$

$$\therefore \quad R = rn^{1/3}$$

Decrease in surface area $= 4\pi r^2 n - 4\pi R^2 = 4\pi r^2\left(n - n^{2/3}\right)$
Energy released = (decrease in surface area) (surface tension)

$$\Delta W = 4\pi r^2 \left(n - n^{2/3}\right) \text{ s}$$

Also $\quad \Delta W = 4\pi R^2 S \left(n^{1/3} - 1\right)$

Then there will be a rise in temperature as energy is converted into heat. Energy conservation gives

$$mc\,\Delta\theta = 4\pi R^2 s \left(n^{1/3} - 1\right)$$

$$\frac{4\pi}{3} R^3 c\rho\,\Delta\theta = 4\pi s R^3 \left[\frac{1}{r} - \frac{1}{R}\right]$$

$$\therefore \quad \Delta\theta = \frac{3s}{\rho c}\left[\frac{1}{r} - \frac{1}{R}\right]$$

10.65 $W = 2 \times 4\pi \left(r_2^2 - r_1^2\right) S$

The factor of 2 arises as there are two surfaces.

$$W = 8\pi \left((0.05)^2 - (0.01)^2\right) \times 0.03$$

$$= 0.0018\,\text{J}$$

10.66 The excess pressure must be equal to the pressure due to the water column of depth h before the water leaks into the vessel.

$$\frac{2S}{r} = \rho\, gh$$

$$\therefore \quad r = \frac{2s}{\rho gh} = \frac{2 \times 0.073}{1000 \times 9.8 \times 0.45} = 0.033 \times 10^{-3}\,\text{m} = 0.033\,\text{mm}$$

10.67 Balancing the excess pressure in the bubble with the pressure due to a water column of depth h

$$\frac{2S}{r} = \rho gh$$

$$\therefore \quad h = \frac{2s}{r\rho g} = \frac{2 \times 0.072}{0.3 \times 10^{-3} \times 1000 \times 9.8} = 0.049\,\text{m} = 4.9\,\text{cm}$$

10.68 $h = \dfrac{2s}{r\rho g} = \dfrac{2 \times 0.073}{0.2 \times 10^{-3} \times 1000 \times 9.8} = 0.07449\,\text{m} = 7.45\,\text{cm}$

The tube is inadequate as it is only 6 cm long. Water will not overflow. But the radius of meniscus r_1 would now increase such that the following condition is satisfied:

$h_1 r_1 = hr$, where r_1 is the radius of the meniscus.

$$r_1 = \frac{hr}{h_1} = \frac{7.45 \times 0.2}{6.0} = 0.248\,\text{mm}$$

10.69 When a bubble is charged, the charges stick to the bubble's surface and due to mutual repulsion tend to expand the surface while the surface tension tends to decrease the surface. An equilibrium is reached with a smaller excess of pressure.
Pressure due to electric charge is

$$P = \frac{\sigma^2}{2\varepsilon_0}$$

where σ is the charge density and ε_0 is the permittivity.
If the excess pressure due to surface tension is neutralized by the electric charges then

$$\frac{\sigma^2}{2\varepsilon_0} = \frac{4S}{r} \quad \text{or} \quad \sigma = \sqrt{\frac{8\varepsilon_0 S}{r}}$$

The charge

$$q = 4\pi r^2 \sigma = 8\pi \sqrt{2\varepsilon_0 s r^3}$$
$$= 8\pi \sqrt{2 \times 8.85 \times 10^{-12} \times 0.03 \times (0.02)^3} = 2.06 \times 10^{-9}\,\text{C}$$

10.70 Under isothermal conditions

$$P_1 V_1 + P_2 V_2 = PV \tag{1}$$

$$P_1 = \frac{4S}{r_1},\ P_2 = \frac{4S}{r_2},\ P = \frac{4S}{r} \tag{2}$$

$$V_1 = \frac{4\pi}{3} r_1^3,\ V_2 = \frac{4\pi}{3} r_2^3,\ V = \frac{4\pi}{3} r^3 \tag{3}$$

Using (2) and (3) in (1) and simplifying we get

$$r = \sqrt{r_1^2 + r_2^2} \tag{4}$$

Chapter 11
Electrostatics

Abstract *Chapter 11* deals with electrostatics comprising electric field and potential for various configurations, electric dipole and quadrupole moments, Helmholtz coils, electrostatic energy, Gauss' law in integral and differential forms, capacitors, parallel plates, cylindrical and spherical, various arrangements of capacitors.

11.1 Basic Concepts and Formulae

Coulomb's law for charges q_1 and q_2 separated by r

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \qquad \text{(Electric force)} \tag{11.1}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/N/m^2}$$

$$\frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9\,\mathrm{Nm^2/C^2}$$

Electric field (E) for point charge q at distance r

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \tag{11.2}$$

E for a sphere of radius R of uniform charge distribution

$$\begin{aligned} E &= \frac{1}{4\pi\varepsilon_0}\cdot\frac{qr}{R^2} \quad (r \le R) \\ &= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2} \quad (r \ge R) \end{aligned} \tag{11.3}$$

E for hollow sphere of uniform charge distribution

$$\begin{aligned} E &= 0\,(r < R) \\ &= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2} \quad (r > R) \end{aligned} \tag{11.4}$$

E for an infinite non-conducting sheet

$$E = \frac{\sigma}{2\varepsilon_0} \tag{11.5}$$

E for an infinite conducting sheet

$$\begin{aligned} E &= \frac{\sigma}{\varepsilon_0} \qquad \text{(outside the sheet)} \\ &= 0 \qquad \text{(inside the sheet)} \end{aligned} \tag{11.6}$$

Electric Potential

When a test charge q_0 is moved from point A (at potential V_A) to the point B (at V_B) then the difference in electric potential is defined by the work done W_{AB} by the relation

$$V_B - V_A = W_{AB}/q_0 \tag{11.7}$$

If $V_B > V_A$, W_{AB} is positive and if $V_B < V_A$, W_{AB} is negative. By convention $V_A = 0$ when A is at infinite distance. Then the work required to move the test charge q_0 from infinity to the field point is

$$V = W/q_0 \tag{11.8}$$

Potential and field strength

$$V_B - V_A = W_{AB}/q_0 = Ed \tag{11.9}$$

where d is the distance of separation

Potential due to a point charge

$$V = Er \tag{11.10}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \tag{11.11}$$

Potential due to a group of charges

$$V = \sum\nolimits_n V_n = V = \frac{1}{4\pi\varepsilon_0}\sum \frac{q_0}{r_n} \tag{11.12}$$

Potential due to uniform charge distribution in a non-conducting sphere

$$V = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \qquad (r \leq R) \tag{11.13}$$

$$= \frac{q}{4\pi\varepsilon_0 r} \qquad (r \geq R) \tag{11.14}$$

Calculation of E from V

$$E = -\frac{\mathrm{d}V}{\mathrm{d}r} \tag{11.15}$$

Surface charge density

$$\sigma = \frac{q}{4\pi r^2} \tag{11.16}$$

Electric potential energy (U)

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \tag{11.17}$$

Capacitors

$$C = \frac{q}{V} \tag{11.18}$$

where C is the capacitance.

Capacitance of a sphere

$$C = \frac{q}{V} = 4\pi\varepsilon_0 R \tag{11.19}$$

The parallel plate capacitor

$$C = \frac{\varepsilon_0 K A}{d} \tag{11.20}$$

where A is the area of each plate, d is the distance of plates and K is the dielectric constant.

Equilibrium of an Oil Drop

Let a charge $-q$ be acquired by a small oil drop placed between two charged plates, the upper one being positively charged. The drop is under the joint action of two forces, the electric force acting upwards and the gravitational force acting downwards. If the drop is to remain suspended, then the condition for equilibrium is

$$qE = mg \tag{11.21}$$

Speed of a charged particle falling through a PD, V

$$v = \sqrt{\frac{2qV}{m}} \tag{11.22}$$

The **electric dipole** consists of two equal and opposite charges separated by distance d.

The electric field on the perpendicular bisector of the dipole $p = qd$

$$E = \frac{P}{2\pi\varepsilon_0 x^3} \qquad (x >> d) \tag{11.23}$$

Potential due to a dipole

$$V = \frac{1}{4\pi\varepsilon_0}\frac{P\cos\theta}{r^2} \tag{11.24}$$

where θ is the angle made by the vector $\boldsymbol{r}$ with the axis of the dipole, r being the distance of the field point from the middle point of the dipole.

A Dipole in an Electric Field

Suppose a dipole is placed at a positive angle θ with the electric field $\boldsymbol{E}$ in the plane of the page. Then the torque about the centre of the dipole is given by

$$\boldsymbol{\tau} = \boldsymbol{P} \times \boldsymbol{E} \tag{11.25}$$

its direction being perpendicular to the plane of the page and into the page.

The potential energy of the dipole is given by

$$U = -\boldsymbol{P} \cdot \boldsymbol{E} \tag{11.26}$$

Gauss' Law

The flux (φ_E) of the electric field (E)

$$\varphi_{\mathrm{E}} = \sum \boldsymbol{E} \cdot \boldsymbol{S} \tag{11.27}$$

where S is the surface area.

Gaussian surface is an imaginary closed surface. If infinitesimal areas are considered then the summation in (11.27) can be replaced by an integral over the surface:

$$\varphi_{\mathrm{E}} = \oint E \cdot \mathrm{d}s \tag{11.28}$$

Gauss' law which relates the total flux φ_{E} through this surface to the net charge q enclosed by the surface can be stated as

$$\varepsilon_0 \varphi_{\mathrm{E}} = q \tag{11.29}$$

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = q \qquad \text{(Gauss' law)} \tag{11.30}$$

Capacitors in series

$$\frac{1}{c} = \sum\nolimits_n \frac{1}{c_n} \tag{11.31}$$

Capacitors in parallel

$$C = \sum\nolimits_n C_n \tag{11.32}$$

Energy of a charged capacitor

$$W = \frac{1}{2} qV = \frac{1}{2} \frac{q^2}{C} = \frac{1}{2} CV^2 \tag{11.33}$$

Parallel Plate Capacitor with Dielectric

If a dielectric slab of thickness t and dielectric constant K is introduced in a parallel plate air capacitor, whose plates have area A and are separated by a distance d, the capacitance becomes

$$C = \frac{\varepsilon_0 A}{d - t\left(1 - \frac{1}{K}\right)} \tag{11.34}$$

If a metal of thickness t is introduced in the air capacitor, the effective distance between the plates is reduced and the capacitance becomes

$$C = \frac{\varepsilon_0 A}{d - t} \tag{11.35}$$

Energy Loss in the Combined System of Capacitors

If the positive end of a capacitor of capacitance C charged to potential difference V is connected in parallel with the positive end of the capacitor of capacitance C charged to potential difference V, then common potential difference will be

$$V = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2} \tag{11.36}$$

and the energy loss will be

$$\Delta W = \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} (V_1 - V_2)^2 \tag{11.37}$$

If the positive end is joined to the negative end the common potential will be

$$V = \frac{C_1 V_1 - C_2 V_2}{C_1 + C_2} \tag{11.38}$$

and the energy loss will be

$$\Delta W = \frac{1}{2}\frac{C_1 C_2}{C_1 + C_2}(V_1 + V_2)^2 \tag{11.39}$$

Dielectric Strength

Every dielectric material is characterized by dielectric strength which is the maximum value of the electric field that can be tolerated without breakdown resulting in a conducting path between the plates of the capacitor.

Energy density (u) is defined as the electrostatic energy (U) per unit volume. For a parallel plate capacitor of area A and plate separation d, the volume enclosed between the plates is Ad.

$$u = \frac{U}{Ad} = \frac{1}{2}\frac{CV^2}{Ad} \tag{11.40}$$

But $C = \varepsilon_0 A/d$ and $V = Ed$

$$u = \frac{1}{2}\varepsilon_0 E^2 \tag{11.41}$$

Force of attraction between the capacitor plates is given by

$$F = -\frac{1}{2}\varepsilon_0 \frac{AV^2}{d^2} \tag{11.42}$$

Coalescing of Charged Drops

Let n identical droplets, each of charge q, coalesce to form a large drop of charge Q. If the droplets are assumed to be incompressible, referring the parameters of the drop primed and the droplets unprimed, the following relations will hold good:

$$\text{Charge } Q = q' = nq \tag{11.43}$$

$$\text{Surface charge density } \sigma' = n^{1/3}\sigma \tag{11.44}$$

$$\text{Capacitance } C' = n^{1/3}C \tag{11.45}$$

$$\text{Potential } V' = n^{2/3}V \tag{11.46}$$

$$\text{Energy stored } W' = n^{5/3}W \tag{11.47}$$

11.2 Problems

11.2.1 Electric Field and Potential

11.1 **(a)** Figure 11.1 shows two point clusters of charge situated in free space placed on a line that is called the x-axis. The first, with a positive charge of $Q_1 = +8\mathrm{e}$, is at the origin. The second, with a negative charge of $Q_2 = -4\mathrm{e}$, is to the right at a distance equal to 0.2 m.

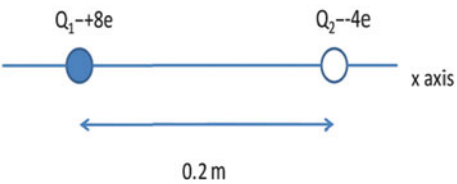
(i) What is the magnitude of the force between them?
(ii) Where would you expect to find the position of zero electric field: to the left of Q_1, between Q_1 and Q_2 or to the right of Q_2? Briefly explain your choice and then work out the exact position.

(b) The electron in a hydrogen atom orbits the proton at a radius of 5.3×10^{-11} m.

(i) What is the proton's electric field strength at the position of the electron?
(ii) What is the magnitude of the electric force on the electron?

[University of Aberystwyth, Wales]

Fig. 11.1



11.2 **(a)** A tiny ball of mass 0.6 g carries a charge of magnitude 8 μC. It is suspended by a thread in a downward electric field of intensity 300 N/C. What is the tension in the thread if the charge on the ball is

(i) positive?
(ii) negative?

(b) A uniform electric field is in the negative x-direction. Points a and b are on the x-axis, a at $x = 2$ m and b at $x = 6$ m.

(i) Is the potential difference $V_\mathrm{b} - V_\mathrm{a}$ positive or negative?
(ii) If the magnitude of $V_\mathrm{b} - V_\mathrm{a}$ is 10^5 V, what is the magnitude E of the electric field?

[University of Aberystwyth, Wales 2005]

11.3 Show that the electric potential a distance z above the centre of a horizontal circular loop of radius R, which carries a uniform charge density per unit length λ, is given by

$$V = \frac{\lambda R}{2\varepsilon_0} \frac{1}{(z^2 + R^2)^{1/2}}$$

Obtain an expression for the electrostatic field strength as a function of z.

[University of Aberystwyth, Wales 2007]

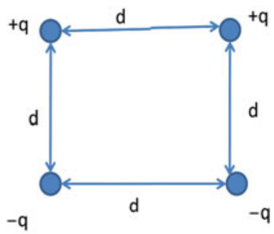
11.4 **(a)** Starting from Coulomb's law, show that the electric potential a distance r from a point charge q is given by

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

(b) Four point charges are assembled as shown in Fig. 11.2. Calculate the potential energy of this configuration (you may assume that the charges are isolated and in a perfect vacuum). Does the potential energy depend upon the order in which the charges are assembled?

(c) Is the charge configuration in (b) stable?

Fig. 11.2



11.5 A spherical liquid drop has a diameter of 2 mm and is given a charge of 2×10^{-15} C.

(i) What is the potential at the surface of the drop?

(ii) If two such drops coalesce to form a single drop, what is the potential at the surface of the drop so formed?

[Indian Institute of Technology 1973]

11.6 A pendulum bob of mass 80 mg carries a charge of 2×10^{-8} C at rest in a horizontal uniform electric field of 20,000 V/m. Find the tension in the thread of the pendulum and the angle it makes with the vertical.

[Indian Institute of Technology 1979]

11.7 An infinite number of charges, each equal to q, are placed along the x-axis at $x = 1$, $x = 2$, $x = 4$, $x = 8$, etc. Find the potential and the electric field at the point $x = 0$ due to the set of charges.

[Indian Institute of Technology 1974]

11.8 In prob. (11.7) what will be the potential and electric field in the above set-up if the consecutive charges have opposite sign?

[Indian Institute of Technology 1974]

11.9 A thin fixed ring of radius 1 m has a positive charge 1×10^{-5} C uniformly distributed over it. A particle of mass 0.9 g and having negative charge of 1×10^{-6} C is placed on the axis, at a distance of 1 cm from the centre of the ring. Show that the motion of the negative charge is approximately simple harmonic. Calculate the time period of oscillation.

[Indian Institute of Technology 1982]

11.10 Three charges, each of value q, are placed at the corners of an equilateral triangle. A fourth charge Q is placed at the centre of the triangle.

(i) If $Q = -q$ will the charges at the corners move towards the centre or fly away from it?
(ii) For what value of Q will the charges remain stationary?

[Indian Institute of technology 1978]

11.11 Two identically charged spheres are suspended by strings of equal length. The strings make an angle 30° with each other. When suspended in a liquid of density 0.8 g/cm^3, the angle remains the same. What is the dielectric constant of the liquid? The density of the material of the sphere is 1.6 g/cm^3.

[Indian Institute of Technology 1976]

11.12 At the corner A of square ABCD of side 10 cm a charge 6×10^{-8} C is placed. Another charge of -3×10^{-8} C is located at the centre of the square. Find the work done in carrying a charge 5×10^{-9} C from the corner C to the corner B of the square.

[Indian Institute of Technology 1972]

11.13 A pith ball carrying a charge of 3×10^{-10} C is suspended by an insulated thread of length 50 cm. When a uniform electric field is applied in a horizontal direction, the ball is found to deflect by 2 cm from the vertical. If the mass of the ball is 0.5 g what is the magnitude and direction of the electric field?

[Indian Institute of Technology 1973]

11.14 A positively charged oil droplet remains in the electric field between two horizontal plates, separated by a distance 1 cm. If the charge on the drop is 3.2×10^{-19} C and the mass of the droplet is 10^{-14} kg what is the potential difference between the plates? Now if the polarity of the plates is reversed what is the instantaneous acceleration of the droplet?

[Indian Institute of Technology 1974]

11.15 Suppose equal amount of charge of the same sign is placed on the earth and the moon, what would be its magnitude if the gravitational attraction between the two bodies may be nullified? Take mass of the earth and moon to be 6×10^{24} and 7.4×10^{22} kg, respectively.

11.16 A spark is produced between two insulated surfaces, maintained at a constant difference of 5×10^6 V. If the energy output is 10^{-5} J, calculate the charge transferred. How many electrons have flowed?

[Indian Institute of Technology 1974]

11.17 A rod 25 cm long has a uniform linear charge density (charge per unit length) $\lambda = 200\,\mu\text{C/m}$. Calculate the electric field (in N/C) at 10 cm from one end along the axis of the rod.

11.18 A disc of radius R is uniformly charged to Q and placed in the xy-plane with its centre at the origin. Find the electric field along the z-axis.

11.19 Electronic charge e may be determined by Millikan's oil drop method. Oil drops of radius r acquire a terminal speed v_1 with downward electric field E and a speed v_2 with the upward electric field. Derive an expression for e in terms of E, v_1, v_2, r and η, the viscosity of oil in air.

11.20 A circular wire of radius r has a uniform linear charge density $\lambda = \lambda_0 \cos^2\theta$. Show that the total charge on the wire is $\pi\lambda_0 r$.

11.21 The distance between the electron and the proton in the hydrogen atom is about 0.53 Å. By what factor is the electrical force stronger than the gravitational force? Does the distance matter?

11.22 The combined charge on two small spheres is $+15\,\mu$C. If each sphere is repelled by the other by a force of 5.4 N when the spheres are 30 cm apart, find the charges on the spheres.

11.23 Charges are placed at the four corners of a square of side a, as in Fig. 11.3. Find the magnitude and direction of the electric field at the centre of the square.

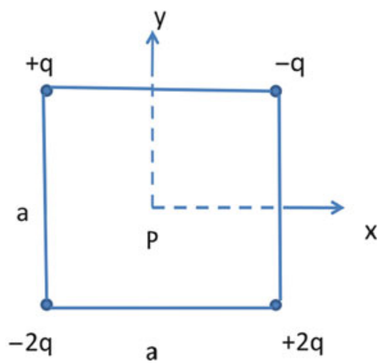


Fig. 11.3

11.24 A thin, non-conducting rod of length L carries a total charge $+Q$ spread uniformly along it. Find the electric field at point p distant y from the axis of the rod on the perpendicular bisector.

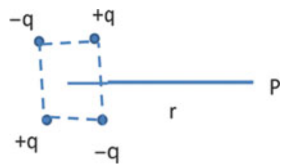
11.25 A thin non-conducting rod is bent to form an arc of a circle of radius r and subtends an angle θ_0 at the centre of the circle. If a total charge q is spread uniformly along the rod, find the electric field at the centre of the circle.

11.26 A ring of radius r located in the xy-plane is given a total charge $Q = 2\pi R\lambda$. Show that E is maximum when the distance $z = r/\sqrt{2}$.

11.27 A total charge q is spread uniformly over the inner surface of a non-conducting hemispherical cup of inner radius a. Calculate (a) the electric field and (b) the electric potential at the centre of the hemisphere.

11.28 The quadrupole consists of four charges $q, q, -q, -q$ located at the corners of a square on side a (Fig. 11.4). Show that at a point p, distant r from the centre of the charges and in the same plane, the electric field varies inversely as the fourth power of r, where $r >> a$.

Fig. 11.4



11.29 Show that the electrical and gravitational force between two bodies each of mass m and charge q will be equal at any distance r if the ratio $q/m = 8.6 \times 10^{-10}$ C/kg.

11.30 Two small, equally charged spheres, each of mass m, are suspended from the same point by silk threads of length L. Initially, the spheres are separated by distance $x << L$. As the charge leaks out at the rate $\mathrm{d}q/\mathrm{d}t$, the spheres approach each other with relative velocity $v = a/\sqrt{x}$, where a is a constant. Find the rate at which charge leaks out.

Show that

$$\mathrm{d}q/\mathrm{d}t = \frac{3}{2}a\sqrt{\frac{2\pi\varepsilon_0 mg}{L}}$$

11.31 A charge q is uniformly distributed over a thin ring of radius R. A very long uniformly charged thread with linear charge density λ is placed on the axis of the ring with one end coinciding with the centre of the ring. Show that the force of interaction F will be equal to $\frac{q\lambda}{4\pi\varepsilon_0 R}$.

11.32 A very long wire with uniform charge density λ is placed along the x-axis with one end of the thread coinciding with the origin. Show that the electric field is given by $E = \frac{\sqrt{2}\lambda}{4\pi\varepsilon_0 y}$ at 45° with the x-axis at a distance y from the end of the thread.

11.33 If the electric potential is given by $\varphi = cxy$, calculate the electric field.

11.34 Charges Q and $-2Q$ are placed at a fixed distance of separation. Show that the locus of points in the plane of charges, where the potential is zero, will be a circle.

[Indian Administrative Services]

11.35 Two identical thin rings, each of radius R, are coaxially placed a distance R apart. If Q_1 and Q_2 are, respectively, the charges uniformly spread on the two rings, find the work done in moving a charge q from the centre of one ring to that of the other.

[Indian Institute of Technology 1992]

11.36 A thin rod of length $2a$ is placed along the y-axis in the xy-plane. The rod carries a charge density λ (Fig. 11.5). The point P_1 is located at $(0, 2a)$ and P_2 at $(x, 0)$.

(a) Find x if the potentials at P_1 and P_2 are equal.
(b) Find the corresponding potential.

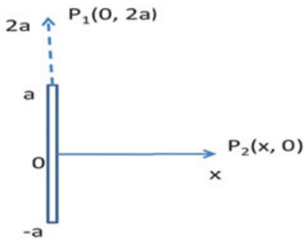


Fig. 11.5

11.37 Three charges $+q$, $+q$ and Q are located at the vertices of a right-angled isosceles triangle, Fig. 11.6. If the total interaction energy is zero what should be the value of Q?

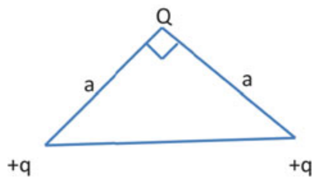
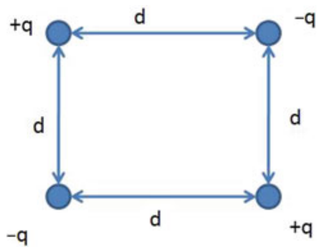


Fig. 11.6

11.38 Four charges each of magnitude q are located at the four corners of a square of side a such that like charges occupy the corners across the diagonals (Fig. 11.7). Calculate the work done in assembling these charges.

Fig. 11.7

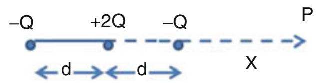


11.39 Calculate the total potential energy of sphere of radius R carrying a uniformly distributed charge q.

11.40 A linear quadrupole (Fig. 11.8) consists of charge $+2Q$ at the origin and two charges $-Q$ at $(-d, 0)$ and $(+d, 0)$.

- **(i)** Write down the magnitude of the electric field at P on the x-axis where $x > d$.
- **(ii)** If $x >> d$ show that the field varies inversely as the fourth power of distance from the origin.
- **(iii)** If $Q = 2\,\mu\text{C}$ and $d = 0.01$ mm, calculate the field at $x = 20$ cm.

Fig. 11.8 Linear quadrupole



11.41 **(a)** If the breakdown field strength of air is 5×10^6 V/m how much charge can be placed on a sphere of radius 1 mm? **(b)** What would be the corresponding electrical potential?

11.42 An electron is released from a distance 120 cm from a stationary point charge $+2 \times 10^{-9}$ C. Calculate the speed of the electron when it is 18 cm from the point charge.

11.43 Figure 11.8 shows the linear quadrupole. Show that the electric potential $V(r)$ at a distance $r >> d$ from the central charge and in a direction normal to the axis of the quadrupole varies inversely as the third power of r.

11.44 An electron of mass $m_e = 9.1 \times 10^{-31}$ kg is accelerated in the uniform electric field $\boldsymbol{E}$ between two parallel charged plates, as shown in Fig. 11.9. There is no electric field outside of the plates. The electric field has a magnitude $E = 2.0 \times 10^3$ N/C and electron charge $e = -1.6 \times 10^{-19}$ C. The separation of the plates is 1.5 cm and the electron is accelerated from rest near the negative plate and passes through a tiny hole in the positive plate. Assume the hole is so small that it does not affect the uniform field between the plates.

(i) What is the force on the electron while it is between the plates?
(ii) What is its acceleration and with what speed does it leave the hole?
(iii) What is the force on the electron outside of the plates?

[University of Aberystwyth, Wales 2005]

11.45 What is the electric potential V at a distance r from a point charge Q? Write down an expression describing the electric potential due to a continuous charge distribution.

Consider a disk of radius R which carries a uniform surface charge distribution.

(a) Find the total charge on the disc.
(b) Find the potential at a point on the axis of the disc lying at a distance x from the disc.
(c) What is the form of the potential when x becomes much larger than R? Comment on your result.

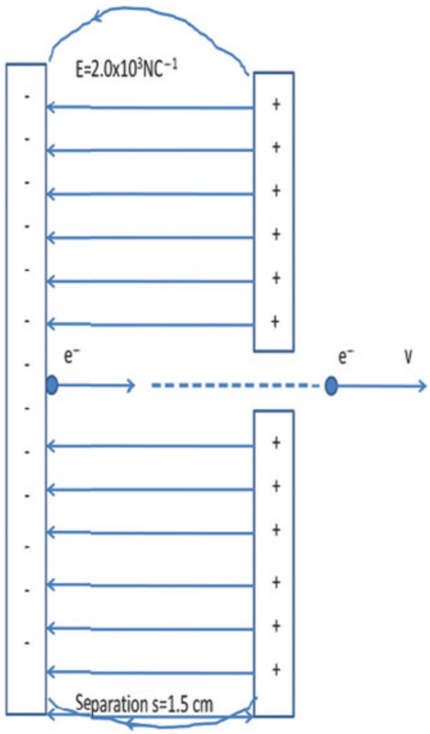


Fig. 11.9

11.46 In the Bohr's hydrogen atom model, show that the orbital motion of the electron obeys Kepler's third law of motion, that is, $T^2 \propto r^3$.

11.47 Equal charges (Q) are placed at the four corners of a square of side a. Show that the force on any charge due to the other three charges is given by 1.914 $Q^2/4\pi\varepsilon_0 a^2$.

11.48 **(a)** Calculate the electric field due to a dipole on its perpendicular bisector.
(b) Show that for the distance $x >> d/2$, where d is the distance between the charges, the field varies inversely as the cube of distance.
(c) A molecule has a dipole moment of 6×10^{-30} cm. Calculate the difference in potential energy when the dipole is placed parallel to the electric field of 3×10^6 V/m and then antiparallel to the field.

11.2.2 Gauss' Law

11.49 **(a)** State Gauss' law of electrostatics in mathematical form.
(b) Use Gauss' law to show that the electric field magnitude due to an infinite sheet of charge, carrying a surface density σ, is given by

$$E = \frac{\sigma}{2\varepsilon_0}$$

(c) A small sphere of mass 2 mg carries a charge of 5×10^{-8} C. It hangs by a silk thread attached to a vertical uniformly charged sheet such that, under the influence of both gravity and the electric force, it makes an angle of 10° with the sheet. Calculate the surface charge density of the sheet.

11.50 **(a)** State Gauss' law in differential and integral form.
(b) Show that the electric field outside a charged sphere is $Q/4\pi\varepsilon_0 r^2$, where r is the distance from the centre of the ball.
(c) Show that the electric field inside a uniformly charged solid sphere, with total charge Q and radius R, is $Qr/4\pi\varepsilon_0 R^3$.

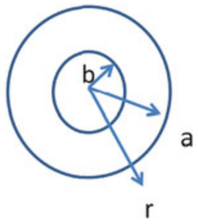
11.51 In prob. (11.50) the central part of the sphere is hollowed by creating a cavity of radius $\frac{1}{2}R$ concentric with the original sphere. If the charge density of the hollowed sphere remains unchanged show that the electric field at the surface is now 7/8 of the original value on the surface.

11.52 In prob. (11.50) show that the electric potential (a) varies as that to simple harmonic motion for $r < R$. (b) $V(0) = \dfrac{3}{2}V(R)$ where $V(\infty) = 0$.

11.53 Figure 11.10 shows a non-conducting hollow sphere with inner radius b and outer radius a. A total charge Q is uniformly distributed in the material $b < r < a$. Find the electric field for (a) $r < b$; (b) $b < r < a$; (c) $r > a$.

11.54 A charge Q is uniformly distributed in a long cylinder of radius R and charge density ρ. Find the electric field for the regions (a) $r > R$; (b) $r < R$.

Fig. 11.10



11.55 **(a)** The electric field on the surface of a thin spherical shell of radius 0.5 m is measured to be 800 N/C and points radially towards the centre of the sphere. What is the net charge within the sphere's surface?
(b) An electric field of 120 N/C points down over a football field. Calculate the surface charge density on the field.
(c) What would be the total electric flux if the field is $100 \times 75\,\text{m}^2$.

11.56 **(a)** Using Gauss' law derive Coulomb's formula for the electric field due to an isolated point charge q.
(b) A positive charge Q is uniformly distributed in a non-conducting sphere of radius R. Calculate the electric flux passing through the spherical surface of radius r concentric with the sphere for (i) $r < R$; (ii) $r > R$.

11.57 How is electric flux related to the electric field **E**? How is the total electric flux over a closed surface related to the charge enclosed within the surface?

A thin spherical shell of radius R_1 carries a total charge Q_1 that is uniformly distributed on its surface. A second, larger concentric thin shell of radius R_2 carries a charge Q_2 also uniformly distributed over the surface of the shell. Use Gauss' law to find the electric field in the regions.

(a) $r < R_1$,
(b) $R_1 < r < R_2$, and
(c) $R_2 < r$.

The electric charges are such that $Q_1 > 0$ and the electric field is zero for $r > R_2$. Find the ratio Q_1/Q_2.

11.58 Two insulated spheres of radii 1 and 3 cm at a considerable distance apart are each charged positively with 3×10^{-8} C. They are brought into contact and separated by the same distance as before. Compare the forces of repulsion before and after contact.

[Northern Universities of UK]

11.59 What is the maximum charge that can be given to a sphere of diameter 10 cm if the breakdown voltage of air is 2×10^4 V/cm.

11.60 **(a)** Show that the capacitance, C, of a conducting sphere of radius a is given by $C = 4\pi\varepsilon_0 a$.
(b) Two isolated conducting spheres, both of radius a, initially carry charges of q_1 and q_2 and are held far apart. The spheres are connected together by a conducting wire until equilibrium is reached, whereupon the wire

is removed. Show that the total electrostatic energy stored in the spheres decreases by an amount ΔU, given by

$$\Delta U = \frac{1}{16\pi\varepsilon_0 Q}(q_1 - q_2)^2$$

What happens to this energy?

11.61 Two spherical conductors of radii R_1 and R_2 and charges Q_1 and Q_2, respectively, are brought in contact and separated. Show that their charge densities will be inversely proportional to their radii.

11.62 A light spherical balloon is made of conducting material. It is suggested that it could be kept spherical simply by connecting it to a high-voltage source. The balloon has a diameter of 100 mm.

(a) What is the voltage of the source if the electric field on the balloon surface is 5×10^6 V/m?
(b) What gas pressure inside the balloon would produce the same effect?
(c) The voltage source is removed and the balloon remains at the same voltage. Calculate the total electrostatic energy of the balloon.

11.63 A soap bubble of radius R_1 is given a charge q. Due to mutual repulsion of the surface charges the radius is increased to R_2, the pressure remaining constant. Show that the charge is given by

$$q = \left[\frac{32}{3}\pi^2\varepsilon_0 p R_1 R_2 \left(R_1{}^2 + R_1 R_2 + R_2{}^2\right)\right]^{1/2}$$

11.64 An insulating spherical shell of inner radius r_1 and outer radius r_2 is charged so that its volume charge density is given by

$$\rho(r) = 0 \text{ for } 0 \le r \le r_1$$
$$\rho(r) = \frac{A}{r} \text{ for } r_1 \le r \le r_2$$
$$\rho(r) = 0 \text{ for } r > r_2$$

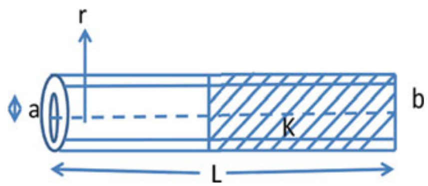
where A is a constant and r is the radial distance from the centre of the shell. Find the electric field due to the shell throughout all space.

11.65 **(a)** Show that electrostatic field is conservative.
(b) An isolated soap bubble of radius 1 cm is at a potential of 10 V (assume the bubble material is a conductor); calculate the total charge on the bubble. If it collapses to a drop of radius 1 mm (no charge loss or gain during the process), what is the change of the electrostatic energy of the system?

[University of Aberystwyth, Wales 2008]

11.66 A long cylinder of charge q has a radius a. The charge density within its volume, ρ, is uniform (Fig. 11.11). Describe the form of the electric field generated by the cylinder. Find the electric field strength at a distance r from the axis of the cylinder in the regions (i) $r > a$ and (ii) $0 < r < a$.
If a non-relativistic electron moves in a circle at a constant distance R from the axis of the cylinder, where $R > a$, find an expression for its speed.
[University of Manchester 2006]

Fig. 11.11



11.67 Consider an isolated non-conducting sheet with charge density σ. The electric field at 25 cm from the sheet is found to be 200 V/m, directed towards the sheet. Calculate σ on the sheet. What electric field is expected at 2 cm from its surface? How are the values of σ and E changed if a conducting sheet is substituted.

11.2.3 Capacitors

11.68 Calculate the capacitance of a parallel plate capacitor of area A and thickness d if a dielectric slab of thickness t, area A and dielectric constant k is inserted. How is the capacitance modified if a metal of thickness t is introduced?

11.69 Two capacitors C_1 and C_2 are connected in parallel and their combined capacitance is measured as 9 μF. When they are combined in series their capacitance is 2 μF. What are the individual capacitances?

11.70 Find the energy which may be stored in capacitors of 2 and 4 μF when taken (a) singly, (b) in series and (c) in parallel when a potential difference of 100 V is available.
[University of New Castle]

11.71 An air capacitor with plates 1 m^2 and 0.01 m apart is charged with 10^{-6} C of electricity. Calculate the change in energy which results when the capacitor is submerged in oil of relative permittivity 2.0.
[University of Manchester]

11.72 Two parallel plates, each of area 1 m^2, are separated by a distance 0.001 m and have a capacity of 0.1 μF. What must then the dielectric constant of the material separating the plates be?

[University of Newcastle]

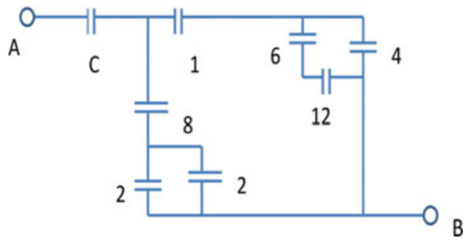
11.73 A capacitor of capacitance 5 μF is charged up to a PD of 250 V. Its terminals are then connected to those of an uncharged capacitor of capacitance 20 μF. What would be the resulting voltage?

[Northern Universities of UK]

11.74 From Fig. 11.12 find the value of capacitance C if the equivalent capacitance between points A and B is 1 μF. All the capacitances are in microfarads.

[Indian Institute of Technology 1977]

Fig. 11.12



11.75 Three capacitors of capacitance 4, 3 and 2 μF, respectively, are connected in series to a battery of 240 V. Calculate (a) the charge, (b) the potential and (c) the electrostatic energy associated with each of the three capacitors, stating in each case the units in which the results are expressed.

[Northern Universities of U.K.]

11.76 Each of the two capacitors A and B of capacitances 1.0 and 2.0 μF, respectively, are charged initially by connecting them in turn to a 12 V battery. What is the final potential difference of the combination if the capacitors are later connected in parallel such that

(a) the positive plate of one is connected to the positive plate of the other;
(b) the positive plate of one is connected to the negative plate of the other.

[Indian Institute of Technology 1971]

11.77 Two capacitors of capacitances C_1 and C_2 charged to potential difference V_1 and V_2 are connected in parallel. Calculate the energy loss when (a) the positive ends are joined and (b) the positive end of one is joined to the negative end of the other.

11.78 A capacitor of capacitance C is charged to potential V by connecting it to a battery. Let q be the charge on it, E the electric field within the plates and U the energy stored. When a dielectric of constant K is introduced filling completely the space between the plates, how will the following quantities change (i) V, (ii) E, (iii) q, (iv) C and (v) U, when (a) the battery remains connected and (b) the battery is disconnected?

11.79 In prob. (11.78) if the plate separation is increased, how would the following quantities change (i) V, (ii) E, (iii) C, (iv) q and (v) U when (a) the battery remains connected and (b) the battery is disconnected?

11.80 Show that the force of attraction between the plates of a parallel plate capacitor is given by $F = \frac{1}{2}\frac{\varepsilon_0 A V^2}{d}$, where A is the area, d the distance of separation, V the voltage to which the plates are charged and ε_0 the permittivity.

11.81 Let n identical droplets, each of radius r and charge q, coalesce to form a large drop of radius R and charge Q. Assuming that the droplets are incompressible, show that (a) the radius $R = n^{1/3}r$; (b) the capacitance C' of the large drop is $C' = n^{1/3}C$, where C is the capacitance of the droplet; (c) the potential V' of the large drop is given by $V' = n^{2/3}V$, where V is the potential of the droplet; (d) the surface charge density $\sigma' = n^{1/3}\sigma$; (e) the energy U' stored in the large drop is given by $U' = n^{5/3}U$ where U is the energy stored in the droplet.

11.82 A cylindrical capacitor has radii a and b. Show that half of the stored electrical potential energy lies within a cylinder whose radius is $\sqrt{ab}$.

11.83 A capacitor of capacitance $C_1 = 3.0\,\mu\text{F}$ withstands the maximum voltage $V_1 = 4.0\,\text{kV}$, while a capacitor of capacitance $C_2 = 6.0\,\mu\text{F}$ the maximum voltage $V_2 = 3.0\,\text{kV}$. If they are connected in series what maximum voltage can the system withstand?

11.84 A Geiger–Muller tube consists of a thin uniform wire of radius 'a' of length L surrounded by a concentric hollow metal cylinder of radius b with a gas of dielectric constant K between them. Apply Gauss' law to calculate the capacitance of the tube.

11.85 Two spherical metallic shells of radii a and b ($b > a$) constitute a capacitor with the outer shell grounded and contact is made with the inner one through a hole in the outer one. Show that the capacitance is given by $C = \frac{4\pi\varepsilon_0 ab}{b - a}$.

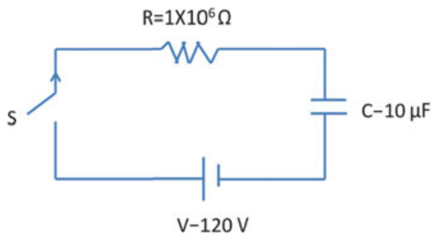
11.86 Show that for two concentric shells of radii a and b ($b \approx a$), the capacitance reduces to that of a parallel plate capacitor

11.87 In an R–C circuit the emf supplied by the battery is 120 V, $R = 1 \times 10^6\,\Omega$ and $C = 10\,\mu\text{F}$. The switch S is closed at $t = 0$. Find

(i) the time taken for the charge to reach 90% of its final value;
(ii) the energy stored in the capacitor at one time constant;
(iii) the Joule heating in the resistor at one time constant.

11.88 After how many time constants will the energy stored in the capacitor in Fig. 11.13 reach one-half of its equilibrium value?

Fig. 11.13



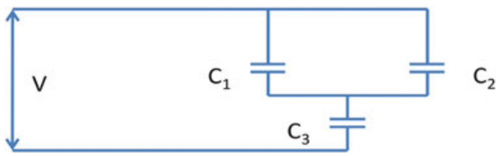
11.89 Two square metal plates measuring 'a' on the side are used as a parallel plate capacitor with the plates slightly inclined at an angle θ. If the smaller gap between the plates is D, then show that the capacitance is given by

$$C = \frac{\varepsilon_0 a^2}{D}\left(1 - \frac{a\theta}{2D}\right)$$

11.90 Capacitors $C_1 = 8\,\mu\text{F}$, $C_2 = 4\,\mu\text{F}$ and $C_3 = 3\,\mu\text{F}$ are arranged as in Fig. 11.14. A voltage of $V = 100\,\text{V}$ is applied. Determine

(a) the potential difference across C_1, C_2 and C_3.
(b) the charge q_1, q_2 and q_3 on C_1, C_2 and C_3.
(c) the energy U_1, U_2 and U_3 stored in the capacitors.

Fig. 11.14



11.91 Capacitors $C_1 = 8\,\mu\text{F}$, $C_2 = 4\,\mu\text{F}$ and $C_3 = 3\,\mu\text{F}$ are arranged as in Fig. 11.15. A voltage of $V = 100\,\text{V}$ is applied. Determine

(a) the charges q_1, q_2 and q_3 on C_1, C_2 and C_3, respectively.
(b) the potential difference across C_1, C_2 and C_3.
(c) the energy U_1, U_2 and U_3 stored in the capacitors.

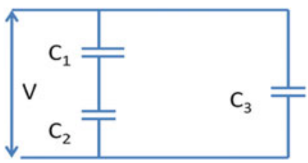


Fig. 11.15

11.92 Find the effective capacitance between points a and b in Fig. 11.16. Assume that $C_1 = C_2 = C_3 = C_4 = 2\,\mu\text{F}$ and $C_5 = 1\,\mu\text{F}$.

Fig. 11.16

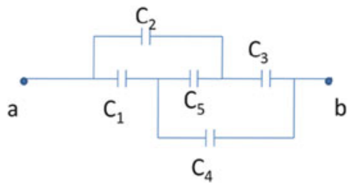
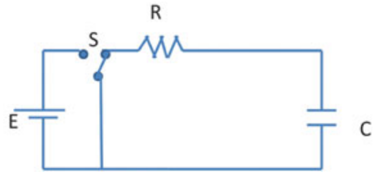


Fig. 11.17



11.93 Consider a circuit consisting of a resistor R and a capacitor C in series with a battery of emf ξ and a switch (Fig. 11.17). The capacitor is initially uncharged and the switch is closed at time $t = 0$. By considering the potential drop across each of the components of the circuit, verify that the charge Q on the capacitor has the form

$$Q = C\xi\left(1 - \mathrm{e}^{-\dfrac{r}{RC}}\right)$$

(a) What is the current flowing in the circuit?
(b) What is the power supplied by the battery as a function of t?
(c) What is the power dissipated in the resistor as a function of t?
(d) What is the rate at which energy is stored in the capacitor as a function of t?

[University of Durham 2000]

11.94 For the circuit shown in Fig. 11.18,

(i) What is the initial battery current immediately after switch S is closed?
(ii) What is the battery current a long time after switch S is closed?
(iii) If the switch has been closed for a long time and is then opened, find the current through the 600 kΩ resistor as a function of time.

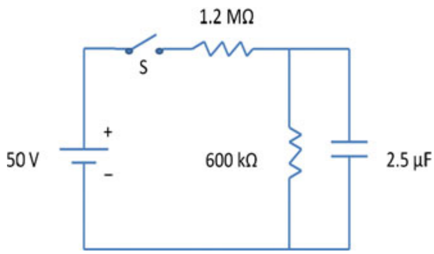


Fig. 11.18

11.95 A capacitor of capacitance $C = 500\,\mu\text{F}$ is charged to a voltage of 900 V and is then discharged through a resistance $R = 200\,\text{k}\Omega$ when a switch is closed.

(i) Find the initial charge stored in the capacitor.
(ii) Find the initial discharge current when the switch is closed.
(iii) Find the voltage across the capacitor in a time $t = 25$ s after the start of discharge.
(iv) Find the time constant of this capacitor resistor network combination.
(v) Work out an equation to show the time it takes for the charge in the capacitor to drop by one-half of its starting value and find this time.

[University of Aberystwyth, Wales 2008]

11.96 Charge $q = 10^{-9}$ C is uniformly distributed in a sphere of radius $R = 1$ m.

(i) Find the divergence of the electric field inside the sphere.
(ii) A proton is moved from infinity to $r = 0.8$ m from the centre of the sphere. Find the electric force experienced by the proton at $r = 0.8$ m.
(iii) Find the work done by the electric field of the charged sphere when the proton is moved from infinity to its current position ($r = 0.8$ m).

11.97 **(a)** Write down the integral and differential forms of Gauss' law in a dielectric, defining all quantities used.
(b) A parallel plate capacitor is completely filled with a non-conducting dielectric. Show that the electric displacement, D, is uniform between the plates and calculate its value. (You may assume that the plates each have area A and are separated by a small distance d. Each plate carries a surface charge density σ C/m^2.)
(c) The dielectric has a non-uniform relative permittivity

$$K(x) = ax + b$$

where a and b are constants and x is the perpendicular distance from one plate. Using Gauss' law, show that the electric field between the plates satisfies

$$E(x) = \frac{E_0}{K(x)}$$

where E_0 is a constant. Find the value of E_0.
(d) Show that the voltage across the capacitor is given by

$$V = \frac{E_0}{\varepsilon_0 a} \ln\left(1 + \frac{Qd}{b}\right)$$

and calculate the capacitance.
(e) Find the volume polarization charge density in the dielectric.

11.98 Both gravitational field and electric field obey inverse square law. Using this analogy show that the differential Gauss' law for gravitation is given by $\nabla \cdot g = -\rho/G$, where ρ is the mass density.

11.3 Solutions

11.3.1 Electric Field and Potential

11.1 **(a)**

(i) $$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} = \frac{(8 \times 1.6 \times 10^{-19})(-4 \times 1.6 \times 10^{-19})}{4\pi \times 8.85 \times 10^{-12} \times (0.2)^2}$$
$$= -1.8432 \times 10^{-25}\,\text{N}$$

(ii) For the position of zero electric field the forces due to the two charges must be equal in magnitude but oppositely directed. Clearly the neutral point must be on the x-axis. On the left of Q_1, the forces will be oppositely directed but cannot be equal as $|Q_1| > |Q_2|$. Between Q_1 and Q_2, the forces are exerted in the same direction. On the right of Q_2 conditions are favourable for a null point. Let the zero electric field be situated at a distance x from Q_2 on the right.

$$\frac{8e}{(x + 0.2)^2} - \frac{4e}{x^2} = 0$$

whence $x = 0.4828$ on the right of Q_2.

(b) $$E = \frac{e}{4\pi\varepsilon_0 r^2} = \frac{1.6 \times 10^{-19}}{4\pi \times 8.85 \times 10^{-12} \times (5.3 \times 10^{-11})^2}$$
$$= 5.12 \times 10^{18}\,\text{N/C}$$

Force $F = Ee = 5.12 \times 10^{18} \times 1.6 \times 10^{-19} = 81.92\,\text{N}$

11.2 **(a)** As the electric field is downwards, the force on the positive charge will be downwards and the force on the negative charge will be upwards.

(i) $q = +8\,\mu\text{C}$

$$F_q = qE = +8 \times 10^{-6} \times 300 = 2.4 \times 10^{-3}\,\text{N}$$
$$F_g = mg = 0.6 \times 10^{-3} \times 9.8 = 5.88 \times 10^{-3}\,\text{N}$$

$\therefore$ Tension in the thread

$$T = F_g + F_q = 5.88 \times 10^{-3} + 2.4 \times 10^{-3} = 8.28 \times 10^{-3}\,\text{N}$$

(ii) $q = -8\,\mu\text{C}$

$F_q = qE = -2.4 \times 10^{-3}\,\text{N}$

Tension in the thread

$$T = 5.88 \times 10^{-3} - 2.4 \times 10^{-3} = 3.48 \times 10^{-3}\,\text{N}$$

(b) As the electric field is in the negative x-direction, point b will be at a higher potential than a.

(i) Therefore $V_\text{b} - V_\text{a}$ will be positive

(ii) $E = \dfrac{V_\text{b} - V_\text{a}}{d} = \dfrac{10^5}{(6-2)} = 2.5 \times 10^4\,\text{N/C}$

11.3 The total charge on the circular loop $Q = 2\pi R\lambda$; the distance of the point $P(0, 0, Z)$ from the loop is $r = (Z^2 + R^2)^{1/2}$. Therefore, the electric potential P will be

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{\lambda R}{2\varepsilon_0 (Z^2 + R^2)^{1/2}}$$

$$E = -\frac{\partial V}{\partial z} = \frac{\lambda R Z}{2\varepsilon_0 (Z^2 + R^2)^{3/2}}$$

11.4 **(a)** $F = \dfrac{q}{4\pi\varepsilon_0 r^2}\ (Q = 1)$

$$V = -\int F\,\mathrm{d}r = -\frac{q}{4\pi\varepsilon_0}\int \frac{\mathrm{d}r}{r^2} = \frac{q}{4\pi\varepsilon_0 r} + C$$

When $r = \infty$, $V = 0$. Therefore $C = 0$

(b) $U = \dfrac{q^2}{4\pi\varepsilon_0}\left[\dfrac{1}{d} - \dfrac{1}{d} + \dfrac{1}{d} - \dfrac{1}{d} - \dfrac{\sqrt{2}}{d} - \dfrac{\sqrt{2}}{d}\right]$

For six pairs of charges

$$U = -\frac{q^2}{\sqrt{2}\pi\varepsilon_0 d}$$

The potential energy does not depend on the order in which the charges are assembled.

(c) Consider the forces on the top left-hand charge due to the three other charges.

$$E_x = E_y = -\frac{q^2}{8\,\pi\varepsilon_0 d^2}$$

$$\therefore \quad E = \sqrt{E_x^2 + E_y^2} = \sqrt{2}\,\frac{q^2}{8\,\pi\varepsilon_0 d^2} \neq 0$$

Therefore the charge is not in equilibrium. Same thing is true for the other three charges.

11.5 **(i)** $V_1 = \dfrac{q}{4\pi\varepsilon_0 r} = \dfrac{2\times10^{-15}\times9\times10^9}{10^{-3}} = 0.018\,\text{V}$

(ii) If n droplets each of radius r coalesce to form a large drop of radius R, then assuming that the droplets are incompressible, the volume does not change. Therefore

$$\frac{4}{3}\pi R^3 = n\frac{4}{3}\pi r^3$$

$$\text{or}\qquad R = n^{1/3} r \tag{1}$$

As charge is conserved

$$Q = nq \tag{2}$$

where Q and q are the charges on the drop and droplet, respectively. Then the potential of the droplet is

$$V_2 = \frac{Q}{4\pi\varepsilon_0 R} = \frac{nq}{4\pi\varepsilon_0 n^{1/3} r} = \frac{n^{2/3}q}{4\pi\varepsilon_0 r}$$
$$= n^{2/3}\,V_1 = 2^{2/3}\,V_1 = 0.0286\,\text{V}$$

where $n = 2$, and $V_1 = 0.018\,\text{V}$ by (i).

11.6 Electric force

$$F = qE = 2\times10^{-8}\times20,000 = 4\times10^{-4}$$

Gravitational force $mg = 80\times10^{-6}\times9.8 = 7.84\times10^{-4}\,\text{N}$
Balancing the horizontal and vertical components of forces, Fig. 11.19

$$T\sin\theta = F \tag{1}$$
$$T\cos\theta = mg \tag{2}$$

where T is the tension in the thread.

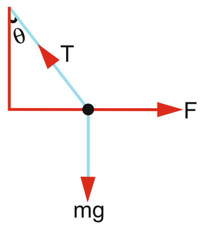


Fig. 11.19

$$\tan\theta = \frac{F}{mg} = \frac{4\times 10^{-4}}{7.84\times 10^{-4}} = 0.51$$
$$\theta = 27^\circ$$

Squaring (1) and (2) and adding and extracting the square root

$$T = \sqrt{F^2 + (mg)^2} = \sqrt{(4\times 10^{-4})^2 + (7.84\times 10^{-4})^2} = 8.8\times 10^{-4}\,\mathrm{N}$$

11.7 Electric potential

$$\begin{aligned} V &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{1} + \frac{q}{2} + \frac{q}{4} + \frac{q}{8} + \cdots\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) \end{aligned}$$

The terms in brackets form a geometric progression of infinite terms whose sum is

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$
$$\therefore\quad V = \frac{q}{2\pi\varepsilon_0}$$

Electric field

$$\begin{aligned} E &= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1} + \frac{q}{2^2} + \frac{q}{4^2} + \frac{q}{8^2} + \cdots\right] \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(1-\frac{1}{4}\right)} = \frac{q}{3\pi\varepsilon_0} \end{aligned}$$

11.8
$$\begin{aligned} V &= \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{2} + \frac{q}{4} - \frac{q}{8} + \frac{q}{16} - \frac{q}{32} + \cdots\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) - \frac{q}{8\pi\varepsilon_0}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{2}\right)\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= \frac{q}{8\pi\varepsilon_0}\frac{1}{1-\frac{1}{4}} = \frac{q}{6\pi\varepsilon_0} \end{aligned}$$

$$
\begin{aligned}
E &= \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{2^2} + \frac{q}{4^2} - \frac{q}{8^2} + \frac{q}{16^2}\cdots\right)\\
&= \frac{q}{4\pi\varepsilon_0}\left(1 + \frac{1}{4^2} + \frac{1}{16^2} + \cdots\right) - \frac{q}{16\pi\varepsilon_0}\left[1 + \frac{1}{4^2} + \frac{1}{16^2} + \cdots\right]\\
&= \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{4}\right)\left(1 + \frac{1}{4^2} + \frac{1}{16^2} + \cdots\right)\\
&= \frac{3q}{16\pi\varepsilon_0}\frac{1}{1 - \frac{1}{16}} = \frac{q}{5\pi\varepsilon_0}
\end{aligned}
$$

11.9 By prob. (11.3), $E = \dfrac{Qx}{4\pi\varepsilon_0\left(x^2 + R^2\right)^{3/2}}$, where we have replaced z by x and substituted $\lambda = Q/2\pi R$. As $x << R$ and $F = qE$,

$$
\begin{aligned}
F &= \frac{Qqx}{4\pi\varepsilon_0 R^3} = -kx, \text{ where } q \text{ is negative and } \quad k = \frac{Qq}{4\pi\varepsilon_0 R^3}\\
&= \frac{10^{-5} \times 10^{-6} \times 9 \times 10^9}{1^3} = 0.09
\end{aligned}
$$

Thus, the motion of the negatively charged particle is approximately simple harmonic with angular frequency

$$
\begin{aligned}
\omega &= \sqrt{\frac{k}{m}} = \sqrt{\frac{0.09}{0.9 \times 10^{-3}}} = 10\\
T &= \frac{2\pi}{\omega} = \frac{2\pi}{10} = 0.628\,\text{s}
\end{aligned}
$$

11.10 Let a be the side of the equilateral triangle. The forces on the charge q placed at C due to the charges at A and B are repulsive and represented by CE and CD, respectively, each given by $q^2/4\pi\varepsilon_0 a^2$. The resultant of these two forces is given by CP, the diagonal of the parallelogram CDPE, Fig. 11.20

$$
\text{CP} = 2\text{CD}\cos 30^\circ = \frac{2q^2}{4\pi\varepsilon_0 a^2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}q^2}{4\pi\varepsilon_0 a^2}
$$

The force on q at C due to Q at the centre of the triangle is

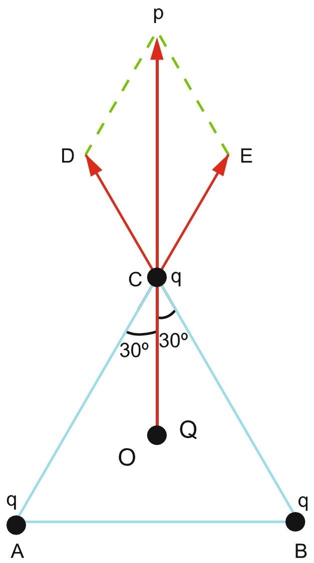
$$
\frac{\text{Q}q}{4\pi\varepsilon_0(\text{OC})^2} = \frac{3\,\text{Q}q}{4\pi\varepsilon_0 a^2}
$$

i. If $Q = -q$, this force will be attractive and will be directed along CO. As the attractive force due to $-q$ is greater than the combined repulsive force due to charges $+q$ at A and B, the charge at C will be attracted towards O. Same is true for the charges placed at A and B.

ii. For equilibrium, the attractive force must balance the repulsive force:

$$\frac{\sqrt{3}q^2}{4\pi\varepsilon_0 a^2} - \frac{3Qq}{4\pi\varepsilon_0 a^2} = 0 \rightarrow Q = -q/\sqrt{3}$$

Fig. 11.20



11.11 Referring to Fig. 11.21, the Coulomb force between the spheres in air is

$$F = \frac{q^2}{4\pi\varepsilon_0 x^2}$$

In liquid $F' = \dfrac{q^2}{4\pi\varepsilon_0\, K\, x^2}$; weight of the sphere in air $= mg$

Apparent weight of the sphere in liquid $= mg\left(1 - \dfrac{\rho'}{\rho}\right)$, where ρ is the density of the material of sphere and ρ' that of the liquid. For equilibrium the vertical and horizontal components of the force must separately balance. If T is the tension in the string when the sphere is in air and T' when it is in the liquid,

$$T\sin\theta = F, T\cos\theta = mg$$

$$T'\sin\theta = F',\; T'\cos\theta = mg\left(1 - \frac{\rho'}{\rho}\right)$$

$$\tan\theta = \frac{F}{mg} = \frac{F'}{mg\left(1 - \dfrac{\rho'}{\rho}\right)}$$

$$\frac{F}{F'} = K = \frac{1}{1 - \dfrac{\rho'}{\rho}} = \frac{\rho}{\rho - \rho'} = \frac{1.6}{1.6 - 0.8} = 2$$

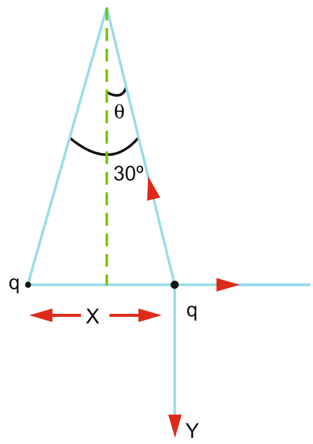


Fig. 11.21

11.12 From the geometry of Fig. 11.22, AB $= 0.1$ m, OB $=$ OC $= 0.05\sqrt{2}$ m, AC $= (0.1)\sqrt{2}$ m.

$$\text{Potential energy of } q \text{ at B is } U(\mathrm{B}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{q_1}{\mathrm{AB}} + \frac{q_2}{\mathrm{OB}}\right]$$

$$\text{Potential energy of } q \text{ at C is } U(\mathrm{C}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{q_1}{\mathrm{AC}} + \frac{q_2}{\mathrm{OC}}\right]$$

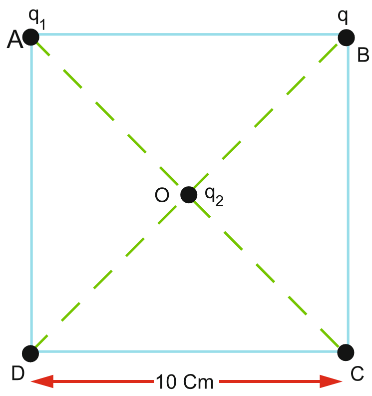


Fig. 11.22

Work done in carrying the charge q from C to B will be

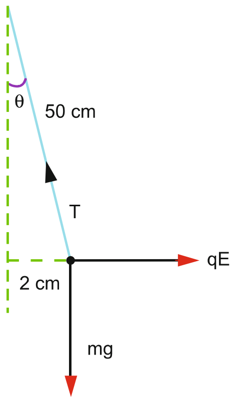
$$\begin{aligned} W_{\mathrm{CB}} &= U(\mathrm{B}) - U(\mathrm{C}) = \frac{qq_1}{4\pi\varepsilon_0}\left[\frac{1}{\mathrm{AB}} - \frac{1}{\mathrm{AC}}\right] \\ &= 5 \times 10^{-9} \times 6 \times 10^{-8} \times 9 \times 10^{9}\left(\frac{1}{0.1} - \frac{1}{(0.1)\sqrt{2}}\right) \\ &= 7.9 \times 10^{-6}\,\mathrm{J} \end{aligned}$$

11.13 The problem is similar to prob. (11.6), Fig. 11.23. For equilibrium $T \sin\theta = qE$, $T\cos\theta = mg$, $\tan\theta = \dfrac{qE}{mg} \sim \dfrac{2}{50} = 0.04$

$$\therefore \qquad E = \frac{0.04\,mg}{q} = \frac{(0.04)(0.5 \times 10^{-3})(9.8)}{3 \times 10^{-10}} = 6.53 \times 10^{5}\,\mathrm{N/C}$$

which is directed away from the equilibrium position.

Fig. 11.23



11.14 The electric field $E = V/d$ where V is the PD and d is the distance of separation of plates. The electric force on the droplet is $F = qE = qV/d$. If the upper plate is negative then the condition for equilibrium against gravitational force acting downwards is

$$\frac{qV}{d} = mg$$

$$V = \frac{mgd}{q} = \frac{10^{-14} \times 9.8 \times 0.01}{3.2 \times 10^{-19}} = 3062.5\,\mathrm{V}$$

If the polarity of the plates is reversed, both the electric and gravitational forces would act down. The net force would become

$$F' = qE + mg = 2mg$$

$$\text{Acceleration } a = \frac{F'}{m} = 2g = 2 \times 9.8 = 19.6\,\text{m/s}^2$$

11.15 Let q be the charge on each body. Electric force = gravitational force

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{GMm}{r^2}$$

$$q = \sqrt{4\pi\varepsilon_0 GMm}$$

$$= \left[\frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 7.4 \times 10^{22}}{9 \times 10^9}\right]^{1/2} = 5.736 \times 10^{13}\,\text{C}$$

11.16 Energy $W = qV$

$$q = \frac{W}{V} = \frac{10^{-5}}{5 \times 10^6} = 2 \times 10^{-12}\,\text{C}$$

$$\text{Number of electrons flowed out} = \frac{q}{e} = \frac{2 \times 10^{-12}}{1.6 \times 10^{-19}} = 1.25 \times 10^7$$

11.17 Consider an element dx of the rod at distance x from the point P on the axis of the rod. In the length dx the charge is $\mathrm{d}q = \lambda \mathrm{d}x$, Fig. 11.24.
The field at P due to dq will be

$$\mathrm{d}E = \frac{\lambda\,\mathrm{d}x}{4\pi\varepsilon_0 x^2}$$

The total electric field will be

$$E = \int \mathrm{d}E = \int_{0.1}^{0.35} \frac{\lambda\,\mathrm{d}x}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{1}{x}\Big|_{0.35}^{0.1}$$

$$= 200 \times 10^{-6} \times 9 \times 10^9 \times \left(\frac{1}{0.1} - \frac{1}{0.35}\right) = 1.286 \times 10^7\,\text{N/C}$$

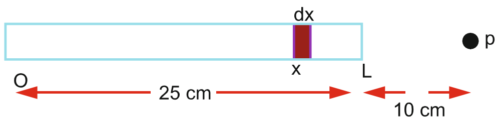


Fig. 11.24

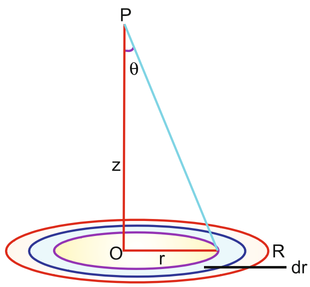
11.18 Let p be the field point on the axis of the disc at distance z from the origin. Consider a ring of radius r and width dr. The charge on the ring is $\mathrm{d}q = 2\pi r\,\mathrm{d}r\,\sigma$ where σ is the charge density (charge per unit area), Fig. 11.25.

The electric field at P can be resolved into a component along the z-axis and perpendicular to it. The perpendicular components when added become zero for reasons of symmetry. The components along the z-axis are added

$$dE_{||} = \frac{dq \cos\theta}{4\pi\varepsilon_0(z^2 + r^2)} = \frac{2\pi r \, dr \, \sigma}{4\pi\varepsilon_0(z^2 + r^2)} \cdot \frac{z}{(z^2 + r^2)^{1/2}}$$

$$E = \int dE_{||} = \frac{\sigma}{2\varepsilon_0} z \int_0^R \frac{r dr}{(z^2 + r^2)^{3/2}} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{(z^2 + R^2)^{1/2}}\right)$$

Fig. 11.25



11.19 Equations of motion are

$$eE + mg - 6\pi\eta v_1 \mathrm{r} = 0 \quad \text{(downward field)}$$
$$eE - mg - 6\pi\eta v_2 \mathrm{r} = 0 \quad \text{(upward field)}$$

Adding and solving for e

$$e = \frac{3\pi\eta}{E}(v_1 + v_2)r$$

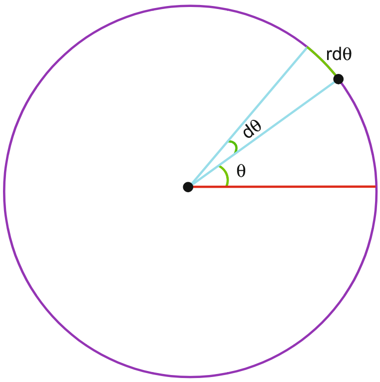
11.20 Consider an element of the circular wire ds (Fig. 11.26). Then $dq = \lambda ds$.

$$\text{Now } ds = r \, d\theta$$

$$\therefore \quad dq = (\lambda \cos^2\theta)(r d\theta)$$

$$\therefore \quad q = \int dq = \lambda_0 r \int_0^{2\pi} \cos^2\theta d\theta = \pi\lambda_0 r$$

Fig. 11.26



11.21 Electric force $F_e = \dfrac{q^2}{4\pi\varepsilon_0 r^2}$ (11.48)

Gravitational force $F_g = \dfrac{GMm}{r^2}$ (11.49)

$$\frac{F_e}{F_g} = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{GMm} = \frac{(1.6\times10^{-19})^2(9\times10^9)}{(6.67\times10^{-11})(1.66\times10^{-27})(9.1\times10^{-31})} \quad (11.50)$$

$$= 2.29\times10^{39} \quad (11.51)$$

The distance is immaterial. Note that the gravitational force at the atomic and sub-atomic levels is small simply because the masses are small.

11.22 $q_1 + q_2 = 15\,\mu\text{C}$ (1)

$$F = \frac{q_1q_2}{4\pi\varepsilon_0 r^2} = \frac{9\times10^9}{(0.3)^2}q_1q_2 = 5.4$$

or $\quad q_1\,q_2 = 54\,(\mu\text{C})^2$ (2)

Solving (1) and (2), $q_1 = 6\,\mu\text{C}$, $q_2 = 9\,\mu\text{C}$.

11.23 At P, the electric field due to $+q$ is $\dfrac{q}{4\pi\varepsilon_0(a/\sqrt{2})^2}$ or $\dfrac{2q}{4\pi\varepsilon_0(a)^2}$ and points towards $+2q$. The field due to $+2q$ is $\dfrac{4q}{4\pi\varepsilon_0 a^2}$ and points towards $+q$. The resultant field due to the pair $(q, 2q)$ is $\dfrac{4q}{4\pi\varepsilon_0 a^2} - \dfrac{2q}{4\pi\varepsilon_0 a^2}$ or $E_1 = \dfrac{2q}{4\pi\varepsilon_0 a^2}$ towards $+q$.

Similarly, the resultant filed due to the pair of charges $(-q, -2q)$ will be $E_2 = -\dfrac{2q}{4\pi\varepsilon_0 a^2}$ towards $-2q$ or $+\dfrac{2q}{4\pi\varepsilon_0 a^2}$ towards $-q$.

Now $E_1 = E_2$ in magnitude and act at right angles (from the geometry of the diagram). The overall field will then be $E = \sqrt{2}\,E_1 = \dfrac{2\sqrt{2}q}{4\pi\varepsilon_0 a^2}$ along positive y-axis.

11.24 Consider an infinitesimal length dx at distance x from O, the centre of the rod. The charge on dx will be d$q = q$ (dx/L). The field at P due to dq shown by an arrow can be resolved into x- and y-components. The x-component of the field will be cancelled by a symmetric charge on the negative side at equal distance. The $y-$components of the field will be added up, Fig. 11.27.

$$\mathrm{d}E_y = \mathrm{d}E\ \cos\theta = \frac{q\mathrm{d}x}{4\pi\varepsilon_0 L(x^2+y^2)}\frac{y}{(x^2+y^2)^{1/2}}$$

$$\therefore \qquad E = \int \mathrm{d}E_y = \frac{qy}{4\pi\varepsilon_0 L}\int\frac{\mathrm{d}x}{(x^2+y^2)^{3/2}}$$

Put $x = y\tan\theta$, d$x = y\sec^2\theta\mathrm{d}\theta$

$$E = \frac{q}{4\pi\varepsilon_0 Ly}\int_{-\alpha}^{\alpha}\cos\theta\mathrm{d}\theta = \frac{q}{2\pi\varepsilon_0 Ly}\sin\alpha = \frac{q}{2\pi\varepsilon_0(4y^2+L^2)^{1/2}}$$

where we have put $\sin\alpha = \dfrac{L/2}{(y^2+L^2/4)^{1/2}}$.

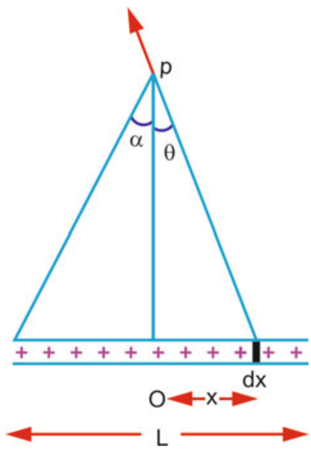


Fig. 11.27

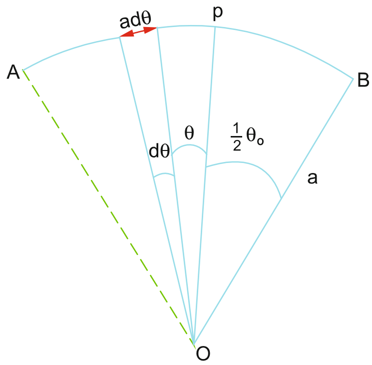
11.25 Consider an element of angle between θ and $\theta+\mathrm{d}\theta$. Let OP be the bisector of angle θ_0 subtended by the arc AB at the centre O. The charge on the element of the arc $a\mathrm{d}\theta$ will be $q\,\dfrac{\mathrm{d}\theta}{\theta_0}$. The electric field at O due to this element of arc can be resolved $E_{||}$ along PO and $E_{\perp}$ perpendicular to it.

The perpendicular components will be cancelled for reasons of symmetry while the parallel components get added up, Fig. 11.28.

$$\mathrm{d}E = \mathrm{d}E_{\parallel} = \frac{q\,\mathrm{d}\theta}{4\pi\varepsilon_0\theta_0 a^2}\ \cos\theta$$

$$E = \int \mathrm{d}E_{\parallel} = \frac{q}{4\pi\varepsilon_0\theta_0 a^2}\int_{-\theta_0/2}^{\theta_0/2}\cos\theta\,\mathrm{d}\theta = \frac{q}{2\pi\varepsilon_0 a^2}\frac{\sin(\theta_0/2)}{\theta_0}$$

Fig. 11.28



11.26 The electric field at distance z from the centre of the ring on the axis of the ring is given by prob. (11.3)

$$E = \frac{\lambda r}{2\varepsilon_0}\frac{z}{(z^2+r^2)^{3/2}}$$

The maximum field is obtained by setting $\dfrac{\partial E}{\partial z} = 0$.

This gives $(z^2+r^2)^{1/2}(r^2-2z^2) = 0$.

Since the first factor cannot be zero for any real value of z, the second factor gives $z = r/\sqrt{2}$.

11.27 Consider a circular strip symmetric about z-axis of radius r and width $a\mathrm{d}\theta$ (Fig. 11.29). The charge on the strip is

$$\mathrm{d}q = q\frac{2\pi r\, a\mathrm{d}\theta}{2\pi\, a^2} = \frac{qr\,\mathrm{d}\theta}{a} = q\ \sin\theta\,\mathrm{d}\theta$$

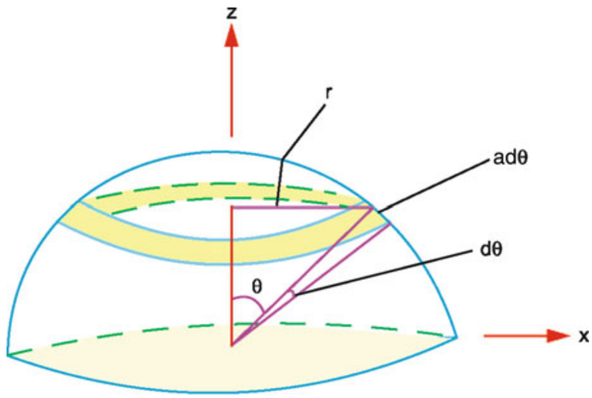
(a) At the centre of the hemisphere, the x-component of the field will be cancelled for reasons of symmetry. The entire field will be contributed by the z-component alone.

$$\mathrm{d}E = \mathrm{d}E_z = \frac{q \sin\theta \mathrm{d}\theta \ \cos\theta}{4\pi\varepsilon_0 a^2}$$

$$\therefore \qquad E = \int \mathrm{d}E_z = \frac{q}{4\pi\varepsilon_0 a^2} \int_0^{\pi/2} \sin\theta \cos\theta \mathrm{d}\theta = \frac{q}{8\pi\varepsilon_0 a^2}$$

(b) $\mathrm{d}V = \dfrac{q \sin\theta \mathrm{d}\theta}{4\pi\varepsilon_0 a}$; $\quad V = \displaystyle\int dv = \frac{q}{4\pi\varepsilon_0 a} \int_0^{\pi/2} \sin\theta \mathrm{d}\theta = \frac{q}{4\pi\varepsilon_0 a}$

Fig. 11.29



11.28 The x-component of the field due to front charges will get cancelled and the y-component is added up to

$$\frac{2q}{4\pi\varepsilon_0} \frac{a/2}{\left[\left(r - \frac{a}{2}\right)^2 + \frac{a^2}{4}\right]^{1/2}}$$

along the negative y-axis, Fig. 11.30.

Similarly the field due to the other two charges will be

$\dfrac{2q \cdot a/2}{4\pi\varepsilon_0 \left[\left(r + \frac{a}{2}\right)^2 + \frac{a^2}{4}\right]^{3/2}}$ along the positive y-axis.

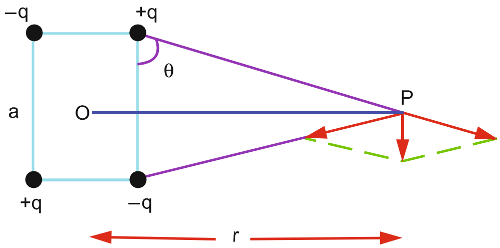
Neglecting terms of the order of a^2, the net field will be

$$E = \frac{qa}{4\pi\varepsilon_0 r^3} \left[\left(1 - \frac{a}{r}\right)^{-3/2} - \left(1 + \frac{a}{r}\right)^{-3/2}\right]$$

Using the binomial expansion up to retaining terms linear in a,

$$E = \frac{3qa^2}{4\pi\varepsilon_0 r^4}. \text{ Then } E \ \alpha \ \frac{1}{r^4}$$

Fig. 11.30



11.29 The electric force $F_{\mathrm{e}} =$ the gravitational force

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{Gm^2}{r^2}$$

$$\frac{q}{m} = (4\pi\varepsilon_0 G)^{1/2} = \left(\frac{6.67 \times 10^{-11}}{9 \times 10^9}\right)^{1/2} = 8.65 \times 10^{-9}\,\mathrm{C/kg}$$

11.30 Consider the equilibrium of one of the spheres, Fig. 11.31. If T is the tension in the string then

$$T\cos\theta = mg$$

$$T\ \sin\theta = \frac{q^2}{4\pi\varepsilon_0 x^2}$$

$$\therefore \qquad \tan\theta \simeq \theta = \frac{x}{2L} = \frac{q^2}{4\pi\varepsilon_0 mgx^2}$$

$$\therefore \qquad q = \left(\frac{2\pi\varepsilon_0 mg}{L}\right)^{1/2} x^{3/2}$$

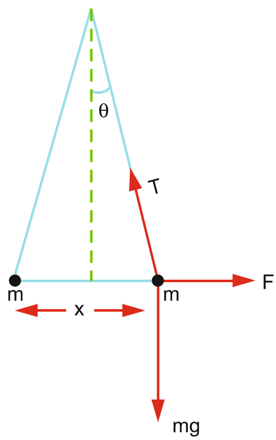


Fig. 11.31

$$\therefore \qquad \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{3}{2}\sqrt{\frac{2\pi\varepsilon_0 mg}{L}}\sqrt{x}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}x}v = \frac{\mathrm{d}q}{\mathrm{d}x}\frac{a}{\sqrt{x}} = \frac{3}{2}a\sqrt{\frac{2\pi\varepsilon_0 mg}{L}}$$

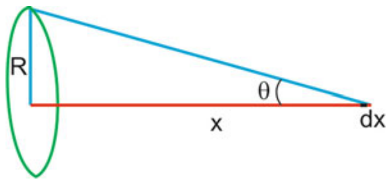
11.31 Consider an element of length dx of the thread at distance x from the centre of the ring. The force between the ring and the element dx can be resolved into x- and y-components, Fig. 11.32. The y-component will get cancelled for reasons of symmetry. The field is entirely contributed by the x-component. The charge in length dx is $\lambda \mathrm{d}x$. The electric force between the wire and the ring is given by

$$F = F_x = \int \frac{1}{4\pi\varepsilon_0}\frac{q\,\lambda\,\mathrm{d}x\cos\theta}{(R^2 + x^2)} = \frac{q\lambda}{4\pi\varepsilon_0}\int_0^\infty \frac{x\mathrm{d}x}{(x^2 + R^2)^{3/2}}$$

Put $x = R\cot\theta$, $\mathrm{d}x = -R\operatorname{cosec}^2\theta\,\mathrm{d}\theta$

$$F = -\frac{q\lambda}{4\pi\varepsilon_0 R}\int_{\pi/2}^{0}\cos\theta\,\mathrm{d}\theta = \frac{q\lambda}{4\pi\varepsilon_0 R}$$

Fig. 11.32



11.32 Consider an element of wire dx at distance x from O, Fig. 11.33. The charge in dx will be $\lambda \mathrm{d}x$. The x-component of the electric field will be

$$E_x = \int E\sin\theta = \int \frac{\lambda\mathrm{d}x\,\sin\theta}{4\pi\varepsilon_0(x^2 + y^2)} = \frac{\lambda}{4\pi\varepsilon_0}\int_0^\infty \frac{x\,\mathrm{d}x}{(x^2 + y^2)^{3/2}}$$

Put $\quad x^2 + y^2 = z^2, x\,\mathrm{d}x = z\,\mathrm{d}z$

where $y =$ constant.

$$E_x = \frac{\lambda}{4\pi\varepsilon_0}\int_y^\infty \frac{\mathrm{d}z}{z^2} = \frac{\lambda}{4\pi\varepsilon_0 y}$$

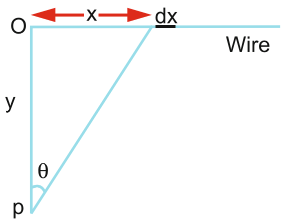
Similarly, $\quad E_y = \int E\,\cos\theta = \dfrac{\lambda}{4\pi\varepsilon_0 y}$

$$E = \sqrt{E_x^2 + E_y^2} = \frac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 y}$$

$$\tan\alpha = \frac{E_y}{E_x} = 1 \to \alpha = 45^\circ$$

Thus $\vec{E}$ makes an angle 45° with the y-axis.

Fig. 11.33



11.33 $\phi = Cxy$

$$E_x = -\frac{\partial\phi}{\partial x} = -cy, \quad E_y = \frac{\partial\phi}{\partial y} = -cx$$

$$\therefore \quad \vec{E} = -c(y\hat{i} + x\hat{j})$$

11.34 Let the charges Q and $-2Q$ be located on the x-axis at distance x on the opposite side of the y-axis. Let the point $P(x, y)$ be at distance r_1 from Q and at r_2 from $-2Q$, Fig. 11.34. By problem

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{2Q}{r_2}\right] = 0$$

or $\quad r_2 = 2r_1 \qquad (1)$

Writing $\quad r_1^2 = (x + a)^2 + y^2 \qquad (2)$

and $\quad r_2^2 = (a - x)^2 + y^2 \qquad (3)$

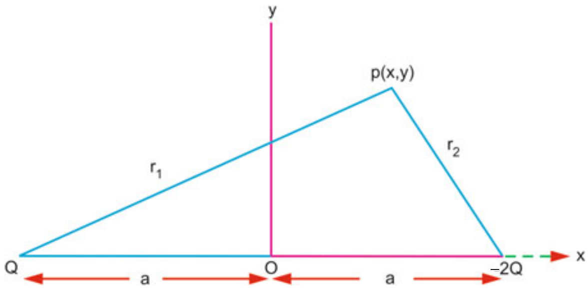
and eliminating r_1 and r_2 in (1), (2) and (3) and simplifying

$$3x^2 + 3y^2 + 10xa + 3a^2 = 0$$

or $\quad \left(x + \frac{5}{3}a\right)^2 + y^2 = \frac{16}{9}a^2$

which is the equation to a circle.

Fig. 11.34



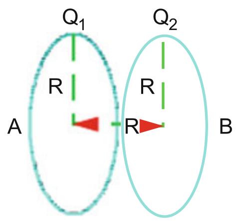
11.35 Let the charge q be taken from the centre of ring A to the centre of ring B, Fig. 11.35

$$\text{At A,}\quad U_A = \frac{qQ_1}{4\pi\varepsilon_0 R} + \frac{q\,Q_2}{4\pi\varepsilon_0(\sqrt{2}\,R)}$$

$$\text{At B,}\quad U_B = \frac{qQ_2}{4\pi\varepsilon_0 R} + \frac{q\,Q_1}{4\pi\varepsilon_0(\sqrt{2}\,R)}$$

$$\text{Work done,}\quad W = U_B - U_A = \frac{(\sqrt{2}-1)\,q\,(Q_2 - Q_1)}{4\sqrt{2}\pi\varepsilon_0 R}$$

Fig. 11.35



11.36 **(a)** Consider an infinitesimal length of the rod, at distance y from the origin, Fig. 11.36. The charge in $\mathrm{d}y$ will be $\lambda \mathrm{d}y$. The distance of P_1 from $\mathrm{d}y$ will be $2a - y$. The potential at P_1 is

$$V_1 = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a} \frac{\lambda\,\mathrm{d}y}{(2a-y)} = \frac{\lambda}{4\pi\varepsilon_0}\ln 3$$

The potential at P_2 is

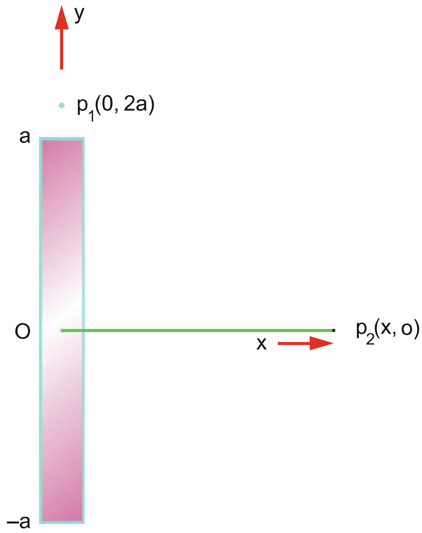
$$V_2 = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a} \frac{\lambda \mathrm{d}y}{\sqrt{y^2+x^2}} = \frac{2\lambda}{4\pi\varepsilon_0}\ln\left(\frac{a+\sqrt{a^2+x^2}}{x}\right)$$

By problem $V_2 = V_1$

$$\therefore \quad \left(\frac{a + \sqrt{a^2 + x^2}}{x}\right)^2 = 3 \rightarrow x = \sqrt{3}\,\mathrm{a}$$

$$V_1 = V_2 = \lambda \times 9 \times 10^9 \times \ln 3$$
$$= 9.89 \times 10^9\,\mathrm{V}$$

Fig. 11.36



11.37 $U = U_{Qq} + U_{Qq} + U = \frac{1}{4\pi\varepsilon_0}\left[\frac{Qq}{a} + \frac{Qq}{a} + \frac{q^2}{\sqrt{2}a}\right] = 0$ by problem.

Therefore $Q = -\frac{q}{2\sqrt{2}}$.

11.38 Work done $W = U_{12} + U_{23} + U_{34} + U_{41} + U_{13} + U_{24}$ where charges 1 and 3 are positive and 2 and 4 negative assuming that the potential energy is zero for infinite separation of charges.

$$W = \frac{q^2}{4\pi\varepsilon_0}\left[-\frac{1}{a} - \frac{1}{a} - \frac{1}{a} - \frac{1}{a} + \frac{1}{\sqrt{2}a} + \frac{1}{\sqrt{2}a}\right]$$
$$= \frac{-q^2}{4\pi\varepsilon_0 a}\left(4 - \sqrt{2}\right)$$

11.39 **(a)** The charge density is given by $\rho = \frac{3q}{4\pi R^3}$. Consider a shell of radius r and thickness dr concentric with the sphere. The volume of the shell is $4\pi r^2 \mathrm{d}r$ and the charge in it will be $\mathrm{d}q = 4\pi r^2 \mathrm{d}r \rho$, Fig. 11.37. The charge of the sphere of radius r is $\mathrm{d}q' = \frac{4}{3}\pi r^3 \rho$ and may be considered to be concentrated at the centre. The interaction energy between the shell and the sphere of radius r will be

$$\mathrm{d}U = \frac{1}{4\pi\varepsilon_0}\frac{(\mathrm{d}q)(\mathrm{d}q')}{r} = \frac{1}{4\pi\varepsilon_0}\frac{(4\pi r^2\mathrm{d}r\rho)\left(\frac{4}{3}\pi r^3\rho\right)}{r} = \frac{4\pi\rho^2 r^4\mathrm{d}r}{3\varepsilon_0}$$

Total interaction energy

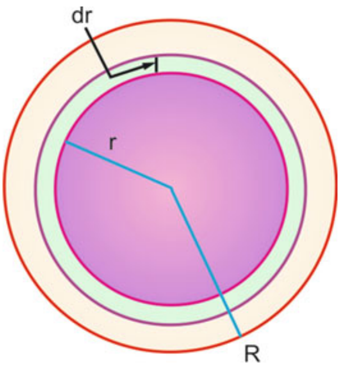
$$U = \int \mathrm{d}U = \frac{4\pi\rho^2}{3\varepsilon_0}\int_0^R r^4\mathrm{d}r = \frac{4\pi\rho^2 R^5}{15\varepsilon_0} = \frac{3q^2}{20\pi\varepsilon_0 R}$$

where we have substituted the value of ρ.

(b) $U = \frac{3}{5}\frac{\times 9\times 10^9\times(92\times 1.6\times 10^{-19})^2}{1.5\times(238)^{1/3}\times 10^{-15}} = 1.259\times 10^{-11}\,\mathrm{J}$

$= 78.7\,\mathrm{MeV}$

Fig. 11.37



11.40 **(i)** $E = \frac{Q}{4\pi\varepsilon_0}\left[\frac{2}{x^2} - \frac{1}{(x+d)^2} - \frac{1}{(x-d)^2}\right]$

$$= -\frac{2Q\,d^2(3x^2-d^2)}{4\pi\varepsilon_0 x^2(x^2-d^2)^2}$$

(ii) For $x >> d,\ 3x^2 - d^2 \simeq 3x^2$ and $x^2 - d^2 \simeq x^2$

$$E = -\frac{6Q\,d^2}{4\pi\varepsilon_0 x^4}$$

(iii) $E = -\frac{6\times 9\times 10^9\times 2\times 10^{-6}\times(10^{-4})^2}{(0.2)^2} = 0.675\,\mathrm{N/C}$

11.41 **(a)** $q = CV = 4\pi\varepsilon_0 r\, V = \dfrac{5 \times 10^6 \times 10^{-3}}{9 \times 10^9} = 5.5 \times 10^{-7}\,\mathrm{C}$

(b) $V = \dfrac{q}{4\pi \varepsilon\, r} = \dfrac{5.5 \times 10^{-7} \times 9 \times 10^9}{1 \times 10^{-3}} = 1.65 \times 10^6\,\mathrm{V}$

11.42 Initial potential energy $U_1 = -\dfrac{qe}{4\pi\varepsilon_0 r_1}$

Final potential energy $U_2 = -\dfrac{qe}{4\pi\varepsilon_0 r_2}$

$$\Delta U = U_1 - U_2 = \frac{qe}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

By work–energy theorem, gain in kinetic energy = loss in potential energy.

$$\frac{1}{2}mv^2 = \frac{qe}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$v = \left[\frac{2qe}{4\pi\varepsilon_0 m}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right]^{1/2}$$

$$= \left[\frac{2 \times 2 \times 10^{-9} \times 1.6 \times 10^{-19} \times 9 \times 10^9}{9.1 \times 10^{-31}}\left(\frac{1}{0.18} - \frac{1}{1.2}\right)\right]^{1/2}$$

$$= 5.467 \times 10^6\,\mathrm{m/s}$$

11.43 $V(r) = \dfrac{Q}{4\pi\varepsilon_0}\left[\dfrac{2}{r} - \dfrac{1}{\sqrt{r^2+d^2}} - \dfrac{1}{\sqrt{r^2+d^2}}\right]$

$$= \frac{2Q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{r}\left(1 + \frac{d^2}{r^2}\right)^{-1/2}\right] \simeq \frac{Qd^2}{4\pi\varepsilon_0 r^3}$$

Thus $V(r)\, \alpha \dfrac{1}{r^3}$

11.44 **(i)** $F = qE = (1.6 \times 10^{-19})(2 \times 10^3) = 3.2 \times 10^{-16}\,\mathrm{N}$

(ii) Acceleration $a = \dfrac{F}{m} = \dfrac{3.2 \times 10^{-16}}{9.1 \times 10^{-31}} = 3.516 \times 10^{14}\,\mathrm{m/s^2}$

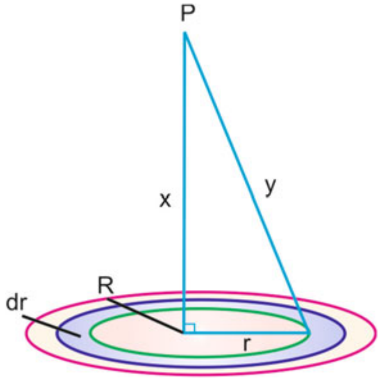
$$v = \sqrt{2as} = \sqrt{2 \times 3.516 \times 10^{14} \times 0.015}$$
$$= 3.25 \times 10^6\,\mathrm{m/s}$$

(iii) Outside the plates there is no force on the electron as there is no electric field.

11.45 $V(r) = \dfrac{Q}{4\pi\varepsilon_0 r}$ (1)

For continuous distribution of charge, each element dq can be treated as point charge so that the contribution dV to the potential can be written according to (1), Fig. 11.38:

Fig. 11.38



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r} \tag{2}$$

For the potential due to the entire distribution of all the elements, (2) is integrated:

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int \frac{dQ}{r} \tag{3}$$

In case of uniform charge distribution we can write $dQ = \lambda\, ds$, $dQ = \sigma dA$ or $dQ = \rho dV$ depending on the geometry of the problem. Here λ is the linear charge density, σ is the surface charge density and ρ is the volume charge density.

(a) $q = \pi R^2 \sigma$

(b) Consider a ring of radius r and width dr concentric with the disc of radius R. The charge on the ring is $dq = \sigma 2\pi r dr$. The potential at P, at a distance x on the axis of the disc, will be

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{y} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi r\, dr}{\sqrt{r^2 + x^2}} \tag{4}$$

Potential due to the disc will be

$$V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r\, dr}{\sqrt{r^2 + x^2}} \tag{5}$$

Put $r^2 + x^2 = y^2$, $r\, dr = y\, dy$, then (5) becomes

$$V = \frac{\sigma}{2\varepsilon_0}\int dy = \frac{\sigma}{2\varepsilon_0} y \Big|_x^{\sqrt{x^2+R^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{x^2 + R^2} - x\right] \tag{6}$$

(c) If $x >> R$, (6) can be expanded binomially,

$V \rightarrow \frac{\sigma}{2\varepsilon_0}\left[x\left(1+\frac{R^2}{x^2}\right)^{1/2} - x\right] = \frac{\sigma R^2}{4\varepsilon_0 x} = \frac{q}{4\pi\varepsilon_0 x}$, an expression which is appropriate for the point charge. This result is reasonable since at very large distances the disc appears as a point.

11.46 For circular motion of electron, the speed

$$v = \frac{2\pi r}{T} \tag{1}$$

For a stable orbit, the centripetal force = electric force.

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \tag{2}$$

Eliminating v between (1) and (2)

$$T^2 = 16\pi^3\varepsilon_0 m r^3$$

or $\quad T^2 \,\alpha\, r^3 \qquad$ (Kepler's third law)

11.47 Figure 11.39 shows the forces on charge 2 due to the charges 1, 3 and 4. The forces

$$F_{12} = F_{32} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a^2}$$

$$F_{42} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{\left(\sqrt{2}a\right)^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(2a^2)}$$

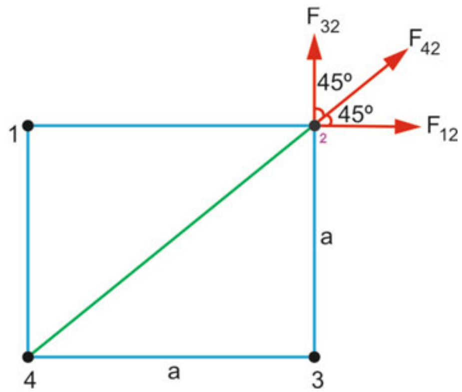


Fig. 11.39

Now $|\boldsymbol{F}_{32} + \boldsymbol{F}_{12}| = \sqrt{2}\, F_{12}(\because\ \boldsymbol{F}_{12}$ and $\boldsymbol{F}_{32}$ act at right angle and are equal in magnitude)

Further, $\boldsymbol{F}_{42}$ acts in the same direction as the combined force of $\boldsymbol{F}_{32}$ and $\boldsymbol{F}_{12}$.

$$\therefore \qquad |\boldsymbol{F}_{12} + \boldsymbol{F}_{32} + \boldsymbol{F}_{42}| = \frac{Q^2}{4\pi\varepsilon_0 a^2}\left(\sqrt{2} + \frac{1}{2}\right) = \frac{1.914\, Q^2}{4\pi\varepsilon_0 a^2}$$

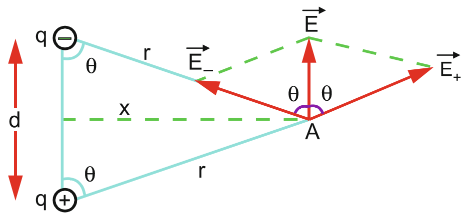
11.48 **(a)** A dipole consists of two equal and opposite charges. To find the electric field at A, on the perpendicular bisector of the dipole, at distance x. As the point A is equidistant from the two charges, the magnitudes E_+ and E_- are equal. The net electric field $\boldsymbol{E}$ at A is given by the vector addition of $\boldsymbol{E}_+$ and $\boldsymbol{E}_-$ (Fig. 11.40)

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- \qquad (1)$$

$$E_+ = E_- = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{[x^2 + (d/2)^2]} \qquad (2)$$

Since both E_+ and E_- are equal and equally inclined to the y-axis, their x-components gets cancelled and the combined field is contributed by the y-component alone.

Fig. 11.40



$$E = E_y = E_+\cos\theta + E_-\cos\theta = 2E_+\cos\theta$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{qd}{[x^2 + (d/2)^2]^{3/2}} \qquad (3)$$

(b) For $x >> d/2$, $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{p}{x^3}$, where $p = qd$ is the dipole. Thus the electric field at large distance varies inversely as the third power of distance, which is much more rapid than the inverse square dependence for point charge.

(c) Potential energy $U = -\boldsymbol{P}\cdot\boldsymbol{E} = -pE\cos\theta$, for parallel alignment, $\theta = 0$, and

$$U_1 = -pE = -(6\times 10^{-32})(3\times 10^6) = -1.8\times 10^{-25}\ \text{J}$$

For antiparallel arrangement, $\theta = 180°$ and $U_2 = +PE = +1.8 \times 10^{-25}$ J.
Therefore the difference in the potential energy $\Delta U = U_2 - U_1 = 3.6 \times 10^{-25}$ J.

11.3.2 Gauss' Law

11.49 **(a)** If ϕ_E is the electric flux, $\boldsymbol{E}$ the electric field, q the charge enclosed and $\mathrm{d}\boldsymbol{A}$ the element of area then $q = \varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A}$. The integration is to be carried over the entire surface. The circle on the integral sign indicates that the surface of integration is a closed surface.

(b) Figure 11.41 shows a portion of a thin non-conducting infinite sheet of charge of constant charge density σ (charge per unit area). To calculate the electric field at points close to the sheet construct a Gaussian surface in the form of a closed cylinder of cross-sectional area A, piercing the plane of the sheet, Fig. 11.41. From symmetry, it is obvious that $\boldsymbol{E}$ points are at right angle to the end caps, away from the plane, and are positive at both the end caps. There is no contribution to the flux from the curved wall of the cylinder as $\boldsymbol{E}$ does not pierce. By Gauss law

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = q$$

$$\varepsilon_0(EA + EA) = q$$

where σA is the enclosed charge. Thus $E = \sigma/2\varepsilon_0\cdot$

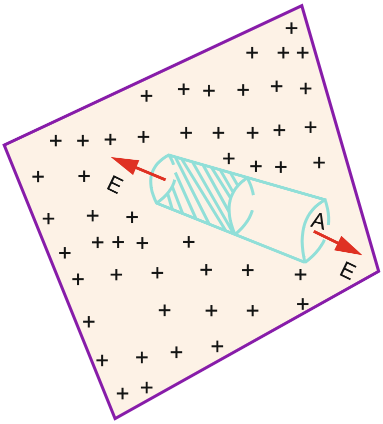


Fig. 11.41

For a non-conducting sheet the field

$$E = \frac{\sigma}{2\varepsilon_0} \tag{1}$$

The electric force acting on the sphere is

$$F = qE = \frac{q\sigma}{2\varepsilon_0} \tag{2}$$

(c) The sphere is held in equilibrium under the joint action of three forces:

(1) Weight acting vertically down,
(2) Electric force F acting horizontally, and
(3) Tension in the thread acting along the thread at an angle θ with the vertical.

From Fig. 11.42, $F/mg = \tan\theta$ (3)
Combining (2) and (3)

$$\sigma = \frac{2\varepsilon_0 mg\ \tan\theta}{q} = \frac{2 \times 8.9 \times 10^{-12} \times 2 \times 10^{-6} \times 9.8 \times \tan 10°}{5 \times 10^{-8}}$$

$$= 2.15 \times 10^{-11}\ \mathrm{C/m^2}$$

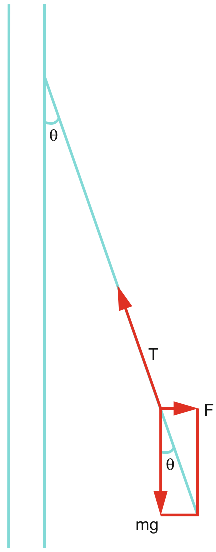


Fig. 11.42

11.50 **(a)** Integral form of Gauss' law:

$$\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \frac{\Sigma Q}{\varepsilon_0}$$

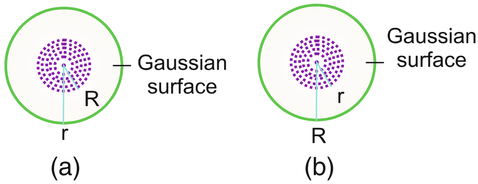
Differential form: $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\varepsilon_0}$

(b) $r > R$. Construct a Gaussian surface in the form of a sphere of radius $r > R$, concentric with the charged sphere of radius R, Fig. 11.43a. By Gauss' law

$$\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = (E)\, 4\pi\, r^2 = \frac{Q}{\varepsilon_0}$$

$$\therefore \quad E = \frac{Q}{4\pi\varepsilon_0 r^2} \tag{1}$$

Fig. 11.43



(c) $r < R$. Construct a Gaussian surface in the form of a sphere of radius $r < R$, concentric with the charged sphere of radius R, Fig. 11.43b. Let charge q' reside inside the Gaussian surface. Then by Gauss' law

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = (\varepsilon_0 E)\,(4\pi r^2) = q'$$

$$E = \frac{q'}{4\pi\varepsilon_0 r^2} \tag{2}$$

Now the charge outside the sphere of radius r does not contribute to the electric field. Assuming that ρ is constant throughout the charge distribution,

$$\frac{q'}{q} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

$$or \qquad q' = q\,\frac{r^3}{R^3} \tag{3}$$

Using (3) in (2), $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qr}{R^3}$ (4)

Thus for $r < R$, E varies linearly with r. Note that at $r = R$, both (1) and (4) give the same value as they should.

11.51 The electric field on the surface of the original sphere is

$$E_{\mathrm{R}} = \frac{q}{4\pi\varepsilon_0 R^2}$$

After creating the cavity, the charge in the remaining part of the sphere will be

$$q' = \left[\frac{\frac{4}{3}\pi R^3 \rho - \frac{4}{3}\pi \left(\frac{R}{2}\right)^3 \rho}{\frac{4}{3}\pi R^3 \rho}\right] q = \frac{7}{8} q$$

$\therefore$ The electric field on the surface is now

$$E'_{\mathrm{R}} = \frac{q'}{4\pi\varepsilon_0 R^2} = \frac{7}{8}\frac{q}{4\pi\varepsilon_0 R^2} = \frac{7}{8} E$$

11.52 **(a)** $E = \dfrac{q}{4\pi\varepsilon_0 r}\dfrac{r}{R^3}$

$$V(r) = -\int E\mathrm{d}r = -\frac{q}{4\pi\varepsilon_0 R^3}\int r\,\mathrm{d}r = -\frac{qr^2}{8\pi\varepsilon_0 R^3} + C$$

Now the potential at the surface ($r = R$) is

$$V(R) = \frac{q}{4\pi\varepsilon_0 R} = -\frac{q}{8\pi\varepsilon_0 R} + C$$

$$\therefore \quad C = \frac{3q}{8\pi\varepsilon_0 R}$$

$$\therefore \quad V(r) = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

(b) At the centre $r = 0$. From the result of ***(a)***

$$V(0) = \frac{3q}{8\pi\varepsilon_0 R} = \frac{3}{2} V(R)$$

11.53 **(a)** $E = 0$ when $r < b$ as no charge exists in this region.

(b) Region $b < r < a$: Consider a Gaussian surface, a sphere of radius r, where $b < r < a$. Charge at distance a between b and r only will contribute to the field. The charge residing in the shell of radii b and r is

$$Q' = \frac{\frac{4}{3}\pi\,(r^3 - b^3)Q}{\frac{4\pi}{3}\,(a^3 - b^3)} = \frac{(r^3 - b^3)}{(a^3 - b^3)}Q$$

$$E = \frac{Q'}{4\pi\varepsilon_0 r^2} = \frac{Q'}{4\pi\varepsilon_0 r^2}\,\frac{(r^3 - b^3)}{(a^3 - b^3)}$$

(c) Region $r > a$: $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

11.54 **(a)** Construct a Gaussian surface in the form of a cylinder of radius $r > R$ and height h. By Gauss' law

$$(\varepsilon_0 E)\,(2\pi r h) = Q = \rho\pi\,R^2 h$$

$$\therefore \qquad E = \frac{1}{2}\frac{\rho}{\varepsilon_0}\frac{R^2}{r}$$

(b) Construct a Gaussian surface in the form of a cylinder of height h and radius $r < R$ coaxial with the cylinder of radius R and height h. By Gauss' law

$$(\varepsilon_0 E)\,(2\pi r h) = Q' = \rho\pi\,R^2 h$$

$$\therefore \qquad E = \frac{1}{2}\frac{\rho}{\varepsilon_0}\frac{R^2}{r} \quad (r < R)$$

11.55 **(a)** $E = \dfrac{q}{4\,\pi\varepsilon_0 r^2}$

$$\therefore \qquad q = 4\,\pi\varepsilon_0 r^2 E = \frac{1}{9 \times 10^9} \times (0.5)^2 \times 800 = 2.22 \times 10^{-8}\,\mathrm{C}$$

(b) $E = \dfrac{\sigma}{\varepsilon_0}$

$$\therefore \qquad \sigma = E\varepsilon_0 = 120 \times 8.85 \times 10^{-12} = 1.062 \times 10^{-9}\,\mathrm{C/m^2}$$

(c) $q = \varepsilon_0\phi$

$$\phi = EA = 120 \times (100 \times 75) = 9 \times 10^5\,\mathrm{N\,m^2/C}$$

11.56 **(a)** $E_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = q \qquad \text{(Gauss' law)} \qquad (1)$

Consider an isolated positive point charge. Construct a Gaussian surface, a sphere of radius r centred at the charge. At every point on the spherical surface, the field is perpendicular to the surface. As both $\boldsymbol{E}$ and d$\boldsymbol{A}$ are directed radially outwards, the angle θ between them is zero so that the dot product $\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A}$ reduces to $E\mathrm{d}A$ and (1) can be written as

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = \varepsilon_0 \int E\mathrm{d}A = q \tag{2}$$

As E is constant, it can be factored out of the integral:

$$\varepsilon_0 E \int \mathrm{d}A = (\varepsilon_0 E)\,(4\pi\, r^2) = q$$

$$\therefore \qquad E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \text{(Coulomb's law)} \tag{3}$$

(b) **(i)** $r < R$

By prob. (11.50), $E = \dfrac{Qr}{4\pi\varepsilon_0 R^3}$

$$\phi = EA = \frac{Q\,r}{4\pi\varepsilon_0 R^3} 4\pi\, r^2 = \frac{Q\,r^3}{\varepsilon_0\, R^3}$$

(ii) $r > R$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\phi = EA = \frac{Q}{4\pi\varepsilon_0 r^2}\, 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

11.57 $\phi = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A}, \qquad q = \varepsilon_0 A$

(a) Construct a Gaussian surface, a sphere of radius r_1 concentric with the spherical shells. Since no charge is enclosed by the Gaussian surface with $r < R_1$, $E = 0$.

(b) Here the net charge enclosed by the Gaussian surface is Q_1. As $\boldsymbol{E}$ is normal to the spherical surface by Gauss' law

$$(\varepsilon_0 E)\,(4\pi\, r^2) = Q_1 \quad \text{or} \quad E = \frac{Q_1}{4\pi\varepsilon_0 r^2}$$

(c) Here the net charge enclosed by the Gaussian surface is $Q_1 + Q_2$ and $\boldsymbol{E}$ is normal to the spherical surface (Fig. 11.44). By Gauss' law

$$(\varepsilon_0 E)(4\pi r^2) = Q_1 + Q_2 \quad \text{or} \quad E = \frac{Q_1 + Q_2}{4\pi\varepsilon_0 r^2}$$

In (c) if $Q_1 + Q_2 = 0$, then $Q_1/Q_2 = -1$.

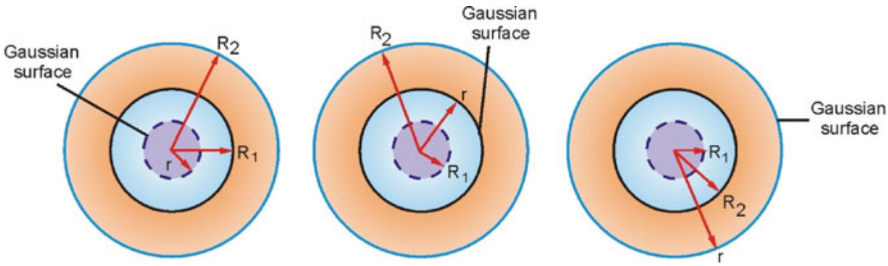


Fig. 11.44

11.58 Let the charges be $q_1 = q_2$ on the two spheres before contact.

$$\text{Force } F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} = \frac{q_1^2}{4\pi\varepsilon_0 r^2} \tag{1}$$

When the spheres are brought into contact and separated they are at a common potential. Let the new charges be q_1' and q_2', respectively

$$q_1' + q_2' = q_1 + q_2 = 2q_1 \qquad \text{(from charge conservation)} \tag{2}$$

$$\text{Also} \quad V = \frac{q_1'}{4\pi\varepsilon_0 R_1} = \frac{q_2'}{4\pi\varepsilon_0 R_2} \tag{3}$$

$$\text{or} \quad \frac{q_1'}{q_2'} = \frac{R_1}{R_2} \tag{4}$$

Solving (2) and (4)

$$q_1' = \frac{2q_1 R_1}{R_1 + R_2}, \quad q_2' = \frac{2q_1 R_2}{R_1 + R_2} \tag{5}$$

$$\text{After separation, force } F' = \frac{q_1' q_2'}{4\pi\varepsilon_0 r^2} \tag{6}$$

$$\frac{F}{F'} = \frac{q_1^2}{q_1' q_2'} = \frac{(r_1 + r_2)^2}{4r_1 r_2} = \frac{(1+3)^2}{4 \times 1 \times 3} = \frac{4}{3} \tag{7}$$

11.59 $V = \dfrac{q}{4\pi\varepsilon_0 r}$

$$\therefore \quad q = 4\pi\varepsilon_0 r V = \frac{(5)(2 \times 10^4)}{9 \times 10^9} = 1.11 \times 10^{-5}\,\text{C}$$

11.60 **(a)** $V = \dfrac{q}{4\pi\varepsilon_0 a}$

$$\therefore \text{ Capacitance } C = \frac{q}{V} = 4\pi\varepsilon_0 a$$

(b) $q_1 = 4\pi\varepsilon_0 a V_1, \quad q_2 = 4\pi\varepsilon_0 a V_2$

$$U_1 = \frac{1}{2} q_1 V_1 = \frac{1}{2} \frac{q_1^2}{4\pi\varepsilon_0 a}, U_2 = \frac{1}{2} \frac{q_2^2}{4\pi\varepsilon_0 a}$$

Initially total energy $U = U_1 + U_2 = \dfrac{\left(q_1^2 + q_2^2\right)}{8\pi\varepsilon_0 a}$ (1)

When the spheres are connected they reach a common potential and when disconnected let the charges be q_1' and q_2'.

$$q_1' = 4\pi\varepsilon_0 a V, \quad q_2' = 4\pi\varepsilon_0 a V$$

$$\therefore \quad q_2' = q_1'$$

Final energy in the spheres

$$U_1' = \frac{1}{2} q_1' V = \frac{q_1'^2}{8\pi\varepsilon_0 a}, \ U_2' = \frac{q_2'^2}{8\pi\varepsilon_0 a}$$

Total final energy $U' = U_1' + U_2' = \dfrac{q_1'^2 + q_2'^2}{8\pi\varepsilon_0 a}$ (2)

$q_1 + q_2 = q_1' + q_2' = 2q_1'$ (charge conservation) (3)

$$\Delta U = U_1 - U_2 = \frac{1}{8\pi\varepsilon_0 a} \left(q_1^2 + q_2^2 - 2q_1'^2\right) \tag{4}$$

Eliminating q_1' between (3) and (4) and simplifying

$$\Delta U = \frac{1}{16\pi\varepsilon_0 a} (q_1 - q_2)^2 \tag{5}$$

This energy is dissipated in Joule heating of the wire.

11.61 When the spheres are brought into contact they reach a common potential, say V. If the charges on them are now Q_1' and Q_2'

$$V = \frac{Q_1'}{4\pi\varepsilon_0 R_1} = \frac{Q_2'}{4\pi\varepsilon_0 R_2} \tag{1}$$

$$\sigma_1' = \frac{Q_1'}{4\pi R_1^2} = \frac{\varepsilon_0 V}{R_1} \tag{2}$$

where we have used (1).

Similarly

$$\sigma_2' = \frac{\varepsilon_0 V}{R_2} \tag{3}$$

Thus $\sigma' \, \alpha \, \dfrac{1}{R}$.

11.62 **(a)** $V = Er = 5 \times 10^6 \times 0.1 = 5 \times 10^5\,\text{V}$

(b) Pressure $p = \dfrac{1}{2}\dfrac{\sigma^2}{\varepsilon_0} = \dfrac{1}{2\varepsilon_0}(E\varepsilon_0)^2 = \dfrac{1}{2}\varepsilon_0 E^2$

$$= \frac{1}{2} \times 8.85 \times 10^{-12} \times (5 \times 10^6)^2 = 110.6\,\text{N/m}^2$$

(c) If q is the charge and C the capacitance then the electrostatic energy

$$U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 r}$$

$$\text{Now } q = 4\pi\varepsilon_0 r^2 E = \frac{(0.1)^2 \times 5 \times 10^6}{9 \times 10^9} = 5.55 \times 10^{-6}\,\text{C}$$

$$\therefore \qquad U = \frac{1}{2} \times 9 \times \frac{10^9}{0.1} \times (5.55 \times 10^{-6})^2 = 1.386\,\text{J}$$

11.63 Work done in the isobaric expansion (constant pressure) is

$$W = P\Delta v = \frac{4\pi}{3}\left(R_2^3 - R_1^3\right) p \tag{1}$$

where we have written υ for volume.
Increase in electrostatic energy

$$\Delta U = \frac{1}{2}\frac{q^2}{C_1} - \frac{1}{2}\frac{q^2}{C_2} = \frac{q^2}{2}\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2}$$

where C is the capacitance of the spherical bubble. Equating (1) and (2) and simplifying, we obtain

$$q = \left[\frac{32}{3}\pi^2\varepsilon_0 p\, R_1 R_2 \left(R_1^2 + R_1 R_2 + R_2^2\right)\right]^{1/2} \tag{3}$$

11.64 $$q = \int_{r_1}^{r_2} \rho(r)\,\mathrm{d}\upsilon = \int \frac{A}{r} \cdot 4\pi r^2 \mathrm{d}r = 2\pi\, A\left(r_2^2 - r_1^2\right) \quad (1)$$

Region (i), $0 \le r < r_1$, $\rho(r) = 0$

As no charge is enclosed, $E = 0$ (2)

Region (ii), $r_1 \le r \le r_2$, $\rho(r) = \frac{A}{r}$

By Gauss' law

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_{r_1}^{r} \rho(r)\mathrm{d}\upsilon = \int_{r_1}^{r} \frac{A}{r} 4\pi r^2 \mathrm{d}r = 2\pi\, A\left(r^2 - r_1^2\right)$$

$$\therefore \quad (\varepsilon_0 E)(4\pi r^2) = 2\pi\, A\left(r_2^2 - r_1^2\right)$$

$$\therefore \quad E = \frac{A}{2\varepsilon_0 r^2}\left(r^2 - r_1^2\right) \quad (3)$$

Region (iii), $r > r_2$, $\rho(r) = 0$

$$\varepsilon_0 \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = q = 2\pi\, A\left(r_2^2 - r_1^2\right) \text{ by (1)}$$

$$\therefore \quad (\varepsilon_0 E)(4\pi r^2) = 2\pi\, A\left(r_2^2 - r_1^2\right)$$

$$\therefore \quad E = \frac{A}{2\varepsilon_0 r^2}\left(r_2 - r_1^2\right) \quad (4)$$

11.65 **(a)** In order to show that the electric field is conservative, it is sufficient to establish the existence of a potential. Now, if potential V exists, it must be such that

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\mathrm{d}V$$

where $\boldsymbol{F} = f(r)\boldsymbol{e}_r$ is the central force and $\boldsymbol{e}_r$ is the unit vector along the radius vector $\boldsymbol{r}$

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = f(r)\boldsymbol{e}_r \cdot \mathrm{d}\boldsymbol{r} = f(r)\,\mathrm{d}r$$

$$\therefore \quad -\mathrm{d}V = f(r)\,\mathrm{d}r$$

$$\text{or} \quad V = -\int f(r)\mathrm{d}r$$

We conclude that the field is conservative and V represents the potential given by the above relation.

(b) Initial electrostatic energy

$$U_1 = \frac{1}{2}qV = \frac{1}{2}CV^2 = \frac{1}{2}4\pi\varepsilon_0 r_1 V^2$$
$$= \frac{1}{2} \times \frac{1}{9 \times 10^9} \times 0.01 \times 10^2 = 5.55 \times 10^{-11}\,\text{J}$$

Final electrostatic energy

$$U_2 = \frac{1}{2}4\pi\varepsilon_0 r_2 V^2 = \frac{1}{2} \times \frac{1}{9 \times 10^9} \times 0.001 \times 10^2$$
$$= 5.55 \times 10^{-12}\,\text{J}$$

$\therefore$ Energy decrease $= U_1 - U_2 = (5.55 - 0.555) \times 10^{-11} = 5 \times 10^{-11}$ J

11.66 Construct a Gaussian cylindrical surface of radius r and length L coaxial with the cylinder.

(i) $r > a$. The charge enclosed is $q = \pi a^2 L\rho$. By Gauss' law

$$\varepsilon_0 \oint E \cdot \mathrm{d}A = q$$
$$\therefore \quad \varepsilon_0 E\,(2\pi r L) = \pi a^2 L\rho$$
$$\therefore \quad E = \frac{a^2\rho}{2\varepsilon_0 r}$$

(ii) $0 < r < a$. By Gauss' law

$$\varepsilon_0 \oint E \cdot \mathrm{d}A = q'$$
$$(\varepsilon_0 E)(2\pi r L) = \pi(a^2 - r^2)L\rho$$
$$\therefore \quad E = \frac{(a^2 - r^2)\rho}{2\varepsilon_0 r}$$

Centripetal force = Electric force

$$\frac{mv^2}{r} = Ee = \frac{a^2\rho e}{2\varepsilon_0 r}$$
$$\therefore \quad v = \sqrt{\frac{a^2\rho e}{2\varepsilon_0 m}}$$

11.67 For an infinite non-conducting charge sheet $E = \sigma/2\varepsilon_0$

$$\therefore \qquad \sigma = 2E\varepsilon_0 = 2 \times 200 \times 8.85 \times 10^{-12} = 3.54 \times 10^{-9}\,\mathrm{C/m^2}$$

The electric field is independent of the distance.
For an infinite conducting sheet

$$\sigma = E\varepsilon_0 = 200 \times 8.85 \times 10^{-12} = 1.77 \times 10^{-9}\,\mathrm{C/m^2}$$

11.3.3 Capacitors

11.68 The field strength E_0 between the plates of a parallel plate capacitor in vacuum is

$$E_0 = \frac{V}{d} = \frac{\sigma}{\varepsilon_0} \tag{1}$$

where V is the applied voltage, σ the charge density (charge per unit area) and ε_0 the permittivity in vacuum. Now $\sigma = q/A$. Therefore the capacitance in air or vacuum will be

$$C_0 = \frac{q}{V} = \frac{A\varepsilon_0}{d} \tag{2}$$

With the introduction of the dielectric slab the electric field in the slab will be E_0/K, and the potential across the capacitor becomes

$$V = E_0(d - K) + \frac{E_0 t}{K} = E_0\left[(d-t) + \frac{t}{K}\right]$$

$$= \frac{q}{A\varepsilon_0}\left[(d-t) + \frac{t}{K}\right]$$

$$\therefore \quad C = \frac{q}{V} = \frac{\varepsilon_0 A}{d - t\left(1 - \dfrac{1}{K}\right)} \tag{3}$$

If a metal of thickness t is to be introduced, the effective distance between the capacitor plates is reduced and the capacitance becomes

$$C = \frac{\varepsilon_0 A}{d - t} \tag{4}$$

a result which is obtained by putting $K = \infty$ in (3).

11.69 $C_1 + C_2 = 9 \qquad \text{(parallel)} \qquad (1)$

$$\frac{C_1 C_2}{C_1 + C_2} = 2 \qquad \text{(series)} \qquad (2)$$

$$(C_1 - C_2)^2 = (C_1 + C_2)^2 - 4C_1C_2 = (C_1 + C_2)^2 - 8(C_1 + C_2) = 9$$

where we have used (1)

$$\therefore \qquad C_1 - C_2 = 3 \qquad (3)$$

Solving (1) and (3), $C_1 = 6\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$.

11.70 **(a)** $U_1 = \frac{1}{2}C_1V^2 = \frac{1}{2} \times 2 \times 10^{-6} \times (100)^2 = 0.01\,\text{J}$

$$U_2 = \frac{1}{2}C_2V^2 = \frac{1}{2} \times 4 \times 10^{-6} \times (100)^2 = 0.02\,\text{J}$$

(b) $C = \frac{C_1C_2}{C_1 + C_2} = \frac{2 \times 10^{-6} \times 4 \times 10^{-6}}{(2+4) \times 10^{-6}} = \frac{4}{3} \times 10^{-6}\,\text{F}$

$$U = \frac{1}{2}CV^2 = \frac{1}{2} \times \frac{4}{3} \times 10^{-6} \times (100)^2 = 0.0067\,\text{F}$$

(c) $C = C_1 + C_2 = (2+4) \times 10^{-6} = 6 \times 10^{-6}\,\text{F}$

$$U = \frac{1}{2}CV^2 = \frac{1}{2} \times 6 \times 10^{-6} \times (100)^2 = 0.03\,\text{J}$$

11.71 $C_0 = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times 1}{0.01} = 8.85 \times 10^{-10}\,\text{F}$

$$U_0 = \frac{1}{2}\frac{Q^2}{C_0} = \frac{1}{2} \times \frac{(10^{-6})^2}{8.85 \times 10^{-10}} = 5.65 \times 10^{-4}\,\text{J}$$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{C_0K} = \frac{U_0}{K} = \frac{5.65 \times 10^{-4}}{2} = 2.83 \times 10^{-4}\,\text{J}$$

The energy is decreased by $\Delta U = U_0 - U = (5.65 - 2.83) \times 10^{-4} = 2.82 \times 10^{-4}\,\text{J}$, that is, by a factor of 2.

11.72 $C = \frac{\varepsilon_0 K A}{d}$

$$\therefore \quad K = \frac{Cd}{\varepsilon_0 A} = \frac{0.1 \times 10^{-6} \times 0.001}{8.85 \times 10^{-12} \times 1.0} = 11.3$$

11.73 $Q = C_1 V = 5 \times 10^{-6} \times 250 = 1.25 \times 10^{-3}\,\mathrm{C}$
For the parallel connection

$$C = C_1 + C_2 = 5 \times 10^{-6} + 20 \times 10^{-6} = 25 \times 10^{-6}\,\mathrm{F}$$

The resulting voltage

$$V' = \frac{Q}{C} = \frac{1.25 \times 10^{-3}}{25 \times 10^{-6}} = 50\,\mathrm{V}$$

11.74 The combination of 2 µF and 2 µF in parallel is equivalent to 4 µF. This in series with 8 µF gives a combined capacitance of 8/3 µF; 12 µF and 6 µF in series gives an equivalent capacitance of 4 µF. 4 µF with 4 µF in parallel gives 8 µF which in series with 1 µF yields 8/9 µF.

Combination of 8/9 µF and 8/3 µF in parallel gives 32/9 µF.

Effective value of C with 32/9 µF in series gives

$$\frac{\frac{32}{9}C}{C + \frac{32}{9}} = 1, \quad \text{by problem}$$

$$\therefore \qquad C = 1\frac{9}{23}\,\mu\mathrm{F}$$

11.75 Combined capacitance C for the three capacitors in series:

$$C = \frac{C_1 C_2 C_3}{C_1 C_2 + C_2 C_3 + C_3 C_1} = \frac{(4 \times 3 \times 2) \times 10^{-18}}{(4 \times 3 + 3 \times 2 + 2 \times 4) \times 10^{-12}}$$
$$= 0.923 \times 10^{-6}\,\mathrm{F}$$

(a)
$$q = CV = 0.923 \times 10^{-6} \times 260 = 240 \times 10^{-6}\,\mathrm{C}$$
$$\therefore \qquad q_1 = q_2 = q_3 = 240 \times 10^{-6}\,\mathrm{C}$$

(b)
$$V_1 = \frac{q}{C_1} = \frac{240 \times 10^{-6}}{4 \times 10^{-6}} = 60\,\mathrm{V}$$
$$V_2 = \frac{q}{C_2} = \frac{240 \times 10^{-6}}{3 \times 10^{-6}} = 80\,\mathrm{V}$$
$$V_3 = \frac{q}{C_3} = \frac{240}{2} = 120\,\mathrm{V}$$

(c)
$$W_1 = \frac{1}{2}C_1 V_1^2 = \frac{1}{2} \times 4 \times 10^{-6} \times (60)^2 = 0.0072\,\mathrm{J}$$
$$W_2 = \frac{1}{2}C_2 V_2^2 = \frac{1}{2} \times 3 \times 10^{-6} \times (80)^2 = 0.0096\,\mathrm{J}$$
$$W_3 = \frac{1}{2}C_3 V_3^2 = \frac{1}{2} \times 2 \times 10^{-6} \times (120)^2 = 0.0144\,\mathrm{J}$$

11.76 Charge on the first capacitor

$$q_1 = C_1 V = 1 \times 10^{-6} \times 12 = 12 \times 10^{-6}\,\mathrm{C}$$

Charge on the second capacitor

$$q_2 = C_2 V = 2 \times 10^{-6} \times 12 = 24 \times 10^{-6}\,\mathrm{C}$$

Capacitance for the parallel combination

(a) $C = C_1 + C_2 = (1 + 2) \times 10^{-6} = 3 \times 10^{-6}\,\mathrm{F}$

$$q = q_1 + q_2 = (12 + 24) \times 10^{-6} = 36 \times 10^{-6}\,\mathrm{C}$$

$$V = \frac{q}{C} = \frac{36 \times 10^{-6}}{3 \times 10^{-6}} = 12\,\mathrm{V}$$

(b) $q' = q_2 - q_1 = (24 - 12) \times 10^{-6} = 12 \times 10^{-6}\,\mathrm{C}$

$$V = \frac{q'}{C} = \frac{12 \times 10^{-6}}{3 \times 10^{-6}} = 4\,\mathrm{V}$$

11.77 **(a)** If the positive end of a capacitor of capacitance C_1, charged to potential difference V_1, is connected in parallel to the positive end of the capacitor of capacitance C_2 charged to potential difference V_2, then conservation of charge gives the equation

$$(C_1 + C_2)V = C_1 V_1 + C_2 V_2 \tag{1}$$

$$\therefore \quad V = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2} \quad \text{(common potential)} \tag{2}$$

The energy loss

$$\Delta W = \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2 - \frac{1}{2} (C_1 + C_2)\, V^2$$

$$= \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} (V_1 - V_2)^2 \tag{3}$$

where we have used (2).

(b) If the positive end is joined to the negative end, the common potential difference will be

$$V = \frac{C_1 V_1 - C_2 V_2}{C_1 + C_2} \tag{4}$$

and the energy loss will be

$$\Delta W = \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} (V_1 + V_2)^2 \tag{5}$$

11.78 **(a)** Battery remains connected

(i) $V' = V$; potential remains unchanged.
(ii) $E' = E$; electric field is unchanged.
(iii) $q' = Kq$; charge is increased by a factor K. The additional charge $(K - 1)q$ is moved from the negative to the positive plate by the battery, as the dielectric slab is inserted.
(iv) $C' = KC$; capacitance is increased by a factor K.
(v) $U' = \frac{1}{2}q'V' = \frac{1}{2}KqV = KU$

Energy is increased by a factor K.

(b) The battery is disconnected

(i) $V' = \frac{V}{K}$; potential is decreased by a factor K

(ii) $E' = \frac{E}{K}$; electric field is decreased by a factor K. Both (i) and (ii) follow from the fact that $q' = q$ so that $C'V' = CV$ and $V' = \frac{CV}{C'} = \frac{V}{K}$. Same reasoning holds good for E'.

(iii) $q' = q$; charge remains unchanged as there is no path for charge transfer.
(iv) $C' = KC$; capacitance is increased by a factor K.
(v) $U' = \frac{1}{2}q'V' = \frac{1}{2}\frac{qV}{K} = \frac{U}{K}$

The energy is lowered by a factor K.

11.79 **(a)** The battery remains connected

(i) $V' = V$; potential remains unchanged.
(ii) $E' < E$; the electric field is decreased since $E = V/d$, and V is constant.
(iii) $C' < C$; capacitance is reduced since $C \propto 1/d$.
(iv) $q' < q$; the charge is reduced since $q = CV$, with C decreasing and V remaining constant. Some charge is transferred from the capacitor to the charging battery.
(v) $U' < U$; the energy is decreased since $U = \frac{1}{2}qV$, with q decreasing and V remaining constant.

(b) Battery is disconnected

(i) $V' > V$, the potential increases because $q = CV$, with C decreasing and q remaining constant.
(ii) $E' = E$, the electric field is constant because $q = CV = \frac{\varepsilon_0 AV}{d} = \varepsilon_0 AE$, with q remaining constant.
(iii) $C' < C$; the capacitance is decreased since $C \propto 1/d$.

(iv) $q' = q$; the charge remains constant.

(v) $U' > U$; energy increases because $U = \frac{1}{2}qV$, with V increasing and q remaining constant.

11.80 As the plates carry equal but opposite charges, the force of attraction, which is conservative, is given by

$$F = -\frac{dU}{dx}$$

But $U = \frac{1}{2}\frac{q^2}{C}$

For the parallel plate capacitor,

$$C = \frac{\varepsilon_0 A}{x}$$

where x is the distance of separation. Combining the above equations

$$F = -\frac{d}{dx}\left[\frac{1}{2}\frac{q^2 x}{\varepsilon_0 A}\right] = -\frac{q^2}{2\varepsilon_0 A} = -\frac{1}{2}\frac{\varepsilon_0 A V^2}{d^2}$$

where we have put $x = d$.

11.81 **(a)** As the drops are assumed to be incompressible, the volume does not change.

$$\frac{4}{3}\pi R^3 = n\,\frac{4}{3}\pi r^3$$

$$\therefore \quad R = n^{1/3}\,r$$

(b) $C' = 4\pi\varepsilon_0 R = 4\pi\varepsilon_0 r\,n^{1/3}$

$\therefore \quad C' = n^{1/3}\,C$

(c) $Q = nq$ (charge conservation)

$$V' = \frac{Q}{4\pi\varepsilon_0 R} = \frac{nq}{4\pi\varepsilon_0 n^{1/3} r} = \frac{n^{2/3} q}{4\pi\varepsilon_0 r} = n^{2/3}\,V$$

(d) $\sigma' = \dfrac{Q}{4\pi R^2} = \dfrac{nq}{4\pi\, r^2 n^{2/3}} = n^{1/3}\,\sigma$

(e) $U' = \dfrac{1}{2}\,QV' = \dfrac{1}{2}\,nqn^{2/3}V = n^{5/3}U$

11.82 The electric field for a cylindrical capacitor is

$$E = \frac{q}{2\pi\varepsilon_0 l r} \tag{1}$$

where l is the length and r the radius. The energy density (energy/unit volume)

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{q^2}{8\pi^2\varepsilon_0 l^2 r^2} \tag{2}$$

where we have used (1).
The energy stored between the coaxial cylinders of length l and radii R and a is

$$U = \int u\,\mathrm{d}v = \int_a^R u\,(2\pi r l)\,\mathrm{d}r \tag{3}$$

where $\mathrm{d}v = (2\pi r\,\mathrm{d}r)l$ is the volume element. Using (2) in (3)

$$U = \frac{q^2}{4\pi\varepsilon_0 l}\int_a^R \frac{\mathrm{d}r}{r} = \frac{q^2}{4\pi\varepsilon_0 l}\ln\frac{R}{a}$$

Similarly, the energy stored between the coaxial cylinders of radii b and a is

$$U_0 = \frac{q^2}{4\pi\varepsilon_0 l}\ln\frac{b}{a}$$

$$\therefore \quad \frac{U}{U_0} = \frac{\ln(R/a)}{\ln(b/a)}$$

$$\text{Set} \quad \frac{U}{U_0} = \frac{1}{2}$$

$$\therefore \quad \frac{\ln(R/a)}{\ln(b/a)} = \frac{1}{2} \to \ln\frac{b}{a} = 2\ln\frac{R}{a} = \ln\frac{R^2}{a^2}$$

$$\text{or} \quad \frac{b}{a} = \frac{R^2}{a^2} \to R = \sqrt{ab}$$

11.83 The charge on C_1 is

$$q_1 = C_1 V_1 = 3\times 10^{-6}\times 4000 = 0.012\,\mathrm{C}$$

The charge on C_2 is

$$q_2 = C_2 V_2 = 6 \times 10^{-6} \times 3000 = 0.018\,\mathrm{C}$$

The combined capacitance

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{3 \times 6}{3 + 6} \times 10^{-6} = 2 \times 10^{-6}\,\mathrm{F}$$

Take the lower charge to find the maximum voltage V:

$$V = \frac{q_1}{C} = \frac{0.012}{2 \times 10^{-6}} = 6000\,\mathrm{V}$$

11.84 If the dielectric is present, Gauss' law gives

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = q - q' = \frac{q}{K} \tag{1}$$

where $-q'$ is the induced surface charge, q is the free charge and K is the dielectric constant. Construct a Gaussian surface in the form of a coaxial cylinder of radius r and length l, closed by end caps. Applying (1),

$$\varepsilon_0 E\,(2\pi r l) = \frac{q}{K}$$

$$\text{or} \quad E = \frac{q}{2\pi\varepsilon_0 r l k} \tag{2}$$

In (1) the integral is contributed only by the curved surface and not the end caps. The potential difference between the central rod and the surrounding tube is given by

$$V = -\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b E\,\mathrm{d}r = \int_a^b \frac{q}{2\pi\varepsilon_0 l K} \frac{\mathrm{d}r}{r} = \frac{q}{2\pi\varepsilon_0 l K} \ln \frac{b}{a}$$

The capacitance is given by

$$C = \frac{q}{V} = \frac{2\pi\varepsilon_0 l K}{\ln(b/a)}$$

11.85 The field at point P is caused entirely by the charge Q on the inner sphere, Fig. 11.45, and has the value

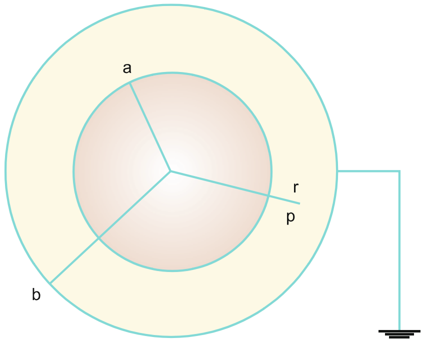
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

The potential difference between the two spheres is given by

$$V = -\int_b^a \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{\mathrm{d}r}{r^2} = \frac{Q\,(b-a)}{4\pi\varepsilon_0 ab}$$

whence $C = \dfrac{Q}{V} = \dfrac{4\pi\varepsilon_0 ab}{b-a}$

Fig. 11.45



11.86 By prob. (11.85)

$$C = \frac{4\pi\varepsilon_0 ab}{b-a} \tag{1}$$

Let $b = a + \Delta$ where Δ is a small quantity. Then (1) can be written as

$$C = \frac{4\pi\varepsilon_0 a(a+\Delta\, a)}{\Delta a} \simeq \frac{4\pi\varepsilon_0 a^2}{\Delta a} \tag{2}$$

Now the surface area $A = 4\pi a^2$ and $\Delta a = d$, the distance between the surfaces, so that

$$C \simeq \frac{\varepsilon_0 A}{d} \qquad \text{(parallel plate capacitor)}$$

11.87 **(i)** $q = q_0\,\mathrm{e}^{-t/RC}$

$$\mathrm{e}^{-t/RC} = \frac{q}{q_0} = \frac{90}{100}$$

$$\therefore \qquad t = RC\ln\frac{10}{9} = 1\times10^6\times10\times10^{-6}\times0.1056 = 1.056\,\mathrm{s}$$

(ii) At one time constant $t = RC$

$$V = V_0\,\mathrm{e}^{-1} = \frac{120}{2.718} = 44.15\,\mathrm{V}$$
$$U = \frac{1}{2}CV^2 = \frac{1}{2} \times 10 \times 10^{-6} \times (44.15)^2 = 0.097\,\mathrm{J}$$

(iii) $H = i^2\,Rt = \dfrac{V^2}{R}t = V^2C = (44.15)^2 \times 10 \times 10^{-6} = 0.0195\,\mathrm{J}$

11.88 Equilibrium energy $U_0 = \frac{1}{2}CV_0^2$
Energy at time t

$$U = \frac{1}{2}CV^2 = \frac{1}{2}CV_0^2\left(1 - \mathrm{e}^{-t/RC}\right)^2 = U_0\left(1 - \mathrm{e}^{-t/RC}\right)^2$$
$$\frac{U}{U_0} = \frac{1}{2} = \left(1 - \mathrm{e}^{-t/RC}\right)^2$$

Solving $t = 1.228RC$.
Thus after 1.228 time constants the energy stored in the capacitor will reach half of its equilibrium value.

11.89 Let the capacitor be divided into differential strips which are practically parallel. Consider a strip at distance x of length a perpendicular to the plane of paper and of width $\mathrm{d}x$ in the plane of paper, the area of the strip being $\mathrm{d}A = a\mathrm{d}x$, Fig. 11.46. At the distance x, the separation of the plates is seen to be $t = D + x\theta$. The capacitance due to the differential strip facing each plate is

$$\mathrm{d}C = \frac{\varepsilon_0 \mathrm{d}A}{D} = \frac{\varepsilon_0 a \mathrm{d}x}{D + x\,\theta}$$

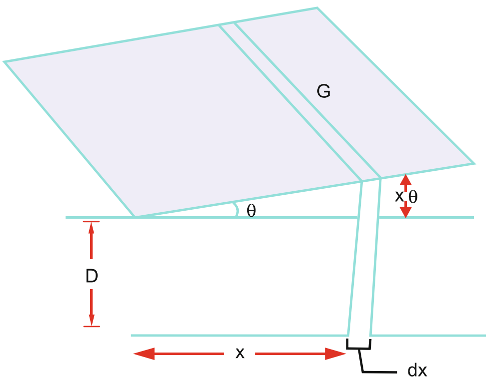


Fig. 11.46

The capacitance is given by

$$C = \int \mathrm{d}C = \int_0^a \frac{\varepsilon_0 a\,\mathrm{d}x}{D + x\,\theta} = \varepsilon_0 a \int_0^a \frac{\mathrm{d}x}{D + x\,\theta} = \frac{\varepsilon_0 a}{D} \int_0^a \left(1 + \frac{x\,\theta}{D}\right)^{-1} \mathrm{d}x$$

$$= \frac{\varepsilon_0 a}{D} \int_0^a \left(1 - \frac{x\theta}{D} + \cdots\right) \mathrm{d}x \simeq \frac{\varepsilon_0 a}{D} \left(x - \frac{x^2\theta}{2D}\right)\Bigg|_0^a = \frac{\varepsilon_0 a^2}{D} \left(1 - \frac{a\theta}{2D}\right)$$

Note that for $\theta = 0$, capacitance reduces to that for the parallel plate capacitor.

11.90 **(a)** The equivalent capacitance of C_1 and C_2 in parallel is $C_{12} = 8 + 4 = 12\,\mu\mathrm{F}$.

The combined capacitance of C_{12} and C_3 in series is $C = \dfrac{C_3 C_{12}}{C_3 + C_{12}} = \dfrac{3 \times 12}{3 + 12} = 2.4\,\mu\mathrm{F}$.

Applied charge $q = CV = 2.4 \times 100 = 240\,\mu\mathrm{C}$. Therefore charge on C_3 will be $q_3 = 240\,\mu\mathrm{C}$. PD across C_3 will be $V_3 = \dfrac{q_3}{C_3} = \dfrac{240}{3} = 80\,\mathrm{V}$. The PD across C_1 and C_2 will be equal.

$$V_1 = V_2 = (V - V_3) = (100 - 80) = 20\,\mathrm{V}$$

(b) Now $\dfrac{q_1}{C_1} = \dfrac{q_2}{C_2} \qquad (\because\ V_1 = V_2)$

$$\therefore \qquad q_1 = \frac{C_1 q_2}{C_2} = \frac{8}{4} q_2 = 2q_2$$

Also $q_1 + q_2 = 240$

$$\therefore \qquad q_1 = 160\,\mu\mathrm{C} \text{ and } q_2 = 80\,\mu\mathrm{C}$$

(c) $U_1 = \dfrac{1}{2} C_1 V_1^2 = \dfrac{1}{2} \times 8 \times 10^{-6} \times 20^2 = 0.0016\,\mathrm{J}$

$$U_2 = \frac{1}{2} C_2 V_2^2 = \frac{1}{2} \times 4 \times 10^{-6} \times 20^2 = 0.0008\,\mathrm{J}$$

$$U_3 = \frac{1}{2}C_3V_3^2 = \frac{1}{2} \times 3 \times 10^{-6} \times 80^2 = 0.0096\,\text{J}$$

Note that $U_1 + U_2 + U_3 = 0.012\,\text{J} = U = \frac{1}{2}CV^2$

11.91 **(a)** The combination of C_1 and C_2 in series yields $C_{12} = \frac{8 \times 4}{8+4} = 2.667 \times 10^{-6}\,\text{F}$.

$$C = C_{12} + C_3 = (2.667 + 3.0) \times 10^{-6} = 5.667 \times 10^{-6}\,\text{F}$$

$$q = CV = 5.667 \times 10^{-6} \times 100 = 5.667 \times 10^{-4}\,\text{C}$$

$$q_3 = C_3V = 3 \times 10^{-6} \times 100 = 3 \times 10^{-4}\,\text{C}$$

$$q_1 = q_2 = q - q_3 = (5.667 - 3.0) \times 10^{-4} = 2.667 \times 10^{-4}\,\text{C}$$

(b) $V_3 = 100\,\text{V}$

$$V_1 = \frac{q_1}{C_1} = \frac{2.667 \times 10^{-4}}{8 \times 10^{-6}} = 33.33\,\text{V}$$

$$V_2 = \frac{q_2}{C_2} = \frac{2.667 \times 10^{-4}}{4 \times 10^{-6}} = 66.66\,\text{V}$$

(c) $U_1 = \frac{1}{2}C_1V_1^2 = \frac{1}{2} \times 8 \times 10^{-6} \times (33.33)^2 = 0.00444\,\text{J}$

$$U_2 = \frac{1}{2}C_2V_2^2 = \frac{1}{2} \times 4 \times 10^{-6} \times (66.66)^2 = 0.00889\,\text{J}$$

$$U_3 = \frac{1}{2}C_3V_3^2 = \frac{1}{2} \times 3 \times 10^{-6} \times 100^2 = 0.015\,\text{J}$$

Note that $U_1 + U_2 + U_3 = U = \frac{1}{2}CV^2$, as it should.

11.92 Let the effective capacitance between points a and b be C. Apply a potential difference V between a and b and let C be charged to q, Fig. 11.47.
Let the charge across C_1 and C_5 be q_1 and q_5, respectively; the charges across various capacitors are shown in Fig. 11.47.

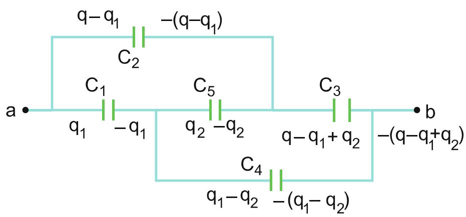


Fig. 11.47

The potential drop across C_1 plus that across C_4 must be equal to the potential drop across C_2 plus that across C_3.

$$V_1 + V_4 = V_2 + V_3 = V \tag{1}$$

$$\therefore \quad \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_4} = \frac{q - q_1}{C_2} + \frac{q - q_1 + q_2}{C_3} \tag{2}$$

By problem $C_1 = C_2 = C_3 = C_4$.

$$\therefore \quad 2q_1 - q_2 = C_1 V \tag{3}$$

$$2q - 2q_1 + q_2 = C_1 V \tag{4}$$

Adding (3) and (4)

$$2q = 2C_1 V \rightarrow C = \frac{q}{V} = C_1 = 2\,\mu\text{F}$$

11.93 Applying the loop theorem to the circuit, traversing clockwise from the negative terminal of the battery we have the equation

$$\xi - iR - \frac{q}{C} = 0 \tag{1}$$

where ξ is the emf of the battery and the second and the third terms represent the potential drop across the resistor and the capacitor.

$$\text{Now} \qquad i = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{2}$$

Using (2) in (1)

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = \xi \tag{3}$$

This differential equation describes the time variation of the charge on the capacitor. Re-arranging (3)

$$\frac{\mathrm{d}q}{C\xi - q} = \frac{\mathrm{d}t}{RC} \tag{4}$$

$$\text{Integrating (4)} \qquad -\ln(C\xi - q) = \frac{t}{RC} + A \tag{5}$$

where A is the constant of integration which can be determined from the initial condition.
At $t = 0$, $q = 0$ since the capacitor was uncharged.

$$\therefore \qquad A = -\ln \xi C \tag{6}$$

Using (6) in (5) and re-arranging

$$q = C\xi(1 - \mathrm{e}^{-t/RC})$$

(a) $i = \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\xi}{R}e^{-t/RC}$

(b) $P = i\xi = \dfrac{\xi^2}{R}e^{-t/RC}$

(c) $H = i^2R = \dfrac{\xi^2}{R}e^{-2t/RC}$

(d) $\dfrac{\mathrm{d}U}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\dfrac{q^2}{e}\right) = \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\left[C^2\xi^2\left(1 - e^{-t/RC}\right)^2\right] = \dfrac{C}{R}\xi^2 e^{-t/RC}$

11.94 **(i)** By prob. (11.93)

$$q = C\xi(1 - \mathrm{e}^{-t/RC})$$

$$\therefore \quad i = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\xi}{R}\,\mathrm{e}^{-t/RC}$$

$$R = (1200 + 600) \times 10^3 = 1.8 \times 10^6\,\Omega$$

$$RC = 1.8 \times 10^6 \times 2.5 \times 10^{-6} = 4.5$$

$$\text{At} \quad t = 0,\ i = \frac{\xi}{R} = \frac{50}{1.8 \times 10^6} = 27.8 \times 10^{-6}\,\mathrm{A} = 27.8\,\mu\mathrm{A}$$

(ii) At $t = \infty, i = 0$

(iii) $i = -\dfrac{\xi\,\mathrm{e}^{-t/RC}}{R} = -\dfrac{50 \times \mathrm{e}^{-t/4.5}}{1.8 \times 10^6} = -27.8\,\mathrm{e}^{-0.222t}\ \mu\mathrm{A}$

11.95 **(i)** Time constant, $RC = 200 \times 10^3 \times 500 \times 10^{-6} = 100$
When the switch is closed there is no emf in the circuit, and (3) in prob. (11.93) reduces to

(ii) $R\dfrac{\mathrm{d}q}{\mathrm{d}t} + \dfrac{q}{C} = 0 \qquad (1)$

or $\dfrac{\mathrm{d}q}{q} = -\dfrac{\mathrm{d}t}{RC} \qquad (2)$

Integrating, $\ln q = -t/RC + A$
where A is the constant of integration. When $t = 0$, $q = q_0$. Therefore $A = \ln q_0$.

$$\therefore \quad \ln \frac{q}{q_0} = -\frac{t}{RC}$$

$$\therefore \quad q = q_0 \mathrm{e}^{-t/RC} \tag{3}$$

$$\frac{q}{q_0} = \frac{1}{2} = \mathrm{e}^{-t/RC}$$

$$\therefore \quad t = RC \ln 2 = 100 \ln 2 = 69.3\,\mathrm{s}.$$

(iii) In (3) put $t = 0$. Then

$$q = q_0 = C\xi = 500 \times 10^{-6} \times 200 \times 10^3 = 100\,\mathrm{C}$$

(iv) Differentiating (3) with respect to time

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\xi}{R}\,\mathrm{e}^{-t/RC} \tag{4}$$

The negative sign shows that the current in the discharging process flows opposite to that in the charging process. At $t = 0$

$$i = -\frac{\xi}{R} = -\frac{900}{200 \times 10^3} = -4.5 \times 10^{-3}\,\mathrm{A}$$

(v) From (4) $V = iR = -\xi \mathrm{e}^{-t/RC}$

At $\quad t = 25\,\mathrm{s},\ V = -900\,\mathrm{e}^{-25/100} = -701\,\mathrm{V}.$

11.96 **(i)** By (4), prob. (11.50) the electric field inside the sphere is given by

$$\boldsymbol{E} = \frac{q\boldsymbol{r}}{4\pi\varepsilon_0 R^3} \tag{1}$$

$$\therefore \quad \mathrm{div}\,\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 R^3}\,\mathrm{div}\,\boldsymbol{r} \tag{2}$$

$$\text{Now} \quad \mathrm{div}\,\boldsymbol{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 1 + 1 + 1 = 3$$

$$\therefore \quad \mathrm{div}\,\boldsymbol{r} = \frac{3q}{4\pi\varepsilon_0 R^3} = \frac{3 \times 10^{-9} \times 9 \times 10^9}{1^3} = 27$$

(ii) $$F = QE = \frac{Qqr}{4\pi\varepsilon_0 R^3}$$

$$= 1.6 \times 10^{-19} \times 10^{-9} \times 9 \times 10^9 \times \frac{0.8}{1^3} = 1.152 \times 10^{-18}\,\mathrm{N}$$

(iii) By prob. (11.52), $V(r) = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$. The potential energy of the proton at r will be $U(r) = QV(r) = \frac{Qq}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$.

$$\therefore \qquad U(r = 0.8\,\text{m}) = \frac{1.6 \times 10^{-19} \times 10^{-9} \times 9 \times 10^{9}}{2 \times 1.0}\left(3 - \left(\frac{0.8}{1.0}\right)^2\right)$$
$$= 1.7 \times 10^{-19}\,\text{J}$$

Since $U(r = \infty) = 0$, work done $= 1.7 \times 10^{-19}$ J.

11.97 **(a)** If the dielectric is present, Gauss' law gives

$$\varepsilon_0 \oint \boldsymbol{E} \cdot d\boldsymbol{s} = \varepsilon_0 E A = q - q' \qquad \text{(Integral form)}$$

$$\text{or} \quad E = \frac{1}{\varepsilon_0 A}(q - q')$$

where q is the free charge and $-q'$ the induced charge.

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon \qquad \text{(Differential form)}$$

(b) The displacement vector

$$D = \frac{q}{A} \tag{1}$$

where A is the area

$$E = \frac{E_0}{K} = \frac{q}{K\varepsilon_0 A} \tag{2}$$

Combining (1) and (2)

$$D = K\varepsilon_0 E \tag{3}$$

As $\boldsymbol{E}$ is uniform in a parallel plate capacitor, $\boldsymbol{D}$ will be also uniform via (3)

(c) By Gauss law

$$\varepsilon_0 \oint K(x)\,\boldsymbol{E}(x) \cdot d\boldsymbol{s} = q = \varepsilon_0 \oint \boldsymbol{E}_0 \cdot d\boldsymbol{s}$$

$$\therefore \quad E(x) = \frac{E_0}{K(x)} \tag{4}$$

$$E_0 = \frac{q}{\varepsilon_0 A} = \frac{\sigma A}{\varepsilon_0 A} = \frac{\sigma}{\varepsilon_0} \quad (\because\ \sigma = q/A) \tag{5}$$

(d) $V = \int E(x)\mathrm{d}x = E_0 \int\limits_0^d \frac{\mathrm{d}x}{ax+b}$ (6)

Put $y = ax + b$, $\mathrm{d}x = \mathrm{d}y/a$. Then (6) becomes

$$V = \frac{E_0}{a} \int\limits_b^{ad+b} \frac{\mathrm{d}y}{y} = \frac{E_0}{a} \ln\ y|_b^{ad+b} = \frac{\sigma_0}{\varepsilon_0 a} \ln\left(1 + \frac{ad}{b}\right) \tag{7}$$

where we have used (5).

The capacitance $C = \dfrac{q}{V} = \dfrac{A\varepsilon_0 a}{\ln\left(1 + \dfrac{ad}{b}\right)}$ (8)

where we have used (7) and q $= \sigma$A.

(e) Vacuum polarization charge density

$$P(x) = \varepsilon_0(k(x) - 1)E = \frac{\varepsilon_0 E_0}{k(x)}(k(x) - 1)$$
$$= \varepsilon_0 E_0 \left[1 - \frac{1}{ax+b}\right] \tag{9}$$

11.98 Newton's law of gravitation is

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{e}_r = m\boldsymbol{g}$$

The Gauss' law for gravitation may be written as

$$\oint \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{s} = -\frac{GM}{r^2} 4\pi r^2 = -4\pi GM$$

The divergence theorem gives

$$\oint \nabla \cdot \boldsymbol{g}\, \mathrm{d}^3 r = \oint \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{s}$$
$$\therefore \quad \nabla \cdot g \frac{4}{3}\pi r^3 = -4\pi GM$$
$$\therefore \quad \nabla \cdot g = -4\pi\ G\rho m$$

This is analogous to the law for electric field

$$\nabla \cdot E = \frac{\rho_q}{\varepsilon_0}$$

Chapter 12
Electric Circuits

Abstract *Chapter 12* is mainly concerned with the analysis of electric network employing Kirchhoff's laws. Problems are solved under resistivity, Joule heating, emf, internal resistance, arrangement of cells, electric instruments such as ammeter, voltmeter, potentiometer and Wheatstone bridge.

12.1 Basic Concepts and Formulae

Electric current (i) is defined as the rate at which the net charge q passes through a cross-section of a conductor

$$i = q/t \tag{12.1}$$

The instantaneous current is defined by

$$i = \mathrm{d}q/\mathrm{d}t$$

The current density (j) is given by

$$j = i/A \tag{12.2}$$

where A is the cross-sectional area.

The Drift Velocity

$$v_\mathrm{d} = i/nAe = j/ne \tag{12.3}$$

where n is the number of electrons per unit volume

Resistance (R) and Resistivity (ρ)

$$R = \rho L/A \tag{12.4}$$

where L is the length of the conductor. Unit of resistivity is Ωm.

Electrical Conductance (σ)

$$\sigma = 1/\rho \tag{12.5}$$

The units of conductance are mho/m.

Variation of Resistance and Resistivity with Temperature

$$R_{\mathrm{t}} = R_0(1 + \alpha T) \tag{12.6}$$

$$\rho_{\mathrm{t}} = \rho_0(1 + \alpha T) \tag{12.7}$$

where α is the temperature coefficient of resistance or resistivity.

Resistors in Series

$$R = \sum\nolimits_n R_n \tag{12.8}$$

Resistors in Parallel

$$\frac{1}{R} = \sum\nolimits_n \frac{1}{R_n} \tag{12.9}$$

Joule's law: The power (P) developed is given by

$$P = i^2 R = iV = V^2/R \tag{12.10}$$

Cells

Cells in series: If n cells each of emf ξ and internal resistor r are connected in series then the current in the circuit is given by

$$i = \frac{n\xi}{R + nr} \tag{12.11}$$

where R is the external resistance.

Cells in parallel: In this case the total emf is that of a single cell ξ. As the internal resistances of the cells are in parallel, the equivalent internal resistance is r/n, with the external resistance R in series. The current is

$$i = \frac{\xi}{R + r/n} \tag{12.12}$$

Mixed Grouping of Cells

Let there be m rows of cells, with each row containing n cells in series. The emf for each row of cells would be $n\xi$ and the equivalent internal resistance nr. The effective emf for m rows would again be $n\xi$, but since the rows are in parallel, the effective internal resistance would be nr/m. The total resistance then becomes $R + (nr/m)$ (Fig. 12.1):

$$i = \frac{n\xi}{R + \left(\frac{nr}{m}\right)} = \frac{mn\xi}{mR + nr} = \frac{N\xi}{mR + nr} \tag{12.13}$$

where $N = m \times n =$ total number of cells.

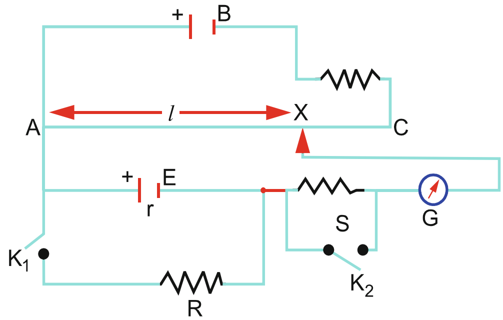
Instruments

Potentiometer may be used to measure the internal resistance of a cell: B = battery, E = cell of internal resistance r, S = resistor, R = resistor, G = galvanometer, AC = potentiometer, AX = l = balancing length.

If l_1 = balancing length with K_1 open and K_2 closed and l_2 with K_1 closed,

$$r = R\frac{(l_1 - l_2)}{l_2} \tag{12.14}$$

Fig. 12.1



Wheatstone bridge consists of a network of four resistors P, Q, R and S, battery E and galvanometer G, Fig. 12.2. Of the four resistors P, R and S are known whose values are adjustable while Q is unknown. When the bridge is balanced, i.e. the galvanometer shows null deflection:

$$\frac{P}{Q} = \frac{R}{S} \text{ or } Q = \frac{PS}{R} \tag{12.15}$$

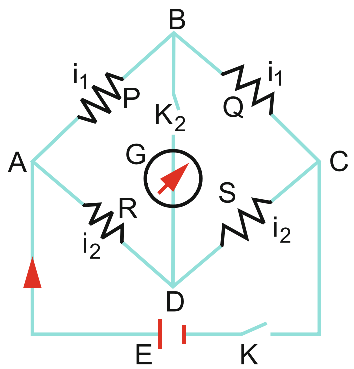


Fig. 12.2

Kirchhoff's Laws

1. *Junction theorem*: At any junction of an electric network (branched circuit), the algebraic sum of the currents flowing towards that junction is zero, i.e. the total current flowing towards the junction is equal to the total current flowing away from it:

$$\sum i = 0 \tag{12.16}$$

2. *The loop theorem*: The sum of the changes in the potential, encountered in traversing a loop (closed circuit) in a particular direction (clockwise or counterclockwise), is zero.

   (i) If a resistor is traversed in the direction of current, the change in the potential is $-iR$, while in the opposite direction it is $+iR$.
   (ii) If a seat of emf is traversed in the direction of the emf, the change in potential is $+\xi$, while in the opposite direction it is $-\xi$.

12.2 Problems

12.2.1 Resistance, EMF, Current, Power

12.1 All resistors in Fig. 12.3 are in ohms. Find the effective resistance between the points A and B.

[Indian Institute of Technology 1979]

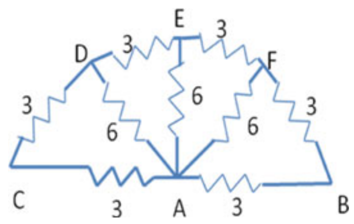


Fig. 12.3

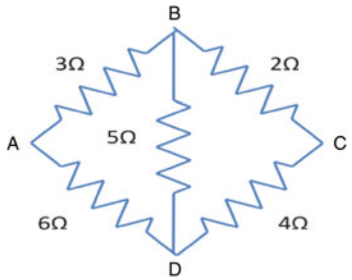
12.2 If a copper wire is stretched to make it 0.1% longer, what is the percentage change in its resistance?

[Indian Institute of Technology 1978]

12.3 The equivalent resistance of the series combination of two resistors is p. When they are joined in parallel, the equivalent resistance is q. If $p = nq$, find the minimum possible value of n.

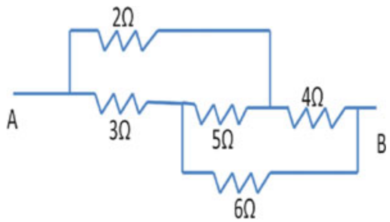
12.4 Five resistors are connected as in Fig. 12.4. Find the equivalent resistance between A and C.

Fig. 12.4



12.5 Five resistors are arranged as in Fig. 12.5. Find the effective resistance between A and B.

Fig. 12.5



12.6 Each of the resistances in the network, Fig. 12.6, is equal to R. Find the resistance between the terminals A and B.

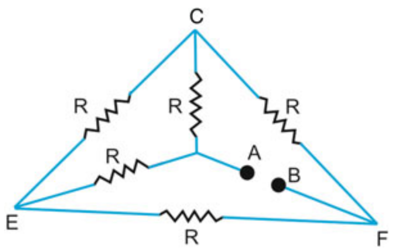
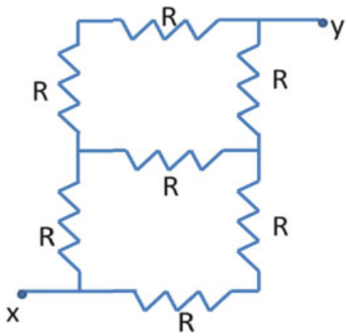


Fig. 12.6

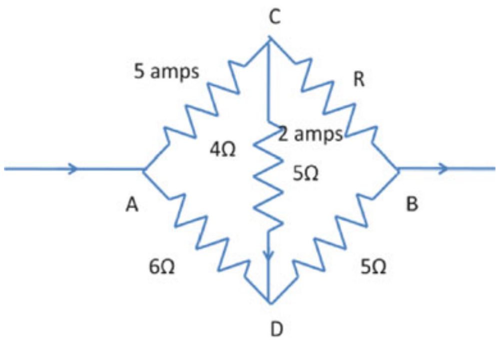
12.7 Find the equivalent resistance between the terminals x and y of the network shown in Fig. 12.7.

Fig. 12.7



12.8 A circuit is set up as shown in Fig. 12.8, in which certain resistors are known; the current in some of the branches has been measured by ammeter. Calculate

Fig. 12.8



(i) The resistance R in CB
(ii) The potential difference between A and B
(iii) The heat developed per second between A and B.

[Northern Universities of U.K.]

12.9 Five resistances are connected as shown in Fig. 12.9. Find the equivalent resistance between the points A and B.

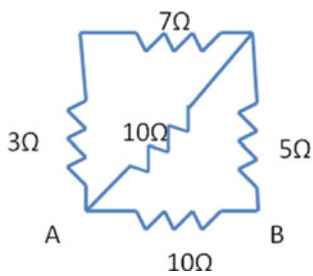
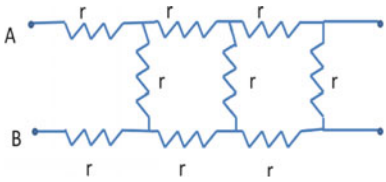


Fig. 12.9

12.10 A network of infinite resistors is shown in Fig. 12.10. Find the effective resistance of the network between terminal points A and B.

Fig. 12.10



12.11 What equal length of an iron wire and a constantan wire, each 1 mm diameter, must be joined in parallel to give an equivalent resistance of 2 Ω? (resistivity of iron and constantan are 10 and 49 μΩ cm, respectively).

[University of London]

12.12 A coil of wire has a resistance of 20 Ω at 25°C and 25.7 Ω at 100°C. Calculate the temperature coefficient.

[University of London]

12.13 A wire of resistance 0.1 Ω/cm is bent to form a square ABCD of side 10 cm. A similar wire is connected between the corners B and D to form the diagonal BD. Find the effective resistance of this combination between A and C. A battery of negligible internal resistance is connected across A and C. Calculate the total power dissipated.

[Indian Institute of Technology 1971]

12.14 A 60 W-100 V tungsten lamp has a resistance of 20 Ω at air temperature (0°C). What is the rise in temperature of the filament under normal working conditions? The temperature coefficient of resistance of tungsten is 0.0052/°C.

[University of London]

12.15 A skeleton cube is made of wires soldered together at the corners of the cube, the resistance of each wire being 10 Ω. A current of 6 A enters at one corner

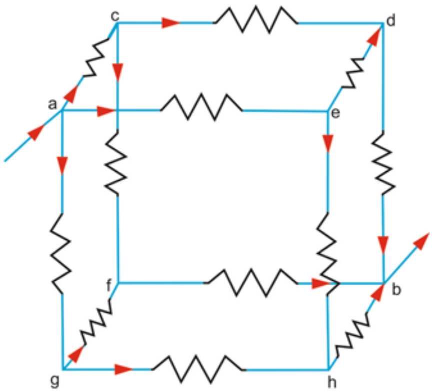


Fig. 12.11

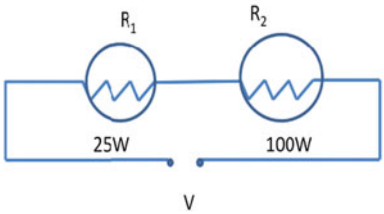
and leaves the diagonally opposite corner. Calculate the equivalent resistance of the network and the fall of potential across it (Fig. 12.11).

[University of London]

12.16 A 25 W bulb and a 100 W bulb are joined in series and connected to the mains (Fig. 12.12). Which bulb will glow brighter?

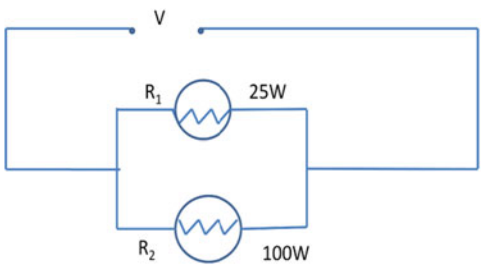
[Indian Institute of Technology 1979]

Fig. 12.12



12.17 A 25 W bulb and a 100 W bulb are joined in parallel and connected to the mains (Fig. 12.13). Which bulb will glow brighter?

Fig. 12.13

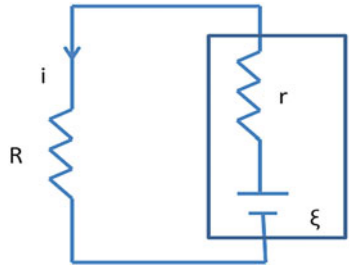


12.18 Three resistors of 4, 6 and 12 Ω are connected together in parallel. This parallel arrangement is then connected in series with a 1 and 2 Ω resistors. If a potential difference of 120 V is applied across the end of the circuit, what will be the potential drop across the part of the circuit connected in parallel?

[University of Newcastle]

12.19 In the given circuit, Fig. 12.14, show that the maximum power delivered to the external resistor R is $P = \xi^2/4r$ where r is the internal resistance of the battery of emf ξ.

Fig. 12.14



12.20 Two heater coils of power P_1 and P_2 (resistance R_1 and R_2, respectively) take individually time t_1 and t_2 to boil a fixed quantity of water. Find the time t in terms of t_1 and t_2, when they are connected to the mains in **(a)** series and **(b)** parallel to boil the same quantity of water.

12.21 A battery having an emf 24 V and a resistance 2 Ω is connected to two resistances arranged (a) in series and (b) in parallel. If the resistances are 4 and 6 Ω, respectively, calculate the watts expended in each resistance, in each of the two cases.

[University of London]

12.22 Power at the rate of 10^4 kW has to be supplied through 30 km of cable of resistance 0.7 Ω/km. Find the rate of energy loss, if the power is transmitted at 100 kV.

[University of Dublin]

12.23 Three equal resistors, connected in series across a source of emf together, dissipate 10 W of power. What would be the power dissipated, if the same resistors are connected in parallel across the same source of emf?

[Indian Institute of Technology 1972]

12.24 A heater is designed to operate with a power of 1000 W in a 100 V line. It is connected in combination with a resistance of 100 Ω and a resistance R to a 100 V mains as shown in Fig. 12.15. What should be the value of R so that the heater operates with a power of 62.5 W?

[Indian Institute of Technology 1978]

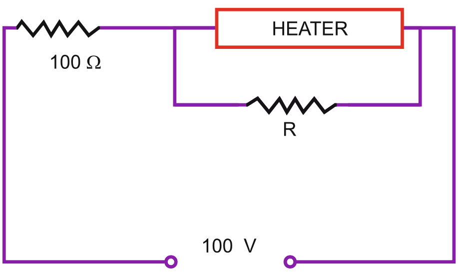


Fig. 12.15

12.2.2 Cells

12.25 Twelve cells having the same emf are connected in series and are kept in a closed box. Some of the cells are wrongly connected. This battery is connected in series with an ammeter and two cells identical with others. The current is 3 A when the cells and the battery aid each other and is 2 A when the cells and the battery oppose each other. How many cells in the battery are wrongly connected?

[Indian Institute of Technology 1976]

12.26 Let there be m rows of cells with each row containing n cells in series, each cell having internal resistance r. Show that maximum current in the external resistance R will be available when $R = nr/m$.

12.27 Two cells with the same emf and internal resistances r_1 and r_2 are connected in series to an external resistance R. Find the value of R so that the potential difference across the first cell is zero.

12.28 A certain circuit consists of three resistors connected in parallel across 200 V mains. The rate of production of heat in them is in the ratio of 5:3:2 and together they generate heat at the rate of 1 kW h in 2 h. Find the power used if the three resistances are connected in series across 248 V mains.

[Northern Universities of UK]

12.29 A battery of emf 2 V and internal resistance 0.1 Ω is being charged with a current of 5 A. In what direction will the current flow inside the battery? What is the potential difference between the two terminals of the battery?

[Indian Institute of Technology 1980]

12.30 A 6 V battery of negligible internal resistance is connected in series with a 3 Ω and a 5 Ω resistance. A further resistance of 2 Ω is connected in parallel with the 5 Ω resistance.

(a) Find the current flowing in each resistance.
(b) Find the power dissipated in each resistance.
(c) Compare the total value for the power dissipated in the resistances with the value for the power supplied by the battery.

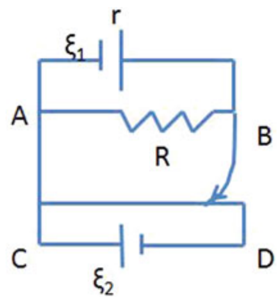
[University of Newcastle]

12.2.3 Instruments

12.31 The terminals of a cell are connected to a resistance and the fall of potential across R is balanced against the fall across the potentiometer wire. When R is 20 and 10 Ω, respectively, the corresponding lengths on the potentiometer are 150 and 120 cm. Calculate the internal resistance of the cell (Fig. 12.16).

[University of London]

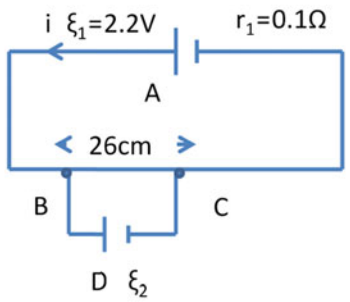
Fig. 12.16



12.32 A thin uniform wire 50 cm long and of 1 Ω resistance is connected to the terminals of an accumulator of emf 2.2 V and the internal resistance 0.1 Ω (Fig. 12.17). If the terminals of another cell can be connected to two points 26 cm apart on the wire without altering the current in the wire, what is the emf of the cell?

[Northern Universities of UK]

Fig. 12.17



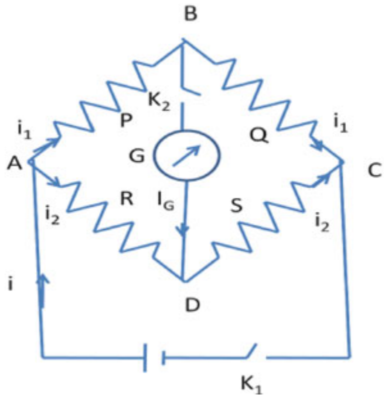
12.33 In a Wheatstone bridge, four resistors P, Q, R and S are arranged as in Fig. 12.18. Show that

(a) condition for null deflection in the galvanometer G is $\dfrac{P}{Q} = \dfrac{R}{S}$

(b) if a non-zero current i_g flows through the galvanometer then

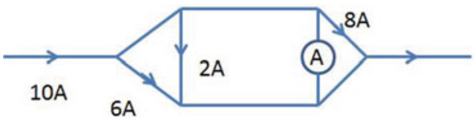
$$\frac{i_g}{i} = \frac{QR - PS}{G(Q + S) + (P + R)(G + Q + S)}$$

Fig. 12.18



12.34 Figure 12.19 shows a network carrying various currents. Find the current through the ammeter A.

Fig. 12.19



12.35 A galvanometer together with an unknown resistance in series is connected across two identical batteries, each of 1.5 V. When the batteries are connected in series, the galvanometer records a current of 1 A and when the batteries are in parallel the current is 0.6 A. What is the internal resistance of the battery?
[Indian Institute of Technology 1973]

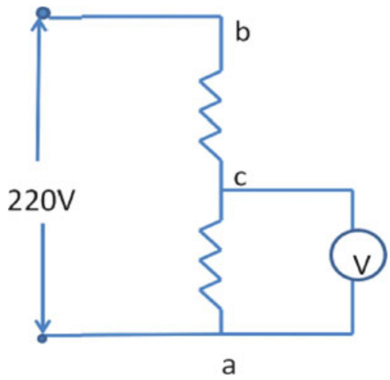
12.36 In a potentiometer experiment, it is found that no current passes through the galvanometer when the terminals of the cell are connected across 52 cm of the potentiometer wire. If the cell is shunted by a resistance of 5 Ω, a balance is found when the cell is connected across 40 cm of the wire. Find the internal resistance of the cell.

12.37 A potentiometer wire of length 100 cm has a resistance of 10 Ω. It is connected in series with a resistance and cell of emf 2 V and of negligible internal resistance. A source of emf 10 mV is balanced against a length of 40 cm of the potentiometer wire. What is the value of the external resistance?
[Indian Institute of Technology 1976]

12.38 A potential difference of 220 V is maintained across a 12,000 Ω rheostat ab (see Fig. 12.20). The voltmeter V has a resistance of 6000 Ω and point c is at one-fourth the distance from a to b. What is the reading in the voltmeter?
[Indian Institute of Technology 1977]

12.39 The balance point in a meter bridge experiment is obtained at 30 cm from the left. If the right-hand gap contains resistance of 3.5 Ω, what is the value of the resistance in the left-hand gap?

Fig. 12.20



12.2.4 Kirchhoff's Laws

12.40 A moving coil meter has a full scale reading of 1 mA and a resistance of 80 Ω. How could the meter be used to measure (a) 100 mA full scale and (b) 80 V full scale?

[University of Manchester]

12.41 A pocket voltmeter has a resistance of 120 Ω. What will it read when connected to a battery of emf 9 V and an internal resistance 15 Ω?

[University of Oxford]

12.42 When a galvanometer is shunted with a 1 Ω resistance, only 1% of the main current passes through it. What is the resistance of the galvanometer?

12.43 A 10 V battery, having an internal resistance of 1.0 Ω, is joined in parallel with another of 20 V and internal resistance of 2 Ω. Calculate the current flowing through each battery, and the rates of expenditure of energy in the two batteries and the 30 Ω resistance (Fig. 12.21).

[University of Cambridge]

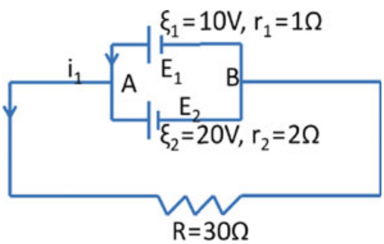


Fig. 12.21

12.44 A battery of emf 1 V and internal resistance 2 Ω is connected to another battery of emf 2 V and internal resistance 1 Ω in parallel with an external resistance of 10 Ω. Find the currents?

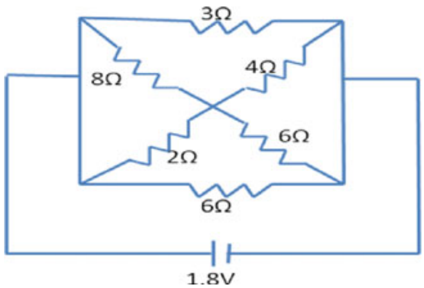
12.45 The electric current of 5 A is divided into three branches, forming a parallel combination. The lengths of the wire in the three branches are in the ratio 2, 3 and 4; their diameters are in the ratio 3, 4 and 5. Find the current in each branch, if the wires are of the same material.

[Indian Institute of Technology 1975]

12.46 Calculate the current through the 3 Ω resistor and the power dissipated in the entire circuit shown in Fig. 12.22. The emf of the battery is 1.8 V and its internal resistance is 2/3 Ω.

[Indian Institute of Technology 1971]

Fig. 12.22



12.47 A series circuit is made up of two cells of emf 1.5 and 3 V, respectively, and two coils each of resistance 10 Ω, arranged in the order cell, coil, cell, coil. The centre points of the two coils are joined by a sensitive galvanometer which shows no deflection. If the cell of 1.5 V has an internal resistance of 5 Ω, calculate the internal resistance of the other cell.

[University of London]

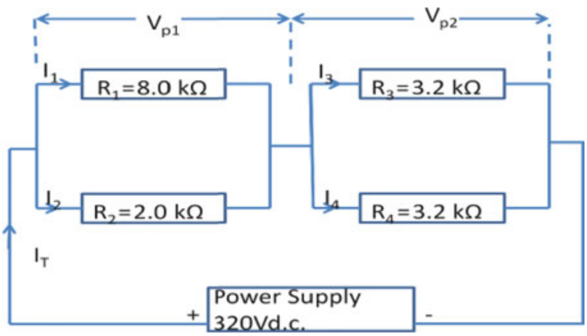
12.48 **(a)** Figure 12.23 shows a series parallel resistive circuit connected to a 320 V d.c. supply. For the circuit shown work out the following:

(i) The total equivalent resistance R_T of the circuit and the total current I_T.
(ii) The voltage V_{p1} across resistors R_1 and R_2.
(iii) The voltage V_{p2} across resistors R_3 and R_4.
(iv) The currents I_1 and I_3.
(v) The total power for the whole circuit and the power dissipated in resistor R_3.

(b) Consider the case of a heavy duty battery whose emf $\xi = 24$ V and internal resistance of $r = 0.01\ \Omega$. If the terminals were accidentally short circuited by a heavy copper bar of negligible resistance what power would be dissipated within the battery?

[University of Aberystwyth, Wales 2005]

Fig. 12.23



12.49 A battery with emf $\xi = 24\,\mathrm{V}$ has internal resistance $r = 0.02\,\Omega$. A load resistor $R = 140\,\Omega$ is connected to the terminals of a battery:

(i) Find the current flowing in the circuit under load conditions.
(ii) Find the terminal voltage of the battery under load conditions.
(iii) Find the power dissipated in the resistor R and in the battery's internal resistance r.
(iv) Find the open circuit voltage of the battery under no load conditions and explain your answer.

12.50 Figure 12.24 shows a series parallel resistive circuit connected to a dc supply.

(i) Find the total equivalent resistance of the circuit.
(ii) Find currents I_T, I_1 and I_3.
(iii) Find voltages V_1, V_2 and V_3.
(iv) Find power dissipated in resistor R_5 and the total power dissipated in the circuit.

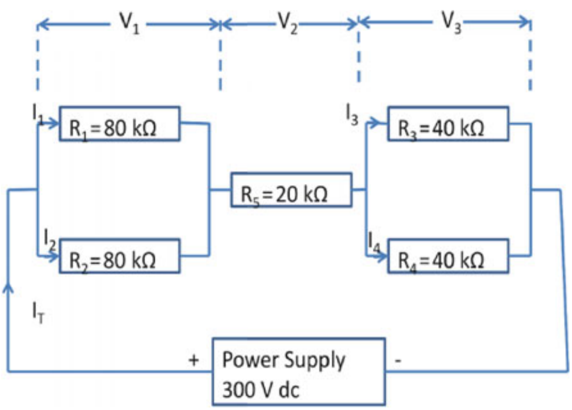
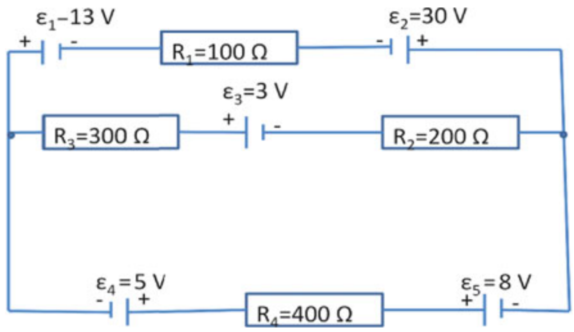


Fig. 12.24

12.51 Apply Kirchhoff's rules to the circuit shown in Fig. 12.25 to produce three equations with three unknown branch currents. You do not have to solve these equations for individual current.

[University of Aberystwyth, Wales 2008]

Fig. 12.25

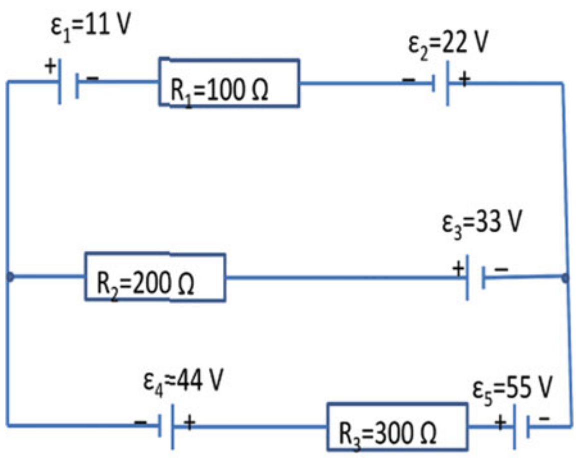


12.52 **(i)** State Kirchhoff's two rules

(ii) Apply Kirchhoff's rules to the circuit shown in Fig. 12.26 to produce three equations with three unknown branch currents. You do not have to solve these equations for individual current.

[University of Aberystwyth, Wales 2007]

Fig. 12.26



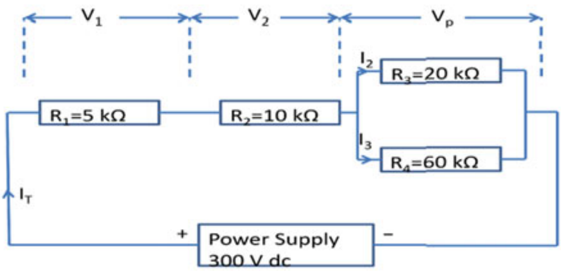
12.53 Figure 12.27 shows a series parallel resistive circuit connected to a dc supply. For the circuit shown work out the following:

(i) The voltages V_1, V_2 across resistors R_1 and R_2.

(ii) The voltage V_p across resistors R_3 and R_4.

(iii) The currents I_T, I_2 and I_3.

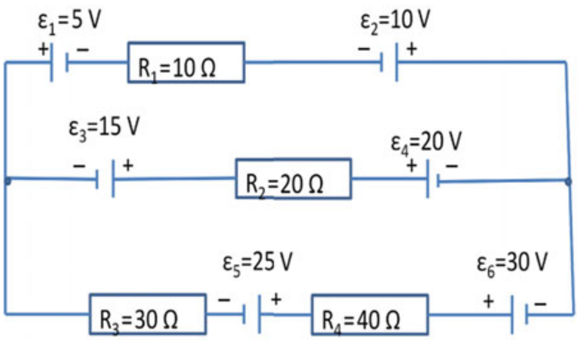
Fig. 12.27



12.54 Apply Kirchhoff's rules to the circuit shown in Fig. 12.28 to produce three equations with three unknown branch currents. You do not have to solve these equations for individual I.

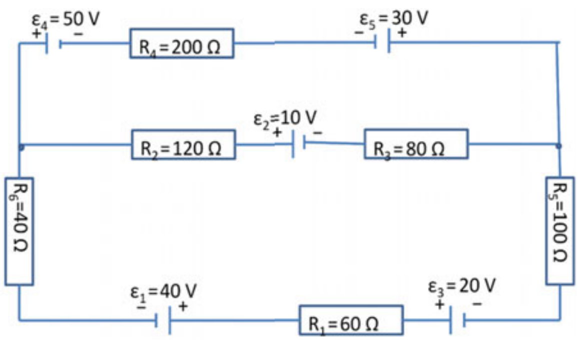
[University of Aberystwyth, Wales 2006]

Fig. 12.28



12.55 Apply Kirchhoff's rules to the circuit shown in Fig. 12.29 and present the simultaneous equations necessary to calculate the currents in each of the branches of the circuit. You do not have to solve these equations for the branch currents.

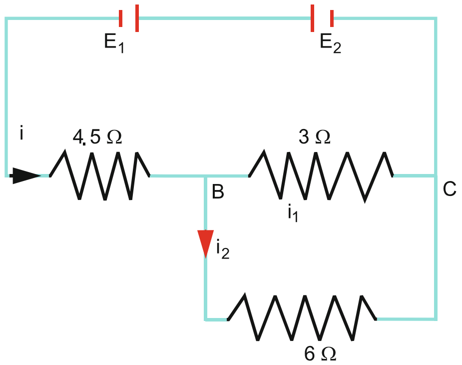
Fig. 12.29



12.56 In the circuit shown in Fig. 12.30, the cells E_1 and E_2 have emfs 4 and 8 V and internal resistances 0.5 and 1 Ω, respectively. Calculate the current in each resistor and the potential difference across each cell.

[Indian Institute of Technology 1973]

Fig. 12.30



12.3 Solutions

12.3.1 Resistance, EMF, Current, Power

12.1 In the segment ACD, the two 3 Ω resistances give 6Ω, which with 6 Ω in parallel yields $\frac{6 \times 6}{6+6} = 3\,\Omega$. This together with 3 Ω, across DE in series, gives 6 Ω which together with 6 Ω across AE in parallel gives 3 Ω. By a similar reasoning resistance along AFB is 6 Ω, which with 3 Ω, across AB in parallel yields the effective resistance across AB:

$$R_{AB} = \frac{6 \times 3}{6+3} = 2\,\Omega$$

12.2 $R = \frac{\rho l}{A} = \frac{\rho l^2}{Al} = \frac{\rho l^2}{v_0}$

where v_0 is the constant volume. Change in the resistance

$$\Delta R = 2\rho l \frac{\Delta l}{v_0}$$

$$\therefore \quad \frac{\Delta R}{R} \equiv 2\frac{\Delta l}{l} \equiv 2 \times \frac{0.1}{100} = \frac{0.2}{100} \quad \text{or} \quad 0.2\%$$

12.3 Let the resistances be R_1 and R_2

$$R_1 + R_2 = p \qquad \text{(series)} \tag{1}$$

$$\frac{R_1 R_2}{R_1 + R_2} = q \qquad \text{(parallel)} \tag{2}$$

Combining (1) and (2)

$$R_1 - R_2 = \pm\sqrt{n(n-4)}$$

Since R_1 and R_2 are real, $n \geq 4$.

12.4 Let a current i enter at A and leave at C. Currents in various branches are given by the junction theorem, Fig. 12.31. The potential difference

$$V_{\text{AB}} + V_{\text{BD}} = V_{\text{AD}}$$

$$\therefore \quad 3i_1 + 5i_2 = 6(i - i_1)$$

$$\text{or} \quad 9i_1 + 5i_2 = 6i \tag{1}$$

$$V_{\text{AC}} = V_{\text{AB}} + V_{\text{BC}} = V_{\text{AD}} + V_{\text{DC}}$$

$$\therefore \quad 3i_1 + 2(i_1 - i_2) = 6(i - i_1) + 4(i - i_1 + i_2)$$

$$\text{or} \quad 15i_1 - 6i_2 = 10i \tag{2}$$

Solving (1) and (2), $i_2 = 0$. Thus the middle branch BD is rendered ineffective.

Two resistances of 3 and 2 Ω in series in the upper branch are to be combined in parallel with two other resistances of 6 and 4 Ω in series in the lower branch to obtain the effective resistance between A and C. This is given by

$$R_{\text{eff}} = \frac{(3+2)(6+4)}{(3+2)+(6+4)} = 3.33\,\Omega$$

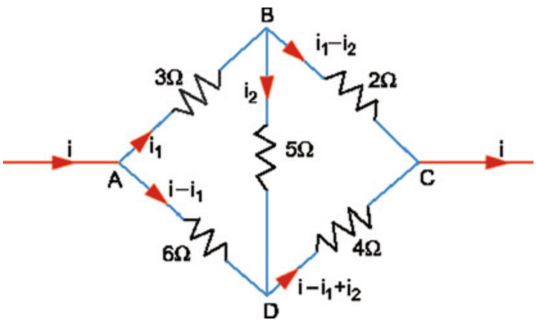


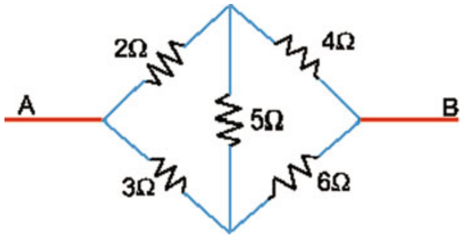
Fig. 12.31

This particular problem could have been easily solved by noticing that Wheatstone bridge balance requirement is fulfilled since $P/Q = R/S$; here $3/2 = 6/4$, in which case the current in the middle branch is zero (see prob. 12.33).

12.5 The network in Fig. 12.3 can be recast as shown in Fig. 12.32. Here again the balancing condition for Wheatstone bridge is satisfied: $\frac{P}{Q} = \frac{R}{S}, \frac{2}{4} = \frac{3}{6}$. Therefore the middle branch resistor of 5 Ω is rendered ineffective. The total resistance in the upper branch is $2 + 4 = 6\,\Omega$ and in the lower branch $3 + 6 = 9\,\Omega$. The equivalent resistance for 6 and 9 Ω in parallel will be

$$R_{\text{eq}} = \frac{6 \times 9}{6 + 9} = 3.6\,\Omega$$

Fig. 12.32



12.6 For convenience the given network can be recast as in Fig. 12.33. It is seen that this network is exactly identical with that in prob. (12.5) in which the Wheatstone bridge condition is satisfied. Therefore, the resistance in the middle branch CE is rendered ineffective. The total resistance in the upper branch is obviously equal to $2R$ which is also the case for the lower branch.

Therefore, the effective resistance between D and F will be

$$R_{\text{eff}} = \frac{2R \times 2R}{2R + 2R} = R$$

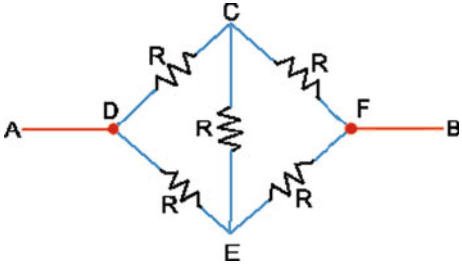


Fig. 12.33

12.7 Let R_{eq} be the equivalent resistance of the circuit. Let a current i enter at X and emerge at Y. The distribution of currents in various branches is shown in Fig. 12.34. The potential drop across X and Y

$$V_{xy} = i\, R_{eq} \tag{1}$$

Now $\quad V_{xy} = V_{XA} + V_{AY}$

$$= R(i - i_1) + 2R(i - i_1 - i_2) = 3Ri - 3Ri_1 - 2Ri_2 \tag{2}$$

From (1) and (2)

$$i\, R_{eq} = R(3i - 3i_1 - 2i_2) \tag{3}$$

i_1 and i_2 can be expressed in terms of i:

$$V_{XB} = V_{XC} + V_{CB} = V_{XA} + V_{AB}$$

$$\therefore \quad 2Ri_1 = R(i - i_1) + Ri_2$$

$$\text{or} \quad 3i_1 - i_2 = i \tag{4}$$

Also, $\quad V_{AY} = V_{AB} + V_{BY} = V_{AD} + V_{DY}$

$$\therefore \quad Ri_2 + R(i_1 + i_2) = 2R(i - i_1 - i_2)$$

$$\text{or} \quad 3i_1 + 4i_2 = 2i \tag{5}$$

Solving (4) and (5), $i_1 = 2i/5$ and $i_2 = i/5$.

Using the values of i_1 and i_2 in (3) we find $R_{eq} = 7R/5$.

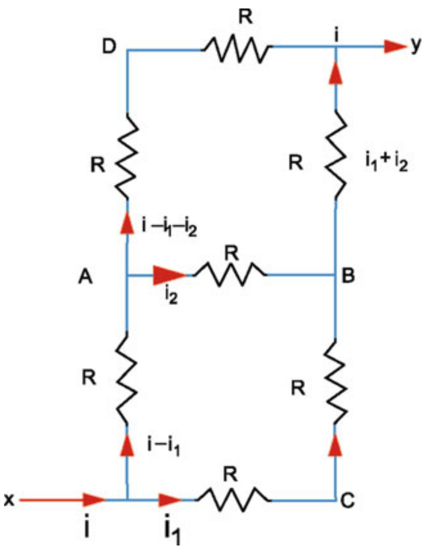


Fig. 12.34

12.8 **(i)** Let the current in AD, DE and R be i_1, i_2 and i_3, respectively:

$$i_3 = 5 - 2 = 3\,\text{A} \qquad \text{(by junction theorem)}$$
$$V_{\text{AD}} = V_{\text{AC}} + V_{\text{CD}}$$
$$\therefore \quad i_1 \times 6 = 5 \times 4 + 5 \times 2$$
$$\therefore \quad i_1 = 5\,\text{A}$$
$$\therefore \quad i_2 = 2 + 5 = 7\,\text{A}$$
$$V_{\text{CB}} = V_{\text{CD}} + V_{\text{DB}}$$
$$3R = 2 \times 5 + 7 \times 5$$
$$\therefore \quad R = 15\,\Omega$$

(ii) $V_{\text{AB}} = V_{\text{AC}} + V_{\text{CB}} = 4 \times 5 + 15 \times 3 = 65\,\text{V}$

(iii) Heat developed per second = power = $\sum i_n^2 R_n$

$$= 5^2 \times 4 + 2^2 \times 5 + 5^2 \times 6 + 3^2 \times 15 + 7^2 \times 5$$
$$= 650\,\text{J/s} = 650/4.18\,\text{Cal/s} = 155.5\,\text{Cal/s}$$

12.9 The combination of 3 and 7 Ω resistance in series is 10 Ω. This in parallel with 10 Ω resistance yields 5 Ω. This in series with another 5 Ω resistance gives the combined resistance of 10 Ω. This being in parallel with 10 Ω resistance across A and B gives the effective resistance of 5 Ω across A and B.

12.10 Let the effective resistance between A and B be R. Then by adding one more section to infinite sections of resistors, the effective resistance will not change, Fig. 12.35.
The middle r is in parallel with R and the other two r's are in series. Then

$$R = r + \frac{Rr}{R + r} + r$$

Simplifying $R^2 - 2Rr - 2R^2 = 0$
whose solution is $R = r\left(1 + \sqrt{3}\right)$.
The second solution is ignored since R must be positive.

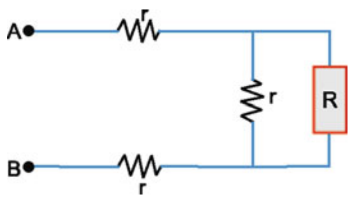


Fig. 12.35

12.11 $R_{\mathrm{I}} = \dfrac{\rho l}{A} = \dfrac{10 \times 10^{-6}\, l}{\pi (0.05)^2} = 1.274 \times 10^{-3} l$

$$R_{\mathrm{C}} = \frac{\rho' l}{A} = \frac{49 \times 10^{-6} l}{\pi (0.05)^2} = 6.24 \times 10^{-3} l$$

In parallel arrangement

$$R = \frac{R_{\mathrm{I}} R_{\mathrm{C}}}{R_{\mathrm{I}} + R_{\mathrm{C}}} = \frac{(12.739 \times 62.42) \times 10^{-6} l^2}{(12.739 + 62.42) \times 10^{-3} l}$$
$$= 10.58 \times 10^{-3} l = 2\,\Omega$$
$$\therefore \quad l = 1890\,\mathrm{cm} = 18.9\,\mathrm{m}$$

12.12 $R_2 = R_0(1 + \alpha t_2)$

$$R_1 = R_0(1 + \alpha t_1)$$
$$\therefore \quad \frac{R_2}{R_1} = \frac{1 + \alpha t_2}{1 + \alpha t_1}$$
$$\therefore \quad \alpha = \frac{R_2 - R_1}{R_1(t_2 - t_1)} = \frac{25.7 - 20.0}{20 \times (100 - 25)} = 3.8 \times 10^{-3}/^\circ\mathrm{C}$$

12.13 Resistance of each side = $10 \times 0.1 = 1\,\Omega$. Resistance of the diagonal = $\sqrt{2}\,\Omega$. The P.D., $V_{\mathrm{AB}} = V_{\mathrm{AD}}$ as $R_{\mathrm{AB}} = R_{\mathrm{AD}}$. Hence no current flows through the diagonal BD, Fig. 12.36. The effective resistance of the network is obtained by combining the resistance of AB and BC in series (1 + 1) in parallel with that of AD and DC in series (1 + 1):

$$R_{\mathrm{eff}} = \frac{2 \times 2}{2 + 2} = 1\,\Omega$$

Power dissipated $P = \dfrac{\xi^2}{R} = \dfrac{2^2}{1} = 4\,\mathrm{W}$.

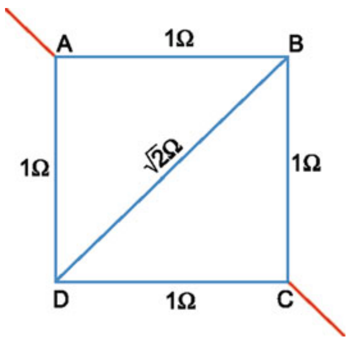


Fig. 12.36

12.14 $R = \dfrac{V^2}{P} = \dfrac{(100)^2}{60} = 166.7\,\Omega$

$R = R_0(1 + \alpha t)$

$t = \dfrac{R - R_0}{R_0\alpha} = \dfrac{166.7 - 20}{20 \times 0.0052} = 1410°\text{C}.$

12.15 Let a current of 6 A be sent through the corner a. Let a potential difference V_{ab} be established between a and b and current 6 A flow out from b. The currents in various branches are indicated in Fig. 12.37 from symmetry considerations. If R_{eq} is the equivalent resistance of this network across the body diagonal ab

$V_{ab} = 6\,R_{eq}$

But $\quad V_{ab} = V_{ac} + V_{cd} + V_{db}$

$= 2R + R + 2R = 5R$

$\therefore \quad 6\,R_{eq} = 5R$

or $\quad R_{eq} = \dfrac{5}{6}R = \dfrac{5}{6} \times 10 = 8.33\,\Omega$

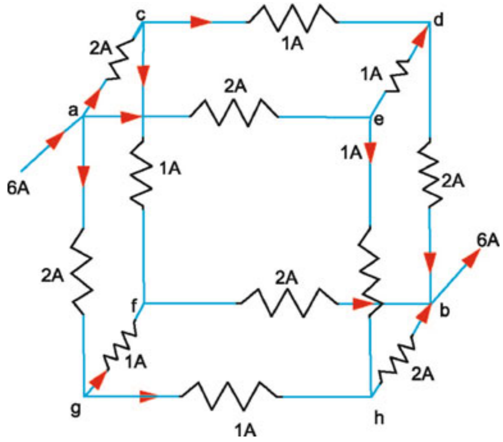


Fig. 12.37

12.16 $W = V^2/R$

$R_1 = \dfrac{V^2}{W_1} = \dfrac{(220)^2}{25} = 1936\,\Omega$

$R_2 = \dfrac{V^2}{W_2} = \dfrac{(220)^2}{100} = 484\,\Omega$

Joule heat in R_1, $H_1 = i^2 R_1 = 1936\, i^2$
Joule heat in R_2, $H_2 = i^2 R_2 = 484\, i^2$
Therefore, 25 W bulb glows brighter.

12.17 By prob. (12.16), $R_1 = 1936\,\Omega$ and $R_2 = 484\,\Omega$:

$$H_1 = i_1 V = \frac{iV\, R_2}{R_1 + R_2}$$
$$H_2 = i_2 V = \frac{iV R_1}{R_1 + R_2}$$

Therefore, 100 W bulb glows brighter.

12.18 Effective resistance in parallel is given by (Fig. 12.38)

$$\frac{1}{R} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$$
$$\therefore \quad R = 2\,\Omega$$

Total resistance, $R_{AB} = 2 + 1 + 2 = 5\,\Omega$
Current in the circuit

$$i = \frac{V_{AB}}{R_{AB}} = \frac{120}{5} = 24\,\text{A}$$
$$V_{AC} = i R_{AC} = 24 \times 2 = 48\,\text{V}.$$

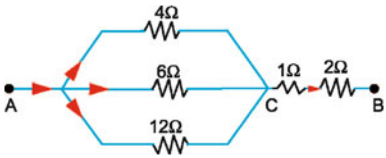


Fig. 12.38

12.19 $i = \dfrac{\xi}{R + r}$

$$P = i^2 R = \frac{\xi^2 R}{(R + r)^2}$$

Maximum power delivered to R is obtained by setting $\dfrac{\partial P}{\partial R} = 0$. This gives $R = r$:

$$\therefore \qquad P_{max} = \frac{\xi^2 r}{(r + r)^2} = \frac{\xi^2}{4r}$$

12.20 $H = P_1 t_1 = \dfrac{V^2}{R_1} t_1$

where H is joule heat and t is time

$$\therefore \quad t_1 = \frac{H}{V^2} R_1 \quad \text{(coil 1)}$$

$$t_2 = \frac{H}{V^2} R_2 \qquad \text{(coil 2)}$$

(a) $t = \dfrac{H}{V^2} R = \dfrac{H}{V^2} (R_1 + R_2) = t_1 + t_2 \quad$ (coils in series)

(b) $\dfrac{1}{t_1} = \dfrac{V^2}{H R_1} \qquad$ (coil 1)

$\dfrac{1}{t_2} = \dfrac{V^2}{H R_2} \qquad$ (coil 2)

$$\therefore \qquad \frac{1}{t_1} + \frac{1}{t_2} = \frac{V^2}{H} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{V^2}{H R} = \frac{1}{t}$$

or $\qquad t = \dfrac{t_1 t_2}{t_1 + t_2} \qquad$ (coils in parallel)

12.21 **(a)** $i = \dfrac{\xi}{R_1 + R_2 + r} = \dfrac{24}{4 + 6 + 2} = 2\,\text{A}$

$P_1 = i^2 R_1 = 2^2 \times 4 = 16\,\text{W}$

$P_2 = i^2 R_2 = 2^2 \times 6 = 24\,\text{W}$

(b) Effective resistance of R_1 and R_2 in parallel is

$$R = \frac{R_1 R_2}{R_1 + R_2} = \frac{4 \times 6}{4 + 6} = 2.4\,\Omega$$

$$i = \frac{\xi}{R + r} = \frac{24}{2.4 + 2} = 5.45\,\text{A}$$

$$i_1 = \frac{i R_2}{R_1 + R_2} = \frac{5.45 \times 6}{4 + 6} = 3.27\,\text{A}$$

$$i_2 = \frac{i R_1}{R_1 + R_2} = \frac{5.45 \times 4}{4 + 6} = 2.18\,\text{A}$$

$$P_1 = i^2{}_1 R_1 = (3.27)^2 \times 4 = 42.8\,\text{W}$$

$$P_2 = i^2{}_2 R_2 = (2.18)^2 \times 6 = 28.5\,\text{W}$$

12.22 Total resistance of the cable

$$R = 0.7 \times 30 = 21\,\Omega$$

Voltage $V = 100\,\text{kV} = 10^5\,\text{V}$

Power $P = 10^4\,\text{kW} = 10^7\,\text{W}$

Current $I = \dfrac{P}{V} = \dfrac{10^7}{10^5} = 100\,\text{A}$

Power dissipated $= i^2R = (100)^2 \times 21 = 2.1 \times 10^5\,\text{W}$

$$\text{Fractional power loss} = \frac{2.1 \times 10^5}{10^7} = 0.021 \text{ or } 2.1\%$$

12.23 Let each of the three resistances be r. In the series arrangement the effective resistance, $R_1 = 3r$:

$$P_1 = \frac{\xi^2}{R_1} = \frac{\xi^2}{3r} = 10$$

$$\therefore \qquad \frac{\xi^2}{r} = 30 \tag{1}$$

In the parallel arrangement the effective resistance $R_2 = r/3$:

$$P_2 = \frac{\xi^2}{R_2} = \frac{3\xi^2}{r} = 3 \times 30 = 90\ \text{W}$$

where we have used (1).

12.24 If the heater resistance is R_0,

$$R_0 = \frac{V^2}{P} = \frac{(100)^2}{1000} = 10\,\Omega$$

The combined resistance of R_0 and R in parallel is $\dfrac{10R}{R+10}$. As this is in series with $10\,\Omega$, the effective resistance of the circuit

$$R_e = 10 + \frac{10R}{R+10} = \frac{20R+100}{R+10}$$

If P' is the power of the heater

$$R_e = \frac{20R+100}{R+10} = \frac{V^2}{P'} = \frac{(100)^2}{62.5}$$

Solving for R, we find $R = 5\,\Omega$.

12.3.2 Cells

12.25 Let n cells of emf ξ and internal resistance r be wrongly connected. The effective emf of the battery is $(12–2n)\ \xi$. When the two cells and the battery aid each other, the net emf is $(12-2n)\xi+2\xi$ or $(14-2n)\xi$. The total internal resistance is $14r$. By problem, when the two cells and battery aid each other

$$(14 - 2n)\xi = 3 \times 14r \tag{1}$$

and when the two cells and battery oppose each other, the net emf is $(12-2n)\xi - 2\xi$ or $(10-2n)\xi$, the total internal resistance being $14r$. By problem

$$(10 - 2n)\xi = 2 \times 14r \tag{2}$$

Dividing (1) by (2)

$$\frac{14 - 2n}{10 - 2n} = \frac{3}{2}$$

whence $n = 1$.

12.26 The total number of cells is $N = m \times n$. The emf for each row of cells will be $n\xi$ and the combined internal resistance nr (Fig. 12.39). The effective emf for m rows would again be $n\xi$, but because the rows are in parallel, the effective internal resistance would become nr/m. The total resistance then becomes $R + nr/m$. The current through R will be

$$i = \frac{n\xi}{R + \frac{nr}{m}} = \frac{mnr}{Rm + nr} = \frac{N\,r}{Rm + nr} \tag{1}$$

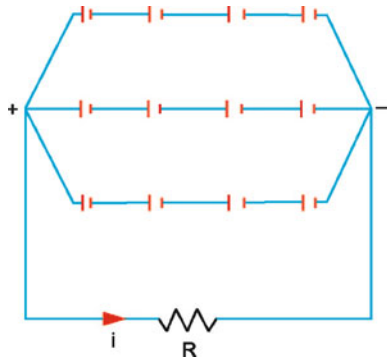
Writing $m = N/n$ in (1) and holding N as constant, maximum current i is found by setting $\frac{\partial i}{\partial n} = 0$. This gives

$$n^2 = \frac{RN}{r} = \frac{R\,m\,n}{r}$$

$$\text{or} \qquad R = \frac{n\,r}{m}$$

But the right-hand side is equal to the total internal resistance of the cells. Thus the current is maximum when the cells are arranged such that their total internal resistance is equal to the external resistance. In particular, for a single cell, $m = n = 1$, and the condition for maximum current is $R = r$, a result which is identical with that of prob. (12.19).

Fig. 12.39



12.27 Let ξ be the emf of each cell, i the current flowing in the circuit, r_1 and r_2 be the internal resistance of the first and the second cells, respectively. The potential drop across the first cell will be

$$V_1 = \xi - ir_1 = 0 \qquad \text{(by problem)} \tag{1}$$

$$i = \frac{2\xi}{r_1 + r_2 + R} \tag{2}$$

Combining (1) and (2)

$$\xi = ir_1 = \frac{2\xi r_1}{r_1 + r_2 + R}$$

$$\therefore \qquad R = r_1 - r_2$$

12.28 Total power $P = P_1 + P_2 + P_3 = \dfrac{10^3\,\mathrm{W\,h}}{2\,\mathrm{h}} = 500\,\mathrm{W}$

$$P_1 : P_2 : P_3 = 5 : 3 : 2$$

$$\therefore \quad P_1 = 250\,\mathrm{W}, \qquad P_2 = 150\,\mathrm{W}, \qquad P_3 = 100\,\mathrm{W}$$

$$R_1 = \frac{V^2}{P_1} = \frac{(200)^2}{250} = 160\,\Omega, \qquad R_2 = \frac{V^2}{P_2} = \frac{(200)^2}{150} = 267\,\Omega,$$

$$R_3 = \frac{V^2}{P_3} = \frac{(200)^2}{100} = 400\,\Omega$$

In series total resistance $R' = R_1 + R_2 + R_3 = 160 + 267 + 400 = 827\,\Omega$. Required power for the series arrangement

$$P' = \frac{V'^2}{R'} = \frac{(248)^2}{827} = 74.4\ \mathrm{W}$$

12.29 The current flows from positive to negative terminal inside the battery. The potential difference between the two terminals of the battery would be

$$V = \xi + ir = 2 + 5 \times 0.1 = 2.5\,\text{V}$$

12.30 **(a)** The effective resistance in the circuit (Fig. 12.40) from 3 Ω in series with 5 and 2 Ω in parallel

$$R_0 = R + \frac{R_1 R_2}{R_1 + R_2} = 3 + \frac{2 \times 5}{2 + 5} = 4.43\,\Omega$$
$$i = \frac{\xi}{R_0} = \frac{6}{4.43} = 1.35\,\text{A}$$

(b)
$$i_1 = \frac{i\,R_2}{R_1 + R_2} = \frac{1.35 \times 2}{5 + 2} = 0.386$$
$$P_1 = {i_1}^2 R_1 = (0.386)^2 \times 5 = 0.74\,\text{W}$$
$$i_2 = i - i_1 = 1.35 - 0.386 = 0.964\,\text{A}$$
$$P_2 = {i_2}^2 R_2 = (0.964)^2 \times 2 = 1.86\,\text{W}$$
$$P = i^2 R = (1.35)^2 \times 3 = 5.47\,\text{W}$$

(c) Total power dissipated by the resistances
$= P_1 + P_2 + P = 0.74 + 1.86 + 5.47 = 8.07\,\text{W} \simeq 8.1\,\text{W}$

Power supplied by the battery $= \xi i = 6 \times 1.35 = 8.1\,\text{W}$

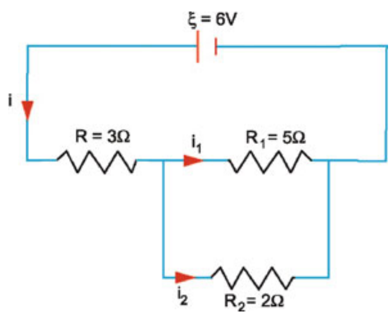


Fig. 12.40

12.3.3 Instruments

12.31 P.D across AB is $\dfrac{\xi R}{R + r} = \dfrac{l}{l_0}\,\xi_0$

$$\frac{\xi \times 20}{20+r} = \frac{150}{l_0}\xi_0$$
$$\frac{\xi \times 10}{10+r} = \frac{120}{l_0}\xi_0$$

Dividing the last two equations

$$\frac{2\,(10+r)}{20+r} = \frac{5}{4}$$

whence $r = 6.67\,\Omega$.

12.32 For the section ABC

$$i = \frac{\xi_1}{R+r} = \frac{2.2}{1+0.1} = 2\,\text{A}$$

In the section BCD also $i = 2\,\text{A}$.
The resistance of 26 cm wire $= \frac{26}{50} \times 1 = 0.52\,\Omega$.
Neglecting the internal resistance of the second cell

$$\xi_2 = 2 \times 0.52 = 1.04\,\text{V}$$

as no current flows through the galvanometer

12.33 **(a)** $V_{\text{AB}} = V_{\text{AD}}; \qquad V_{\text{BC}} = V_{\text{DC}}$

$$\therefore \quad i_1 P = i_2 R \tag{1}$$
$$i_1 Q = i_2 S \tag{2}$$

Dividing (1) by (2)

$$\frac{P}{Q} = \frac{R}{S} \tag{3}$$

(b) Assume that a non-zero current flows through the galvanometer of resistance G. Applying the junction theorem at A

$$i = i_1 + i_2 \tag{4}$$

Applying the loop theorem to the loop ABDA and noting that there is no emf in this loop

$$i_g G + i_1 P - i_2 R = 0 \tag{5}$$

Applying the loop theorem to the loop BCDB, which also does not have an emf

$$(i_1 - i_g)Q - i_g G - (i_2 - i_g)s = 0 \tag{6}$$

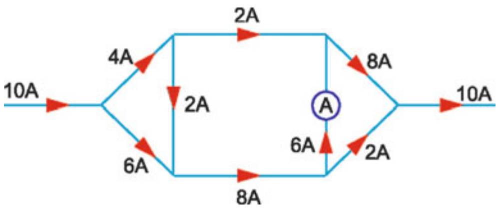
Combining (4), (5) and (6)

$$\frac{i_g}{i} = \frac{QR - PS}{G(Q+S) + (P+R)(G+Q+S)}$$

Note that $i_g = 0$ if $QR - PS = 0$, which is identical with the condition (3).

12.34 Apply the junction theorem to obtain currents in various branches as indicated in Fig. 12.41. Current flowing through the ammeter is 6 A.

Fig. 12.41



12.35 Let the internal resistance of each battery be r, galvanometer resistance G and the external resistance R. Then in the series arrangement, total resistance in the circuit (Fig. 12.42)

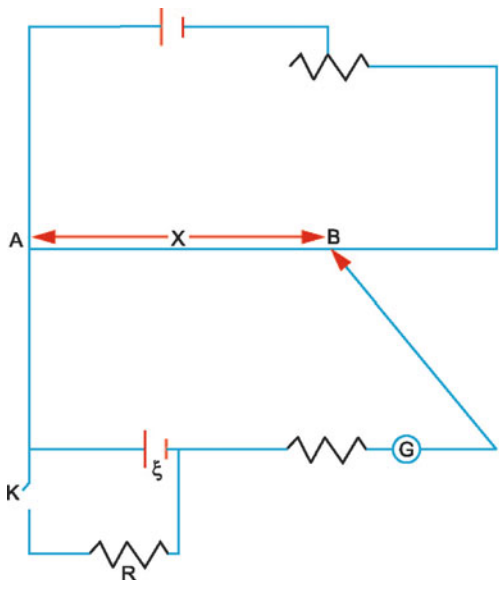


Fig. 12.42

$$R_{\text{eff}} = G + R + 2r$$

$$\text{Effective emf, } \xi_{\text{eff}} = 2 \times 1.5 = 3\,\text{V} \tag{12.17}$$

$$\xi_{\text{eff}} = i(G + R + 2r)$$

$$\therefore \qquad 3 = 1 \times (G + R + 2r) \tag{1}$$

In the parallel arrangement, $\xi_{\text{eff}} = 1.5\,\text{V}$ and the combined internal resistance is $\dfrac{r \times r}{r + r} = 0.5\,r$.
Total resistance in the circuit

$$R_{\text{eff}} = G + R + 0.5\,r$$

$$1.5 = 0.6 \times (G + R + 0.5r)$$

$$\text{or} \qquad 2.5 = G + R + 0.5r \tag{2}$$

Subtracting (2) from (1), $r = 0.333\,\Omega$.

12.36 When the key is closed P.D across R is

$$V = i\,R$$

and the emf of the cell is

$$\xi = i(R + r)$$

where r is the internal resistance of the cell.

$$\frac{\xi}{V} = \frac{R + r}{R} = 1 + \frac{r}{R}$$

When the key is open, let the balancing length be X_1 cm from the end A against the emf ξ. When the key is closed, let the balancing length be X_2 against the P.D. of V volts:

$$\frac{\xi}{V} = \frac{X_1}{X_2} = 1 + \frac{r}{R}$$

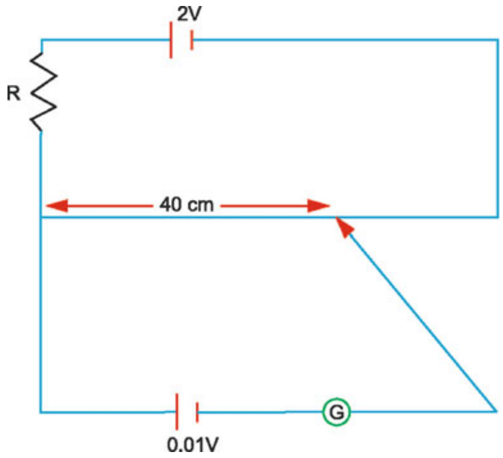
$$\frac{52}{40} = 1 + \frac{r}{5}$$

$$\therefore \qquad r = 1.5\,\Omega.$$

12.37 Resistance (Fig. 12.43) of 40 cm of potentiometer wire $= \dfrac{40}{100} \times 10 = 4\,\Omega$

P.D. across 40 cm wire due to 2 V cell is $V = \dfrac{2 \times 4}{10 + R} = \dfrac{8}{10 + R}$

Fig. 12.43



This is balanced by 0.01 V due to the second cell:

$$\therefore \qquad \frac{8}{10+R} = 0.01$$
$$\therefore \qquad R = 790\,\Omega$$

12.38 Resistance across ac (Fig. 12.20) is $\frac{1}{4} \times 12,000 = 3000\,\Omega$. The combined resistance of voltmeter ($6000\,\Omega$) in parallel with $3000\,\Omega$ resistance is $\dfrac{6000 \times 3000}{6000+3000} = 2000\,\Omega$.
Resistance across bc is $\frac{3}{4} \times 12,000 = 9000\,\Omega$. Effective resistance of the circuit $= 9000 + 2000 = 11,000\,\Omega$.
P.D. across ac is

$$V = \frac{220 \times 2000}{11,000} = 40\,\text{V}$$

Thus the voltmeter reads 40 V.

12.39 $$\frac{R}{3.5} = \frac{l}{100-l} = \frac{30}{100-30} = \frac{3}{7}$$
$$\therefore \quad R = 1.5\,\Omega$$

12.40 **(a)** $$N = \frac{i}{i_1} = \frac{100\,\text{mA}}{1\,\text{mA}} = 100$$
$$S = \frac{G}{N-1} = \frac{80}{100-1} = 0.808\,\Omega$$

A shunt of $0.808\,\Omega$ should be provided for the moving coil meter.

(b) Initially $V = iG = 1 \times 10^{-3} \times 80 = 0.08\,\text{V}$

$$N = \frac{80}{0.08} = 1000$$
$$R = (N-1)G = (1000-1) \times 80 = 79,920\,\Omega$$

A resistance of 79,920 Ω must be connected in series with the moving coil galvanometer to give the required full scale deflection.

12.41 $\xi = i(R + r)$ (by Ohm's law)

$$9 = i(120 + 15)$$
$$\therefore \quad i = \frac{1}{15}\,\text{A}$$

The voltmeter would read

$$V = \xi - ir = 9 - \frac{1}{15} \times 15 = 8\,\text{V}$$

12.42 $S = \dfrac{G}{N-1}$

$$\frac{i_1}{i} = \frac{1}{N} = \frac{1}{100}$$
$$\text{or} \quad N = 100$$
$$\therefore \quad G = (N-1)S = (100-1) \times 1 = 99\,\Omega$$

12.3.4 Kirchhoff's Laws

12.43 By the junction theorem

$$i = i_1 + i_2 \tag{1}$$

Applying the loop theorem to the loop $BE_1\,AE_2B$, traversing clockwise

$$\xi_2 - i_2 r_2 - \xi_1 + i_1 r_1 = 0$$
$$\therefore\ 2i_2 - i_1 = 20 - 10 = 10 \tag{2}$$

Applying the loop theorem to the loop BE_1ARB

$$\xi_1 = i_1 r_1 + iR$$

$\therefore \qquad 10 = i_1 + 30i$

or $\qquad 31i_1 + 30i_2 = 10 \qquad (3)$

where we have used (1). Solving (2) and (3)

$i_1 = -3.04\,\text{A}, \; i_2 = 3.48\,\text{A}$

Power dissipated through E_1 is $i^2{}_1 r_1 = (3.04)^2 \times 1 = 9.24\,\text{W}$
Power dissipated through E_2 is $i^2{}_2 r_2 = (3.48)^2 \times 2 = 24.2\,\text{W}$
Power dissipated through R is $i^2 R = (0.44)^2 \times 30 = 5.8\,\text{W}$

$(\because \; i = i_1 + i_2 = -3.04 + 3.48 = 0.44\,\text{A})$

12.44 Referring to Fig. 12.44

$i_1 + i_2 = i \qquad$ (junction theorem) $\qquad (1)$

Traversing the loop BE_2AE_1B counterclockwise the loop theorem gives

$\xi_2 - r_2 i_2 - (\xi_1 - r_1 i_1) = 0$

$\therefore \qquad 1 - 2i_1 - (2 - 1i_1) = 0$

or $\qquad i_1 - 2i_2 = 1 \qquad (2)$

For the loop BE_1ARB

$\xi_1 - i_1 r_1 - iR = 0$

$\therefore \qquad 2 - 1i_1 - 10i = 0$

or $\qquad 11i_1 + 10i_2 = 2 \qquad (3)$

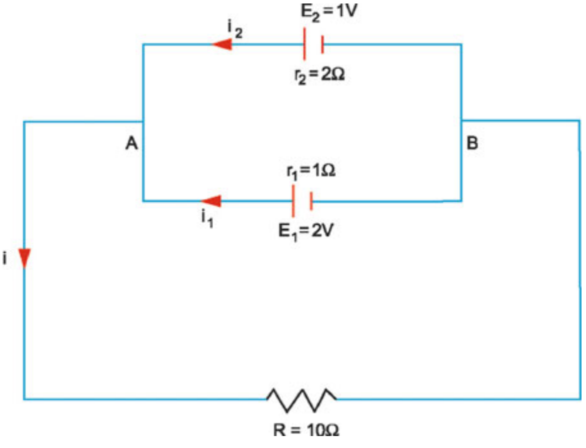


Fig. 12.44

where we have used (1). Application of Kirchhoff's laws to the loop BE_2ARB does not yield anything extra. From (1), (2) and (3) we find $i_1 = 0.44\,\text{A}$, $i_2 = -0.28\,\text{A}$ and $i = 0.16\,\text{A}$. The negative sign of i_2 shows that its direction is opposite to that has been assumed.

12.45 $i_1 + i_2 + i_3 = i = 5\,\text{A}$ (1)

As $\rho = \text{const}$ and $R = \pi\,\text{d}^2/4$ (Fig. 12.45)

$$R_1 : R_2 : R_3 = \frac{l_1}{d_1^2} : \frac{l_2}{d_2^2} : \frac{l_3}{d_3^2}$$

$$\therefore \quad R_1 : R_2 : R_3 = \frac{2}{3^2} : \frac{3}{4^2} : \frac{4}{5^2} = \frac{2}{9} : \frac{3}{16} : \frac{4}{25} \tag{2}$$

Since P.D across all the resistors is identical,

$$i_1 R_1 = i_2 R_2 = i_3 R_3 \tag{3}$$

$$\therefore \quad i_2 = i_1 \frac{R_1}{R_2} = \frac{2}{9} \times \frac{16}{3} i_1 = \frac{32}{27} i_1 \tag{4}$$

$$i_3 = i_1 \frac{R_1}{R_3} = \frac{2}{9} \times \frac{25}{4} = \frac{25}{18}\, i_1 \tag{5}$$

$$\therefore \quad i_1 + \frac{32}{27} i_1 + \frac{25}{18} i_1 = i = 5 \tag{6}$$

$$\therefore \quad i_1 = 1.4\,\text{A},\ i_2 = 1.66\,\text{A},\ i_3 = 1.94\,\text{A}$$

where we have used (1), (4) and (5).

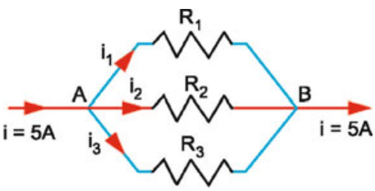


Fig. 12.45

12.46 The P.D across 8 and 2 Ω resistors are equal, Fig. 12.46:

$$2i_2 = 8i_3 \tag{1}$$

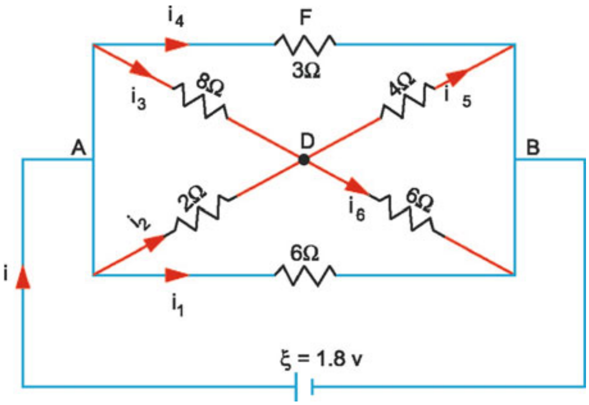
As P.D across 3 and 6 Ω resistors are equal

$$3i_4 = 6i_1 \tag{2}$$

As P.D across 4 and 6 Ω resistors are equal

$$6i_6 = 4i_5 \tag{3}$$

Fig. 12.46



Applying junction theorem

$$i = i_1 + i_2 + i_3 + i_4 \tag{4}$$

$$i_2 + i_3 = i_5 + i_6 \tag{5}$$

$$V_{AB} = V_{AD} + V_{DB}$$

$$\therefore \quad 3i_4 = 8i_3 + 4i_5 \tag{6}$$

Applying the loop theorem to CAFBC

$$\xi - ir - 3i_4 = 0$$

$$\frac{2}{3}i + 3i_4 = 1.8 \tag{7}$$

Solving (1), (2), (3), (4), (5), (6) and (7), $i_4 = 0.4$ A and $i = 0.9$ A. Applying the loop theorem to the entire circuit

$$\xi - ir - iR = 0 \tag{8}$$

where R is the equivalent resistance of the circuit

$$R = \frac{\xi}{i} = \frac{1.8}{0.9} - \frac{2}{3} = \frac{4}{3}$$

Power dissipated in the entire circuit is

$$P = i^2(R + r) = (0.9)^2\left(\frac{4}{3} + \frac{2}{3}\right) = 1.62\,\text{W}$$

12.47 Let A and B be the midpoints of the coils. As no current flows through the galvanometer, P.D across AB is zero. Applying the loop theorem to the main circuit, Fig. 12.47

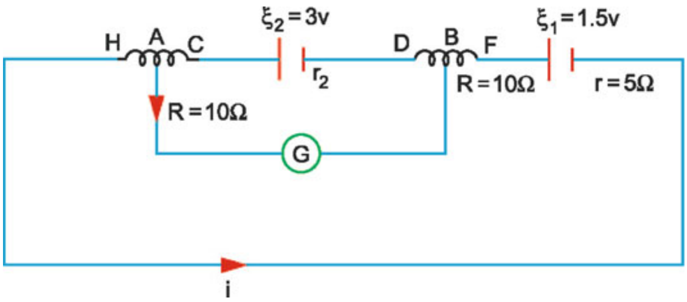


Fig. 12.47

$$\xi_1 + \xi_2 - i(R + R + r_1 + r_2) = 0$$

$$\therefore \quad i = \frac{\xi_1 + \xi_2}{2R + r_1 + r_2} = \frac{1.5 + 3}{2 \times 10 + 5 + r_2} = \frac{4.5}{r_2 + 25} \tag{1}$$

The P.D of the point B with respect to the negative terminal of the first cell is

$$V_B = \xi_1 - \left(\frac{10}{2} + r_1\right) i = 1.5 - 10\,i \tag{2}$$

The P.D of the point A with respect to the negative terminal of the first cell is

$$V_A = \xi_1 + \xi_2 - \left(\frac{10}{2} + r_2 + 10 + r_1\right) i$$
$$= 1.5 + 3 - (5 + r_2 + 10 + 5)i = 4.5 - (20 + r_2)$$

But $V_B = V_A$ ($\because$ no current flows through the galvanometer)

$$\therefore \quad 1.5 - 10i = 4.5 - (20 + r_2)i$$

$$\therefore \quad i = \frac{3}{10 + r_2} \tag{3}$$

From (1) and (2), $r_2 = 20\,\Omega$

12.48 **(a)** **(i)** $R_{eq} = \dfrac{R_1 R_2}{R_1 + R_2} + \dfrac{R_3 R_4}{R_3 + R_4} = \dfrac{8 \times 2}{8 + 2} + \dfrac{3.2 \times 3.2}{3.2 + 3.2} = 3.2\,\text{k}\Omega$

(ii) $V_{p_1} = 320 \times \dfrac{1.6}{3.2} = 160\,\text{V}$

(iii) $V_{p_2} = 320 \times \dfrac{1.6}{3.2} = 160\,\text{V}$

(iv) $I_T = \dfrac{\xi}{R_{eq}} = \dfrac{320}{3.2 \times 10^3} = 0.1\,\text{A}$

$$I_1 = I_T \times \frac{R_2}{R_1 + R_2} = 0.1 \times \frac{2}{8+2} = 0.02\,\text{A}$$

$$I_3 = I_T \times \frac{R_4}{R_3 + R_4} = 0.1 \times \frac{3.2}{3.2+3.2} = 0.05\,\text{A}$$

(v) $W = I_T^2\, R_{eq} = (0.1)^2 \times 3.2 \times 10^3 = 32\,\text{J}$

$$W_3 = I_3^2\, R_3 = (0.05)^2 \times 8 \times 10^3 = 20\,\text{J}$$

(b) $P = \dfrac{\xi^2}{r} = \dfrac{(24)^2}{0.01} = 5.76 \times 10^4\,\text{J}$

12.49 **(i)** $i = \dfrac{\xi}{R+r} = \dfrac{24}{140 + 0.02} = 0.1714\,\text{A}$

(ii) $V = \xi - ir = 24 - 0.1714 \times 0.02 = 23.9966\,\text{V}$

(iii) $P_R = i^2\text{R} = (0.1714)^2 \times 140 = 4.113\,\text{W}$

$$P_r = i^2 r = (0.1714)^2 \times 0.02 = 5.87 \times 10^{-4}\,\text{W}$$

(iv) $V = \xi = 24\,\text{V}$. Full voltage is available in the absence of load resistance.

12.50 **(i)** $R_{eq} = \dfrac{R_1 R_2}{R_1 + R_2} + R_5 + \dfrac{R_3 R_4}{R_3 + R_4}$

$$= \frac{80 \times 80}{80 + 80} + 20 + \frac{40 \times 40}{40 + 40} = 80\,\text{k}\Omega$$

(ii) $I_T = \dfrac{V}{R_{eq}} = \dfrac{300}{80 \times 10^3} = 3.75 \times 10^{-3}\,\text{A}$

$$I_1 = I_T \times \frac{R_2}{R_2 + R_1} = 3.75 \times 10^{-3} \times \frac{80}{80+80} = 1.875 \times 10^{-3}\,\text{A}$$

$$I_3 = I_T \times \frac{R_4}{R_4 + R_1} = 3.75 \times 10^{-3} \times \frac{40}{40+40} = 1.875 \times 10^{-3}\,\text{A}$$

(iii) $V_1 = \dfrac{V}{R_{eq}} \times \dfrac{R_1 R_2}{(R_1 + R_2)} = \dfrac{300}{80} \times \dfrac{80 \times 80}{(80+80)} = 150\,\text{V}$

$$V_2 = \frac{V\,R_5}{R_{eq}} = 300 \times \frac{20}{80} = 75\,\text{V}$$

$$V_3 = \frac{V}{R_{eq}} \times \frac{R_3 R_4}{(R_3 + R_4)} = \frac{300}{80} \times \frac{40 \times 40}{(40+40)} = 75\,\text{V}$$

(iv) $P_5 = I^2{}_T\, R_5 = (3.75 \times 10^{-3})^2 \times 20 \times 10^3 = 0.281\,\text{W}$

$$P = I^2{}_T\, R_{eq} = (3.75 \times 10^{-3})^2 \times 80 \times 10^3 = 1.125\,\text{W}$$

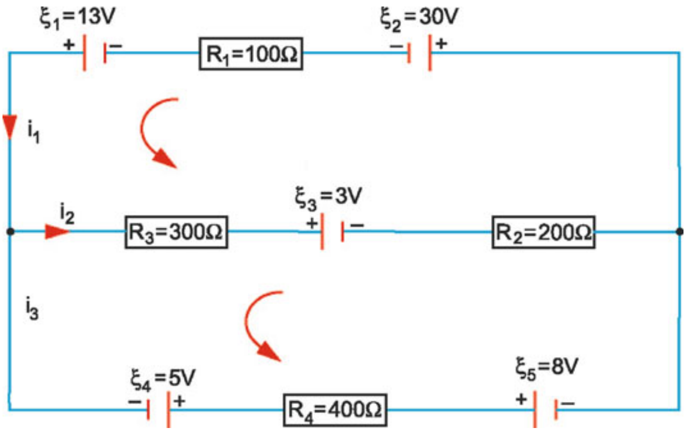


Fig. 12.48

12.51 $i_1 = i_2 + i_3$ (junction theorem) (1)

Traversing the top loop counterclockwise (Fig. 12.48)

$\xi_1 - i_2 R_3 - \xi_3 - i_2 R_2 - \xi_2 - i_1 R_1 = 0$ (loop theorem)

$\therefore \quad 13 - 300\, i_2 - 3 - 200\, i_2 - 30 - 100\, i_1 = 0$

or $\quad 100\, i_1 + 500 i_2 + 20 = 0$ (2)

Traversing the bottom loop counterclockwise

$\xi_4 - i_3 R_4 - \xi_5 + i_2 R_2 - \xi_3 + i_2 R_3 = 0$

$\therefore \quad 5 - 400\, i_3 - 8 + 200\, i_2 - 3 + 300\, i_2 = 0$

or $\quad 500\, i_2 - 400\, i_3 - 6 = 0$ (3)

Required equations for the unknown currents are (1), (2) and (3).

12.52 **(i)** Kirchhoff's rule 1 (junction theorem) At any junction of an electric network (branched circuit) the algebraic sum of the currents flowing towards that junction is zero (Fig. 12.49).
Kirchhoff's rule 2 (loop theorem)
Sum of the changes in the potential encountered in traversing a loop (closed circuit) in a particular direction (clockwise or counterclockwise) is zero.
If a resistor is traversed in the direction of the current, the change in the potential is $-iR$, while in the opposite direction it is $+iR$.
If a seat of emf is traversed in the direction of emf, the change in potential is $+\xi$, while in the opposite direction it is $-\xi$.

(ii) $i_1 = i_2 + i_3$ (junction theorem) (1)

Traversing the top loop counterclockwise

$$-\xi_1 - i_2 R_2 + \xi_3 + \xi_2 - i_1 R_1 = 0 \text{ (loop theorem)}$$
$$\therefore \quad -11 - 200\, i_2 + 33 + 22 - 100\, i_1 = 0$$
$$\text{or} \quad 100\, i_1 + 200 i_2 - 44 = 0 \tag{2}$$

Traversing the bottom loop counterclockwise

$$-\xi_4 - i_3 R_3 + \xi_5 - \xi_3 + i_2 R_2 = 0$$
$$\therefore \quad -44 - 300\, i_3 + 55 - 33 + 200\, i_2 = 0$$
$$\text{or} \quad 200\, i_2 - 300\, i_3 - 22 = 0 \tag{3}$$

Equations (1), (2) and (3) are the required equations in the three unknown currents i_1, i_2 and i_3.

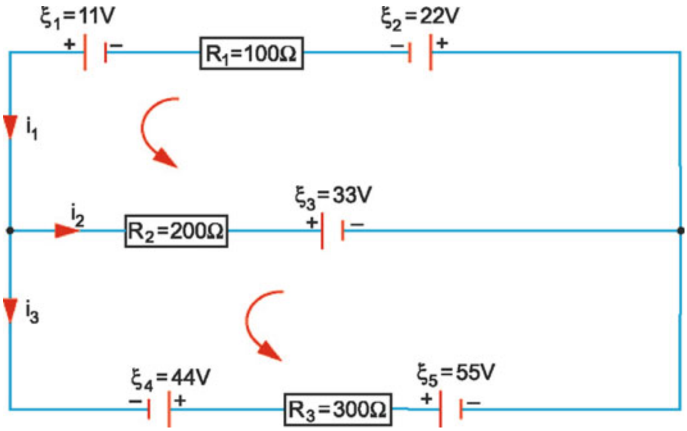


Fig. 12.49

12.53 **(i)** The equivalent resistance of the circuit is

$$R_{\text{eq}} = R_1 + R_2 + \frac{R_3 R_4}{R_3 + R_4}$$
$$= 5 + 10 + \frac{20 \times 60}{20 + 60} = 30\,\text{k}\Omega$$
$$V_1 = V \times \frac{R_1}{R_{\text{eq}}} = 300 \times \frac{5}{30} = 50\,\text{V}$$
$$V_2 = V \times \frac{R_2}{R_{\text{eq}}} = 300 \times \frac{10}{30} = 100\,\text{V}$$

(ii) $V_{\mathrm{p}} = V \times \dfrac{R_3 R_4}{R_{\mathrm{eq}}(R_3 + R_4)} = 300 \times \dfrac{20 \times 60}{30(20 + 60)} = 150\,\mathrm{V}$

(iii) $I_{\mathrm{T}} = \dfrac{V}{R_{\mathrm{eq}}} = \dfrac{300}{30 \times 10^3} = 0.0\,1\,\mathrm{A}$

$$I_2 = I_{\mathrm{T}} \frac{R_4}{R_4 + R_3} = 0.01 \times \frac{60}{60 + 20} = 0.0075\,\mathrm{A}$$

$$I_4 = I_{\mathrm{T}} \frac{R_3}{R_4 + R_3} = 0.01 \times \frac{20}{60 + 20} = 0.0025\,\mathrm{A}$$

12.54 $i_1 - i_2 - i_3 = 0$ (junction theorem) (1)

Traversing clockwise the top loop (Fig. 12.50)

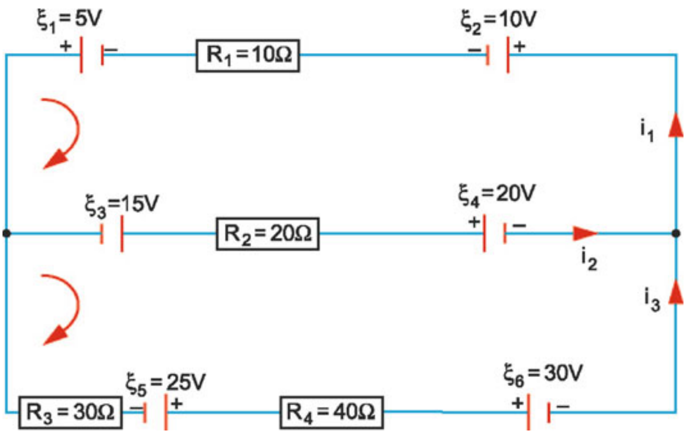


Fig. 12.50

$$\xi_1 + i_1 R_1 - \xi_2 - \xi_4 + i_2 R_2 + \xi_3 = 0 \quad \text{(loop theorem)}$$

$$\therefore \quad 5 + 10 i_1 - 10 - 20 + 20 i_2 + 15 = 0$$

$$\text{or} \quad i_1 + 2 i_2 - 1 = 0 \tag{2}$$

Traversing clockwise the bottom loop

$$-\xi_3 - i_2 R_2 + \xi_4 - \xi_6 + i_3 R_4 + \xi_5 + i_3 R_3 = 0 \quad \text{(loop theorem)}$$

$$\therefore \quad -15 - 20 i_2 + 20 - 30 + 40 i_3 + 25 + 30 i_3 = 0$$

$$\text{or} \quad 7 i_3 - 2 i_2 = 0 \tag{3}$$

The required equations are (1), (2) and (3) in three unknown currents.

12.55 $i_1 - i_2 - i_3 = 0$ (junction theorem) (1)

Traversing the top loop clockwise (Fig. 12.51)

$\xi_4 - i_1 R_4 - \xi_5 - i_2 R_3 - \xi_2 - i_2 R_2 = 0$

$\therefore \quad 50 - 200 i_1 - 30 - 80 i_2 - 10 - 120 i_2 = 0$

or $\quad 20 i_1 + 20 i_2 - 1 = 0$ (2)

Traversing the outer loop clockwise

$\xi_4 - i_1 R_4 - \xi_5 - i_3 R_5 - \xi_3 - i_3 R_1 + \xi_1 - i_3 R_6 = 0$

$50 - 200 i_1 - 30 - 100 i_3 - 20 - 60 i_3 + 40 - 40 i_3 = 0$

or $\quad 5 i_1 + 5 i_2 - 1 = 0$ (3)

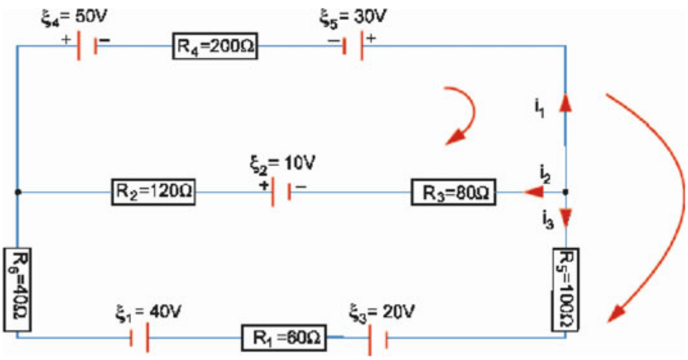


Fig. 12.51

The required equations are (1), (2) and (3) in three unknown currents i_1, i_2 and i_3. Note that by considering the bottom loop no new information is provided as it is already contained in the other two loops. In general it is sufficient to consider any two loops out of three.

12.56 Traversing the loop ABCDA clockwise

$$-iR - i_1 R_1 + \xi_2 - i r_2 - \xi_1 - i r_1 = 0$$

$i_1 = 2i/3, \; i_2 = i/3$

$$\therefore \quad -4.5 i - 3 \times \frac{2i}{3} + 8 - i - 4 - 0.5 i = 0$$

$$\therefore \quad i = 0.5\,\text{A}, \; i_1 = 0.33\,\text{A}, \; i_2 = 0.165\,\text{A}$$

P.D over E_1

$$V_1 = \xi_1 + i r_1 = 4 + 0.5 \times 0.5 = 4.25\,\text{V}$$

P.D over E_2

$$V_2 = \xi_2 - i r_2 = 8 - 0.5 \times 1 = 7.5\,\text{V}$$

Chapter 13
Electromagnetism I

Abstract *Chapters 13 and 14* are devoted to electromagnetism concerned with motion of charged particles in electric and magnetic fields, Lorentz force, cyclotron and betatron, magnetic induction, magnetic energy and torque, magnetic dipole moment, Faraday's law, Hall Effect, RLC circuits, resonance frequency, Maxwell's equations, Electromagnetic waves, Poynting vector, phase velocity and group velocity, dispersion relations, waveguides and cut-off frequency.

13.1 Basic Concepts and Formulae

Motion of Charged Particles in Electric and Magnetic Fields

Assuming that the magnetic field (B) acts perpendicular to the plane of orbit of a particle of charge q and mass m moving with velocity v, the radius of curvature (r) is given by

$$r = \frac{mv}{qB} \tag{13.1}$$

the angular velocity by

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{13.2}$$

the frequency by

$$f = \frac{\omega}{2\pi} = \frac{qB}{2\pi m} \qquad \text{(cyclotron frequency)} \tag{13.3}$$

the kinetic energy (K) by

$$K = \frac{1}{2}q^2\frac{r^2B^2}{m} \tag{13.4}$$

Null deflection: If the electric and magnetic fields are crossed, i.e. arranged at right angles then the charged particle is undeflected. The condition is

$$v = E/B \tag{13.5}$$

Magnetic Induction (*B*)

B at the Centre of a Coil of *N* Turns and of Radius *r*, Carrying Current *i*

$$B = \frac{\mu_0 N i}{2r} \qquad \text{(centre of coil)} \tag{13.6}$$

The field into the page is indicated by a cross X and out of page by a dot.

B at Distance *r* from a Long Straight Wire

$$B = \frac{\mu_0 i}{2\pi r} \qquad \text{(long wire)} \tag{13.7}$$

B at Distance *r* from a Straight Wire of Finite Length

$$B = \frac{\mu_0 i}{4\pi r} \qquad (\cos\theta_1 + \cos\theta_2) \qquad \text{(finite wire)} \tag{13.8}$$

where θ_1 and θ_2 are the inner angles subtended by the field point at the extremities of the wire.

B on the Axis of a Solenoid

$$B = \frac{\mu_0 N i}{l} \qquad \text{(solenoid)} \tag{13.9}$$

where N is the number of turns over the axial length l.

Magnetic Induction due to a Long Cylindrical Conductor of Radius *R*

$$\begin{aligned} B &= \frac{\mu_0 i r}{2\pi R^2} \qquad (r < R) \\ &= \frac{\mu_0 i}{2\pi r} \qquad (r > R) \end{aligned} \tag{13.10}$$

B due to a Hollow Cylindrical Shell of Radii *a* and *b* ($a < b$)

$$
\begin{aligned}
B &= \frac{\mu_0 i (r^2 - a^2)}{2\pi r (b^2 - a^2)} && (a < r < b) \\
&= 0 && (r < a) \\
&= \frac{\mu_0 i}{2\pi r} && (r > b)
\end{aligned}
\tag{13.11}
$$

where r is measured from the axis.

B due to a Loop of *N* Turns and Radius *r* on the Axis

$$B = \frac{\mu_0 i N R^2}{2(R^2 + z^2)^{3/2}} \tag{13.12}$$

Magnetic Force on a Current-Carrying Wire

The force on the current-carrying wire is directed perpendicular to both the length of the wire and the field direction. The direction of motion of the wire is given by the left-hand rule. If the current-carrying wire makes an angle θ with the field direction, then the force on the wire would be

$$F = ilB \sin\theta \tag{13.13}$$

Force on Two Parallel Wires Each of Length *l* Carrying Current i_1 and i_2 and Separated by Distance *d*

$$F = \frac{\mu_0 l i_1 i_2}{2\pi d} \tag{13.14}$$

The two forces (F_1 due to wire 1 on 2 and F_2 due to wire 2 on 1) form an action–reaction pair. For parallel currents the wires attract each other and for antiparallel currents the wires repel each other.

Magnetic Dipole Moment, Magnetic Energy

Magnetic Material, Hall Effect

The magnetic moment produced by a circular current i enclosing an area A is given by

$$\mu = Ai \tag{13.15}$$

Magnetic energy density

$$u_B = \frac{1}{2\mu_0}B^2 \tag{13.16}$$

Magnetic material

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) \tag{13.17}$$

where H is the magnetic field strength and M is the magnetization.

Hall effect: When a strip of conductor carries a dc current along its length, a magnetic field set up in a perpendicular direction produces a magnetic field sideways. The charge build up on one side establishes a potential across the width of the strip, known as Hall potential and the phenomenon is called Hall effect.

If R_{H} is the Hall coefficient, σ the electrical conductivity, then the mobility μ is given by

$$\mu = R_{\mathrm{H}}\sigma \tag{13.18}$$

Lorentz Force

$$F = q\boldsymbol{E} + qv \times \boldsymbol{B} \tag{13.19}$$

Faraday's Law

An emf is induced in a conductor when there is a change in the number of lines 'linking it' (passing through it) or when it cuts across field lines.

Consider a flat wire loop of any shape and of area A in a magnetic field B, the field B making an angle θ perpendicular to the loop. The magnetic flux φ through the loop is defined by

$$\varphi = \boldsymbol{B}.\boldsymbol{A} = BA\cos\theta \tag{13.20}$$

The unit of flux is the weber (Wb), while that of the magnetic field is tesla (T). $1\,\mathrm{T} = 10^4\,\mathrm{G}$, G standing for Gauss.

The electromotive force ξ in such a wire loop is equal to the rate of change of flux through it:

$$\xi = -\frac{\Delta\varphi}{\Delta t} \qquad \text{(Faraday's law)} \tag{13.21}$$

where $\Delta\varphi$ is the change in flux that occurs in time interval Δt.

Lenz law: The reason for the minus sign in (13.21) is given by Lenz law which states 'the induced current will appear in such a direction that it opposes the change that produced it'.

If the coil forms a closed circuit then only the induced current can be present, otherwise in the case of an open circuit one can only speak of induced emf and its direction.

In a coil of N turns

$$\xi = -N\frac{\Delta\varphi}{\Delta t} \tag{13.22}$$

If the circuit is complete current will appear and will be given by

$$i = \frac{\xi}{R} = -\frac{N}{R}\frac{\Delta\varphi}{\Delta t} \tag{13.23}$$

where R is the resistance of the circuit. The corresponding charge flowing is given by

$$\Delta q = i\,\Delta t = -\frac{N}{R}\Delta\varphi \tag{13.24}$$

Consider a conducting rod of length l moving sideways in the plane of paper over a U-shaped metal frame at constant speed v at right angles to a uniform magnetic field of flux density B into the paper. Then the emf induced across the ends of the rod is given by

$$\xi = -Blv \tag{13.25}$$

13.2 Problems

13.2.1 Motion of Charged Particles in Electric and Magnetic Fields

13.1 Calculate the cyclotron frequency to accelerate alpha particles in a magnetic field of 10^4 G. The mass of $^4He_2$ is 4.002603 u.

13.2 If the pole pieces of a cyclotron are 50 cm in diameter, a flux density of 15,000 G, find approximate values for the energies to which (a) protons and (b) α-particles could be accelerated. What oscillator frequency would be required in each case?

[University of London]

13.3 In a mass spectrometer, the velocity filter employs electric field E and a perpendicular magnetic field B. The deflection magnetic field, perpendicular to a beam is B'. Ions with similar charges q and mass numbers m_1 and

m_2 pass through the filter. Show that the separation between them will be $\dfrac{2E(m_2 - m_1)}{qBB'}$

[Indian Administrative Services]

13.4 A singly charged particle of known velocity 2.5×10^7 m/s but unknown mass moves in a bubble chamber in a circular path of radius 0.2 m in a field of 0.2 T acting perpendicular to the path. Determine the mass of the particle and identify it.

13.5 A particle of mass m and charge q travelling with a velocity v along the x-axis enters a uniform electric field $\boldsymbol{E}$ directed along the y-axis. Show that the trajectory will be a parabola.

13.6 Find the radius of a circular orbit of an electron of energy 5 keV in a field of 10^{-2} T.

[Osmania University 1992]

13.7 An electric field of 1500 V/m and a magnetic field act on an electron moving with a speed of 3000 m/s. If the resultant field is to be zero what should be the strength of the magnetic field (in Wb/m^2).

[Osmania University 1987]

13.8 An electron moves in a circle of radius 1.9 m in a magnetic field of 3×10^{-5} T. Calculate (a) the speed of electrons and (b) time taken to move round the circle.

13.9 A cyclotron is powered by a 50,000 V 5 Mc/s radio frequency source. If its diameter is 1.524 m, what magnetic field satisfies the resonance condition for deuterons?. Also what energies will they attain? Take the mass of deuteron as 2.0141 u.

13.10 Deuterons are accelerated in a conventional cyclotron. Given the resonance frequency was 11.5 Mc/s and radius of the dee 30″, calculate the resonance frequency of protons and the maximum energy of protons that is obtainable using the same magnetic field. (In a cyclotron the vacuum chamber is partitioned into two D-shaped components)

13.11 A cyclotron has a magnetic field of 15,000 G. The extraction radius is 50 cm. Calculate (a) the frequency of the *rf* necessary for accelerating deuterons and (b) the energy of the extracted beam.

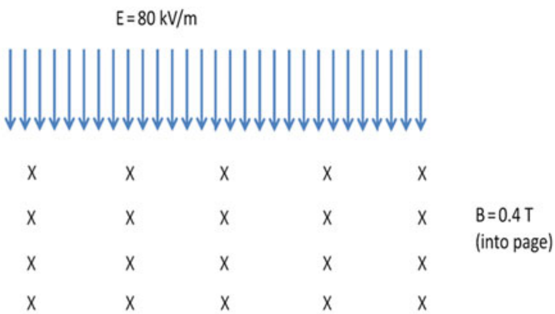
[University of Liverpool]

13.12 In the Bohr model of hydrogen atom the electron revolves in a circular orbit of radius 0.53 Å with a time period of 1.5×10^{-16} s. Find the corresponding current.

13.13 As shown in Fig. 13.1, a beam of particles of charge q enters a region where an electric field is uniform and directed downwards. Its value is 80 kV/m. Perpendicular to $\boldsymbol{E}$ and directed into the page is a magnetic field $\boldsymbol{B} = 0.4\,\text{T}$.

- **(i)** If the speed of the particles is properly chosen, the particles will not be deflected by these crossed electric and magnetic fields. What speed is selected in this case?
- **(ii)** If the electric field is cut off and the same magnetic field is maintained, the charged particles move in the magnetic field in a circular path of radius 1.14 cm. Determine the ratio of the electric charge to the mass of the particles.

Fig. 13.1



13.14 **(a)** Write an expression for the force acting on a charge q moving with velocity v in an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$.

(b) A charged particle of mass m and charge q is accelerated through a potential difference of V and then injected into a region with a magnetic field B perpendicular to the plane in which the charge moves. Derive an expression for the radius of curvature, r, of the path of the particle when in the magnetic field.

[University of Durham 2004]

13.15 In a certain mass spectrometer the magnetic field has a magnitude of 0.2 T. It is intended that this spectrometer be used to separate two isotopes of uranium, ${}^{235}_{92}\text{U}$(mass 3.90×10^{-25} kg) and ${}^{238}_{92}\text{U}$(mass 3.95×10^{-25} kg). In order to be separated the radii of curvature described by singly charged (charge $+e$) ions must differ by 2 mm. Calculate the electric potential through which the ions must be accelerated in order to achieve this.

13.16 **(a)** Write down an expression for the force experienced by a particle with charge q moving with velocity v in a magnetic field B. Under what circumstances does the particle mass m, of describe a circle of radius r?

(b) A coil of cross-sectional area A composed of N turns is placed perpendicular to a magnetic field which is uniform in space, with a strength that varies in time according to $B = B_0 \cos(15t)$. Calculate the electromotive force induced in the coil.

[University of Manchester 2007]

13.17 A water droplet of radius 1 μm is charged such that the electric field on its surface is 5.8 mV/m. (a) How many electrons does the droplet carry? (b) How strong a vertical electric field is required to prevent it from falling?

13.18 An electron of energy 1 eV enters an infinitely large region containing only a homogeneous magnetic field of 10^{-3} T, at an angle of 60° to the direction of the field. Calculate its subsequent motion assuming no energy losses. What type of energy losses will occur even in complete vacuum?

[University of Manchester 1972]

13.19 A uniform electric field is established between the plates of a parallel plate capacitor by holding one plate at ground and the other at a positive potential V as shown. A uniform magnetic field B is established perpendicular to the electric field (Fig. 13.2).
A charge $-q$ is released from rest from the lower plate.

(i) Write down the equations of motion for the velocity components of the charge.
(ii) Show that, at some time t later, the velocity of the electron in the x-direction is related to the distance y moved along the y-axis by

$$v_x = \omega y$$

(iii) By applying the conservation of energy or otherwise to determine the square of the velocity in the x–y-plane, show that

$$v_y^2 = \left(\frac{2qv}{md}y - \omega^2 y^2\right)$$

[University of Aberystwyth, Wales]

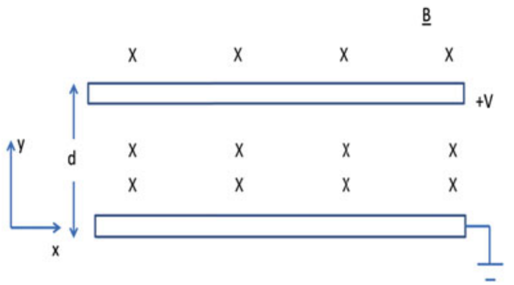


Fig. 13.2

13.20 Electrons are liberated with zero velocity from the negative plate of a parallel plate condenser, in which there is a constant magnetic field B parallel to the plates. If the separation of the plates is d and the potential across them is V, show that the electrons only arrive at the positive plate if $d^2 < \dfrac{2mV}{eB^2}$

[University of Durham 1962]

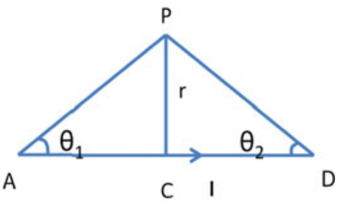
13.2.2 Magnetic Induction

13.21 The magnetic field at 40 cm from a long straight wire is 10^{-6} T. What current is carried by the wire?

13.22 A current I flows through a straight wire AB of finite length.

(a) Find the magnetic field B at distance r from the wire, the ends of the wire making inner angles θ_1 and θ_2 with P, Fig. 13.3.
(b) Obtain the limit value for B for a very long wire.

Fig. 13.3



13.23 The magnetic field at the centre of a circular current loop is 10^{-5} T. If the radius of the loop is 50 cm, find the current.

13.24 A square conducting loop, of side a carries a current I. Calculate the magnetic field at the centre of the loop.

13.25 Two wires are bent into semicircles of radius a, as in Fig. 13.4. The upper half has resistance $R\ \Omega$ and the lower half resistance $4R\ \Omega$. Find the magnetic induction at the centre of the circle.

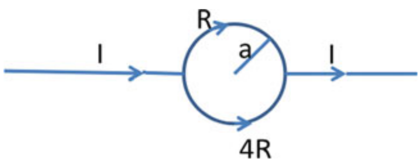
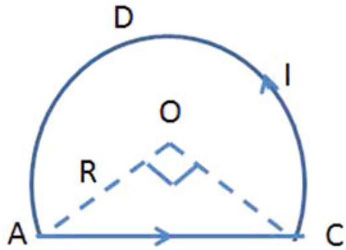


Fig. 13.4

13.26 A current I is sent through a thin wire as in Fig. 13.5. The radius of the curved part of the wire is R. Show that the magnetic induction at the point O will be $B = \frac{\mu_0 i}{2\pi R}\left(1 + \frac{3\pi}{4}\right)$

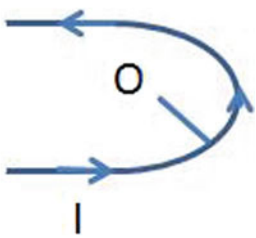
Fig. 13.5



13.27 **(a)** A current I is sent through a thin wire as in Fig. 13.6. The straight wires are very long and the radius of the curved part of the wire is R. Show that the magnetic induction at the point O will be

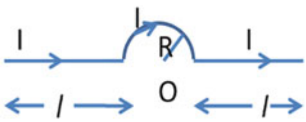
$$B = \frac{\mu_0 I(\pi + 2)}{4\pi R}$$

Fig. 13.6



(b) A wire shown in Fig. 13.7 carries current I. Find the field of induction B at the centre O.

Fig. 13.7



13.28 **(a)** A wire in the form of a polygon of n sides is circumscribed by a circle of radius a. If the current through the wire is i, show that the magnetic induction at the centre of the circle is given by

$$B = \frac{\mu_0 n i}{2\pi a} \tan\left(\frac{\pi}{n}\right)$$

(b) Show that in the limit $n \to \infty$ you get the expected result.

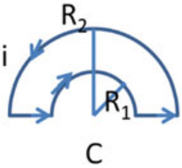
13.29 A wire of length l can form a circle or a square. A current i is set up in both the structures. Show that the ratio of magnetic induction at the centres of these structures will be approximately 0.87.

13.30 **(a)** C is the common centre of the circular arcs of the circuit-carrying current i, its arcs cutting a sector of angle, θ. Show that the magnetic induction at C is

$$B = \frac{\mu_0 i \theta}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

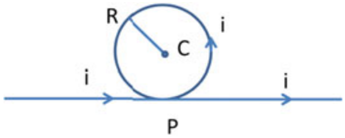
(b) C is the common centre of the semicircular arc of radii R_1 and R_2, Fig. 13.8, carrying current i. Show that the magnetic induction at C is $B = \frac{\mu_0 i}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

Fig. 13.8



13.31 A long wire is bent into the shape as in Fig. 13.9, without any cross-contact at P. The current flows as indicated with the circular portion having radius R. Show that the magnetic induction at C, the centre of the circle is $B = \frac{\mu_0 i}{2R}\left(1 + \frac{1}{\pi}\right)$

Fig. 13.9



13.32 A current I flows through a ring of radius r placed in the xy-plane. Show that the magnetic induction at a point along the z-axis passing through the centre of the ring is given by

$$B(z) = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}$$

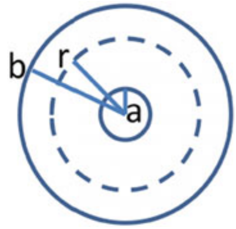
13.33 Five hundred turns of a wire are wound on a thin tube 1 m long. If the wire carries a current of 5 A, determine the field in the tube.

13.34 Two parallel wires, a distance d apart, carry equal currents I in opposite directions. Calculate the magnetic induction B for points between the wires at a distance x from one wire.

[Adapted from Hyderabad Central University 1993]

13.35 A long hollow copper cylinder with inner radius a and outer radius b carries a current I. Calculate the magnitude of the magnetic field at a point P ($a < r < b$) (Fig. 13.10).

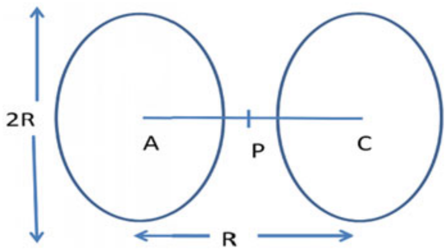
Fig. 13.10



13.36 Helmholtz coils consist of a pair of loops each with N turns and radius R. They are placed coaxially at distance R and the same current I flows through the loops but in the opposite sense. Show that the magnetic field at P, midway between the centres A and C, Fig. 13.11, is given by

$$B = \frac{8N\mu_0 I}{5^{3/2}R}$$

Fig. 13.11



13.37 A plastic of radius R has charge q distributed over its surface. If the disc rotates at an angular frequency about its axis, show that the induction B at the centre is given by

$$B = \frac{\mu_0 \omega q}{2\pi R}$$

13.38 Show that in the case of Helmholtz coils (prob. 13.36), the magnetic induction in the vicinity of the midpoint P is fairly uniform.

13.39 Consider a long straight rod of copper wire. It has a radius of 3×10^{-2} m and 100 A flowing uniformly through it. Find a value for the magnetic field (i) 1 m away and (ii) 6×10^{-3} m away from the central axis of the rod.

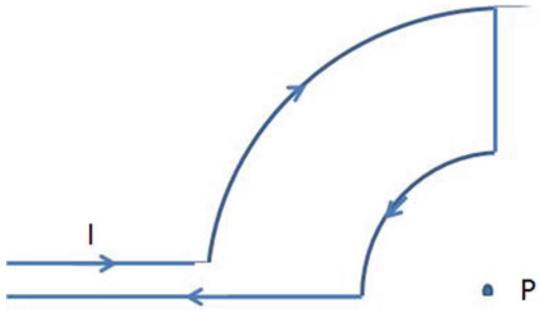
[University of Durham 2004]

13.40 Use Ampere's law to calculate the magnetic field for a long cylindrical conductor of radius R and a current I flowing through it at a distance r from the central axis of the conductor when (a) $r > R$ and (b) $r < R$

13.41 Current I flows in two concentric circular arcs of radii r and $2r$, Fig. 13.12. Both arcs are quarter of a circle with P as the centre. Determine B at P.

[University of Durham]

Fig. 13.12



13.42 **(a)** A current I flows in a straight wire of length L. Show that the magnitude of the magnetic field at a perpendicular distance x from the midpoint of the wire is given by

$$|B| = \frac{\mu_0 I}{4\pi} \frac{L}{x\sqrt{(L/2)^2 + x^2}}$$

where μ_0 is the permeability of free space. What is the direction of the B field?

(b) A loop of wire of length l carries a current I. Compare the magnetic fields at the centre of the loop when it is bent into (a) a square and (b) an equilateral triangle.

[University of Durham 2000]

13.43 Two identical, parallel co-axial coils of radius r, and having N turns, are separated by a distance r along their common axis. They both carry a current I in the same direction. Derive an expression for the magnetic field on the axis at the mid-point in between the coils. Evaluate the field when $N = 100$, $r = 20\,\text{cm}$ and $I = 2\,\text{A}$.

13.44 State the relationship between the tangential component of the magnetic field B summed around a closed curve C and the current I_c passing through the area enclosed by the curve.

A long cylinder conductor of radius R carries a current I along its length. The current is uniformly distributed throughout the cross-section of the conductor. Calculate the magnetic field at a distance $r = R/2$ from the axis of the conductor. Find the distance $r > R$ from the axis of the conductor where the magnitude of the magnetic field is the same as at $r = R/2$.

[University of Durham]

13.45 **(a)** Show that (ignoring edge effects) the self-inductance, L, of a solenoid with n turns per unit length, length l and cross-sectional area A, is given by

$$L = \mu_0 n^2 A l$$

(b) A solenoid with 100 turns, length 10 cm and of radius 1 cm, carries a current of 5A. Calculate the magnetic energy stored in the solenoid.

(c) The current in the solenoid of part (b) is reduced to zero at a uniform rate over 5 s. Calculate the emf induced in the coil.

[University of Durham 2005]

13.46 **(a)** What is the magnetic field at a point 50 mm from a long straight wire carrying a current of 3 A?

(b) A small current element $I\,\mathrm{d}\boldsymbol{l}$ with $\mathrm{d}\boldsymbol{l} = 2\hat{k}$ and $I = 2$ A is centred at the origin. Find the magnetic field $\mathrm{d}\boldsymbol{B}$ at the following points:

(i) on the x-axis at $x = 3$ m,
(ii) on the x-axis at $x = -6$ m,
(iii) on the z-axis at $z = 3$ m,
(iv) on the y-axis at $y = 3$ m.

[University of Aberystwyth, Wales 2005]

13.47 A thin torus, of radius 0.1 m, is wound uniformly with 100 turns of wire. If a current of 2.0 A flows through the wire, what are the magnitudes of the B and H fields generated within the torus if it contains (i) a vacuum and (ii) a material with relative permeability 500? What is the magnetization in the material (Fig. 13.13)?

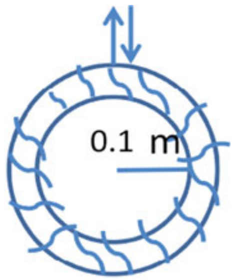


Fig. 13.13

13.48 It is believed that the earth's magnetic field is produced by circulating current in the core. If the mean radius of such currents is 1000 km, what is the order of magnitude of current required to account for the earth's dipolar magnetic field of magnitude 6×10^{-5} T at the north magnetic pole?

[University of Manchester 1972]

13.49 A pair of circular coils each having 50 turns of radius 50 cm are separated by 50 cm. A current of 10 A passes through the coils which are connected in series. Midway between the coils, a flat metal disc of radius 10 cm, is revolving at 1000 rpm What is the emf generated between the centre and the rim of the disc$(\mu_0 = 4\pi \times 10^{-7}$ H/m)?

[University of Manchester 1959]

13.50 A conductor 1 m long moves with a velocity given by $(3\hat{i} + 2\hat{j} + \hat{k})$ m/s through a magnetic field given by $(\hat{i} + 2\hat{j} + 3\hat{k})$ Wb/m^2 How will the voltage developed across the ends of the conductor vary with its orientation? For what orientation will the voltage be zero?

[University of Durham 1962]

13.51 A proton travelling with velocity of $\boldsymbol{v} = (\hat{i} + 3\hat{j})10^4$ m/s is located at $x =$ 2 m and $y = 3$ m at some instant t. Calculate the magnetic field at time t at the position $x = 2$ m, $y = 3$ m.

13.2.3 Magnetic Force

13.52 Two long straight wires lie parallel to each other at a distance 5 cm apart. If one carries a current of 2 A and the other a current of 3 A in the opposite direction, find the force each wire exerts on the other (per metre of wire)?

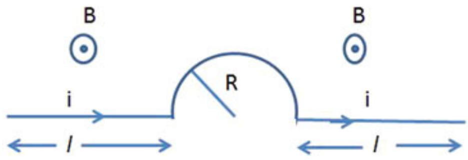
13.53 Two parallel wires 20 cm apart attract each other with a force of 10^{-5} N/m length. If the current in one wire is 10 A, find the magnitude and direction of current in the other wire?

13.54 A 2 m long wire weighs 4 g and carries a 10 A current. It is constrained to move only vertically above another wire carrying 15 A in the opposite direction. At what separation would its weight be supported by magnetic force?

13.55 Three long parallel wires, each carrying 20 A in the same direction, are placed in the same plane with the spacing of 10 cm. What is the magnitude of net force per metre on (a) an outer wire and (b) central wire?

13.56 Calculate the force acting on a bent wire (Fig. 13.14) carrying current i placed in a uniform magnetic field B, normal to the plane of paper in terms of i, B, l and R.

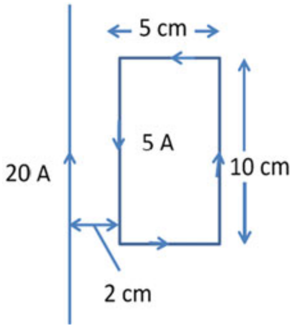
Fig. 13.14



13.57 A long straight wire carries a current of 20 A, as shown in Fig. 13.15. A rectangular coil with two sides parallel to the straight wire has sides 5 and 10 cm with the near side a distance 2 cm from the wire. The coil carries a current of 5 A.

(i) Find the force on each segment of the rectangular coil due to the current in the long straight wire.
(ii) What is the net force on the coil?

Fig. 13.15



13.58 **(a)** A very long straight wire PQ of negligible diameter carries a steady current I_1. A rigid square coil ABCD of side l and n turns is set up with sides AB and DC parallel to the coplanar with PQ as shown in Fig. 13.16. The side of the coil AB is at distance d from the wire PQ. Derive an expression for the resultant force on the coil when a steady current I_2 flows through it. What is the direction of the force?

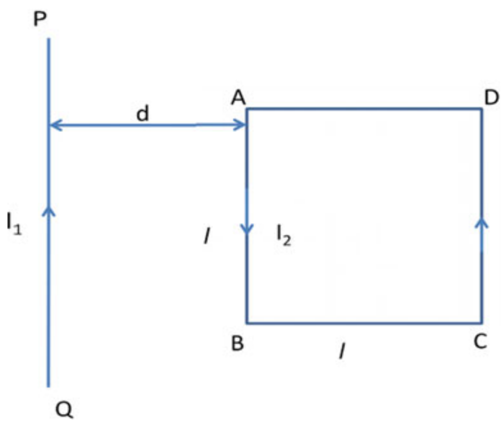


Fig. 13.16

(b) Calculate the magnitude of the force when

$$I_1 = 1\,\mathrm{A},\ I_2 = 4\,\mathrm{A},\ d = 0.10\,\mathrm{m},\ n = 10,\ \text{and}\ l = 0.05\,\mathrm{m}.$$

[University of Durham 2004]

13.59 Two parallel rails of negligible resistance are at distance d apart and are connected by a resistor of resistance R. A conducting rod lies perpendicular to the two rails and is free to slide on the rails. A constant magnetic field B is perpendicular to the loop formed by the rails, rod and resistor. An external agent drags the rod at velocity v along the rails. Find (a) the current flowing in the resistor, (b) the total power delivered to the resistor and (c) the magnitude of the applied force that is needed to move the rod with this velocity.
[University of Durham]

13.2.4 Magnetic Energy, Magnetic Dipole Moment

13.60 In prob. (13.37) show that the magnetic moment of the disc will be $\mu = \dfrac{\omega q R^2}{4}$.

13.61 The earth has a magnetic dipole moment of $6.4 \times 10^{21}\ \mathrm{A/m^2}$. Show that this dipole moment can be produced by passing a current of 5×10^7 A in a single wire going around the magnetic equator.

13.62 Calculate the energy density at the centre of a circular loop of wire 10 cm radius carrying a current of 100 A.

13.63 Given that the magnetic field at the centre of hydrogen atom is $13.5\,\mathrm{Wb/m^2}$, calculate the magnetic energy density at the centre of hydrogen atom due to the circulating electron.

13.64 A wire of length l forms a circular coil. If a current i is set up in the coil show that when the coil has one turn the maximum torque in a given magnetic field developed will be $\dfrac{1}{4\pi} l^2 i B$

13.65 A charge q is uniformly distributed over the volume of a uniform sphere of mass m and radius R, which rotates with an angular velocity ω about the axis passing through its centre. Show that the ratio of the magnetic moment and the angular momentum will be $\dfrac{\mu}{L} = \dfrac{q}{2m}$.

13.66 An electric dipole, whose dipole moment has magnitude 1.6×10^{-29} Cm is placed in a electric field of 1000 V/m. The direction of the dipole moment makes an angle of 30° to the direction of electric field. What is the potential energy of the dipole?

13.2.5 Faraday's Law

13.67 A flexible circular wire expands such that its radius increases linearly with time. It is located in a magnetic field perpendicular to the loop. Show that the emf induced in the wire varies linearly with time.

13.68 An aeroplane, with a wing span of 30 m, is flying horizontally at a speed of 720 km/h, at a point where the vertical component of the earth's field is 0.4 Oe. What is the emf developed between its wing tips.

[University of Durham]

13.69 A metal disc of radius 0.1 m spins about a horizontal axis lying in the magnetic meridian at a speed of 5 rev/s. If the horizontal component of the earth's field is $B = 2 \times 10^{-5}\ \mathrm{Wb/m^2}$, calculate the potential difference between the centre and the outer edge of the disc.

[University of Durham]

13.70 A coil is 30 turns of wire, each of area $10\ \mathrm{cm}^2$, is placed with its plane at right angles to a magnetic field of 0.1 T. When the coil is suddenly withdrawn from the field, a galvanometer in series with the coil indicates that 10^{-5} C passes around the circuit. What is the combined resistance of the coil and the galvanometer?

[University of Cambridge]

13.71 A wire loop of area $0.2\ \mathrm{m}^2$ has a resistance of 20 Ω. A magnetic field, normal to the loop, initially has a magnitude of 0.25 T and is reduced to zero at a uniform rate in 10^{-4} s. Estimate the induced emf and the resulting current.

13.72 A square wire with a loop of resistance 4 Ω, with sides 25 cm rotates 40 times per second about a horizontal axis. The magnetic field is vertical and has a magnitude of 0.5 T. Estimate the amplitude of the induced current.

13.73 A bar slides on rails separated by 20 cm, Fig. 13.17. If the current flowing through the resistor $R = 5\ \Omega$ is 0.4 A and the field $B = 1$ T, what is the speed of the bar?

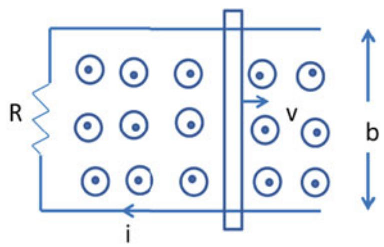


Fig. 13.17

13.74 A uniform magnetic field of induction B fills a cylindrical volume of radius R. A metal rod of length $2l$ is placed as in Fig. 13.18. If $\mathrm{d}B/\mathrm{d}t$ is the rate of change of B show that the emf that is produced by the changing magnetic field that acts at the ends of the rod is given by

$$\xi = -\frac{\mathrm{d}B}{\mathrm{d}t} l\sqrt{R^2 - l^2}$$

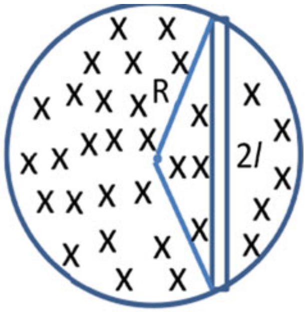


Fig. 13.18 Magnetic induction at the centre of a current-carrying wire made of three-fourths of a circle and a chord

13.75 A square wire of length l, mass m and resistance R slides without friction on parallel conducting resistance rails as in Fig. 13.19. The rails are interconnected at the bottom by resistance rails so that R, the wire and rails form a closed rectangular loop. The plane of the rails is inclined at an angle θ with the horizontal and a vertical uniform magnetic field B exists within the frame. Show that the wire acquires a steady velocity of magnitude

$$v = \frac{mgR\sin\theta}{B^2 l^2 \cos^2\theta}$$

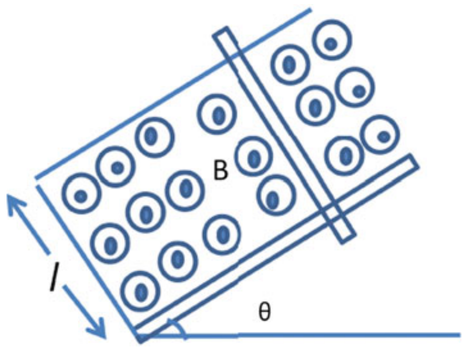


Fig. 13.19

13.76 A copper disc of 10 cm radius makes 1200 rotations per minute with its plane perpendicular to a magnetic field. If the induced emf between the centre and the edge of the disc is 6.28 mV, find the intensity of the field.

[Indian Administrative Services]

13.77 Show that Faraday's law $\xi = \mathrm{d}\varphi_B/\mathrm{d}t$ is dimensionally correct.

13.78 The magnetic field of an electromagnetic wave is given by the relation

$$B = 3 \times 10^{-12} \sin(4 \times 10^6 t)$$

where all quantities are in S.I. units. Find the magnitude of emf induced by the field in a 200-turn coil of 15 cm^2 area placed normal to the field.

13.79 Find the ratio of emf generated in a loop antenna by 100 MHz (typical television frequency) to that of 1 MHz (typical radio frequency) if both have equal field intensities.

13.80 Define magnetic flux and state Faraday's law, describing the relationship between the magnetic flux linked through a circuit and the current induced in the circuit. What is the force on a straight wire of length l carrying a current I in the presence of a magnetic field B?
Two long frictionless and resistanceless parallel rails, separated by a distance a, are connected by a resistanceless wire. A magnetic field B is oriented perpendicular to the plane containing the two rails. A frictionless conduction slider of resistance R and mass m is placed perpendicular to the rails and is given an initial velocity u along the rails. Obtain an expression for the force F on the slider while it moves at velocity v. Hence, find the maximum distance that the slider travels?

13.81 A flat, circular coil has 100 turns of wire, of radius 10 cm. A uniform magnetic field exists in a direction perpendicular to the plane of the coil. This field is increasing at a rate of 0.1 T/s. Calculate the emf induced in the coil.

13.82 A rectangular coil in the plane of the page has dimensions a and b. A long wire that carries a current I is placed directly on the coil, as shown in Fig. 13.20.

(i) Obtain an expression for the magnetic flux through the coil as a function of x for $0 \leq x \leq b$.
(ii) For what value of x is the net flux through the coil a maximum? For what value of x is the net flux a minimum?

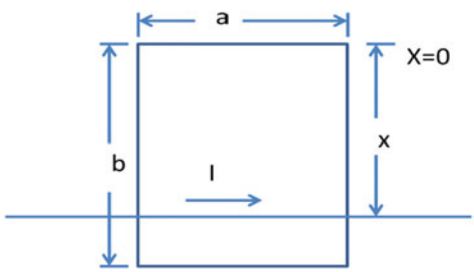


Fig. 13.20

(iii) Obtain an expression for the emf induced in the coil if the wire is placed at $x = b/4$ and the current varies with time according to $I = 2t$.

13.83 Show that in the betatron the magnetic flux Ø linking an electron orbit of radius R is given instantaneously by $\text{Ø} = 2\pi R^2 B$ where B is the instantaneous magnetic field.

[University of Newcastle upon Tyne 1965]

13.2.6 Hall Effect

13.84 What is the Hall effect and what is the significance of a positive Hall coefficient?
A potential difference is applied between the ends of a strip of copper and a current of 100 A flows along its length. The strip is 20 cm long in the x-direction of a rectangular system of coordinates, 2 cm wide in the y-direction and 1 mm thick in the z-direction. A uniform magnetic field of 10 Wb/m^2 is applied across the strip in the positive y- direction and the hall EMF is found to be 5 μV
Derive (a) the magnitude and direction of the Hall field when the current flows in the positive x-direction and (b) the concentration of free electrons.
[University of Manchester 1972]

13.85 The Hall coefficient and electrical conductivity of an n-type silicon are -7.3×10^{-5} m^3/C and 2×10^7 mho/m, respectively. Calculate the magnitude of the mobility of the electrons.

[University of Durham 1962]

13.3 Solutions

13.3.1 Motion of Charged Particles in Electric and Magnetic Fields

13.1 $10^4\,\text{G} = 1\,\text{T}$

$$f = \frac{Bq}{2\pi m} = \frac{1 \times 1.6 \times 10^{-19}}{2\pi \times 4.0026 \times 1.66 \times 10^{-27}} = 3.83 \times 10^6\,\text{Hz} = 3.83\,\text{MHz}$$

13.2 **(a)** $K_\text{p} = \frac{1}{2}\frac{q^2 r^2 B^2}{m_\text{p}} = \frac{1}{2} \times \frac{(1.6 \times 10^{-19})^2 (0.25)^2 (1.5)^2}{1.66 \times 10^{-27}}$

$$= 2.17 \times 10^{-13}\,\text{J} = \frac{2.17 \times 10^{-12}}{1.6 \times 10^{-13}}\,\text{MeV} = 13.56\,\text{MeV}$$

$$f_{\mathrm{p}} = \frac{Bq}{2\pi m_{\mathrm{p}}} = \frac{1.5 \times 1.6 \times 10^{-19}}{2\pi \times 1.66 \times 10^{-27}} = 2.3 \times 10^{7}\,\mathrm{Hz} = 23\,\mathrm{MHz}$$

(b) $$K_{\alpha} \simeq \frac{1}{2}\frac{(2e)^2 r^2 B^2}{4m_{\mathrm{p}}} = K_{\mathrm{p}} = 13.56\,\mathrm{MeV}$$

$$f_{\alpha} = \frac{(B)(2e)}{2\pi \times 4m_{\mathrm{p}}} = \frac{f_{\mathrm{p}}}{2} = 11.5\,\mathrm{MHz}$$

13.3 $$q\,v\,B = E\,q \rightarrow v = \frac{E}{B} \tag{1}$$

$$\frac{mv^2}{r} = q\,v\,B' \rightarrow r = \frac{mv}{qB'} \tag{2}$$

Separation $= 2r_2 - 2r_1 = 2(r_2 - r_1)$

$$= \frac{2E}{qBB'}(m_2 - m_1) \tag{3}$$

where we have used (1) and (2).

13.4 $$\frac{mv^2}{r} = Bev$$

$$m = \frac{Ber}{v} = \frac{0.2 \times 1.6 \times 10^{-19} \times 0.2}{2.5 \times 10^{7}} = 2.56 \times 10^{-28}\,\mathrm{kg}$$

In terms of electron mass

$$m = \frac{2.56 \times 10^{-28}}{9.1 \times 10^{-31}} = 281\,m_{\mathrm{e}}$$

Hence the particle is a pion (π – meson)

13.5 Acceleration $$a = \frac{qE}{m} \tag{1}$$

$$y = \frac{1}{2}at^2 \tag{2}$$

$$x = vt \tag{3}$$

Combining (1), (2) and (3)

$$y = \frac{qEx^2}{2mv^2} \tag{4}$$

which is the equation to a parabola.

13.6 $$r = \frac{\sqrt{2mK}}{qB} = \frac{(2 \times 9.1 \times 10^{-31} \times 5 \times 10^{3} \times 1.6 \times 10^{-19})^{1/2}}{1.6 \times 10^{-19} \times 10^{-2}} = 0.0238\,\mathrm{m}$$

$$= 2.38\,\mathrm{cm}$$

13.7 $qE = q\,v\,B$

$$\therefore \qquad B = \frac{E}{v} = \frac{1500}{3000} = 0.5\,\text{T}$$

13.8 **(a)** $v = \dfrac{rqB}{m} = \dfrac{1.9 \times 1.6 \times 10^{-19} \times 3 \times 10^{-5}}{9.1 \times 10^{-31}} = 10^{7}\,\text{m/s}$

(b) $t = \dfrac{2\pi r}{v} = \dfrac{2\pi \times 1.9}{10^{7}} = 1.19 \times 10^{-6}\text{s} = 1.19\,\mu\text{s}$

13.9 $B = \dfrac{2\pi m f}{q} = \dfrac{2\pi \times 2.0141 \times 1.66 \times 10^{-27} \times 5 \times 10^{6}}{1.6 \times 10^{-19}} = 0.656\,\text{Wb/m}^2$

$$K = \frac{1}{2}\frac{q^2 r^2 B^2}{m} = \frac{1}{2} \times \frac{(1.6 \times 10^{-19})^2 (0.762)^2 (0.325)^2}{2.0141 \times 1.66 \times 10^{-27}} = 4.696 \times 10^{-13}\,\text{J}$$

$$= 2.9\,\text{MeV}$$

13.10 For deuterons

$$B = \frac{2\pi\ f m}{q} = \frac{2\pi \times 11.5 \times 10^{6} \times 2.014 \times 1.66 \times 10^{-27}}{1.6 \times 10^{-19}} = 1.509\,\text{Wb/m}^2$$

For protons

$$f = \frac{qB}{2\pi m} = \frac{1.6 \times 10^{-19} \times 1.509}{2\pi \times 1.66 \times 10^{-27}} = 2.316 \times 10^{7}\,\text{c/s} = 23.16\,\text{Mc/s}$$

13.11 **(a)** $f = \dfrac{Bq}{2\pi m} = \dfrac{1.5 \times 1.6 \times 10^{-19}}{2\pi \times 3.32 \times 10^{-27}} = 11.5 \times 10^{6}\,\text{c/s} = 11.5\,\text{Mc/s}$

(b) $K = \dfrac{1}{2}\dfrac{q^2 r^2 B^2}{m} = \dfrac{1}{2}\dfrac{(1.6 \times 10^{-19} \times 0.5 \times 1.5)^2}{3.32 \times 10^{-27}} = 21.69 \times 10^{-13}\,\text{J} =$ $13.56\,\text{MeV}$

13.12 $i = \dfrac{q}{t} = \dfrac{1.6 \times 10^{-19}}{1.5 \times 10^{-16}} = 1.06 \times 10^{-3}\,\text{A}$

13.13 **(i)** Magnetic force, $F_M = qVB$
Electric force, $F_E = qE$
For no deflection, $F_M = F_E$

$$\therefore \qquad qvB = qE$$

$$\therefore \qquad v = \frac{E}{B} = \frac{80 \times 10^{3}}{0.4} = 2 \times 10^{5}\,\text{m/s}$$

(ii) $\dfrac{mv^2}{r} = q\,v\,B$

$$\therefore \qquad \frac{q}{m} = \frac{v}{Br} = \frac{E}{B^2 r} = \frac{80 \times 10^{3}}{(0.4)^2 (1.14 \times 10^{-2})} = 4.38 \times 10^{7}\,\text{C/kg}$$

13.14 **(a)** $\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}$

(b) In the electric field energy acquired, $K = qV$

$$\therefore \qquad p = \sqrt{2mK} = \sqrt{2mqV}. \tag{1}$$

In the magnetic field

$$p = qBr \tag{2}$$

Combining (1) and (2)

$$r = \sqrt{\frac{2mV}{qB^2}}$$

13.15 Let the isotopes ^{235}U and ^{238}U be called 1 and 2, respectively. In the magnetic field the momenta are given by

$$p_1 = qBr_1, \ \mathrm{p}_2 = \mathrm{q}Br_2$$

$$p_2 - p_1 = \mathrm{q}B\,(r_2 - r_1) = 1.6 \times 10^{-19} \times 0.2 \times 2 \times 10^{-3} \tag{13.26}$$

$$= 6.4 \times 10^{-21}\,\mathrm{kg\,m/s}. \tag{1}$$

In the electric field

$$qV = \frac{p_1^2}{2m_1} = \frac{p_2^2}{2m_2} \tag{2}$$

$$\therefore \quad p_2 = p_1\sqrt{\frac{m_2}{m_1}} = p_1\sqrt{\frac{3.95 \times 10^{-25}}{3.90 \times 10^{-25}}} = 1.00639\,p_1. \tag{3}$$

Combining (1) and (3)

$$p_1 = 1.00159 \times 10^{-19}\,\mathrm{kg\,m/s}. \tag{4}$$

Substituting (4) into (2)

$$V = \frac{(10^{-19})^2}{2 \times 1.6 \times 10^{-19} \times 3.90 \times 10^{-25}} = 8 \times 10^4\,\mathrm{V}$$

13.16 **(a)** $F = q\,v \times B$
If B is perpendicular to v, then the particle would move in a circle.
Centripetal force = magnetic force

$$\frac{mv^2}{r} = qvB \to r = \frac{mv}{qB}$$

(b) $$\xi = -\frac{\mathrm{d}}{\mathrm{d}t}(N\phi) = -\frac{\mathrm{d}}{\mathrm{d}t}(NBA) = -NA\frac{\mathrm{d}B}{\mathrm{d}t}$$
$$= -NA\frac{\mathrm{d}}{\mathrm{d}t}(B_0\cos(15t)) = 15\,NAB_0\sin(15t)$$

13.17 **(a)** $$E = \frac{q}{4\pi\varepsilon_0 r}$$
$$\therefore \quad q = 4\pi\varepsilon_0 rE = 4\pi \times 8.85\times10^{-12}\times10^{-6}\times5.8\times10^{-3}$$
$$= 6.447\times10^{-19}\,\mathrm{C}$$

Number of electrons

$$n = \frac{q}{e} = \frac{6.447\times10^{-19}}{1.6\times10^{-19}} = 4.029 \text{ or } 4$$

(b) Minimum electric field E required to prevent the droplet from falling is conditioned by equating the electric force to the gravitational force:

$$qE = mg$$
$$\therefore \quad E = \frac{mg}{q} = \frac{4}{3}\pi r^3\frac{\rho g}{q} = \frac{4}{3}\pi\times\frac{\left(10^{-6}\right)^3\times1000\times9.8}{6.447\times10^{-19}}$$
$$= 6.37\times10^4\,\mathrm{V/m}$$

13.18 When a charged particle moves at an angle θ to the field direction, the particle will move in a helical path. The vector velocity v of the particle can be resolved into two components, one parallel to B and one perpendicular to it:

$$v_{\parallel} = v\cos\theta \text{ and } v_{\perp} = v\sin\theta \tag{1}$$

The parallel component determines the pitch of the helix, that is, the distance between the adjacent turns. The perpendicular component determines the radius r of the helix:

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2\times1\times1.6\times10^{-19}}{9.1\times10^{-31}}} = 5.93\times10^5\,\mathrm{m/s}$$
$$r = \frac{mv\,\sin\theta}{|q|\,B} = \frac{9.1\times10^{-31}\times5.93\times10^5\sin60^\circ}{1.6\times10^{-19}\times10^{-3}} = 2.92\times10^{-3}\mathrm{m}$$
$$= 2.92\,\mathrm{mm}$$

$$\text{Time period } T = \frac{2\pi\,m}{|q|\,B} = \frac{2\pi\times9.1\times10^{-31}}{1.6\times10^{-19}\times10^{-3}} = 3.57\times10^{-8}\,\mathrm{s}$$

$$\text{Pitch} = (v\cos\theta)T = 5.93\times10^5\times\cos60^\circ\times3.57\times10^{-8} = 0.1\,\mathrm{m}$$

13.19 **(i)** Choose the origin at o, Fig. 13.21. The electric field E acts along the y-direction perpendicular to the plates which are located in the x-direction. The electric force on the electron is directed along the y-axis and the magnetic force along the z-axis. If the component of initial velocity in the direction of B is zero then the path of electron will be contained entirely in the xy-plane.

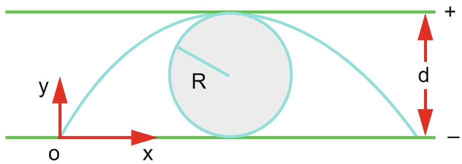


Fig. 13.21 Generating a cycloid

Writing Lorentz force $\boldsymbol{F} = q\boldsymbol{E} + qv\boldsymbol{B}$, in the component form

$$F_y = m\frac{\mathrm{d}v_y}{\mathrm{d}t} = qE - qBv_x \tag{1}$$

$$F_x = m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qBv_y \tag{2}$$

Writing for convenience

$$\omega = \frac{qB}{m} \text{ and } \gamma = \frac{E}{B} \tag{3}$$

Equations (1) and (2) can be rewritten as

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \omega\gamma - \omega v_x \tag{4}$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \omega v_y \tag{5}$$

Differentiating (4) and using (5)

$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\omega^2 v_y$$

$$\text{or} \qquad \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} + \omega^2 v_y = 0 \tag{6}$$

With the initial conditions $v_x = v_y = 0$, at $t = 0$, (6) has the solution

$$v_y = A \sin\omega t \tag{7}$$

where $A =$ constant:

$$\therefore \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = A\omega \cos\omega t = \omega\gamma - \omega v_x$$

At $t = 0$, $v_x = 0$

$$\therefore \quad A\omega = \omega\gamma \to A = \gamma$$

$$\therefore \quad v_y = \gamma \sin \omega t \tag{8}$$

Substituting (8) into (5), integrating and using the initial condition $v_x = 0$ at $t = 0$

$$v_x = \gamma(1 - \cos\, \omega t) \tag{9}$$

(ii) The coordinates x and y at any time t can be found out by integrating separately (8) and (9) with the initial condition $x = y = 0$ at $t = 0$

$$y = \frac{\gamma}{\omega}(1 - \cos \omega t) \tag{10}$$

$$x = \gamma \left(t - \frac{\sin \omega t}{\omega} \right) \tag{11}$$

Using (9) and (10) we get

$$v_x = \omega y \tag{12}$$

(iii) The energy of the particle is unaffected in the static magnetic field. Under the electric field in the y-direction the energy picked up will be

$$\frac{qVy}{d} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2\right)$$

$$\text{or} \qquad v_y^2 = \frac{2qVy}{md} - \omega^2 y^2 \tag{13}$$

where we have used (12).

13.20 Referring to Fig. 13.21 and setting $\theta = \omega t$ and $R = \gamma/\omega$ and (10) and (11) of prob. (13.19) we get the parametric equations of cycloid

$$y = R(1 - \cos \theta) \tag{1}$$

$$x = R(\theta - \sin \theta) \tag{2}$$

These equations define the path generated by a point on the circumference of a circle which rolls along the x-axis. The maximum displacement of electron along the y-axis is equal to the diameter of the rolling circle, that is, $2R$. Identifying $2R = d$

$$\frac{d}{2} = R = \frac{\gamma}{\omega} = \frac{E/B}{eB/m} = \frac{Em}{eB^2} \tag{3}$$

where we have set $q = e$, for the electron charge.

$$\text{Also} \qquad E = \frac{V}{d} \tag{4}$$

Using (4) in (3)

$$d^2 = \frac{2mV}{eB^2} \tag{5}$$

Thus the condition that the electrons are able to arrive at the positive plate is

$$d^2 < \frac{2mV}{eB^2}$$

13.3.2 Magnetic Induction

13.21 $B = \dfrac{\mu_0 i}{2\pi r}$

$$\therefore \quad i = \frac{2\pi Br}{\mu_0} = \frac{2\pi \times 10^{-6} \times 0.4}{4\pi \times 10^{-7}} = 2\,\text{A}$$

13.22 **(a)** Consider a typical current element dx. The magnitude of the contribution dB of this element to the magnetic field at P is found from Biot–Savart law and is given by (Fig. 13.22)

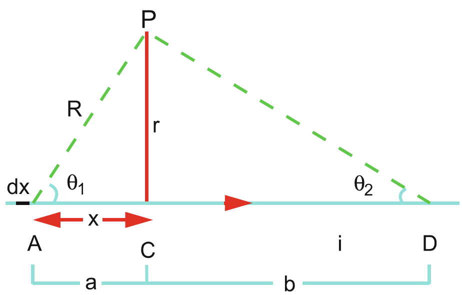


Fig. 13.22 Magnetic induction due to a current-carrying wire of finite length

$$\mathrm{d}B = \frac{\mu_0 i\, \mathrm{d}x\ \sin\theta}{4\pi\ R^2} \tag{1}$$

Since the direction of the contribution dB at point for all such elements is identical, i.e. at right angles to the plane of paper, the resultant field is obtained by integrating dB from A to D in (1). Writing $\sin\theta = \dfrac{r}{R}$

$$B = \int \mathrm{d}B = \frac{\mu_0 i r}{4\pi}\left[\int_{-a}^{b} \frac{\mathrm{d}x}{(x^2 + r^2)^{3/2}}\right] = \frac{\mu_0 i r}{4\pi r^2}\, \frac{x}{(x^2 + r^2)^{1/2}}\Bigg|_{-a}^{b}$$

$$\therefore \qquad B = \frac{\mu_0 i}{4\pi r}(\cos\theta_1 + \cos\theta_2) \tag{2}$$

(b) For infinite wire $\theta_1 \to 0$ and $\theta_2 \to 0$, in this limit (2) becomes

$$B = \frac{\mu_0 i}{2\pi r}$$

which is the expression for a long wire.

13.23 $B = \dfrac{\mu_0 i}{2r}$

$$\therefore \qquad i = \frac{2Br}{\mu_0} = \frac{2 \times 10^{-5} \times 0.5}{4\pi \times 10^{-7}} = 7.96\,\text{A}$$

13.24 Magnetic field B_1 due to current i in one segment is (Fig. 13.23)

$$B_1 = \frac{\mu_0 i}{4\pi R}(\cos\theta_1 + \cos\theta_2)$$

Putting $\theta_1 = \theta_2 = 45°$ and $R = a/2$

$$B_1 = \frac{\mu_0 i}{\sqrt{2}\pi a}$$

Fields due to four sides are equal and additive. Therefore net field

$$B = 4B_1 = \frac{2\sqrt{2}\mu_0 i}{\pi a}$$

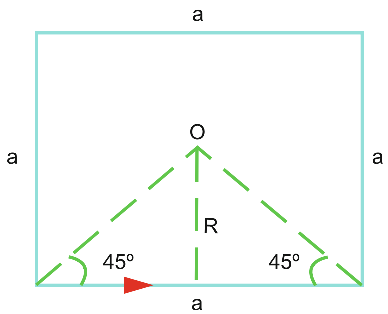


Fig. 13.23 Magnetic induction at the centre of a square conducting loop

13.25 The magnetic fields in the upper branch and lower branch act in the opposite direction. The current in the upper branch is $\dfrac{4I}{5}$ and in the lower branch is $\dfrac{I}{5}$. As the current is flowing through semicircles,

$$B_1 = \frac{\mu_0}{4a}\left(\frac{4I}{5}\right) = \frac{\mu_0 I}{5a}$$

$$B_2 = \frac{\mu_0}{4a}\left(\frac{I}{5}\right) = \frac{\mu_0 I}{20\,a}$$

Net field $B = B_1 - B_2 = \dfrac{3\mu_0 I}{20\,a}$

13.26 Field B at the centre is due to three-fourths of the circle (B_1) added to that due to the straight segment (B_2)

$$B_1 = \frac{3}{4} \times \frac{\mu_0 i}{2R}$$

$$B_2 = \frac{\mu_0 i}{4\pi d}(\cos\theta_1 + \cos\theta_2)$$

From the geometry of Fig. 13.24, $\theta_1 = \theta_2 = 45°$ and $d = \dfrac{R}{\sqrt{2}}$.

Then $$B = B_1 + B_2 = \frac{\mu_0\, i}{2\pi\, R}\left(1 + \frac{3\pi}{4}\right)$$

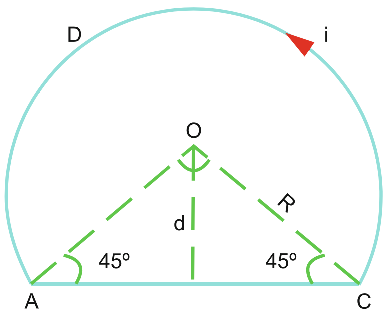


Fig. 13.24 Magnetic induction at the centre of a current-carrying wire made of three-fourths of a circle and a chord

13.27 **(a)** The magnetic induction due to straight wires is

$$B_1 = \frac{\mu_0\, i}{4\pi R} + \frac{\mu_0 i}{4\pi R} = \frac{\mu_0 i}{2\pi R} \tag{1}$$

because straight wires are of infinite length only on left side.
Induction at 0 due to semicircular portion is

$$B_2 = \frac{\mu_0 i}{4R} \tag{2}$$

Total magnetic induction

$$B = B_1 + B_2 = \frac{\mu_0 i}{2\pi R} + \frac{\mu_0 i}{4R} = \frac{\mu_0 i}{4\pi R}(2 + \pi)$$

(b) The straight portions of the wire do not contribute to the field at O as the current is directed towards C and makes an angle $\theta = 0°$, for which the Biot–Savart formula gives $B = 0$. Thus the entire induction comes from the semicircular portion of the wire for which $B = \dfrac{\mu_0 i}{4R}$.

13.28 **(a)** Let the angle θ be subtended at the centre by one side AC of a regular n-sided polygon, Fig. 13.25. Then

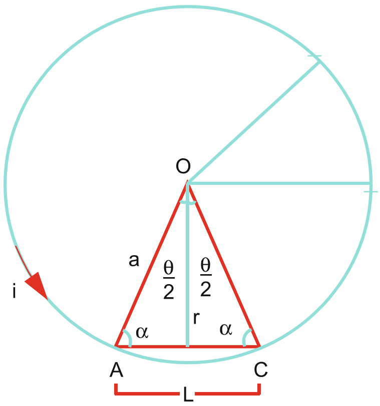


Fig. 13.25 Magnetic induction at the centre of a current-carrying regular n-sided polygon

$$\theta = \frac{2\pi}{n} \text{ or } \frac{\theta}{2} = \frac{\pi}{n} \tag{1}$$

The magnetic induction due to one side AC at the centre O is

$$B_1 = \frac{\mu_0 i}{4\pi r}(\cos\alpha + \cos\alpha) = \frac{\mu_0 i \, \cos\alpha}{2\pi \, r} \tag{2}$$

where r is the distance of O from AC.
The field B due to n sides will be additive and is given by

$$B = nB_1 = \frac{\mu_0 n i \, \cos\alpha}{2\pi \, r} \tag{3}$$

Now $r = a \sin\alpha$, so that in (3)

$$\frac{\cos\alpha}{r} = \frac{\cos\alpha}{a\sin\alpha} = \frac{1}{a}\cot\alpha = \frac{1}{a}\tan\left(\frac{\theta}{2}\right) = \frac{1}{a}\tan\left(\frac{\pi}{n}\right) \tag{4}$$

where we have used (1). Using (4) in (3)

$$B = \frac{\mu_0 n i}{2\pi \, a}\tan\left(\frac{\pi}{n}\right) \tag{5}$$

(b) In the limit $n \to \infty$, $\tan\left(\frac{\pi}{n}\right) \to \frac{\pi}{n}$ and (5) becomes

$$B = \frac{\mu_0 i}{2a}$$

an expression which is identical for B for a circular loop. This is reasonable since as $n \to \infty$, polygon $\to$ circle.

13.29 For square $l = 4a$ and for circle $l = 2\pi r$

$$\therefore \quad \frac{a}{r} = \frac{\pi}{2} \tag{1}$$

At the centre of the circle, $B_c = \frac{\mu_0 i}{2r}$.

At the centre of the square, $B_s = \frac{2\sqrt{2}\,\mu_0 i}{\pi a}$ via prob. (13.24).

$$\therefore \quad \frac{B_c}{B_s} = \frac{\pi}{4\sqrt{2}}\frac{a}{r} = \frac{\pi^2}{8\sqrt{2}} = 0.87$$

13.30 **(a)** The magnetic induction B at the centre of a circular wire is $B = \frac{\mu_0 i}{2r}$. Hence for the arc which subtends an angle θ at the centre

$$B = \frac{\mu_0 i}{2r}\frac{\theta}{2\pi} = \frac{\mu_0\, i\, \theta}{4\pi\, r}$$

Induction at C due to the inner arc is

$$B_1 = \frac{\mu_0 i\theta}{4\pi R_1}$$

and due to the outer arc

$$B_2 = -\frac{\mu_0 i\theta}{4\pi R_2}$$

The negative sign arises due to the fact that the current has reversed. As the radial part of the path points towards C, it does not contribute to B. Therefore, the resultant induction is

$$B = B_1 + B_2 = \frac{\mu_0 i\theta}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Note that for clockwise current we take B as positive and for counterclockwise we take B as negative.

(b) Put $\theta = \pi$ to obtain

$$B = \frac{\mu_0 i}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

13.31 The total induction is given by adding B_s due to the straight conductor which contributes on both sides of P and B_c due to the circular path, both being directed out of page (Fig. 13.9)

$$B_s = \frac{\mu_0 i}{2\pi R}$$
$$B_c = \frac{\mu_0 i}{2R}$$
$$B = B_s + B_c = \frac{\mu_0 i}{2\pi R}(\pi + 1)$$

13.32 $$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 i}{4\pi}\frac{\mathrm{d}\boldsymbol{s} \times \boldsymbol{r}}{r^3} \qquad \text{(Biot–Savart law)} \qquad (1)$$

The magnetic induction d$\boldsymbol{B}$ at P on the z-axis due to an element of length dl of the ring is shown in Fig. 13.26. Resolving d$\boldsymbol{B}$ along z-axis and perpendicular to it, and summing over all such elements it is seen that the perpendicular components vanish for reasons of symmetry and the parallel components get added up.
Writing dl for ds and noting that the angle between $\boldsymbol{R}$ and d$\boldsymbol{l}$ is 90°, (1) can be written as

$$\mathrm{d}\boldsymbol{B}_z = (\mathrm{d}\boldsymbol{B})\cos\alpha = \frac{(\mu_0\, i r \mathrm{d}l)\cos\alpha}{4\pi r^3} \qquad (2)$$

Writing dl = Rdϕ, where ϕ is the azimuth angle and r cos α = R, (2) becomes

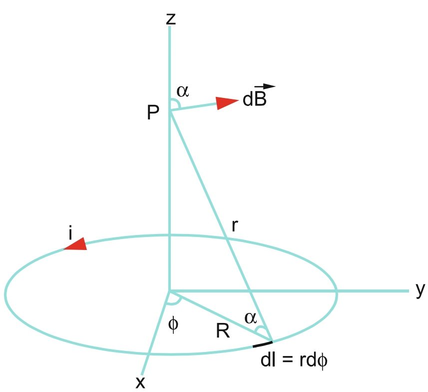


Fig. 13.26 Magnetic induction on the axis of a current-carrying ring

$$dB_z = \frac{\mu_0 i R^2 d\phi}{4\pi r^3} \tag{3}$$

Integrating

$$B_z = B = \int dB_z = \frac{\mu_0 i R^2}{4\pi r^3} \int_0^{2\pi} d\phi = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}} \tag{4}$$

13.33 $B = \frac{\mu_0 N i}{l} = \frac{4\pi \times 10^{-7} \times 500 \times 5}{1.0} = 3.14 \times 10^{-3}\,\mathrm{T}$

13.34 $B = \frac{\mu_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{d - x}\right]$

13.35 Apply Ampere's law inside the hollow cylindrical conductor

$$(B)(2\pi r) = \frac{\mu_0 i \pi (r^2 - a^2)}{\pi (b^2 - a^2)} \tag{1}$$

where the right-hand side includes only the fraction of the current that passes through the surface enclosed by the path of integration. Solving for B,

$$B = \frac{\mu_0 i}{2\pi r} \frac{(r^2 - a^2)}{(b^2 - a^2)}.$$

13.36 Magnetic field at P, due to loop A is

$$B_\mathrm{A} = \frac{\mu_0 I N R^2}{2(R^2 + x^2)^{3/2}} \tag{1}$$

where $x = \mathrm{AP} = \frac{R}{2}$. Similarly the magnetic field B_c due to the second loop is given by an identical expression with $x = \mathrm{CP} = \frac{R}{2}$. As the currents are in the opposite direction these two fields are added:

$$\therefore \qquad B = B_\mathrm{A} + B_\mathrm{C} = \frac{\mu_0 I N R^2}{(R^2 + x^2)^{3/2}} \tag{2}$$

Put $x = R/2$ to find

$$B = \frac{8 N \mu_0 I}{5^{3/2} R} \tag{3}$$

13.37 Consider a ring of radius r, width dr, concentric with the disc. The charge on the ring, Fig. 13.27

$$dq = q\,\frac{2\pi r\,dr}{\pi R^2} = \frac{2qr\,dr}{R^2} \tag{1}$$

The elementary current due to rotation of charge with frequency f is

$$di = f\,dq = \frac{\omega}{2\pi}\,\frac{2qr\,dr}{R^2} = \frac{\omega\,qr\,dr}{\pi\,R^2} \tag{2}$$

The induction at the centre due to the current in the ring is

$$dB = \frac{\mu_0 di}{2r} = \frac{\mu_0}{2r}\,\frac{\omega\,qr\,dr}{\pi\,R^2} = \frac{\mu_0\omega\,q\,dr}{2\pi\,R^2} \tag{3}$$

Total induction from the rotating disc

$$B = \int dB = \int_0^R \frac{\mu_0\omega\,q\,dr}{2\,\pi\,R^2} = \frac{\mu_0\omega\,q}{2\,\pi\,R} \tag{4}$$

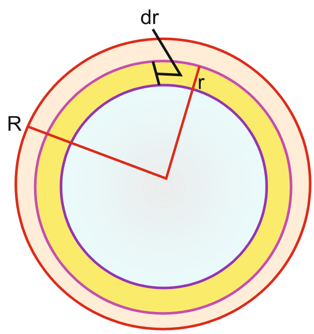


Fig. 13.27 Magnetic induction at the centre of a charged rotating disc

13.38 The field at any point P_1 at distance x from P, the middle point will be

$$B = \frac{\mu_0\,iNR^2}{2}\left\{\frac{1}{\left[R^2 + \left(\frac{1}{2}R + x\right)^2\right]^{3/2}} + \frac{1}{\left[R^2 + \left(\frac{1}{2}R - x\right)^2\right]^{3/2}}\right\} \tag{1}$$

by prob. (13.36).

Differentiate B with respect to x and evaluate $\left(\frac{\partial B}{\partial x}\right)_{x=0}$ to find the first derivative zero. Differentiate B once again to find $\left(\frac{\partial^2 B}{\partial x^2}\right)_{x=0} = 0$. Thus the field around P is seen to be fairly uniform.

13.39 **(i)** $$B = \frac{\mu_0 i}{2\pi r} \qquad (r > R) \qquad (1)$$
$$= \frac{4\pi \times 10^{-7} \times 100}{2\pi \times 1.0} = 2 \times 10^{-5}\,\mathrm{T}$$

(ii) $$B = \frac{\mu_0 i r}{2\pi R^2} \qquad (r < R) \qquad (2)$$
$$= \frac{4\pi \times 10^{-7} \times 100 \times 6 \times 10^{-3}}{2\pi \times (3 \times 10^{-2})^2} = 1.33 \times 10^{-6}\,\mathrm{T}$$

13.40 **(a)** $\Sigma B_t \Delta l = \mu_0 i$ (Ampere's law)

$$\Sigma B\Delta l = B\Sigma \Delta l = B \cdot 2\pi r = \mu_0 i$$

where we enclose the current i by going round once the circle of radius r. The magnetic induction will be tangential to the circle and the summation is simply the circumference of the circle, Fig. 13.28a:

$$(B)(2\pi r) = \mu_0 i$$
$$\therefore \qquad B = \frac{\mu_0 i}{2\pi r} \qquad (r > R)$$

(b) Consider a circular path C at distance $r < R$, Fig. 13.28b. The current i_0 inside a cross-section of radius r is proportional to the cross-sectional area

$$i_0 = i\frac{\pi r^2}{\pi R^2} = \frac{i r^2}{R^2}$$

By Ampere's law

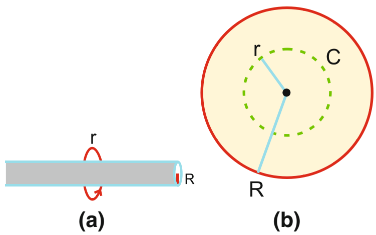


Fig. 13.28 Magnetic induction due to a current-carrying cylinder

$$\Sigma B_t\,\Delta l = (B)\,(2\pi\, r) = \mu_0 i_0 = \mu_0 i\,\frac{r^2}{R^2}$$

$$\therefore \qquad B = \frac{\mu_0 i r}{2\pi\, R^2} \qquad (r < R)$$

13.41 Let the radius of the inner arc of the loop be r and that of the outer arc $2r$. Both the arcs are quarter of a circle. The straight portions do not contribute to B as their directions pass through P. As the currents in the two arcs flow in the opposite sense, B will be down due to current in the outer arc and up due to the current in the inner wire:

$$\therefore \qquad B_{\text{net}} = \frac{1}{4}\frac{\mu_0 I}{2r} - \frac{1}{4}\frac{\mu_0 I}{2\times 2r} = \frac{\mu_0 I}{16\,r}\text{ down}$$

13.42 By prob. (13.22) at P, Fig. 13.29

$$B = \frac{\mu_0 I}{4\pi x}(\cos\,\theta_1 + \cos\theta_2), \tag{1}$$

put $\theta_1 = \theta_2$ then

$$\cos\theta_1 = \cos\theta_2 = \frac{L/2}{\sqrt{(L/2)^2 + x^2}}$$

So that (1) becomes

$$B = \frac{\mu_0 I}{4\pi x}\frac{L}{\sqrt{(L/2)^2 + x^2}} \tag{2}$$

(a) Let the square be of side $a = l/4$. The distance of the centre of square from the side is $a/2$. Put $x = a/2$ and $L = a = l/4$ in (2) to find for one side

$$B_1 = \frac{2\sqrt{2}\mu_0 I}{\pi l}$$

As there are four equal sides, $B = 4B_1 = \dfrac{8\sqrt{2}\mu_0 I}{\pi l}\cdot$

The B-field will be perpendicular to the plane containing the wire and the field point.

(b) Let each side of the equilateral triangle be $a = l/3$. Distance of the centre of the triangle from any side is $x = a/2\sqrt{3}$. Put $L = a = l/3$ and $x = a/2\sqrt{3}$ in (1) to find for one side $B_1 = \dfrac{9\mu_0 I}{2\pi l}$. Hence for three sides

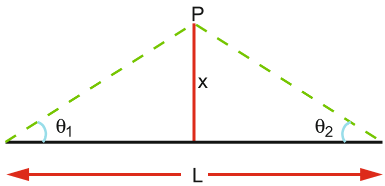


Fig. 13.29 Magnetic field due to a current-carrying straight wire of finite length

$$B = 3B_1 = \frac{27\mu_0 I}{2\pi l}$$

$$\therefore \quad \frac{B\text{ (square)}}{B\text{ (triangle)}} = \frac{27/2}{8\sqrt{2}} = 1.19$$

13.43 By prob. (13.36) $B = \dfrac{8N\mu_0 I}{5^{3/2}R} = \dfrac{(8)(100)(4\pi \times 10^{-7})(2)}{5^{3/2}(0.2)} \simeq 9 \times 10^{-4}$ T

13.44 $\oint \boldsymbol{B}\,\mathrm{d}l = \mu_0 i$ (Ampere's law)

By prob. (13.43)

$$B = \frac{\mu_0 I r}{2\pi R^2}\ (r < R) \tag{1}$$

$$\therefore \quad \text{For } r = \frac{R}{2}, \quad B = \frac{\mu_0 I}{4\pi R} \tag{2}$$

$$\text{Further } B = \frac{\mu_0 I}{2\pi r}\ (r > R). \tag{3}$$

Equating (2) and (3) we find $r = 2R$. Thus at $r = 2R$, the magnetic field is the same as at $r = R/2$.

13.45 **(a)** In the absence of magnetic material the number of flux linkages $N\phi_B$(N being the number of turns) is proportional to the current

$$N\phi_{\mathrm{B}} = Li \tag{1}$$

If n is the number of turns per unit length, A the cross-sectional area and l the length of the solenoid and B the magnetic induction

$$N\phi_{\mathrm{B}} = (nl)(BA) \tag{2}$$

By Ampere's theorem

$$B = \mu_0 n i \tag{3}$$

Combining (1), (2) and (3)

$$L = \mu_0 n^2 A l \tag{4}$$

(b) $$B = \mu_0 ni = \frac{\mu_0 Ni}{l}$$

$$U_B = u_B Al = \frac{B^2}{2\mu_0} \cdot Al = \frac{(\mu_0 Ni)^2 Al}{2\mu_0 l^2} = \frac{\mu_0 N^2 i^2 \pi r^2}{2l}$$

$$= \frac{(4\pi \times 10^{-7})\,(100)^2 (5)^2 \pi\,(0.01)^2}{2 \times 0.1} = 4.93 \times 10^{-4}\,\text{J}$$

(c) $$\phi = BA = \frac{\mu_0 Ni}{l} \cdot \pi r^2 = \frac{4\pi \times 10^{-7} \times 100 \times 5 \times \pi \times (0.1)^2}{0.1}$$

$$= 1.972 \times 10^{-4}\text{Wb}$$

$$\xi = \frac{-\Delta\phi}{\Delta t} = \frac{-1.972 \times 10^{-4}}{5} = -3.94 \times 10^{-5}\,\text{V}$$

13.46 **(a)** $$B = \frac{\mu_0 i}{2\pi r} = \frac{4\pi \times 10^{-7} \times 3}{2\pi \times 50 \times 10^{-3}} = 1.2 \times 10^{-7}\,\text{T}$$

(b) In the vector form the Biot and Savart law can be written as

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 i}{4\pi} \frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3}$$

(i) $$\mathrm{d}\boldsymbol{B} = \frac{(4\pi \times 10^{-7})(2)(2\hat{k} \times 3\hat{i})}{4\pi \qquad 3^3} = 4.44 \times 10^{-8}\hat{j} \qquad (\because\ \hat{k} \times \hat{i} = \hat{j})$$

Thus $\mathrm{d}B = 4.44 \times 10^{-8}$ T along positive y-axis

(ii) $$\mathrm{d}\boldsymbol{B} = \frac{(4\pi \times 10^{-7})(2)(2\hat{k} \times (-6\hat{i}))}{4\pi \qquad 6^3} = -1.11 \times 10^{-8}\hat{j}$$

Thus $\mathrm{d}\boldsymbol{B} = 1.11 \times 10^{-8}$ T along negative y-axis

(iii) $\mathrm{d}\boldsymbol{B} = 0\ (\because\ \hat{k} \times \hat{k} = 0)$

(iv) $$\mathrm{d}\boldsymbol{B} = \frac{(4\pi \times 10^{-7})(2)(2\hat{k} \times 3\hat{j})}{4\pi \qquad 3^3} = -4.44 \times 10^{-8}\hat{i}$$

$\mathrm{d}\boldsymbol{B} = 4.44 \times 10^{-8}$ T along negative x-axis.

13.47 $H = n_0 i = \dfrac{Ni}{2\pi r} = \dfrac{100 \times 2}{2\pi \times 0.1} = 318.47$ A/m for both vacuum and material.

$$B = K\mu_0 H = 1 \times 4\pi \times 10^{-7} \times 318.47 = 4 \times 10^{-4}\,\text{T (vacuum)}$$

$$B = 500 \times 4\pi \times 10^{-7} \times 318.47 = 0.2\,\text{T (material)}$$

$$M = \frac{B - \mu_0 H}{\mu_0} = 0 \qquad \text{(vacuum)}$$

$$M = \frac{B}{\mu_0} - H = \frac{0.2}{4\pi \times 10^{-7}} - 318 = 1.59 \times 10^5 \qquad \text{(material)}$$

13.48 Use the formula for $B(z)$ on the axis of a circular coil of radius r carrying current i:

$$B(z) = \frac{\mu_0 i r^2}{2(r^2 + z^2)^{3/2}} \tag{1}$$

Use the following values:
$B = 6 \times 10^{-5}$ T, $\mu_0 = 4\pi \times 10^{-7}$ A/m, $r = 10^6$ m and z $= 6.4 \times 10^6$ m (distance of the pole from earth's centre) and solve for the current i. We find $i = 2.6 \times 10^{10}$ A. Thus the order of magnitude of current is 10^{10} A.

13.49 The induction midway between Helmholtz coils is (prob. 13.36)

$$B = \frac{8N\mu_0 I}{5^{3/2} R} \tag{1}$$

Given $N = 50$, $I = 10$ A, $R = 0.5$ m and $\mu_0 = 4\pi \times 10^{-7}$ H/m

$$\therefore \qquad B = 8.94 \times 10^{-4}\,\text{T} \tag{2}$$

Emf generated between the centre and the rim of the disc is

$$\xi = \pi r^2 B f = \pi \times (0.1)^2 \times 8.94 \times 10^{-4} \times 16.66 = 468 \times 10^{-6}\,\text{V} = 468\,\mu\,\text{V}.$$

13.50 Given

$$l = 1.0\,\text{m},\ v = 3\hat{i} + 2\hat{j} + 3\hat{k},\ B = \hat{i} + 2\hat{j} + 3\hat{k}$$

Voltage developed

$$\xi = |\boldsymbol{v} \times \boldsymbol{B}|\, l \sin\phi = |\boldsymbol{v}|\,|B|\,(\sin\theta) l \sin\phi$$

where θ is the angle between v and B, ϕ is the angle which l makes with B:

$$|v| = \left(3^2 + 2^2 + 1^2\right)^{1/2} = \sqrt{14}$$
$$|\text{B}| = \left(1^2 + 2^2 + 3^2\right)^{1/2} = \sqrt{14}$$
$$\cos\theta = \frac{v \cdot B}{|v|\,|B|} = \frac{3 + 4 + 3}{\left(\sqrt{14}\right)\left(\sqrt{14}\right)} = \frac{5}{7}$$
$$\therefore \qquad \sin\theta = 0.4898$$
$$\therefore \qquad \xi = \left(\sqrt{14}\right)\left(\sqrt{14}\right)(0.4898)\ (1)\ \sin\phi = 6.857\ \sin\phi$$

ξ will be maximum for $\phi = 90°$ and zero for $\phi = 0$ or $180°$.

13.51 The motion of the proton is equivalent to a current. The current density is given by

$$J = e\boldsymbol{v} \tag{1}$$

The magnetic field due to a current-carrying circuit is given by the Biot–Savart law

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi}\left(\frac{\mathrm{d}l' \times \boldsymbol{R}}{R^3}\right) \tag{2}$$

When $\mathrm{d}l'$ is the circuit element, $\boldsymbol{R}$ is the vector which points from $\mathrm{d}l'$ to the field point. As there is only one proton there is no need to integrate to find B. Replacing the current by the current density J, the Biot–Savart law is modified as

$$B = \frac{\mu_0}{4\pi}\left(\frac{J \times \boldsymbol{R}}{R^3}\right) \tag{3}$$

Substituting (1) into (3)

$$B = \frac{\mu_0 e}{4\pi}\left(\frac{v \times \boldsymbol{R}}{R^3}\right)$$

$$\boldsymbol{R} = \left(\hat{i} + 2\hat{j}\right)\ \mathrm{m}$$

$$v = \left(\hat{i} + 3\hat{j}\right)\ 10^4\ \mathrm{m/s}$$

$$\boldsymbol{v} \times \boldsymbol{R} = 10^4 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -10^4\hat{k}$$

$$B = \frac{4\pi \times 10^{-7}}{4\pi} \times 1.6 \times 10^{-19} \times \frac{\left(-10^4\hat{k}\right)}{\left(\sqrt{5}\right)^3}$$

$$= 1.43 \times 10^{-23}\hat{k}\ \mathrm{T}$$

13.3.3 Magnetic Force

13.52 $$\frac{F}{l} = \frac{\mu_0 i_1 i_2}{2\pi\, d} = \frac{4\pi \times 10^{-7} \times 2 \times 3}{2\pi \times 0.05} = 2.4 \times 10^{-5}\ \mathrm{N}$$

13.53 $$\frac{F}{l} = \frac{\mu_0 i_1 i_2}{2\,\pi\, d}$$

$$\therefore \qquad i_2 = \frac{2\pi d}{\mu_0 i_1}\left(\frac{F}{l}\right) = \frac{2\pi \times 0.2 \times 10^{-5}}{4\pi \times 10^{-7} \times 10} = 1.0\ \mathrm{A}$$

The currents are parallel.

13.54 For equilibrium, magnetic force = gravitational force.

$$F = \frac{\mu_0\, l i_1 i_2}{2\pi\, d} = \text{mg}$$

$$\therefore \quad d = \frac{\mu_0\, l i_1 i_2}{2\pi mg} = \frac{4\pi \times 10^{-7} \times 2 \times 10 \times 15}{2\pi \times 4 \times 10^{-3} \times 9.8} = 1.53 \times 10^{-3}\,\text{m}$$

$$= 1.53\,\text{mm}$$

13.55 **(a)** Net force per metre on the outer wire

$$\frac{F}{l} = \frac{F_1}{l} + \frac{F_2}{l} = \frac{\mu_0\, i_1 i_3}{2\pi\, d} + \frac{\mu_0\, i_2 i_3}{2\pi \times 2d}$$

$$= \frac{\mu_0\, i_3(2i_1 + i_2)}{4\pi\, d} = \frac{4\pi \times 10^{-7} \times 20(2 \times 20 + 20)}{4\pi \times 0.1} = 1.2 \times 10^{-3}\,\text{N}$$

(b) Zero

13.56 $F = ilB + (i)(\pi R)B + ilB = i(2l + \pi R)B.$

13.57 **(i)** The force on the horizontal segments is zero as they are perpendicular to the straight wire. For the vertical segments the force is repulsive for antiparallel currents and attractive for parallel currents, the magnitude being

$$F = \frac{\mu_0 l i_1 i_2}{2\pi d} \tag{1}$$

where d is the distance of separation.
Force on the nearer vertical segment

$$F_1 = -\frac{(4\pi \times 10^{-7})(0.1)(20)(5)}{(2\pi)(0.02)} = -1 \times 10^{-4}\,\text{N}$$

Force on the farther vertical segment

$$F_2 = +\frac{(4\pi \times 10^{-7})(0.1)(20)(5)}{(2\pi)(0.07)} = +2.86 \times 10^{-5}\,\text{N}$$

(ii) The net force on the coil, $F_{\text{net}} = F_1 + F_2 = -7.14 \times 10^{-5}\,\text{N}$

13.58 **(a)** Wire PQ will produce a field of induction B_1 at the segment AB of the coil. The magnitude of B_1 will be

$$B_1 = \frac{\mu_0 I_1}{2\pi\, d} \tag{1}$$

The right-hand rule shows that the direction of B_1 at the segment AB is down, wire AB, which carries a current I_2 finds itself immersed in an external field of magnetic induction B_1. A length l of the segment AB will experience a sideway magnetic force equal to $i\boldsymbol{l} \times \boldsymbol{B}$ whose magnitude is

$$F_2 = I_2 l B_1 = \frac{\mu_0 l I_1 I_2}{2\pi d} \tag{2}$$

The vector rule of signs shows that F_2 lies in the plane of the coil and points to the right. A similar reasoning shows that the force on the segment CD will be

$$F_3 = \frac{\mu_0 l I_1 I_2}{2\pi\ (d + l)} \tag{3}$$

to the left in the plane of the coil. There is no force on the segments BC and AD as they are perpendicular on PQ:

$$F_{\text{net}} = \frac{\mu_0 l I_1 I_2}{2\pi} \left[\frac{1}{d} - \frac{1}{d + l} \right] \tag{4}$$

to the right

$$F_{\text{net}} = \frac{(4\pi\ \times 10^{-7})(0.05)(1)(4)(10)}{2\pi} \left[\frac{1}{0.1} - \frac{1}{0.1 + 0.05} \right]$$
$$= 1.335 \times 10^{-5}\text{N}$$

where we have multiplied (4) by 10 for the number of coils.

13.59 **(a)** The flux ϕ_B enclosed by the loop, Fig. 13.30, is

$$\phi_B = Blx \tag{1}$$

where lx is the area of that part of the loop in which B is not zero. By Faraday's law

$$\xi = -\frac{\mathrm{d}\phi_\mathrm{B}}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(Blx) = -Bl\frac{\mathrm{d}x}{\mathrm{d}t} = Bl\,v \tag{2}$$

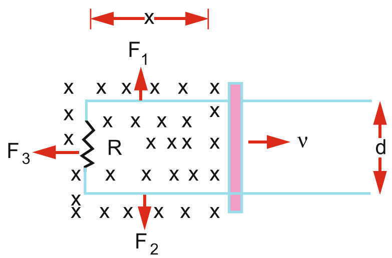


Fig. 13.30

where the speed $v = -\mathrm{d}x/\mathrm{d}t$ for the speed with which the connecting rod is pulled out of the magnetic field. The emf Blv sets up a current in the loop given by

$$i = \frac{\xi}{R} = \frac{Blv}{R} \tag{3}$$

(b) The total power delivered to the resistor is just the Joule heat given by

$$P_I = i^2 R = \frac{B^2 l^2 v^2}{R}$$

Note that energy conservation tells us that for steady motion of the rod the external agent must provide power equal to the Joule heat. That this is so is borne out from

$$P = F_3 v = \frac{B^2 l^2 v^2}{R} \cdot$$

(c) The current the loop produces forces F_1, F_2 and F_3 on the three sides of the loop in accordance with

$$F = i\boldsymbol{l} \times \boldsymbol{B} \tag{4}$$

Because F_1 and F_2 are equal and opposite, they cancel each other, while F_3 which opposes the motion of the sliding rod is given by (4) and (3) in magnitude as

$$F_1 = ilB \sin 90° = \frac{B^2 l^2 v}{R} \tag{5}$$

13.3.4 Magnetic Energy, Magnetic Dipole Moment

13.60 Elementary magnetic moment

$$\mathrm{d}\mu = (\mathrm{d}i)(\mathrm{d}A) = \left(\frac{\omega q r \mathrm{d}r}{\pi R^2}\right)(\pi r^2) = \frac{\omega q r^3 \mathrm{d}r}{R^2}$$

where $\mathrm{d}A$ is the area enclosed by r and the value of $\mathrm{d}i$ is used from (2) of prob. (13.37).
Therefore the magnetic moment of the disc is

$$\mu = \int \mathrm{d}\mu = \int_0^R \frac{\omega q r^3 \mathrm{d}r}{R^2} = \frac{\omega q R^2}{4}$$

13.61 Magnetic moment

$$\mu = NiA = (1)(i)(\pi r^2)$$

$$\therefore \quad i = \frac{\mu}{\pi r^2} = \frac{6.4 \times 10^{21}}{\pi\,(6.4 \times 10^6)^2} = 4.98 \times 10^7\,\mathrm{A}$$

13.62 Magnetic energy density

$$u_\mathrm{B} = \frac{1}{2\mu_0} B^2$$

But $\quad B = \dfrac{\mu_0 i}{2r}$

$$\therefore \quad u_\mathrm{B} = \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2r}\right)^2 = \frac{1}{8}\,\frac{\mu_0 i^2}{r^2} = \frac{4\pi \times 10^{-7} \times (100)^2}{8 \times (0.1)^2}$$

$$= 0.157\,\mathrm{J/m^3}$$

13.63 $u_\mathrm{B} = \dfrac{1}{2\mu_0} B^2 = \dfrac{(13.5)^2}{2 \times 4\pi \times 10^{-7}} = 7.25 \times 10^7\,\mathrm{J/m^3}$

13.64 $\tau_{\max} = i\mathrm{AB} = (i)\,(\pi r^2)\,B$

But $\quad l = 2\pi r \to r = \dfrac{l}{2\pi}$

$$\therefore \quad \tau_{\max} = \pi i \left(\frac{l}{2\pi}\right)^2 B = \frac{l^2 i B}{4\pi}$$

13.65 Let the sphere of radius R rotate about the z-axis, Fig. 13.31. Consider a spherical shell of radius $r (r < R)$ concentric with the sphere. Consider a volume element symmetrical about the z-axis.

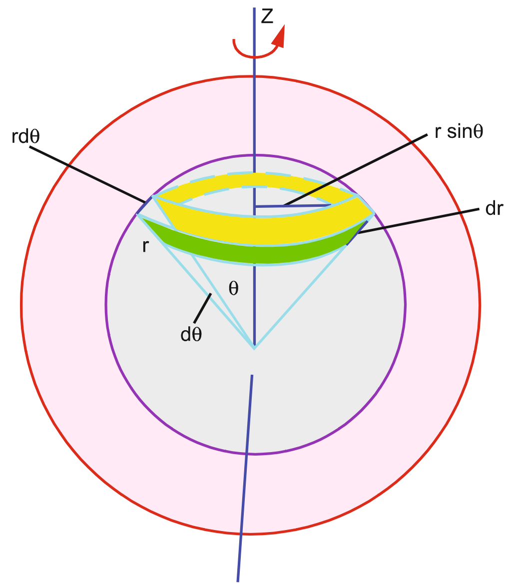
$$\mathrm{d}V = 2\pi r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r \tag{1}$$

where θ is the polar angle and r is the distance of the volume element from the centre. The charge dq residing in the volume element is

$$\mathrm{d}q = \frac{3q}{4\pi R^3}\,\mathrm{d}V \tag{2}$$

The current due to rotation of charge is

Fig. 13.31 Magnetic moment due to a rotating charged sphere



$$di = \frac{\omega}{2\pi}\, dq = \frac{3q\omega}{4\pi}\,\frac{r^2 \sin\theta\, d\theta\, dr}{R^3} \tag{3}$$

where we have used (2) and (1).
The magnetic moment due to di is

$$d\mu = (di)(dA) = \left(\frac{3q\,\omega r^2\, \sin\theta\, d\theta\, dr}{4\pi\, R^3}\right)(\pi\, r^2 \sin^2\theta)$$
$$= \frac{3}{4}\,\frac{\omega\, q r^4 dr\, \sin^3\theta\, d\theta}{R^3} \tag{4}$$

where dA is the area enclosed by the circle of radius $r\,\sin\theta$.
Total magnetic moment

$$\mu = \int d\mu = \frac{3}{4}\,\frac{\omega\, q}{R^3}\int_0^R r^4 dr \int_0^\pi \sin^3\theta\, d\theta = \frac{3}{4}\,\frac{\omega\, q}{R^3}\left(\frac{R^5}{5}\right)\left(\frac{4}{3}\right) = \frac{\omega\, q R^2}{5} \tag{5}$$

The angular momentum of a sphere about a diameter is

$$L = I\,\omega = \frac{2}{5}\, m\, R^2\omega \tag{6}$$

$$\therefore \qquad \frac{\mu}{L} = \frac{q}{2m} \tag{7}$$

13.66 $U = -p \cdot E = -p\,E\,\cos\theta = -(1.6\times10^{-29})(1000)\cos 30° = -1.38\times 10^{-26}$ J.

13.3.5 Faraday's Law

13.67 $$\frac{dr}{dt}\,\alpha\, t \qquad \text{(by problem)} \tag{1}$$

$$\xi = -\frac{d\phi}{dt} = \frac{d}{dt}(BA) \qquad \text{(by Faraday's law)} \tag{2}$$

$$\text{But} \qquad B \propto \frac{1}{r} \text{ and } A = \pi r^2 \tag{3}$$

$$\therefore \qquad \xi \propto \frac{dr}{dt} \tag{4}$$

$$\text{or} \qquad \xi \propto t \tag{5}$$

where we have used (1)

13.68 $0.4\,\text{Oe} = 0.4\times 80\,\text{A/m} = 32\,\text{A/m}$

$$B = \mu_0 H = 4\pi\times10^{-7}\times 32 = 4.02\times10^{-5}\,\text{T}$$
$$v = 720\,\text{km/h} = 200\,\text{m/s}$$
$$\xi = Blv = 4.02\times10^{-5}\times 30\times 200 = 0.24\,\text{V}$$

13.69 $V = \pi r^2 Bf = \pi(0.1)^2\times 2\times10^{-5}\times 5 = 3.14\times10^{-6}\,\text{V}$

13.70 $R = \dfrac{BAN}{q} = \dfrac{0.1\times 0.001\times 30}{10^{-5}} = 300\,\Omega$

13.71 $\xi = -\dfrac{\Delta\phi}{\Delta t} = \dfrac{A\Delta B}{\Delta t} = \dfrac{0.2\times0.25}{10^{-4}} = 500\,\text{V}$

$$i = \frac{\xi}{R} = \frac{500}{20} = 25\,\text{A}$$

13.72 The amplitude of the induced voltage

$$\xi_0 = \omega BA = 2\pi f BA = 2\pi\times 40\times 0.5\times(0.25)^2 = 7.85\,\text{V}$$

Amplitude of the induced current

$$I_0 = \frac{\xi_0}{R} = \frac{7.85}{4} = 1.96\,\text{A}$$

13.73 $\xi = iR = 0.4\times 5 = 2\,\text{V}$

$$\xi = vlB$$

$$\therefore \quad v = \frac{\xi}{lB} = \frac{2}{0.2 \times 1.0} = 10\,\mathrm{m/s}$$

13.74 $\xi = -\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = -\dfrac{A\mathrm{d}B}{\mathrm{d}t}$

A = area of the triangle sandwiched between the ends of the rod and the radii connecting the centre with the ends, Fig. 13.18:

$$A = \frac{1}{2}\,(\text{base})(\text{altitude}) = \frac{1}{2}l\sqrt{R^2 - (l/2)^2}$$

$$\text{Thus} \quad \xi = -\frac{\mathrm{d}B}{\mathrm{d}t}\,\frac{l}{2}\sqrt{R^2 - (l/2)^2}$$

13.75 Gravitational force on the loop

$$F_\mathrm{g} = mg\,\sin\theta$$

Magnetic force on the loop

$$F_\mathrm{m} = (B\,\cos\theta)\,il = B\,\cos\theta\,\frac{\xi\,l}{R}$$

$$= \frac{B\,\cos\theta\,l}{R}\,v\,B\cos\theta \cdot l = \frac{vB^2l^2\cos^2\theta}{R}$$

For steady speed, $F_{net} = F_\mathrm{m} - F_\mathrm{g} = 0$

$$\therefore \quad \frac{v\,B^2l^2\cos^2\theta}{R} - mg\,\sin\theta = 0$$

$$\text{or} \quad v = \frac{mg\,R\,\sin\theta}{B^2l^2\cos^2\theta}$$

13.76 $\xi = \pi R^2 Bf$

$$\therefore \quad B = \frac{\xi}{\pi\,R^2 f} = \frac{6.28 \times 10^{-3}}{3.14 \times (0.1)^2\,(1200/60)} = 0.01\,\mathrm{T}$$

13.77

$$\begin{aligned}
[\text{emf}] &= [\text{electric field}][\text{distance}] \\
&= [\text{force/charge}][\text{distance}] \\
&= [MLT^{-2}\,Q^{-1}][L] = [ML^2T^{-2}Q^{-1}] \\
[\mathrm{d}\phi_B/\mathrm{d}t] &= [\phi_B][T^{-1}] = [B][\text{area}][T^{-1}] \\
&= [\text{force/(velocity) charge}][L^2][T^{-1}] \\
&= [MLT^{-2}/LT^{-1}Q][L^2][T^{-1}]
\end{aligned}$$

$$= [ML^2T^{-2}Q^{-1}]$$

$$\therefore \qquad [\text{emf}] = [\mathrm{d}\phi_B/\mathrm{d}t]$$

13.78 The flux through the coil

$$\phi = B \cdot A = 15 \times 10^{-4} \times 3 \times 10^{-12}\ \sin(4 \times 10^6 t)$$

$$\xi = -N\frac{\mathrm{d}\phi}{\mathrm{d}t} = -200 \times 15 \times 10^{-4} \times 3 \times 4 \times 10^6 \times 10^{-12} \cos(4 \times 10^6 t)$$

$$= -3.6 \times 10^{-6} \cos(4 \times 10^6\, t)\ \mathrm{V}$$

13.79 $\xi \propto \dfrac{\mathrm{d}\phi}{\mathrm{d}t}$ or $\propto \dfrac{\mathrm{d}B}{\mathrm{d}t}$

$$B = B_0 \sin(\omega t + \phi)$$

$$\therefore \qquad \xi \propto \omega B_0 \cos(\omega t + \phi)$$

$$\therefore \qquad \frac{\xi_{\max}\,(\text{television})}{\xi_{\max}\,(\text{radio})} = \frac{\omega_1}{\omega_2} = \frac{f_1}{f_2} = \frac{100}{1} = 100$$

$\because$ B_0 is the same for both the waves.

13.80 The flux ϕ_B enclosed by a loop of area A is given by $\phi_B = BA$, where B is the magnetic field. Faraday's law of induction says that the induced emf ξ is a circuit equal to the negative rate at which the flux through the circuit is changing. In symbols $\xi = -\mathrm{d}\phi_B/\mathrm{d}t$, the current being $I = \xi/R$, where R is the resistance. The magnetic force on a straight wire is given by $F = i\boldsymbol{l} \times \boldsymbol{B}$:

$$\phi_B = BA = Blx\,(B \perp l)$$

By Faraday's law

$$\xi = -\frac{\mathrm{d}\varphi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(Blx) = -Bl\frac{\mathrm{d}x}{\mathrm{d}t} = Blv$$

$$i = \frac{\xi}{R} = \frac{Blv}{R}$$

$$\therefore \qquad \text{Force } F = ilB = \frac{B^2l^2v}{R}$$

If the magnetic force acts as a resisting force then equation of motion will be

$$ma = -\frac{B^2l^2v}{R}$$

$$\text{or} \qquad m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{B^2l^2v}{R}$$

$$\therefore \quad dv = -\left(\frac{B^2l^2}{mR}\right) ds$$

$$v = \int dv = -\frac{B^2l^2s}{mR} + C$$

where C = constant of integration.
When $s = 0$, $v = u$

$$\therefore \quad c = u$$

$$\therefore \quad s = \frac{(u-v)mR}{B^2l^2}$$

$$\therefore \quad s_{max} = \frac{u\,m\,R}{B^2l^2}$$

where we have put $v = 0$.

13.81 $\xi = -N\dfrac{d\phi}{dt} = -N\dfrac{d}{dt}(BA) = -N\pi r^2\dfrac{dB}{dt}$ (Faraday's law)

$$= -100 \times \pi(0.1)^2(0.1) = 0.314\,\text{V}$$

13.82 **(i)** The flux on one side is equal and opposite to that on the other for $0 < r < b-x$, where r is the distance of any point from the long wire. Only the flux through the portion $b-x < r < x$ is not cancelled:

$$d\phi = (a\,dr)db = \frac{\mu_0 I a\,dr}{2\pi\, r}$$

$$\therefore \quad \phi = \frac{\mu_0 I a}{2\pi}\int_{b-x}^{x}\frac{dr}{r} = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{x}{b-x}\right)$$

(ii) $\phi \to \infty$ for $x \to b$ and $\phi = 0$ for $x = b/2$

(iii) For $I = 2t$ and $x = b/4$

$$\phi = \frac{\mu_0 a}{2\pi}\ln\left(\frac{1}{3}\right)2t$$

$$\xi = -\frac{d\phi}{dt} = \frac{\mu_0 a\ln 3}{\pi}$$

13.83 Betatron is a machine to accelerate electrons to high energy. It consists of an evacuated 'doughnut' in which the electrons are made to circulate under the influence of changing magnetic field. If a magnetic flux ϕ changes in an electromagnet, it accelerates the electrons and at the same time holds them in an orbit of fixed radius. The average force acting on the particle during a single rotation is the work (induced voltage multiplied by charge) divided by the distance $2\pi R$. Equating this to the time rate of change of momentum, by Faraday's law of induction

$$e\frac{\mathrm{d}\phi}{\mathrm{d}t} = 2\pi R\frac{\mathrm{d}P}{\mathrm{d}t}$$
$$e\frac{\mathrm{d}\phi/\mathrm{d}t}{2\pi R} = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(BeR)$$

where B is the magnetic field and R is the radius.
Integrating and assuming that initially the flux is zero and R is constant

$$\phi = 2\pi R^2 B$$

13.3.6 Hall Effect

13.84 Consider a strip of a conductor of width W and thickness t carrying a d.c current i in the positive x-direction along its length, Fig. 13.32. A magnetic field B set up in the z-direction into the page produces a deflection force in the positive y-direction, as the drift velocity of electrons is in the negative x-direction. Consequently charge concentration builds up towards the upper edge of the strip. As the charges collect on one side of the strip they set up an electric field that opposes sideways motion of additional charge carriers inside the conductor. This build up of charges establishes a potential V_H across the width of the strip, called Hall potential and the phenomenon is known as Hall effect. Eventually equilibrium conditions are reached and a maximum voltage, known as Hall voltage, is quickly established. The sign of the voltage gives the sign of charge carriers and its magnitude the number density n of charge carriers:

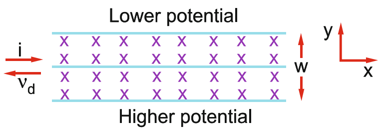


Fig. 13.32 Hall effect

(a) $E_H = v_d B = \dfrac{iB}{ne} = \dfrac{iB}{newb}$

$$= \frac{100 \times 10}{1.25 \times 10^{30} \times 1.6 \times 10^{-19} \times 10^{-3} \times 0.02} = 2.5 \times 10^{-4}\,\mathrm{V/m}$$

The field is along positive y-direction.

(b) When equilibrium is established the Lorentz force is zero:

$$qE + q\boldsymbol{v}_\mathrm{d} \times \boldsymbol{B} = 0$$

$$\text{or} \qquad \boldsymbol{E} = -\boldsymbol{V}_\mathrm{d} \times \boldsymbol{B}$$

$$E = -v_\mathrm{d} B \qquad (\because\ \boldsymbol{v}_\mathrm{d} \perp \boldsymbol{B})$$

$$E = \frac{V_\mathrm{H}}{W} = v_\mathrm{d} B = \frac{jB}{ne} = \frac{iB}{W\,t\,ne}$$

$$\therefore \qquad n = \frac{iB}{etV_\mathrm{H}} = \frac{100 \times 10}{1.6 \times 10^{-19} \times 10^{-3} \times 5 \times 10^{-6}}$$

$$= 1.25 \times 10^{30}/\mathrm{m}^3$$

13.85 If R_H is the Hall coefficient, σ the electrical conductivity, then the mobility μ is given by

$$\mu = R_\mathrm{H}\sigma = (-7.3 \times 10^{-5})\,(2 \times 10^3) = -0.146\,\mathrm{m}^2/\mathrm{V/s}$$

The magnitude is $0.146\,\mathrm{m}^2/\mathrm{V/s}$.

Chapter 14
Electromagnetism II

Abstract *Chapters 13 and 14* are devoted to electromagnetism concerned with motion of charged particles in electric and magnetic fields, Lorentz force, cyclotron and betatron, magnetic induction, magnetic energy and torque, magnetic dipole moment, Faraday's law, Hall effect, RLC circuits, resonance frequency, Maxwell's equations, electromagnetic waves, Poynting vector, phase velocity and group velocity, dispersion relations, waveguides and cut-off frequency.

14.1 Basic Concepts and Formulae

Self-Inductance (*L*)

$$L = \frac{N\varphi}{i} = \frac{\text{Total flux linkage}}{\text{Current linked}} \tag{14.1}$$

$$\xi = -L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{14.2}$$

For a long solenoid or toroid

$$L = \frac{\mu_0 N^2\, A}{l} \tag{14.3}$$

where N is the number of turns, A is the area of cross-section of each turn and l is the length of the coil.

The energy stored in an inductor is

$$W = {}^1\!/_2 L i^2 \tag{14.4}$$

***L–R* Circuit**

$$\text{Differential equation. } L\frac{\mathrm{d}i}{\mathrm{d}t} + iR = \xi \quad \text{(for charging process)} \tag{14.5}$$

$$\text{Solution } i = \frac{\xi}{R}\left(1 - \mathrm{e}^{-\frac{Rt}{L}}\right) \tag{14.6}$$

Inductive time constant

$$\tau_{\mathrm{L}} = L/R \tag{14.7}$$

$$\text{Differential equation} \quad L\frac{\mathrm{d}i}{\mathrm{d}t} + iR = 0 \text{ (discharging process)} \tag{14.8}$$

$$\text{Solution } i = \frac{\xi}{R}\mathrm{e}^{-Rt/L} \tag{14.9}$$

***C–R* Circuit**

$$Ri + q/c = \xi \quad \text{(charging)} \tag{14.10}$$

$$q = q_0(1 - \mathrm{e}^{-t/RC}) \tag{14.11}$$

Capacitive Time Constant

$$\tau = RC \tag{14.12}$$

$$Ri + q/C = 0 \text{ (discharging)} \tag{14.13}$$

$$q = q_0\mathrm{e}^{-t/RC} \tag{14.14}$$

Inductors in Series

$$L_{\mathrm{eq}} = L_1 + L_2 + 2M \quad \text{(coil currents in the same sense)} \tag{14.15}$$

$$L_{\mathrm{eq}} = L_1 + L_2 - 2M \quad \text{(coil currents in the opposite sense)} \tag{14.16}$$

where M is the mutual inductance

$$L_{\mathrm{eq}} = L_1 + L_2 \text{ (inductors well separated)} \tag{14.17}$$

Inductors in Parallel

$$L_{\mathrm{eq}} = L_1L_2/(L_1 + L_2) \tag{14.18}$$

$$M = (L_1L_2)^{1/2} \quad \text{(inductors are closely placed)} \tag{14.19}$$

The Alternating Current (AC)

The effective or root-mean-square value:

$$I_{\mathrm{e}} = I_0/\sqrt{2} \tag{14.20}$$

where I_0 is the peak current

$$\text{Reactance: Inductive } X_\mathrm{L} = \omega L, \text{ Capacitance } X_\mathrm{C} = 1/\omega C \tag{14.21a}$$

$$\text{Impedance} \quad Z = \sqrt{R^2 + (X_\mathrm{L} - X_\mathrm{C})^2} \tag{14.21b}$$

RLC Series Resonance Circuit

$$V_\mathrm{e} = I_\mathrm{e}\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2} \tag{14.22}$$

The phase α is given by the relation

$$\tan\alpha = \frac{X_\mathrm{L} - X_\mathrm{C}}{R} = \frac{\omega L - \frac{1}{\omega C}}{R} \tag{14.23}$$

(i) α is positive if $\omega L > 1/\omega C$, and V leads I.
(ii) α is negative if $\omega L < 1/\omega C$, and V lags behind I.
(iii) α is zero if $\omega L = 1/\omega C$, and V is in phase with I.

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \quad \text{(resonance frequency)} \tag{14.24}$$

$$P = I_\mathrm{e} V_\mathrm{e} \cos\alpha \quad \text{(power)} \tag{14.25}$$

Quality factor (sharpness of resonance):

$$Q = \omega_0 L/R \tag{14.26}$$

Table 14.1 gives the analogues for electrical and mechanical quantities.

Table 14.1 Analogies between mechanical and electrical vibrations

Characteristic	Mechanical	Electrical
Inertia	Mass (m)	Inductance (L)
Stiffness	Stiffness constant (k)	Inverse capacitance ($1/c$)
Force	Force (F)	EMF
Resistance	Frictional factor (r)	Resistance (R)
Kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}Li^2$
Potential energy	$\frac{1}{2}kx^2$	$\frac{1}{2}q^2/C$
Reactance	$m\omega - k/\omega$	$\omega L - 1/\omega C$
Impedance	$\sqrt{r^2 + \left(m\omega - \frac{k}{\omega}\right)^2}$	$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$
Condition for oscillation	$r < 2\sqrt{km}$	$R < 2\sqrt{L/C}$
Resonance frequency	$\frac{1}{2\pi}\sqrt{k/m}$	$\frac{1}{2\pi\sqrt{LC}}$
Quality factor	$\omega_0 m/r$	$\omega_0 L/R$

Parallel Resonance Circuit

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \tag{14.27}$$

When $R = 0$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{14.28}$$

Maxwell's Equations

Differential Form

$$\boldsymbol{\nabla.B} = \mathbf{0} \quad \text{(Gauss' law of magnetostatics)} \tag{14.29}$$

$$\boldsymbol{\nabla.D} = \rho \quad \text{(Gauss' law of electrostatics)} \tag{14.30}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial B}{\partial t} \quad \text{(Faraday's law)} \tag{14.31}$$

$$\nabla \times \boldsymbol{H} = J + \frac{\partial D}{\partial t} \quad \text{(Ampere–Maxwell law)} \tag{14.32}$$

Auxillary Equations

$$D = \varepsilon E,\ D = \varepsilon_0 E + P \tag{14.33}$$

$$B = \mu_0 H,\ B = \mu_0(H + M) \tag{14.34}$$

$$J = \sigma E,\ J = \rho v \tag{14.35}$$

where B = magnetic induction, ρ = charge density, E = electric field, D = displacement vector, H = magnetic intensity, J = current density, σ = conductivity, M = magnetization, v = velocity, P = polarization.

The concept of displacement current can be explained by the working of a parallel plate capacitor placed in a vacuum and connected to a battery. As the capacitor gets charged current flows through the wires but the usual current does not pass between the plates of the capacitor plates. From considerations of continuity Maxwell was led to postulate the existence of a displacement current equivalent to the changing electric field in the space between the plates.

On the theoretical side, Maxwell examined Ampere's law, $\nabla \times \boldsymbol{H} = \boldsymbol{J}$, and noticed that there is something strange about this equation. On taking the divergence of this equation, the left-hand side will be zero, because the divergence of a curl is always zero. This equation then requires that the divergence of J also be zero. But if the divergence of J is zero, then the total flux of current out of closed surface is also zero.

Now the flux of current from a closed surface is the decrease of the charge inside the surface. In general this cannot be zero because charges can be moved from one place to another. This difficulty is avoided by adding the term $\partial D/\partial t$, where $D = \varepsilon_0 E$, on the right-hand side of (32).

Integral Form

$$\varphi_{\mathrm{E}} = \oint \mathbf{B}.\mathrm{d}\boldsymbol{s} = 0 \quad \text{(Gauss' law for magnetostatics)} \tag{14.36}$$

$$\varphi_{\mathrm{E}} = \oint \boldsymbol{E}.\mathrm{d}\boldsymbol{s} = \frac{q_{\mathrm{enc}}}{\varepsilon_0} \quad \text{(Gauss' law for electrostatics)} \tag{14.37}$$

$$\oint \boldsymbol{E}.\mathrm{d}\boldsymbol{s} = -\frac{\mathrm{d}\varphi_B}{\mathrm{d}t} \quad \text{(Faraday's law of induction)} \tag{14.38}$$

$$\oint \mathbf{B}.\mathrm{d}\boldsymbol{s} = \mu_0\varepsilon_0\frac{\mathrm{d}\varphi_{\mathrm{E}}}{\mathrm{d}t} + \mu_0 i_{\mathrm{enc}} \quad \text{(Ampere-Maxwell law)} \tag{14.39}$$

Electromagnetic Waves

The E- and H-waves are transverse to the direction of propagation and are perpendicular to each other.

$$\nabla \times (\nabla \times E) = -\nabla^2 \mathrm{E} + \nabla(\nabla.\mathrm{E}) \tag{14.40}$$

Wave Equation

$$\nabla^2 \mathrm{E} = -\omega^2 \mu_0 \varepsilon_0 \mathrm{E} \tag{14.41}$$

The Intrinsic Impedance

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{14.42a}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega \quad \text{(free space)} \tag{14.42b}$$

The Poynting vector (S) represents the power in the electromagnetic wave and is given by

$$S = E \times H \tag{14.43}$$

and points in the direction of propagation of the wave.

The skin thickness (δ) represents the depth to which an electromagnetic wave of frequency $f = \omega/2\pi$ can penetrate a medium and is given by

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{14.44}$$

where μ is the permeability and σ is the conductivity.

Phase Velocity (v_{ph}) and Group Velocity (v_g)

Phase velocity is just due to the nodes of the wave that are moving and not energy or information. In fact the phase velocity can be greater than c, the velocity of light

$$v_{\mathrm{ph}} = \omega/k \tag{14.45}$$

In order to know how fast signals will travel one must calculate the speed of pulses or modulation caused by the interference of a wave of one frequency with one or more waves of slightly different frequencies. The speed of the envelope of such a group of waves is called the group velocity and is given by

$$v_{\mathrm{g}} = \mathrm{d}\omega/\mathrm{d}k \tag{14.46}$$

Waveguides are hollow metallic structures in which the electromagnetic waves are guided to travel from one place to another without much attenuation. Here, we shall be concerned only with rectangular guide of cross-section of $x = a$ and $y = b$, with the wave propagated in the z-direction.

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \tag{14.47}$$

where c is the velocity of light in free space, m and n are integers. Equation (14.47) is valid both for TE_{mn} and TH_{mn} waves.

For the simplest case $m = 1$ and $n = 0$. For TE_{10} wave.

$$v_{\mathrm{ph}} = \frac{c}{\sqrt{1 - \left(\frac{\lambda_0}{2a}\right)^2}} \tag{14.48}$$

where λ_0 is the free space wavelength.

$$v_{\mathrm{g}} = c\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2} \tag{14.49}$$

$$v_{\mathrm{ph}}.v_{\mathrm{g}} = c^2 \tag{14.50}$$

$$\lambda_{\mathrm{g}} = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{2a}\right)^2}} \tag{14.51}$$

14.2 Problems

14.2.1 The RLC Circuits

14.1 A 2.5 μF capacitor is connected in series with a non-inductive resistor of 300 Ω across a source of PD of rms value 50 V, alternating at $1000/2\pi$ Hz. Calculate

(a) thermal values of the current in the circuit and the PD across the capacitor.
(b) the mean rate at which the energy is supplied by the source.

[Joint Matriculation Board of UK]

14.2 A 3 Ω resistor is joined in series with a 10 mH inductor of negligible resistance, and a potential difference (rms) of 5.0 V alternating at $200/\pi$ Hz is applied across the combination.

(a) Calculate the PD V_R across the resistor and V_L across the inductor.
(b) Determine the phase difference between the applied PD and the current.

[Joint Matriculation Board of UK]

14.3 An inductance stores 10 J of energy when the current is 5 A. Find its value.

14.4 A tuning circuit in a radio transmitter has a 4×10^{-6} H inductance in series with a 5×10^{-11} F capacitance. Find

(a) the frequency of the waves transmitted.
(b) their wavelength.

14.5 A 6 Ω resistor, a 12 Ω inductive reactance and a 20 Ω capacitive reactance are connected in series to a 250 V rms AC generator. (a) Find the impedance. (b) Estimate the power dissipated in the resistor.

14.6 At 600 Hz an inductor and a capacitor have equal reactances. Calculate the ratio of the capacitive reactance to the inductive reactance at 60 Hz.

14.7 A capacitance has a reactance of 4 Ω at 250 Hz. (a) Find the capacitance. (b) Calculate the reactance at 100 Hz. (c) What is the rms current, if it is connected to a 220 V 50 Hz line?

14.8 When an impedance, consisting of an inductance L and a resistance R in series, is connected across a 12 V 50 Hz supply, a current of 0.05 A flows which differs in phase from that of the applied potential difference by 60°. Find the value of R and L. Find the capacitance of the capacitor which when connected in series in the above circuit has the effect of bringing the current into phase with the applied potential difference.

[University of London]

14.9 When a 0.6 H inductor is connected to a 220 V 50 Hz AC line, what is (a) the rms current and (b) peak current?

14.10 A simple alternator, when rotating at 50 revolutions/s, gives a 50 Hz alternating voltage of rms value 24 V. A 4.0 Ω resistance R and a 0.01 H inductance L are connected in series across its terminals. Assuming that the internal impedance of the generator can be neglected, find (a) the rms current flowing; (b) the power converted into heat; (c) the rms potential difference across each component.

14.11 An AC circuit consists of only a resistor $R = 100\,\Omega$ and a source voltage $V = 0.5\,V_{\mathrm{m}}$ at time $t = 1/360\,\mathrm{s}$. Assuming that at $t = 0$, $V = 0$, find the frequency.

14.12 Given that for a series LCR circuit the equation is

$$\frac{\mathrm{d}^2 V}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{1}{LC}V = 0$$

If a similar equation is to be used for a parallel LCR circuit as in Fig. 14.1, then show that

$$R_{\mathrm{p}} = \frac{L}{CR}$$

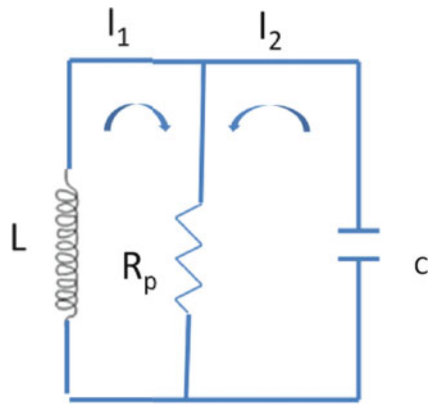


Fig. 14.1

14.13 Verify the equation $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

14.14 Show that in the usual notation the following combinations of physical quantities have the units of time: (a) RC, (b) L/R, (c) $\sqrt{LC}$.

14.15 In an oscillating RLC circuit the amplitude of the charge oscillations drops to one-half its initial value in 4 cycles. Show that the fractional decrement of the resonance frequency is approximately given by $\frac{\Delta\omega}{\omega} = 0.00038$.

14.16 Derive the equation for the current in a damped LC circuit for low damping.

14.17 Set up the equation for the RC circuit in series and show that the input power is the sum of the powers delivered to the inductor and capacitor.

14.18 Set up the equation for the RLC circuit in parallel and show that at any time the Joule heat in the resistor comes from the energy stored in the inductor and capacitor.

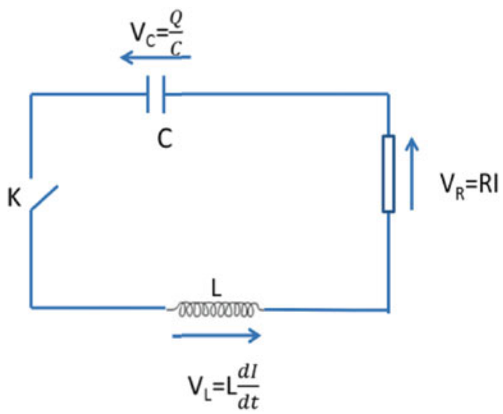
14.19 For the electrical circuit shown below when the switch K is closed the charge and current are observed to oscillate (Fig. 14.2). Show that the differential equation governing the charge of the system can be written as

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}Q}{\mathrm{d}t} + \omega_0^2 Q = 0$$

where γ and ω_0^2 are to be determined in terms of the capacitance, C, inductance, L, and resistance, R.
For a resistance $R = 100\,\Omega$, capacitance $C = 700\,\text{pF}$ and inductance $L = 80\,\text{mH}$ calculate

(a) The natural frequency, f_0, of the oscillation.
(b) The time constant, τ, for the decay.

Fig. 14.2

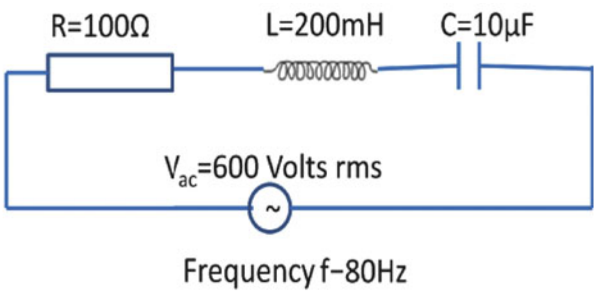


14.20 Solve the differential equation given in prob. (14.19) and obtain the time period for damped harmonic motion.

14.21 A resistor, capacitor and inductor are connected in series across an AC voltage source shown as in Fig. 14.3.

(i) Find the magnitude of the inductive reactance X_L of the inductor and capacitive reactance X_C of the capacitor.
(ii) Find the magnitude of the total impedance Z of the circuit and sketch the impedance phasor diagram for this circuit.
(iii) Find the total current I_T through the circuit.

Fig. 14.3



(iv) Find the phase angle between supply voltage and current through the circuit.

(v) Find the voltages across R, C and L and show these on a phasor diagram.

(vi) What is the condition for resonance to occur in this type of circuit and at what frequency would this occur?

[University of Aberystwyth, Wales 2001]

14.22 A 40 Ω resistor and a 50 μF capacitor are connected in series, and an AC source of 5 V at 300 Hz is applied. What is the magnitude of the current flowing through the circuit?

[University of Manchester 2007]

14.23 For the circuits shown in Fig. 14.4a, b consisting of resistors, capacitors and inductors

(i) Derive the expression to represent the complex impedances for each of the networks.

(ii) Work out the magnitude of the impedance for each of the networks given that the frequency of the supply voltage is 150 Hz.

[University of Aberystwyth, Wales 2006]

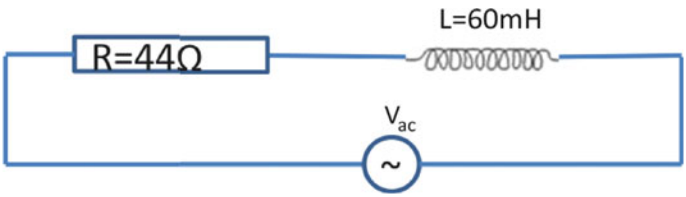


Fig. 14.4a

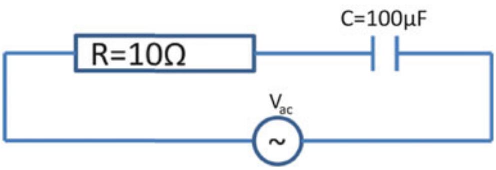


Fig. 14.4b

14.24

(a) Define what is meant by electric current and current density.
(b) When we refer to a quantity of charge we say that the value is quantized. Explain what is meant by quantized.
(c) A thin copper bar of rectangular cross-section of width 5.6 mm and height 50 μm has an electron density of $n = 8.5 \times 10^{28}/\text{m}^3$.

If a uniform current of $i = 2.4 \times 10^{-4}$ A flows through the strip

(i) Find the magnitude of the current density in the strip.
(ii) Find the magnitude of the drift speed of the charge carriers.
(iii) Briefly explain why the current is relatively high for such a small drift speed.

[University of Aberystwyth, Wales 2008]

14.25 Two series resonant circuits with component values L_1C_1 and L_2C_2, respectively have the same resonant frequency. They are then connected in series; show that the combination has the same resonant frequency.

[University of Manchester 1972]

14.26 An inductance and condenser in series have a capacitative impedance of 500 Ω at 1 kHz and an inductive impedance of 100 Ω at 5 kHz. Find the values of inductance and capacitance.

[University of Manchester 1972]

14.27 A condenser of 0.01 μF is charged to 100 V. Calculate the peak current that flows when the charged condenser is connected across an inductance of 10 mH

[University of Manchester 1972]

14.28 An inductance of 1 mH has a resistance of 5 Ω. What resistance and condenser must be put in series with the inductance to form a resonant circuit with a resonant frequency of 500 kHz and a Q of 150?

[University of Manchester 1972]

14.29 A parallel resonant circuit consists of a coil of inductance 1 mH and resistance 10 Ω in parallel with a capacitance of 0.0005 μF. Calculate the resonant frequency and the Q of the circuit.

[University of Manchester 1972]

14.30 The voltage on a capacitor in a certain circuit is given by $V(t) = V_0 e^{-t/RC}$. Find the fractional error in the voltage at $t = 50\,\mu\text{s}$ if $R = 50\,\text{k}\Omega \pm 5\%$ and $C = 0.01\,\mu\text{F} \pm 10\%$.

[University of Manchester 1972]

14.31 A condenser of 10 μF capacitance is charged to 3000 V and then discharged through a resistor of 10, 000 Ω. If the resistor has a temperature coefficient of 0.004/°C and a thermal capacity of 0.9 cal/°C, find **(a)** the time taken for the voltage on the condenser to fall to $1/e$ of its initial value; **(b)** the percentage error which would have been introduced if thermal effects had been ignored.

[University of Manchester 1958]

14.32 Show that the fractional half-width of the resonance curve of an RLC circuit is given by

$$\frac{\Delta\omega}{\omega} = \frac{\sqrt{3}}{Q}$$

where Q is the quality factor given by $Q = \omega \mathrm{L}/\mathrm{R}$.

14.2.2 Maxwell's Equations, Electromagnetic Waves, Poynting Vector

14.33 A plane em wave $E = 100 \cos(6 \times 10^8\, t + 4x)$ V/m propagates in a medium. What is the dielectric constant of the medium?

[Indian Administrative Services]

14.34 An infinite wire with charge density λ and current I is at rest in the Lorentz frame S. Show that the speed of reference frame S' where the electric field is zero, i.e. that frame in which one observes pure magnetic field, is given by $v = \frac{\lambda c^2}{I}$.

14.35 Show that for a magnetic field $\boldsymbol{B}$ the wave equation has the form $\nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$

14.36 Use Maxwell's equation to show that $\nabla . \left(\boldsymbol{j} + \frac{1}{\varepsilon_0} \frac{\partial \boldsymbol{E}}{\partial t}\right) = 0$.

14.37 The free-space wave equation for a medium without absorption is

$$\nabla^2 \boldsymbol{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

Show that this equation predicts that electromagnetic waves are propagated with velocity of light given by $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

14.38 An electromagnetic wave of wavelength 530 nm is incident onto a sheet of aluminium with resistivity $\rho = 26.5 \times 10^{-9}\,\Omega$m. Estimate the depth that the wave penetrates into the aluminium. The expression for the skin depth, δ, is $\delta = \sqrt{2/\mu_0 \sigma \omega}$.

[University of Manchester 2008]

14.39 Consider an electromagnetic wave with its E-field in the y-direction. Apply the relation $\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$ to the harmonic wave

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(kx - \omega t), \boldsymbol{B} = \boldsymbol{B}_0 \cos(kx - \omega t)$$

to show that $E_0 = cB_0$.

14.40 Using Maxwell's equations, show that in a conducting medium, the wave equation can be written as

$$\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} + \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

and a similar expression for the B-field.

[University of Aberystwyth, Wales 2005]

14.41 Let l be the length of the coaxial cylindrical capacitor, a the radius of the central wire and b the radius of the tube. The conductors are connected to a battery of V volts and a current I is passed. Calculate (a) the capacitance per unit length of the cable, (b) the inductance per unit length.

14.42 Consider a coaxial cable with radius a for the central wire and radius b for the tube connected to a resistance R and battery of emf ξ. Calculate (a) $\boldsymbol{E}$; (b) $\boldsymbol{B}$; and (c) $\boldsymbol{S}$ for the region $a < r < b$.

14.43 The general expression for magnetic energy density has the form $u_{\mathrm{B}} = {}^1\!/_2\boldsymbol{B}.\boldsymbol{H}$. Show that in the vacuum the above expression is reduced to $u_B = \frac{B^2}{2\mu_0}$.

14.44 Show that at any point in the electromagnetic field the energy density stored in the electric field is equal to that stored in the magnetic field.

14.45 A current I is passed through a coaxial cable with inner radius a and outer radius b. The cable can function both as a capacitor and as an inductor. If the stored electric and magnetic energy is equal then show that the resistance R is approximately given by

$$R = \frac{377}{2\pi} \ln\left(\frac{b}{a}\right) \Omega$$

14.46 A proton of kinetic energy 20 MeV circulates in a cyclotron with 0.5 m radius. Calculate its energy loss to radiation per orbit and show that it is negligible.

14.47 A 40-W point source radiates equally in all directions. Find the amplitude of the E-field at a distance of 1 m.

14.48 A laser beam has a cross-sectional area of 4.0 mm^2 and a power of 1.2 mW. Find (a) intensity I, (b) E_0, (c)B_0.

14.49 A laser emits a 1-mm diameter highly collimated beam at a power level 314 mW. Calculate the irradiance.

14.50 Beginning with the expression for the Poynting vector show that the time-averaged power per unit area carried by a plane electromagnetic wave in free space is given by

$$S_{\text{ave}} = \frac{E_0^2}{2\mu_0 c}$$

[University of Durham 2003]

14.51 A plane electromagnetic wave has $E_x = E_y = 0$ and $E_z = 50 \sin\left[4\pi \times 10^{14}\left(t - \frac{x}{3\times 10^8}\right)\right]$. Calculate the irradiance (flux density).

14.52 Show that $\boldsymbol{E} \times \boldsymbol{H}$ is in the same direction as the wave propagates and has magnitude equal to $|\boldsymbol{E} \times \boldsymbol{H}| = \frac{|\boldsymbol{E}|^2}{\mu_0 c}$.

[University of Aberystwyth, Wales]

14.53 An electromagnetic plane wave in vacuum has E-field given by

$$E_z = 10 \sin \pi (2 \times 10^6 x - 6 \times 10^{14} t),\ E_x = E_y = 0.$$

Find (a) frequency; (b) wavelength; (c) speed; (d) $\boldsymbol{E}$-field amplitude; (e) polarization.

14.54 Write down the equation for the associated magnetic field for the wave given in prob. (14.53).

14.55 A radar monitors the speed v of approaching cars by sending out waves of frequency ν. If the frequency received is ν'', find the speed of the car. How would the beat frequency change if the car is receding?

14.56 Microwaves of frequency 800 MHz are beamed by a stationary police man towards a receding car speeding at 90 km/h. What beat frequency was registered by the radar?

14.57 State Ampere's law.
A long solid conductor of radius a lies on the axis of a long cylinder of inner radius b and outer radius c. The central conductor carries a current i while the outer conductor carries a current $-i$. The currents are uniformly distributed over the cross-sections of each conductor. By considering the current enclosed by a circular loop of radius r centred on the axis of the inner conductor use Ampere's law to calculate the magnetic field in each of the four regions (a) $r < a$, (b) $a < r < b$, (c) $b < r < c$, (d) $r > c$.

14.58 The CMS experiment at the Large Hadron Collider at CERN uses a large, cylindrical, superconducting solenoid. This magnet is 12.5 m in length with a diameter of 6 m. When powered, it generates a uniform magnetic field of 4 T. Estimate the energy stored in the magnetic field.

14.59 Given that the total power radiated by the sun in the form of electromagnetic radiation is 4×10^{26} W, estimate the electric and magnetic field amplitude at the surface of the sun. (The radius of the sun is 7×10^8 m).

[University of Durham 2003]

14.60 At the orbit of the earth, the power of sunlight is 1,300 $\mathrm{W m^{-2}}$. Estimate the amplitude of the electric field if we assume that all the power arrives on the earth in a monochromatic wave.

[University of Aberystwyth, Wales 2004]

14.61 A typical value for the amplitude of the E-field for sunlight at the surface of Mars is 300 V/m Calculate the amplitude of the corresponding B-field and estimate the flux of radiation at the surface of Mars.

[University of Manchester 2006]

14.62 Show that $\frac{|E|}{|H|} = 377\,\Omega$

[University of Aberystwyth, Wales]

14.63 Calculate the skin depth in copper (conductivity $6 \times 10^7\ \Omega^{-1}/\mathrm{m}$) of radiation of frequency 20 kilocycles/s. Take μ, the relative permeability of copper, as unity.

[University of Newcastle upon Tyne 1964]

14.64 Copper has an electrical conductivity $\sigma = 5.6 \times 10^7\ \Omega^{-1}/\mathrm{m}$ and a magnetic permeability $\mu = 1$. On this basis estimate the order of magnitude of the depth to which radiation at a frequency of 3000 Mc/s can penetrate a large copper screen.

[University of Bristol 1959]

14.65 Show that the skin depth in a good conductor is $\left[\frac{1}{2}\,\omega\sigma\mu\mu_0\right]^{-1/2}$ where the symbols have their usual meaning.

[University of Newcastle upon Tyne 1964]

14.66 If the maximum electric field in a light wave is 10^{-3} V/m, find how much energy is transported by a beam of 1 $\mathrm{cm^2}$ cross-sectional area.

[University of Durham 1962]

14.67 Prove Poynting's theorem, namely

$$\mathrm{div}\,(\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{E}.\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H}.\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{E}.\boldsymbol{j} = 0$$

What is the interpretation of this equation?

[University of Durham 1962][University of New Castle upon Tyne 1965]

14.68 From Maxwell's equations, one can derive a wave equation for a dielectric of the form

$$\nabla^2 E - \mu_0\varepsilon_0\varepsilon_r\frac{\partial^2 E}{\partial t^2} - \mu_0\sigma_N\frac{\partial E}{\partial t} = 0$$

where E is the electric field, t is the time, ε_r is the relative permittivity and σ_N is the electrical conductivity. Hence by substituting a travelling wave solution into the wave equation derive a dispersion relation of the form

$$k^2 = \mu_0 \varepsilon_0 \varepsilon_r \omega^2 + i\mu_0 \sigma_N \omega$$

where k is the wave vector and ω is the angular frequency.

[University of Durham 2006]

14.69 Show that the general identity

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\nabla^2 \boldsymbol{E} + \nabla(\nabla.\boldsymbol{E})$$

is true for the specific vector field $F = x^2 z^3 \hat{i}$.

14.70 Use the Poynting vector to determine the power flow in a coaxial cable by a DC current I when voltage V is applied. Neglect the resistance of the conductors. How are the results affected if this assumption is not made?

14.71 A wire with radius a and of conductivity σ_{E} carries a constant, uniformly distributed current I in the z-direction. Apply Poynting's theorem to show that power dissipated in the wire is given by the familiar expression $I^2 R$ for Joule's heat.

14.72 A super-conductor is a material which offers no DC resistance and satisfies the equation

$$\boldsymbol{B} = -\frac{m_e}{ne^2} \nabla \times \boldsymbol{J}$$

where n is the number of conduction electrons per unit volume, $\boldsymbol{B}$ is the magnetic field and $\boldsymbol{J}$ is the current density. Using Maxwell's equations, show that the equation for a superconductor leads to the relation

$$\nabla^2 \boldsymbol{B} = \frac{\mu_0 n e^2}{m_e} \boldsymbol{B}$$

[University of Durham 2004]

14.73 An oscillating voltage of high frequency is applied to a load by means of copper wire of radius 1 mm. Given that the skin depth is 6.6×10^{-5} m for this frequency, what is the high-frequency resistance per unit length of the wire in terms of its direct current resistance per unit length?

[University of Durham 1966]

14.74 Use Stokes' theorem to derive the expression

$$\mathrm{Curl}\, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

[University of New Castle upon Tyne 1964]

14.75 Explain the difference between the vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ in the theory of magnetism. Derive the expression $B = \mu_0(H + M)$. Indicate briefly how B depends on it in the case of (a) paramagnetics and (b) ferromagnetics.

[University of Durham 1962]

14.76 Show that Gauss' and Ampere's laws in free space, subject to the Lorentz condition, can be expressed in the usual notation as

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon_0} \text{ and } -\nabla^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \mu_0\boldsymbol{J}$$

respectively.

[The University of Aberystwyth, Wales 2005]

14.77 Using Faraday's law, $\nabla \times \boldsymbol{E} = -\partial B/\partial t$, for the propagation of electromagnetic waves travelling along the z-axis, show that

$$E_y = Z_0 H_x$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 376.6\,\Omega$, is the wave impedance of free space.

14.78 **(a)** The electric field of an electromagnetic wave propagating in free space is described by the equation

$$E(z,t) = E_0[\hat{x} sin(kz - \omega t) + \hat{y} cos(kz - \omega t)]$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the x- and y-direction, respectively. What is this wave's direction of propagation? What is the polarization of the wave?

(b) State and prove the boundary conditions satisfied by the magnetic intensity H and the magnetic field B at the boundary between two media with different magnetic properties.

(c) Show that

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

where θ_1 and θ_2 are incident and refraction angles.

14.79 A plane wave is normally incident on a dielectric discontinuity. Use appropriate boundary conditions to calculate R, the reflectance, and T, the transmittance, and show that $T + R = 1$.

14.80 An electromagnetic wave, propagating and linearly polarized in the xz-plane, is incident onto an interface between two non-conducting media as shown in Fig. 14.10. The electric fields and propagation vectors of the incident,

reflected and transmitted waves are denoted by $\boldsymbol{E}_{\mathrm{I}}$, $\boldsymbol{E}_{\mathrm{R}}$, $\boldsymbol{E}_{\mathrm{T}}$, $\boldsymbol{k}_{\mathrm{I}}$, $\boldsymbol{k}_{\mathrm{R}}$ and $\boldsymbol{k}_{\mathrm{T}}$ respectively. The wave is incident onto the interface at the origin and makes an angle θ to the normal. Both incident and reflected waves propagate in the medium with refractive index n_1. The transmitted wave propagates in the medium with refractive index n_2 at refraction angle ϕ.
Use the boundary conditions satisfied by the electric field at the interface to show that the reflectance, R, is given by

$$R = \left(\frac{n_2 \cos\theta - n_1 \cos\phi}{n_2 \cos\theta + n_1 \cos\phi}\right)^2$$

14.81 **(a)** Using the expressions for R in prob. (14.80) and Snell's law, show that the reflectance is zero when $\tan\theta = \frac{n_2}{n_1}$
(b) Calculate this angle for electromagnetic radiation in air incident onto glass, which has a refractive index $n = 1.5$.

14.82 **(a)** For normal incidence the reflection coefficient, R, at the planar surface between two dielectric media is given by

$$R = \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2}$$

where n_1 and n_2 are the refractive indices of the media. Sketch the form of R against n_1/n_2. On the same figure sketch the form of T against n_1/n_2 where T is the transmission coefficient. Indicate numerical values of T and R where appropriate.
(b) At what value of n_1/n_2 does $R = T$?

[University of Durham 2000]

14.83 Consider the solutions of linearly polarized harmonic plane waves

$$\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{i(\omega t - k.r + \varphi)}, \ \boldsymbol{B} = \boldsymbol{B}_0 \mathrm{e}^{i(\omega t - k.r + \varphi)}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ are constant vectors associated with maximum amplitude of oscillations. Show that (a) $\boldsymbol{B}$ is perpendicular to $\boldsymbol{E}$; (b) $\boldsymbol{B}$ is in phase with $\boldsymbol{E}$ and (c) the magnitudes of $\boldsymbol{B}$ and $\boldsymbol{E}$ are related by $B = E/c$ for free space in the SI system.

14.84 An uncharged dielectric cube of material of relative permittivity 6 contains a uniform electric field E of 2 kV/m, which is perpendicular to one of the faces. What is the surface charge density induced on this face?

[University of Manchester 2006]

14.2.3 Phase Velocity and Group Velocity

14.85 Using the results of prob. (14.93) and the following table, estimate the fractional difference between the phase and group velocity in air at a wavelength of 5000 Å

Free space wavelength (Å)	$(n - 1)$ for air
4800	2.786×10^{-4}
5000	2.781×10^{-4}
5200	2.777×10^{-4}

[University of Manchester 1972]

14.86 Given the dispersion relation $\omega = ak^2$, calculate (a) phase velocity and (b) group velocity.

14.87 **(a)** Write down an expression for the phase velocity v_p of an electromagnetic wave in a medium with permittivity ε and permeability μ.

(b) The relative permittivity, ε_r, in an ionized gas is given by

$$\varepsilon_\mathrm{r} = 1 - \frac{D^2}{\omega^2}$$

where D is a constant and ω is the angular frequency.
Find an expression for the refractive index n and thus show that

$$\omega^2 = D^2 + c^2k^2$$

where k is the wavenumber and c is the speed of light in vacuum.

(c) Hence show that $v_\mathrm{p} v_\mathrm{g} = c^2$.

(d) In a particular gas, D has the value 1.2×10^{11}/s. Determine the phase and group velocities at 20 GHz.
Comment on the result.

14.88 Show that the group velocity v_g can be expressed as

$$v_\mathrm{g} = v_\mathrm{p} + k\frac{\mathrm{d}v_p}{\mathrm{d}k}$$

where v_p is the phase velocity and $k = 2\pi/\lambda$.

14.89 Show that the group velocity can be expressed in the form

$$v_\mathrm{g} = \frac{c}{n} + \frac{\lambda c}{n^2}\frac{\mathrm{d}n}{\mathrm{d}\lambda}$$

where n is the refractive index.

14.90 Show that if the phase velocity varies inversely with the wavelength then the group velocity is twice the phase velocity.

14.91 Show that the group velocity can be expressed as

$$v_{\mathrm{g}} = \frac{c}{n + \omega\left(\frac{\mathrm{d}n}{\mathrm{d}\omega}\right)}$$

14.92 For a rectangular guide of width 2.5 cm what free-space wavelength of radiation is required for energy to traverse 50 m of length of the guide in 1 μs. What would be the phase velocity under these conditions?

14.93 Show that for light waves of angular frequency ω in a medium of refractive index n, the group velocity v_{g} and the phase velocity v_{p} are related by the expression

$$\frac{1}{v_{\mathrm{g}}} = \frac{1}{v_{\mathrm{p}}} + \frac{\omega}{c}\frac{dn}{d\omega}$$

where c is the velocity of light in free space.

[University of Manchester 1972]

14.94 Prove that the usual expression for the group velocity of a light wave in a medium can be rearranged as $v_{\mathrm{g}} = c\frac{dv}{d(nv)}$, where c is the phase velocity of the waves in free space, v is the frequency and n is the refractive index of the medium.

[University of Durham 1961]

14.95 Show that the group velocity associated with a free non-relativistic particle is the classical velocity of the particle.

[University of Manchester 1972]

14.96 Calculate the group velocity of light of wavelength 500 nm in glass for which the refractive index μ at wavelength λ (meters) is

$$\mu = 1.420 + \frac{3.60 \times 10^{-14}}{\lambda^2}$$

[University of Manchester 1972]

14.2.4 Waveguides

14.97 For a rectangular guide of width 2.5 cm, calculate (a) the phase velocity; (b) the group velocity; (c) guide wavelength for the free-space wavelength of 4 cm. Assume the dominant mode.

14.98 A rectangular guide has a width $a = 3$ cm. What should be the free-space wavelength if the guide wavelength is to be thrice the free-space wavelength?

14.99 **(a)** Calculate the guide wavelength for a rectangular waveguide of width $a = 5$ cm if the free-space wavelength is 8 cm. **(b)** What is the cut-off wavelength for the guide?

14.100 Calculate the number of states of electromagnetic radiation between 5000 and 6000 Å in wavelength using periodic boundary conditions in a cubical region 0.5 cm on a side.

[University of Manchester 1972]

14.101 Consider a car entering a tunnel of dimensions 15 m wide and 4 m high. Assuming the walls are good conductors, can AM radio waves (530–1600 kHz) propagate in the tunnel?

[The University of Aberystwyth, Wales 2006]

14.102 Calculate the least cut-off frequency for TE_{mn} waves for a rectangular waveguide of dimensions 5 cm × 4 cm.

14.103 Calculate how the wave and group velocities of the TE_{01} wave in a rectangular waveguide with $a = 1$ cm and $b = 2$ cm vary with frequency.

[The University of Wales, Aberystwyth 2004]

14.104 Consider a rectangular waveguide of dimensions $x = a$ and $y = b$, the TM_{mn} wave travelling in the z-direction which is the axis of the guide. Given that the z-component E_z satisfies the equation

$$\left(\frac{\partial^z}{\partial x^z} + \frac{\partial^z}{\partial y^z}\right) E_z = (k^2 - \omega^2 \mu \varepsilon) E_z$$

obtain (a) the solution for E_z and (b) the cut-off frequency.

14.105 Consider a rectangular waveguide of dimensions $x = a$ and $y = b$, the wave travelling along the z-direction, the axis of the guide. Given that the z-component H_z satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_z = (k^2 - \omega^2 \mu \varepsilon) H_z$$

(a) obtain the solution for H_z. (b) Obtain the cut-off frequency. (c) What are the similarities and differences between TM_{mn} mode and TE_{mn} mode?

14.3 Solutions

14.3.1 The RLC Circuits

14.1 **(a)** Reactance of capacitor

$$X_\mathrm{c} = \frac{1}{2\pi f c} = \frac{1}{2\pi \times (1000/2\pi) \times 2.5 \times 10^{-6}} = 400\ \Omega$$

Impedance of the circuit

$$Z = \sqrt{R^2 + X_\mathrm{c}^2} = \sqrt{(300)^2 + (400)^2} = 500\,\Omega$$

The rms current, $I_e = \frac{V_e}{Z} = \frac{50}{500} = 0.1\,\text{A}$

PD across the capacitor, $V_c = I_e X_c = 0.1 \times 400 = 40\,\text{V}$

(b) Power, $P = V_e I_e \cos\alpha = V_e I_e \frac{R}{Z} = 50 \times 0.1 \times \frac{300}{500} = 3.0\,\text{W}$

14.2 **(a)** $X_L = 2\pi f L = 2\pi \times \frac{200}{\pi} \times 10 \times 10^{-3} = 4\,\Omega$

$$Z = \sqrt{R^2 + X^2} = \sqrt{3^2 + 4^2} = 5\,\Omega$$

$$I_e = V_e/Z = 5/5 = 1.0\,\text{A}$$

$$V_R = I_e R = 1.0 \times 3 = 3.0\,\text{V}$$

$$V_L = I_e X_L = 1.0 \times 4 = 4.0\,\text{V}$$

(b) Phase angle between V_e and I_e is given by

$$\tan\alpha = \frac{X_L}{R} = \frac{4}{3} \Rightarrow \alpha = 53^\circ$$

V_e leads I_e by 53°

14.3 $U = \frac{1}{2} L I^2$

$$L = \frac{2U}{I^2} = \frac{2 \times 10}{5^2} = 0.8\,\text{H}$$

14.4 **(a)** $f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{4 \times 10^{-6} \times 5 \times 10^{-11}}} = 1.125 \times 10^{-7}\,\text{Hz}$

(b) $\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1.125 \times 10^7} = 26.67\,\text{m}$

14.5 **(a)** $Z = \sqrt{(X_L - X_C)^2 + R^2} = \sqrt{(12 - 20)^2 + 6^2} = 10\,\Omega$

(b) $P = I_e V_e \frac{R}{Z} = \frac{V_e^2 R}{Z^2} = \frac{(250)^2 \times 6}{10^2} = 3750\,\text{W}$

14.6 At $\omega_0 = 600\,\text{rad/s}$, $X_L = X_C$

$$\therefore \quad \omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \frac{1}{LC} = \omega_0^2$$

At $\omega = 60\,\text{rad/s}$, $\frac{X_C}{X_L} = \frac{1/\omega c}{\omega L} = \frac{1}{\omega^2 LC} = \frac{\omega_0^2}{\omega^2} = \frac{(600)^2}{60^2} = 100$

14.7 **(a)** $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$

$$C = \frac{1}{2\pi f\, X_C} = \frac{1}{2\pi \times 250 \times 4} = 1.57 \times 10^{-4}\,\mathrm{F}$$

(b) $X'_C = \dfrac{1}{2\pi f' C} = \dfrac{1}{2\pi \times 100 \times 1.57 \times 10^{-4}} = 10\,\Omega$

(c) $I_e = \frac{V_e}{X'_C} = \dfrac{220}{10} = 22\,\mathrm{A}$

14.8 $\dfrac{V_e}{I_e} = \sqrt{\omega^2 L^2 + R^2} = \sqrt{4\pi^2 f^2 L^2 + R^2}$

$$\therefore \quad \frac{12}{0.05} = \sqrt{4\pi^2 \times (50)^2 L^2 + R^2} \tag{1}$$

Also $\quad \tan\alpha = \dfrac{\omega L}{R} = \dfrac{2\pi f L}{R}$

$$\therefore \quad \tan 60^\circ = \sqrt{3} = 2\pi \times 50 \frac{L}{R} \tag{2}$$

Solving (1) and (2) we find $R = 120\,\Omega$ and $L = 0.66\,\mathrm{H}$.
When the capacitor of capacitance C is connected in series with the above circuit $\alpha = 0$.

$$\tan\alpha = \tan 0^\circ = \frac{1}{R}\left(\omega L - \frac{1}{\omega c}\right)$$

$$\therefore \quad C = \frac{1}{\omega^2 L} = \frac{1}{(2\pi \times 50)^2 \times 0.66} = 15.37 \times 10^{-6}\,\mathrm{F} = 15.37\,\mu\mathrm{F}$$

14.9 **(a)** The rms current, $I_e = \dfrac{V_e}{\omega L} = \dfrac{V_e}{2\pi f L} = \dfrac{220}{2\pi \times 50 \times 0.6} = 1.17\,\mathrm{A}$

(b) Peak current, $I_0 = \sqrt{2} I_e = 1.414 \times 1.17 = 1.65\,\mathrm{A}$.

14.10 (**a**) $I_e = \dfrac{V_e}{\sqrt{4\pi^2 f^2 L^2 + R^2}} = \dfrac{24}{\sqrt{4\pi^2 \times (50)^2 \times (0.01)^2 + 4^2}} = 4.72\,\mathrm{A}$

(**b**) Power, $P = I_e^2 R = (4.72)^2 \times 4 = 89\,\mathrm{W}$

(**c**) $V_R = I_e R = 4.72 \times 4 = 18.88\,\mathrm{V}$

$V_L = 2\pi f L I_e = 2\pi \times 50 \times 0.01 \times 4.72 = 13.82\,\mathrm{V}$

14.11 $V = V_m \sin\omega t$

$$0.5\, V_m = V_m \sin\left(2\pi f \frac{1}{360}\right)$$

$$\therefore \quad \frac{2\pi f}{360} = \frac{\pi}{6} \Rightarrow f = 30\,\mathrm{Hz}$$

14.12 For the parallel RLC circuit, Fig. 14.1

$$Q = CV,\ I_2 = -\frac{\mathrm{d}Q}{\mathrm{d}t} = -C\frac{\mathrm{d}V}{\mathrm{d}t}$$
$$V = R_\mathrm{p}\,(I_2 + I_1) = -L\frac{\mathrm{d}I_1}{\mathrm{d}t}$$
$$\frac{\mathrm{d}V}{\mathrm{d}t} = R_\mathrm{p}\left(\frac{\mathrm{d}I_2}{\mathrm{d}t} + \frac{\mathrm{d}I_1}{\mathrm{d}t}\right) = -CR_\mathrm{p}\frac{\mathrm{d}^2V}{\mathrm{d}t^2} - R_\mathrm{p}\frac{V}{L}$$

$$\therefore \quad \frac{\mathrm{d}^2V}{\mathrm{d}t^2} + \frac{1}{R_\mathrm{p}C}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{LC} = 0 \qquad \text{(parallel arrangement)}$$

Compare the above equation with the given equation

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{LC} = 0$$
$$\therefore \quad R_\mathrm{p} = \frac{L}{CR}.$$

14.13 We can find the dimensions of μ_0 and ε_0 from the following set of formulae:

$$F = ilB \quad \text{(force on the current-carrying wire)} \tag{1}$$
$$B = \frac{\mu_0 i}{2\pi r} \quad \text{(magnetic field due to a current-carrying wire)} \tag{2}$$
$$F = \frac{Q^2}{4\pi\varepsilon_0 r^2} \quad \text{(electrostatic force between charges)} \tag{3}$$
$$i = Q/t \tag{4}$$

Combining (1)–(4), $[\mu_0\varepsilon_0] = [T^2/L^2]$

$$\text{or} \quad \left[\frac{1}{\sqrt{\mu_0\varepsilon_0}}\right] = \left[\frac{L}{T}\right] = [v] = [c]$$

14.14 We first derive the dimensional formulae for R, C and L from the defining equation Power $= i^2R$

$$[\text{Power}] = [Ml^2T^{-3}] = [A^2R] = [A^2][R]$$
$$\therefore \quad [R] = [Ml^2T^{-3}A^{-2}] \tag{1}$$

Energy of a capacitor $E = \dfrac{1}{2}\dfrac{Q^2}{C}$ and $Q = it$

$$\therefore [C] = [M^{-1}l^{-2}T^4A^2] \tag{2}$$

Energy of an inductance, $E = \frac{1}{2} Li^2$

$$\therefore [L] = [Ml^2T^{-2}A^{-2}] \tag{3}$$

Using (1), (2) and (3), it is observed that the given combinations have the dimension of time and therefore are expressed in seconds

$$[RC] = [L/R] = \left[\sqrt{LC}\right] = [T]$$

14.15 $$\omega = \frac{1}{\sqrt{LC}} \tag{1}$$

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{2}$$

$$\frac{\omega - \omega'}{\omega} = 1 - \frac{\omega'}{\omega} = 1 - \sqrt{1 - \frac{R^2C}{4L}} \simeq 1 - \left(1 - \frac{R^2C}{8L}\right) = \frac{R^2C}{8L} \tag{3}$$

where we have used (1) and (2) and expanded the radical binomially.

$$q = q_{\mathrm{m}}\mathrm{e}^{-Rt/2L}, q/q_{\mathrm{m}} = \frac{1}{2}$$

$$\text{whence } t = \frac{2L}{R} \ln\left(\frac{q_{\mathrm{m}}}{q}\right) = \frac{2L}{R} \ln 2 \tag{4}$$

If n is the number of cycles and ν the oscillating frequency

$$t = \frac{n}{\nu} = 2\pi n\sqrt{LC} = \frac{2L}{R} \ln 2$$

$$\text{or} \quad \frac{CR^2}{L} = \frac{(\ln 2)^2}{(\pi n)^2} \tag{5}$$

Combining (3) and (5)

$$\frac{\omega - \omega'}{\omega} = \frac{\Delta\omega}{\omega} = \frac{(\ln 2)^2}{8\pi^2 n^2} = \frac{0.006085}{n^2} = 0.00038$$

where we have put $n = 4$.

14.16 $q = q_{\mathrm{m}}\mathrm{e}^{-Rt/2L} \cos \omega' t$ (charge oscillation of damped oscillator)
Differentiating with respect to time

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = -q_{\mathrm{m}}\omega'\mathrm{e}^{-Rt/2L}\left(\frac{R}{2L\omega'}\cos\omega' t + \sin\omega' t\right)$$

$$= -q_m\omega' e^{-Rt/2L} \quad (\tan\phi\cos\omega' t + \sin\omega' t)$$

where we have set $R/2L\omega' = \tan\phi$. This gives

$$
\begin{aligned}
i &= \frac{-q_m\omega' e^{-Rt/2L}}{\cos\phi} \quad (\sin\phi\cos\omega' t + \cos\phi\sin\omega' t) \\
&= -q_m\omega' e^{-Rt/2L}\frac{\sin(\omega' t + \phi)}{\cos\phi}
\end{aligned}
$$

But for low damping $\phi \to 0$ as $R/2L\omega' \to 0$.

$$
\begin{aligned}
&\therefore \quad \cos\phi \to 1, \text{ so that} \\
&i = -q_m\omega' e^{-Rt/2L}\sin(\omega' t + \phi)
\end{aligned}
$$

14.17 Equation of the circuit is

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = \xi \tag{1}$$

Multiply (1) by $i = dq/dt$

$$
\begin{aligned}
&Li\frac{di}{dt} + \frac{1}{C}q\frac{dq}{dt} = \xi t \\
&\text{or} \quad \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + \frac{d}{dt}\left(\frac{q^2}{2C}\right) = P_{\text{input}}
\end{aligned} \tag{2}
$$

Thus, the input power is the sum of the powers delivered to the inductor and the capacitor.

14.18 The circuit equation is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \tag{1}$$

Multiply (1) by $i = \dfrac{dq}{dt}$

$$
\begin{aligned}
&Li\frac{di}{dt} + Ri^2 + \frac{1}{C}q\frac{dq}{dt} = 0 \\
&\text{or} \quad \frac{d}{dt}\left(\frac{1}{2}Li^2 + \frac{q^2}{2C}\right) = -Ri^2 \\
&\text{or} \quad \frac{dE}{dt} = -i^2R
\end{aligned} \tag{2}
$$

where $E = \dfrac{1}{2}Li^2 + \dfrac{q^2}{2C}$ = total energy

14.19 If U is the total field energy then

$$U = U_{\mathrm{B}} + U_{\mathrm{E}} = \frac{1}{2}Li^2 + \frac{1}{2}\frac{Q^2}{C} \tag{1}$$

which shows that at any time the energy is stored partly in the magnetic field in the conductor and partly in the electric field in the capacitor. In the presence of the resistance R the energy is transferred to Joule heat, being given by

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -i^2R \tag{2}$$

the minus sign signifying that the stored energy U decreases with time. Differentiating (1) with respect to time and equating the result with (2) gives

$$Li\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{Q}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} = -i^2R \tag{3}$$

Substituting $i = \mathrm{d}Q/\mathrm{d}t$ and $\mathrm{d}i/\mathrm{d}t = \mathrm{d}^2Q/\mathrm{d}t^2$, (3) becomes

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{LC} = 0 \tag{4}$$

Writing $R/L = 2\gamma$ and $1/LC = \omega_0^2$, (4) takes the required form

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}Q}{\mathrm{d}t} + \omega_0^2 Q = 0 \tag{5}$$

(a) $f_0 = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{80\times10^{-3}\times700\times10^{-12}}} = 2.128\times10^4\ \mathrm{Hz}$

(b) $\tau = \dfrac{L}{R} = \dfrac{80\times10^{-3}}{100} = 8\times10^{-4}\ \mathrm{s}$

14.20 $$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}Q}{\mathrm{d}t} + \omega_0^2 Q = 0 \tag{1}$$

Let $Q = \mathrm{e}^{\lambda t}$ so that $\mathrm{d}Q/\mathrm{d}t = \lambda\mathrm{e}^{\lambda t}$ and $\mathrm{d}^2Q/\mathrm{d}t^2 = \lambda^2\mathrm{e}^{\lambda t}$.
The characteristic equation is then

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$$

whose roots are $\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ (2)

Calling $\alpha = \sqrt{\gamma^2 - \omega_0^2}$

$\lambda_1 = -\gamma + \alpha, \lambda_2 = -\gamma - \alpha$

The general solution becomes

$$Q = C_1 e^{(-\gamma+\alpha)t} + C_2 e^{(-\gamma-\alpha)t} \tag{3}$$

The constants C_1 and C_2 are determined from the initial conditions. Suppose at $t = 0$, $Q = Q_0$ and

$$i = \mathrm{d}Q/\mathrm{d}t = 0$$

$$Q_0 = C_1 + C_2 \tag{4}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = C_1(-\gamma + \alpha) - C_2(\gamma + \alpha) = 0 \tag{5}$$

Solving (4) and (5), $C_1 = Q_0(\gamma + \alpha)/2\alpha$ and $C_2 = Q_0(\alpha - \gamma)/2\alpha$
Substituting C_1 and C_2 in (3)

$$Q = \frac{1}{2} Q_0 e^{-\gamma t} \left[(1 + \gamma/\alpha)e^{\alpha t} + (1 - \gamma/\alpha)e^{-\alpha t}\right] \tag{6}$$

For underdamping condition resistance R is small so that $\gamma < \omega_0$ and α is imaginary and may be written as $\alpha = j\omega'$, where j is imaginary. The roots of the characteristic equation are complex conjugate.

$$\omega'^2 = \omega_0^2 - \gamma^2 \tag{7}$$

Equation (6) reduces to

$$Q = Q_0 e^{-\gamma t}[\cos\omega' t + (\gamma/\omega')\sin\omega' t] \tag{8}$$

Calling $\sin\varepsilon = -\gamma/\omega_0$ and $\cos\varepsilon = \omega'/\omega_0$, (8) becomes

$$Q = \left(\frac{\omega_0 Q_0}{\omega'}\right) e^{-\gamma t}\cos(\omega' t + \varepsilon) \tag{9}$$

$$\text{or}\quad Q = A e^{-\gamma t}\cos(\omega' t + \varepsilon) \tag{10}$$

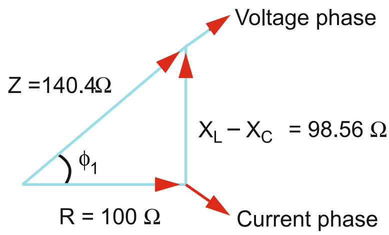
where the amplitude $A = \omega_0 Q_0/\omega'$ and the phase $\varepsilon = \tan^{-1}(-\gamma/\omega')$.
The constants A and ε which are real are determined by initial conditions. Equation (9) represents a damped harmonic motion of period

$$T' = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}} \tag{11}$$

As in the case of an undamped oscillation, the frequency is independent of the amplitude but is always lower than that of the undamped oscillator. The amplitude of oscillations $Ae^{-\gamma t}$ decreases exponentially and is no longer constant.

14.21 Phasor diagram, Fig. 14.5

Fig. 14.5



(i) $X_L = \omega L = 2\pi f L = (2\pi)(80)(0.2) = 100.48\,\Omega$

$$X_C = \frac{1}{\omega c} = \frac{1}{2\pi f c} = \frac{1}{(2\pi)(80)(10 \times 10^{-6})} = 199.04\,\Omega$$

(ii) $Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(100)^2 + (100.48 - 199.04)^2} = 140.4\Omega$

(iii) $I_T = \dfrac{V}{Z} = \dfrac{600}{140.4} = 4.27\,\text{A(rms)}$

(iv) $\cos\varphi = \dfrac{R}{Z} = \dfrac{100}{140.4} = 0.71225 \to \varphi = 44.58°$

(v) $V_R = I_T R = 4.27 \times 100 = 427\, V_{rms}$

$V_C = I_T X_C = 4.27 \times 199.04 = 850\, V_{rms}$

$V_L = I_T X_L = 4.27 \times 100.48 = 429\, V_{rms}$

$V^2 = V_R^2 = (V_L - V_C)^2$

The voltages on R, C and L are shown in the phasor diagram, Fig. 14.6. Here the voltage lags the current as $X_C > X_L$.

(vi) $\omega^2 = \dfrac{1}{LC}$

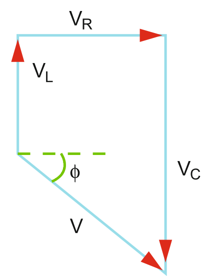


Fig. 14.6

(vii) The circuit will be in resonance when $X_L = X_C$, that is, $\omega L = \dfrac{1}{\omega c}$ or $\omega^2 = \dfrac{1}{LC}$.

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 10 \times 10^{-6}}} = 112.6\,\text{Hz}$$

14.22 $X_\text{C} = \frac{1}{2\pi f C} = \frac{1}{(2\pi)(300)(50 \times 10^{-6})} = 10.6\,\Omega$

$$Z = \sqrt{R^2 + X_\text{C}^2} = \sqrt{(300)^2 + (10.6)^2} = 300.19\,\Omega$$

$$I = \frac{V}{Z} = \frac{5}{300.19} = 0.0166\,\text{A}$$

14.23 Impedance of a capacitor

i. Let an AC emf be applied across a capacitor. The potential difference across the capacitor will be

$$V_\text{C} = V_0 \sin \omega t \tag{1}$$

where V_0 is the amplitude of the AC voltage of angular frequency $\omega t = 2\pi f$, across the capacitor.

$$q_C = CV_\text{C} = CV_0 \sin \omega t \tag{2}$$

The current $i_\text{c} = \frac{\mathrm{d}q_\text{C}}{\mathrm{d}t} = \omega C V_0 \cos\, \omega t = \omega C V_0 \sin(\omega t + 90°)$ (3)

$$\text{or} \quad i_\text{c} = \frac{V_0}{X_\text{c}} \sin\,(\omega t + 90°) \tag{4}$$

$$\text{where} \quad X_\text{C} = 1/\omega C \tag{5}$$

Comparison of (4) with (1) shows that i_c leads V_C by 90° or quarter of a cycle. Further, the current amplitude

$$I_0 = \frac{V_0}{X_\text{C}} \tag{6}$$

By Ohm's law X_C is to be regarded as impedance offered by the capacitor. In complex plane

$$Z_\text{c} = \frac{-\text{j}}{\omega C} \tag{7}$$

where j is imaginary.

Impedance of an inductance

On applying an AC across an inductance the potential difference will be

$$V_L = V_0 \sin \omega t \tag{8}$$

where V_0 is the amplitude of V_L. By Faraday's law of induction ($\xi = -L\,\mathrm{d}i/\mathrm{d}t$) we can write

$$V_L = L\frac{\mathrm{d}i_L}{\mathrm{d}t} \tag{9}$$

Combining (8) and (9)

$$\frac{\mathrm{d}i_L}{\mathrm{d}t} = \frac{V_0}{L}\sin\omega t \tag{10}$$

$$\therefore \quad i_L = \int \mathrm{d}i_L = \frac{V_0}{L}\int \sin\omega t\,\mathrm{d}t = -\left(\frac{V_0}{\omega L}\right)\cos\omega t$$

$$\therefore \quad i_L = \left(\frac{V_0}{X_L}\right)\sin(\omega t - 90°) \tag{11}$$

where $\quad X_L = \omega L \qquad (12)$

is known as the inductive impedance. In complex plane $X_L = j\omega L$, where j is imaginary. Comparison of (11) with (8) shows that the current in the inductance lags behind the voltage by 90° or quarter of a cycle.

ii. $Z_L = \sqrt{R^2 + \omega^2 L^2} = \sqrt{(44)^2 + (2\pi \times 150 \times 0.06)^2} = 71.63\,\Omega$

$$Z_C = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} = \sqrt{10^2 + \frac{1}{(2\pi \times 150 \times 10^{-4})^2}} = 14.58\,\Omega$$

14.24 **(a)** Electric current is the time rate of flow of charge; in symbols $I = \mathrm{d}Q/\mathrm{d}t$.

(b) The charge Q is an integral multiple of the unit of electron's charge e, that is, $Q = ne$, where n is a number. The charge Q is said to be quantized

(c) **(i)** Current density $j = \frac{i}{A} = \frac{2.4 \times 10^{-4}}{(5.6 \times 10^{-3})(50 \times 10^{-6})} = 857\,\mathrm{A/m^2}$

(ii) Drift speed $V_d = \frac{j}{ne} = \frac{857}{(8.5 \times 10^{28})(1.6 \times 10^{-19})} = 6.3 \times 10^{-8}\,\mathrm{m/s}$

(iii) Collisions with atoms and ions of the conductor makes possible large currents to pass.

14.25 By problem $\quad \omega = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}} \qquad (1)$

When the combinations L_1C_1 and L_2C_2 are connected in series, the combination will have inductance L and capacitance C given by

$$L = L_1 + L_2 \tag{2}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{3}$$

Now $\quad LC = (L_1 + L_2)\frac{C_1 C_2}{C_1 + C_2} \qquad (4)$

From (1) we have $L_2 = \dfrac{L_1 C_1}{C_2}$ (5)

Substituting (5) in (4) we get on simplification $LC = L_1 C_1 = \dfrac{1}{\omega^2}$

It follows that $\omega = \dfrac{1}{\sqrt{LC}}$

14.26 $Z_c = \dfrac{1}{\omega c} = \dfrac{1}{2\pi f c}$

$$\therefore \quad C = \frac{1}{2\pi f z_c} = \frac{1}{2\pi \times 1000 \times 500} = 3.18 \times 10^{-7}\,\text{F} = 0.318\,\mu\text{F}$$

$$Z_L = \omega L = 2\pi f L$$

$$\therefore \quad L = \frac{Z_L}{2\pi f} = \frac{100}{2\pi \times 5000} = 3.18 \times 10^{-3}\,\text{H} = 3.18\,\text{mH}$$

14.27 The maximum stored energy in the capacitor must equal the maximum stored energy in the inductor, from the principle of energy conservation.

$$\therefore \quad \frac{1}{2}\frac{q_m^2}{C} = \frac{1}{2}L i_m^2 \qquad (1)$$

where i_m is the maximum current and q_m is the maximum charge. Substituting CV_0 for q_m and solving for i_m in (1)

$$i_m = V_0\sqrt{\frac{C}{L}} = 100\sqrt{\frac{0.01 \times 10^{-6}}{10 \times 10^{-3}}} = 0.1\,\text{A}$$

14.28 For resonance

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad C = \frac{1}{4\pi^2 f^2 L} = \frac{1}{4\pi^2 (5 \times 10^5)^2 \times 10^{-3}} = 1.013 \times 10^{-9}\,\text{F}$$

Quality factor

$$Q = \frac{\omega L}{R}$$

$$\therefore \quad R = \frac{2\pi f L}{Q} = \frac{2\pi (500 \times 10^3)(10^{-3})}{150} = 20.944\,\Omega$$

$\therefore$ Resistance to be included in series is $20.944 - 5.0 = 15.944\,\Omega$.

14.29 For parallel resonance circuit

$$\omega = \omega_0\sqrt{1 - \frac{CR^2}{L}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 5 \times 10^{-6}}} = 1.414214 \times 10^6\,\text{rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{\omega_0}{2\pi}\sqrt{1 - \frac{CR^2}{L}}$$

$$= \frac{1.414214 \times 10^6}{2\pi}\sqrt{1 - \frac{5 \times 10^{-10} \times 10^2}{10^{-3}}}$$

$$= 2.250736 \times 10^5/\text{s}$$

Quality factor

$$Q = \frac{\omega L}{R} = \frac{1.414 \times 10^6 \times 10^{-3}}{10} = 141.4$$

14.30 $V = V_0 \mathrm{e}^{-t/RC}$

$$\ln V = \ln V_0 - \frac{t}{RC}$$

$$\frac{\Delta V}{V} = 0 - \frac{t}{C}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R}\right)\Delta R - \frac{t}{R}\frac{\mathrm{d}}{\mathrm{d}C}\left(\frac{1}{C}\right)\Delta C$$

$$= \frac{t}{CR}\left(\frac{\Delta R}{R} + \frac{\Delta C}{C}\right) = \frac{50 \times 10^{-6}}{10^{-8} \times 5 \times 10^4}\left(\frac{5}{100} + \frac{10}{100}\right) = 0.015$$

14.31 **(a)** $V = \xi\, \mathrm{e}^{-t/RC}$

The voltage on the condensor will fall to l/e of its initial value when the time

$$t = RC = 10^4 \times 10^{-5} = 0.1\,\text{s}$$

(b) Error on t will result from error on R.

$$\Delta t = C\Delta R$$

$$\therefore \quad \Delta t/t = \Delta R/R$$

$$\text{Power } P = i^2 R = \frac{\xi^2}{e^2 R} = \frac{(3000)^2}{(2.718)^2 \times 10^4} = 121.8\,\text{W}$$

$$\text{Energy } U = P.t = 121.8 \times 0.1 = 12.18\,\text{J}$$

$$\text{Heat } H = 12.18/4.18 = 2.914\,\text{cal}$$

Rise in temperature

$$\Delta T = \text{Heat / thermal capacity} = 2.914/0.9 = 3.24° \text{ C}$$

$$R = R_0(1 + \alpha\Delta T) = 10^4(1 + 0.004 \times 3.24)$$

$$= 1.01296 \times 10^4$$

$$\Delta R = R - R_0 = 129.6$$

$$\therefore \quad \frac{\Delta t}{t} = \frac{\Delta R}{R_0} = \frac{129.6}{10^4} = 0.01296$$

$$\text{Percentage error} = \frac{\Delta t}{t} \times 100 = 1.3\%$$

14.32 The amplitude i_m of the current oscillations is given by

$$i_m = \frac{E_m}{\sqrt{(\omega' L - 1/\omega' c)^2 + R^2}}$$

At resonance, $\omega' = \omega$ and $i_m = \dfrac{E_m}{R}$

$$\text{Set} \quad i_m = \frac{E_m}{\sqrt{(\omega L - 1/\omega c)^2 + R^2}} = \frac{1}{2}\frac{E_m}{R}$$

Squaring and simplifying

$$\left(\omega L - \frac{1}{\omega c}\right)^2 = 3\,R^2$$

$$\text{or} \quad \omega^2\,LC \pm \sqrt{3}\omega RC - 1 = 0$$

The only acceptable solutions are

$$\omega_1 = \frac{\sqrt{3}}{2}\frac{R}{L} + \sqrt{\frac{3R^2}{4L^2} + \frac{1}{LC}}$$

$$\omega_2 = -\frac{\sqrt{3}}{2}\frac{R}{L} + \sqrt{\frac{3R^2}{4L^2} + \frac{1}{LC}}$$

Subtracting the last equation from the previous one

$$\Delta\omega = \omega_1 - \omega_2 = \sqrt{3}\frac{R}{L}$$

$$\therefore \quad \frac{\Delta\omega}{\omega} = \frac{\sqrt{3}}{\omega}\frac{R}{L}$$

14.3.2 Maxwell's Equations and Electromagnetic Waves, Poynting Vector

14.33 $E_z = 100\cos(6\times 10^8 t + 4x)$ (by problem) (1)

$E_z = A\cos(\omega t + kx)$ (standard equation) (2)

Comparison of (1) and (2) shows that

$\omega = 6\times 10^8$ and $k = 4$

$$v = \frac{\omega}{k} = \frac{6\times 10^8}{4} = 1.5\times 10^8\ \mathrm{m/s}$$

Dielectric constant, $K = \dfrac{c}{v} = \dfrac{3\times 10^8}{1.5\times 10^8} = 2.0$

14.34 $\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s} = q/\varepsilon_0$ (Gauss' law)

$E(2\pi rl) = q/\varepsilon_0$

$$\therefore \quad E = \frac{\lambda \hat{e}_\mathrm{r}}{2\pi r\,\varepsilon_0} \tag{1}$$

$\oint \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{s} = \mu_0 I$ (Ampere's law)

$2\pi rB = \mu_0 I$

$$\therefore \quad B = \frac{\mu_0 I}{2\pi r}\hat{e}_\phi \tag{2}$$

$E'_\mathrm{r} = \gamma(E_\mathrm{r} - vB_\phi)$ (Lorentz transformation) (3)

$$= \gamma\left(\frac{\lambda}{2\pi r\,\varepsilon_0} - \frac{v\,\mu_0 I}{2\pi r}\right) = \frac{\gamma}{2\pi\,r}\left(\frac{\lambda}{\varepsilon_0} - v\mu_0 I\right)$$

Thus $\quad E'_\mathrm{r} = 0$ if $v\mu_0 I = \dfrac{\lambda}{\varepsilon_0}$

or $\quad v = \dfrac{\lambda}{I\mu_0\,\varepsilon_0} = \dfrac{\lambda c^2}{I}$

14.35 Maxwell's equations in vacuum are

$$\nabla\cdot\boldsymbol{E} = 0 \tag{1}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{2}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

$$\nabla\times\boldsymbol{B} = \mu_0\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t} \tag{4}$$

Use the vector identify

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$$

$$\therefore \quad \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right) = -\nabla^2 B \quad (\because \nabla \cdot \boldsymbol{B} = 0 \quad \text{by}(2)$$

$$\therefore \quad \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}) = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla^2 \boldsymbol{B}$$

$$\therefore \quad \nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

14.36 $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$ (Ampere's law) (1)

Use the vector identity $\boldsymbol{A} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = 0$. Put $\boldsymbol{A} = \nabla$.

$$\therefore \quad \nabla \cdot (\nabla \times \boldsymbol{B}) = 0$$

$$\therefore \quad \nabla \cdot \boldsymbol{j} = 0 \tag{2}$$

More generally, $\nabla \cdot \boldsymbol{j} + \dfrac{\partial \rho}{\partial t} = 0$ (continuity equation) (3)

and $\nabla \cdot \boldsymbol{E} = \varepsilon_0 \rho$ (Gauss' law) (4)

Combining (3) and (4)

$$\nabla \cdot \left(\boldsymbol{j} + \frac{1}{\varepsilon_0}\frac{\partial \boldsymbol{E}}{\partial t}\right) = 0$$

14.37 $\nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \boldsymbol{B}}{\partial t^2}$ (free-space wave equation)

Compare with the standard three-dimensional wave equation

$$\nabla^2 \Psi = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \tag{5}$$

$$\therefore \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7})(8.854 \times 10^{-12})}}$$

$$= 2.998 \times 10^8 \text{ m/s} = \text{c}$$

14.38 $\omega = 2\pi \nu = \dfrac{2\pi c}{\lambda} = \dfrac{2\pi \times 3 \times 10^8}{530 \times 10^{-9}} = 3.55 \times 10^{15}$

$$\sigma = \frac{1}{\rho} = \frac{1}{26.5 \times 10^{-9}} = 3.77 \times 10^7$$

$$\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 3.77 \times 10^7 \times 3.55 \times 10^{15}}}$$

$$= 3.45 \times 10^{-9}\text{ m} = 3.45\text{ nm}$$

14.39 $\boldsymbol{E} = \boldsymbol{E}_0 \cos(kx - \omega t)$ (1)

$$\frac{\partial E_y}{\partial x} = -kE_0 \sin(kx - \omega t) \qquad (2)$$

$$\boldsymbol{B} = \boldsymbol{B}_0 \cos(kx - \omega t) \qquad (3)$$

$$\frac{\partial B_Z}{\partial t} = \omega B_0 \sin(kx - \omega t) \qquad (4)$$

But $\dfrac{\partial E_y}{\partial x} = -\dfrac{\partial E_z}{\partial t}$ (5)

Combining (2), (4) and (5), we get

$$E_0 = \frac{\omega}{k} B_0 = cB_0$$

14.40 Maxwell's equations for a non-ferromagnetic homogeneous isotropic medium can be written as

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon} \qquad (1)$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad (2)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (3)$$

$$\nabla \times \boldsymbol{B} = \mu\sigma \boldsymbol{E} + \mu\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \qquad (4)$$

Taking the curl of (4)

$$\nabla \times (\nabla \times \boldsymbol{B}) = \mu\sigma(\nabla \times \boldsymbol{E}) + \mu\varepsilon \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}) \qquad (5)$$

where the time and space derivatives are interchanged as $\boldsymbol{E}$ is assumed to be a well-behaved function. Expression (3) can be substituted in (5) to obtain

$$\nabla \times (\nabla \times \boldsymbol{B}) = -\mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \qquad (6)$$

Using the vector identity

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla (\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} \qquad (7)$$

By virtue of (2), $\nabla \cdot B = 0$ and (6) becomes

$$\nabla^2 \boldsymbol{B} = \mu\varepsilon \frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} \qquad (8)$$

A similar procedure applied to (3) yields a similar equation for the E-field. Taking the curl of (3)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) \tag{9}$$

Using (4) in (9)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mu\sigma\frac{\partial \boldsymbol{E}}{\partial t} - \mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{10}$$

But $\quad \nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \qquad (11)$

and $\quad \nabla \cdot \boldsymbol{B} = \frac{\rho}{\varepsilon} \qquad (1)$

Combining (10), (11) and (1) we obtain for uncharged medium ($\rho = 0$)

$$\nabla^2 \boldsymbol{E} = \mu\sigma\frac{\partial \boldsymbol{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

14.41 **(a)** $\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = q \quad$ (Gauss' law) $\qquad (1)$

$$\varepsilon_0 E(2\pi r l) = q$$

or $\quad E = q/2\pi\varepsilon_0 r l \qquad (2)$

the flux being entirely through the cylindrical surface and zero through the end caps. The potential difference between the conductors is

$$V = \int_a^b E\mathrm{d}r = \int_a^b \frac{q}{2\pi\varepsilon_0 l}\frac{\mathrm{d}r}{r} = \frac{q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right)$$

Capacitance $C = \frac{q}{V} = \frac{2\pi\varepsilon_0 l}{\ln(b/a)} \qquad (3)$

$\therefore$ Capacitance per unit length of the cable is

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a)} \tag{3a}$$

(b) The magnetic induction between the conductors is

$$B = \frac{\mu_0 i}{2\pi r} \tag{4}$$

Energy density $u = \frac{1}{2\mu_0}B^2 = \frac{\mu_0 i^2}{8\pi^2 r^2} \qquad (5)$

where we have used (4).

Consider a volume element dV for the cylindrical shell of radii r and $r + dr$ and of length l. The energy contained in the volume element is

$$\mathrm{d}U = u\,\mathrm{d}V = \frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi r l\,\mathrm{d}r) = \frac{\mu_0 i^2 l}{4\pi}\frac{\mathrm{d}r}{r} \tag{6}$$

$$U = \int \mathrm{d}U = \frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{\mathrm{d}r}{r} = \frac{\mu_0 i^2 l}{4\pi}\ln(b/a) \tag{7}$$

But $$U = \frac{1}{2}Li^2 \tag{8}$$

Comparing (7) and (8)

$$L = \frac{\mu_0 l}{2\pi}\ln(b/a)$$

$\therefore$ Inductance per unit length of the cable is

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln(b/a)$$

14.42 **(a)** By prob. (14.41) the potential difference between the conductors is

$$V = \frac{q}{2\pi\varepsilon_0 l}\ln(b/a) = \frac{\lambda}{2\pi\varepsilon_0}\ln(b/a) \tag{1}$$

where $\lambda = q/l$ is the charge density.

Now $$E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{\xi}{r\ln(b/a)} \tag{2}$$

where we have put $V = \xi$.

(b) $$B = \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 \xi}{2\pi\, r\, R} \quad (a < r < b) \tag{3}$$

(c) The Poynting vector

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{\mu_0}EB = \frac{\xi^2}{2\pi r^2 R\ln(b/a)}$$

where we have used (2) and (3).

14.43 $$u_\mathrm{B} = \frac{1}{2}B\cdot H \tag{1}$$

$$B = \mu\boldsymbol{H} \tag{2}$$

Substituting (2) in (1)

$$u_B = \frac{1}{2}\,\mu\,H \cdot H = \frac{1}{2}\,\mu\,H^2 = \frac{B^2}{2\mu} \tag{3}$$

In free space $\mu = \mu_0$. Therefore in vacuum

$$u_\mathrm{B} = \frac{B^2}{2\mu_0}$$

14.44 $u_\mathrm{E} = \frac{1}{2}\,\varepsilon_0\,E^2$ (energy density in E-field) (1)

$$u_\mathrm{B} = \frac{1}{2}\,\frac{B^2}{\mu_0} \quad \text{(energy density in } B\text{-field)} \tag{2}$$

The fields for the plane wave are

$$E = E_\mathrm{m}\sin(kx - \omega t) \tag{3}$$

$$B = B_\mathrm{m}\sin(kx - \omega t) \tag{4}$$

Substituting (3) in (1) and (4) in (2)

$$u_\mathrm{E} = \frac{1}{2}\varepsilon_0\,E_\mathrm{m}^2\sin^2(kx - \omega t) \tag{5}$$

$$u_\mathrm{B} = \frac{1}{2}\,\frac{B_\mathrm{m}^2}{\mu_0}\sin^2(kx - \omega t) \tag{6}$$

Dividing (5) by (6)

$$\frac{u_\mathrm{E}}{u_\mathrm{B}} = \frac{\varepsilon_0\mu_0 E_\mathrm{m}^2}{B_\mathrm{m}^2} \tag{7}$$

But $\varepsilon_0\mu_0 = \frac{1}{c^2}$ and $E_\mathrm{m} = cB_\mathrm{m}$

$$\therefore \quad \frac{u_\mathrm{E}}{u_\mathrm{B}} = 1 \quad \text{or} \quad u_\mathrm{E} = u_\mathrm{B}$$

14.45 By prob. (14.41) the magnetic energy stored in a coaxial cable

$$U_\mathrm{B} = \frac{\mu_0 i^2 l}{4\pi}\ln(b/a) \tag{1}$$

where i is the current and l is the length of the cable. Further, its capacitance is given by

$$C = \frac{2\pi\varepsilon_0 l}{\ln(b/a)} \tag{2}$$

The electric energy stored in the cable

$$U_{\text{E}} = \frac{1}{2}\xi^2 C = \frac{1}{2}(i^2 R^2)\left(\frac{2\pi\varepsilon_0 l}{\ln(b/a)}\right) \tag{3}$$

By problem $U_E = U_M$.

$$\therefore \quad \frac{1}{2}(i^2 R^2)\left(\frac{2\pi\varepsilon_0 l}{\ln(b/a)}\right) = \frac{\mu_0 i^2 l}{4\pi}\ln(b/a) \tag{4}$$

$$\therefore \quad R = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\ln(b/a) = \frac{1}{2\pi}\sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}}\ln(b/a)$$

$$= \frac{376.7}{2\pi}\ln(b/a)\,\Omega.$$

14.46 As the kinetic energy of proton (20 MeV) is much smaller than its rest mass energy (938 MeV), non-relativistic calculations will be valid. In the classical picture a charged particle undergoing acceleration 'a' emits electromagnetic radiation. The electromagnetic energy radiated per second is given by

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} \tag{1}$$

Now $a = \dfrac{v^2}{R} = \dfrac{1}{2}mv^2\dfrac{2}{mR} = \dfrac{2K}{mR}$ (2)

Substituting (2) in (1) and putting $q = e$ for the charge of proton

$$P = \frac{2e^2 K^2}{3\pi\varepsilon_0 c^3 m^2 R^2} \tag{3}$$

The energy radiated per orbit is given by multiplying P by this time period $2\pi R/v = 2\pi R\sqrt{m/2K}$

$$\therefore \quad \Delta K = \frac{4e^2 K^{3/2}}{3\sqrt{2}\varepsilon_0 c^3 m^{3/2} R}$$

$$= \frac{4(1.6 \times 10^{-19})^2 (20 \times 1.6 \times 10^{-13})^{3/2}}{3\sqrt{2}(8.85 \times 10^{-12})(3 \times 10^8)^3 (1.67 \times 10^{-27})^{3/2}(0.5)}$$

$$= 1.89 \times 10^{-31}\,\text{J} = 1.18 \times 10^{-12}\,\text{eV}$$

which is quite negligible

14.47 Consider a sphere of radius r with its centre at the point source of power P. Then the intensity of radiation at distance r will be $I = \dfrac{P}{4\pi r^2}$.

$$I = \frac{P}{4\pi r^2} = \frac{c\,\varepsilon_0 E_0^2}{2}$$

$$\therefore \quad E_0 = \left[\frac{P}{2\pi\, r^2 c\varepsilon_0}\right]^{1/2} = \left[\frac{40}{2\pi(1.0)^2(3\times 10^8 \times 8.85\times 10^{-12})}\right]^{1/2} = 49\,\mathrm{V/m}$$

14.48 **(a)** $I_0 = \dfrac{P}{A} = \dfrac{1.2\times 10^{-3}}{4\times 10^{-6}} = 300\,\mathrm{W/m^2}$

(b) $I_0 = \dfrac{c\varepsilon_0 E_0^2}{2} \rightarrow E_0 = \left[\dfrac{2I_0}{c\varepsilon_0}\right]^{1/2} = \left[\dfrac{2\times 300}{3\times 10^8 \times 8.85\times 10^{-12}}\right]^{1/2}$
$= 475\,\mathrm{V/m}$

(c) $B_0 = \dfrac{E_0}{c} = \dfrac{475}{3\times 10^8} = 1.58\times 10^{-6}\ \mathrm{T} = 1.58\,\mu\mathrm{T}$

14.49 $I = \dfrac{P}{A} = \dfrac{P}{\pi r^2} = \dfrac{314\times 10^{-3}}{3.14\times(0.5\times 10^{-3})^2} = 4\times 10^5\ \mathrm{W/m^2}$

14.50 The average value of the Poynting vector $< \boldsymbol{s} >$ over a period of oscillation of the electromagnetic wave is known as the radiant flux density, and if the energy is incident on a surface it is called irradiance.

$$\boldsymbol{E} = \boldsymbol{E}_0\cos(kx-\omega t),\ \boldsymbol{B} = \boldsymbol{B}_0\cos(kx-\omega t)$$

$$\therefore \quad \boldsymbol{S} = c^2\varepsilon_0 \boldsymbol{E}\times\boldsymbol{B} = c^2\varepsilon_0\boldsymbol{E}_0\times\boldsymbol{B}_0\cos^2(kx-\omega t)$$

$$\therefore \quad \langle \boldsymbol{S}\rangle = c^2\varepsilon_0\,|\boldsymbol{E}_0\times\boldsymbol{B}_0|\left\langle\cos^2(kx-\omega t)\right\rangle$$

But $\left\langle\cos^2(kx-\omega t)\right\rangle = \dfrac{1}{T}\displaystyle\int_0^T \cos^2(kx-\omega t)\mathrm{d}t = \dfrac{1}{2}$

Further $\boldsymbol{E}\perp\boldsymbol{B}$ and $E_0 = cB_0$.

$$\therefore \quad < \boldsymbol{s} >= I = \frac{1}{2}c\varepsilon_0 E_0^2 = \frac{1}{2}\frac{E_0^2}{\mu_0 c}$$

14.51 $E_z = 50\sin\left[4\pi\times 10^{14}\left(t - \dfrac{x}{3\times 10^8}\right)\right]$

$$I = \frac{c\varepsilon_0 E_0^2}{2} = \frac{1}{2}(3\times 10^8)(8.85\times 10^{-12})(50^2) = 0.066\,\mathrm{W/m^2}$$

14.52 $|\boldsymbol{E}\times\boldsymbol{H}| = \dfrac{|\boldsymbol{E}\times\boldsymbol{B}|}{\mu_0} = \dfrac{\boldsymbol{E}^2}{\mu_0 c} \quad (\because \boldsymbol{B} = \boldsymbol{E}/c)$

$\boldsymbol{E}\times\boldsymbol{H}$ will be in the direction of $\boldsymbol{S} = (\boldsymbol{E}\times\boldsymbol{B})/\mu_0$, which is the direction of propagation.

14.53 $E_z = 10 \sin \pi \left(2 \times 10^6 x - 6 \times 10^{14} t\right)$

$$= 10 \sin \left(2\pi \times 10^6 x - 6\pi \times 10^{14} t\right)$$

Compare this with the standard equation

$$E_z = E_{z_0} \sin(kx - \omega t)$$

(a) $\omega = 2\pi f = 6\pi \times 10^{14} \rightarrow f = 3 \times 10^{14}\,\text{Hz}$
(b) $k = 2\pi \times 10^6 \rightarrow \lambda = 2\pi/k = 10^{-6}\,\text{m}$
(c) $v = \omega/k = 6\pi \times 10^{14}/2\pi \times 10^6 = 3 \times 10^8\,\text{m/s}$
(d) $E_{z0} = 10\,\text{V/m}$
(e) The wave is linearly polarized in the z-direction and propagates along the x-axis.

14.54 The wave propagates in the x-direction while the E-field oscillates along the z-direction, that is, the E-field is contained in the xz-plane. Since B is normal to both E and the direction of propagation it must be contained in the xy-plane.
Thus, $B_x = B_z = 0$, and $B = B_y(x, t)$. Now $B = E/c$.

$$\therefore \quad B_y(x, t) = 3.33 \times 10^{-8} \sin \pi (2 \times 10^6 x - 6 \times 10^{14} t)\,\text{T}$$

14.55 Let ν be the frequency of the incident microwave beam. Let the car be approaching with speed v towards the observer. Then the frequency seen by the car is given by the formula for Doppler shift

$$\nu' = \nu(1 + v/c) \tag{1}$$

Upon reflection the microwave returns as if it was emitted by a moving source travelling with speed v towards the observer. Therefore the observed frequency is

$$\nu'' = \nu'(1 + v/c) = \nu(1 + v/c)^2 \tag{2}$$

where we have used (1).

$$\Delta\nu = \nu'' - \nu = \nu(1 + v/c)^2 - \nu = 2\nu\frac{v}{c}\left(1 + \frac{v}{2c}\right)$$

Assuming that $v/c << 1$

$$\Delta\nu = 2\nu\,\frac{v}{c} \qquad \text{(beat frequency)} \tag{3}$$

For a receding car, proceeding along similar lines,

$$\Delta\nu = \nu'' - \nu = -2\nu\frac{v}{c} \tag{4}$$

14.56 By prob. (14.55), $\Delta\nu = \nu'' - \nu = -2\nu\frac{v}{c}$

$$\Delta\nu = -\frac{2 \times 800 \times 10^6}{3 \times 10^8} \times \left(\frac{5}{18} \times 90\right) = -133\,\text{Hz}$$

14.57 The relation between the current i and the magnetic field B expressed as $\oint B \cdot \mathrm{d}l = \mu_0 i$ is known as Ampere's law. There are equal and opposite currents i in the conductors.

(a) $r < a$. The net current passing through the conductor bounded by the closed path corresponds to that flowing through the inner conductor, Fig. 14.7. Hence by Ampere's theorem

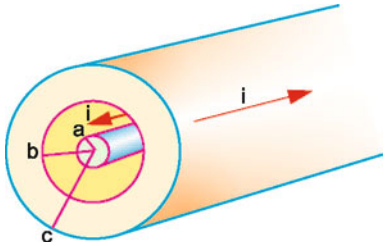


Fig. 14.7 Magnetic field due to current carrying coaxial cylinder

$$\oint B \cdot \mathrm{d}l = (B)(2\pi r) = \mu_0 i \frac{(\pi r^2)}{\pi a^2}$$

$$\text{or} \quad B = \frac{\mu_0 i r}{2\pi a^2}$$

(b) $a < r < b$. Here the current through the outer conductor does not contribute to B.

$$\oint B \cdot \mathrm{d}l = (B)(2\pi r) = \mu_0 i$$

$$\text{or} \quad B = \frac{\mu_0 i}{2\pi r}$$

(c) $b < r < c$. Here currents through both the conductors contribute to B.

$$\oint B \cdot \mathrm{d}l = (B)(2\pi r) = \mu_0 i - \mu_0 i \frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)}$$

$$\text{or} \quad B = \frac{\mu_0 i}{2\pi r} \frac{(c^2 - r^2)}{(c^2 - b^2)}$$

(d) $r > c$. As the net current flowing through the closed path is zero, $B = 0$.

14.58 Magnetic energy density

$$u_{\mathrm{B}} = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\frac{4^2}{4\pi \times 10^{-7}} = \frac{2}{\pi} \times 10^7\ \mathrm{J/m^3}$$

Energy stored in the magnetic field $U_B = u_B \times \text{volume} = u_{\mathrm{B}}\pi r^2 l$

$= 2 \times 10^7 \times 3^2 \times 12.5 = 2.25 \times 10^9\ \mathrm{J}$

14.59 If P_0 is the power radiated by sun of radius r, then using the results of prob. (14.51)

$$P_0 = \langle S \rangle\, 4\pi r^2 = \frac{1}{2\mu_0 c} E_{\mathrm{m}}^2 4\pi r^2$$

$$\therefore \quad E_{\mathrm{m}} = \frac{1}{r}\sqrt{\frac{P_0 \mu_0 c}{2\pi}}$$

$$\therefore \quad E_{\mathrm{m}} = \frac{1}{7 \times 10^8}\sqrt{\frac{4 \times 10^{26} \times 4\pi \times 10^{-7} \times 3 \times 10^8}{2\pi}} = 2.21 \times 10^5\ \mathrm{V/m}$$

$$B_{\mathrm{m}} = \frac{E_{\mathrm{m}}}{c} = \frac{2.21 \times 10^5}{3 \times 10^8} = 7.37 \times 10^{-4}\ \mathrm{T} = 7.37\ \mathrm{G}$$

14.60 By prob. (14.51)

$$\langle S \rangle = \frac{E_{\mathrm{m}}^2}{2\mu_0 c}$$

$$\therefore \quad E_{\mathrm{m}} = \sqrt{2\mu_0 c \langle S \rangle} = \sqrt{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8 \times 1300} = 990\ \mathrm{V/m}$$

14.61 $$B_{\mathrm{m}} = \frac{E_{\mathrm{m}}}{c} = \frac{300}{3 \times 10^8} = 1 \times 10^{-6}\ \mathrm{T}$$

$$\langle S \rangle = \frac{E_{\mathrm{m}}^2}{2\mu_0 c} = \frac{(300)^2}{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8} = 119.4\ \mathrm{W/m^2}$$

14.62 $$\frac{|E|}{|H|} = \frac{E\,\mu_0}{B} = \frac{(Bc)\mu_0}{B} = \frac{\mu_0}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

where we have used the equations $B = \mu_0 H$ and $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 377\ \Omega$$

Units of $\dfrac{|E|}{|H|}$ are $\dfrac{\text{Volt/metre}}{\text{Ampere/metre}} = \dfrac{\text{Volt}}{\text{Ampere}} = \text{Resistance}$

14.63 Skin depth $\delta = \sqrt{\dfrac{2}{2\pi f \sigma \mu_0 \mu}}$

$$= \sqrt{\frac{2}{2\pi \times 20 \times 10^3 \times 6 \times 10^7 \times 4\pi \times 10^{-7} \times 1}}$$

$$= 4.6 \times 10^{-4}\,\text{m} = 0.46\,\text{mm}$$

14.64 $\delta = \sqrt{\dfrac{2}{2\pi f \sigma \mu_0 \mu}}$

$$= \sqrt{\frac{1}{\pi \times 3 \times 10^8 \times 5.6 \times 10^7 \times 4\pi \times 10^{-7} \times 1}} = 1.23 \times 10^{-4}\,\text{m}$$

$$= 0.123\,\text{mm} = 3.88\,\mu\text{m}$$

14.65 Let us begin with the 'curl H' Maxwell's equation

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{1}$$

Assume that all the fields and currents in this equation are sinusoidal at a single frequency ω. In that case (1) can be replaced by

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \mathrm{j}\omega \boldsymbol{D} = \boldsymbol{J} + \mathrm{j}\omega\varepsilon \boldsymbol{E} \tag{2}$$

where J is the current density, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ is the displacement vector and j is imaginary $\left(\sqrt{-1}\right)$.
The 'curl E' Maxwell's equation is obtained in a similar way as (1).

$$\nabla \times \boldsymbol{E} = -\mathrm{j}\omega \boldsymbol{B} \tag{3}$$

Applying the curl operation to each side of (3)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mathrm{j}\omega \nabla \times \boldsymbol{B} \tag{4}$$

Now $\nabla \times \boldsymbol{B}$ can be replaced using (2) and the relation $H = B/\mu$:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mathrm{j}\omega\mu(\boldsymbol{J} + \mathrm{j}\omega\varepsilon \boldsymbol{E}) \tag{5}$$

We now use the vector identity

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) = -\nabla^2 \boldsymbol{E} \tag{6}$$

Conductive materials do not contain any real charge density because any real charge that many exist will repel itself and move outwards until it resides on the material's outer surface. Therefore $\nabla \cdot \boldsymbol{D} = 0$ and so also $\nabla \cdot \boldsymbol{E} = 0$. Thus the first term on the right by (6) vanishes. Therefore (5) becomes

$$\nabla^2 \boldsymbol{E} = \mathrm{j}\omega\mu(\boldsymbol{J} + \mathrm{j}\omega\varepsilon\boldsymbol{E}) \tag{7}$$

Assume that the material under consideration obeys Ohm's law, $\boldsymbol{J} = \sigma_{\mathrm{E}}\boldsymbol{E}$, where σ_{E} is the conductivity. Then (7) becomes

$$\nabla^2 \boldsymbol{E} = \mathrm{j}\omega\mu(\sigma_{\mathrm{E}} + \mathrm{j}\omega\varepsilon)\boldsymbol{E} \tag{8}$$

For simplicity assume that the given material is an excellent conductor, so that $\sigma_{\mathrm{E}} >> |\omega\varepsilon|$. In that case, the displacement current term, masked by the conduction current, can be neglected, yielding

$$\nabla^2 \boldsymbol{E} = \mathrm{j}\omega\mu\sigma_{\mathrm{E}}\boldsymbol{E} \tag{9}$$

Since $\boldsymbol{E} = \boldsymbol{J}/\sigma_{\mathrm{E}}$ we can write

$$\nabla^2 \boldsymbol{J} = j\omega\mu\sigma_{\mathrm{E}}\boldsymbol{J} \tag{10}$$

Suppose the current flows through this material in the z-direction. The current density is independent of x and y. In that case (10) simplifies to

$$\frac{\partial^2 J_z}{\partial z^2} = \mathrm{j}\omega\mu\sigma_{\mathrm{E}} J_z \tag{11}$$

which has the solution

$$\boldsymbol{J}_x = A\mathrm{e}^{-(1+\mathrm{j})z/\delta} + B\mathrm{e}^{(1+\mathrm{j})z/\delta} \tag{12}$$

where A and B are constants, and δ given by

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_{\mathrm{E}}}} \tag{13}$$

is known as the skin depth. Thus the magnitude of current density decreases with depth. This effect is of practical importance as it affects resistive losses accompanying a high-frequency current flow in an electronic circuit.

14.66 The average value of the Poynting vector is

$$\begin{aligned}\langle S \rangle &= \frac{1}{2\mu_0} E_{\mathrm{m}} B_{\mathrm{m}} = \frac{1}{2\mu_0} E_{\mathrm{m}} \left(\frac{E_{\mathrm{m}}}{c}\right) = \frac{E_{\mathrm{m}}^2}{2\mu_0 c} \\ &= \frac{\left(10^{-3}\right)^2}{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8} = 1.327 \times 10^{-9}\,\mathrm{W/m^2} \\ &= 1.327 \times 10^{-13.}\,\mathrm{W/cm^2}.\end{aligned}$$

14.67 Poynting's theorem is a mathematical statement based on Maxwell's equations. The theorem is interpreted as energy balance equation in situations where electromagnetic waves are present. We begin with the vector identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) \tag{1}$$

We now make use of Maxwell's 'curl' equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{3}$$

Substituting (2) and (3) in the RHS of (1)

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{E} \cdot \boldsymbol{J} \tag{4}$$

which is the mathematical statement of Poynting's theorem.

$$\text{Further} \quad \boldsymbol{B} = \mu \boldsymbol{H} \text{ and } \boldsymbol{D} = \varepsilon \boldsymbol{E} \tag{5}$$

$$\frac{\partial}{\partial t}(\boldsymbol{E} \cdot \boldsymbol{E}) = \frac{\partial}{\partial t}|\boldsymbol{E}|^2 = 2\boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \tag{6}$$

Using (5) and (6) in (4) we obtain

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\frac{\partial}{\partial t}\left(\frac{\varepsilon |\boldsymbol{E}|^2}{2}\right) - \frac{\partial}{\partial t}\left(\frac{\mu |\boldsymbol{H}|^2}{2}\right) - \boldsymbol{E} \cdot \boldsymbol{J} \tag{7}$$

We can integrate each side over any arbitrary volume V. Applying the divergence theorem to the integral on the left

$$\int_S (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}S = -\int_v \left[\frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}|\boldsymbol{E}|^2\right) + \frac{\partial}{\partial t}\left(\frac{\mu}{2}|\boldsymbol{H}|^2\right) + \boldsymbol{E} \cdot \boldsymbol{J}\right] \mathrm{d}V \tag{8}$$

and changing the sign of the equation and the order of differentiation and integration we can rewrite (8) as

$$-\int_s (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{S} = \frac{\partial}{\partial t}\int_V \frac{\varepsilon |\boldsymbol{E}|^2}{2} \mathrm{d}V + \frac{\partial}{\partial t}\int_V \frac{\mu |\boldsymbol{H}|^2}{2} \mathrm{d}V + \int_v \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}V \tag{9}$$

The first term on the right represents the rate of increase of electric energy inside the volume V. The second term on the right represents the rate of

increase of magnetic energy. The last term corresponds to the power converted into Joule's heat. Energy conservation demands that these three terms be balanced by flow of energy into the volume and this is accounted for by the term on the left.
The quantity $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, representing the power density, is known as the Poynting vector. The minus sign in (9) means that d$\boldsymbol{s}$ is the outward normal and that $(\boldsymbol{E} \times \boldsymbol{H}) \cdot d\boldsymbol{s}$ represents power flowing outwards rather than inwards for energy balance.

14.68 In one dimension the given equation reduces to

$$\frac{\partial^2 E}{\partial x^2} - \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} - \mu_0 \sigma_N \frac{\partial E}{\partial t} = 0 \qquad (1)$$

Let the travelling wave be given by

$$E = E_0 e^{i(kx - \omega t)} \qquad (2)$$

$$\frac{\partial^2 E}{\partial x^2} = -k^2 E, \ \frac{\partial E}{\partial t} = -i\omega E \text{ and } \frac{\partial^2 E}{\partial t^2} = -\omega^2 E \qquad (3)$$

Inserting (3) in (1) and re-arranging and cancelling E

$$k^2 = \mu_0 \varepsilon_0 \varepsilon_r \omega^2 + i \mu_0 \sigma_N \omega \qquad (4)$$

14.69 $\boldsymbol{F} = x^2 z^3 \hat{i}$

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z^3 & 0 & 0 \end{vmatrix} = 3x^2 z^2 \hat{j}$$

$$\nabla \times 3x^2 z^2 \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 3x^2 z^2 & 0 \end{vmatrix} = -6x^2 z \hat{i} + 6xz^2 \hat{k}$$

$$\therefore \quad \nabla \times (\nabla \times F) = -6x^2 z \hat{i} + 6xz^2 \hat{k} \qquad (1)$$

$$-\nabla^2 F = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) x^2 z^3 \hat{i} = -(6x^2 z + 2z^3)\hat{i} \qquad (2)$$

$$\nabla(\nabla \cdot F) = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^2 z^3 \hat{i})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2xz^3) = 2z^3 \hat{i} + 6xz^2 \hat{k} \qquad (3)$$

Thus $\nabla \times (\nabla \times F) = -6x^2 z\hat{i} + 6xz^2\hat{k}$ (1)

$$-\nabla^2 F + \nabla(\nabla \cdot F) = -(6x^2 z + 2z^3)\hat{i} + 2z^3\hat{i} + 6xz^2\hat{k}$$

$$= -6x^2 z\hat{i} + 6xz^2\hat{k} \tag{4}$$

Comparing (1) and (4), the identity

$\nabla \times (\nabla \times E) = -\nabla^2 E + \nabla(\nabla \cdot E)$ is verified.

14.70 The electric field of the cable is radial and is given by

$$E = E_{\mathrm{r}} = \frac{V}{r\ln(b/a)} \tag{1}$$

where a and b are the radii of the inner and outer cable. The corresponding magnetic intensity is tangential and is given by

$$H = H_\phi = \frac{I}{2\pi r} \tag{2}$$

As the angle between $\boldsymbol{E}$ and $\boldsymbol{H}$ is 90°, the Poynting vector

$$|\boldsymbol{S}| = |\boldsymbol{E} \times \boldsymbol{H}| = EH \tag{3}$$

So that $S = S_z = \dfrac{VI}{2\pi r^2 \ln(b/a)}$ (4)

and the direction of $\boldsymbol{S}$ is that of the current in the positive conductor.
The power flow is confined to the space between the conductors and for any plane perpendicular to the axis of the conductor

$$P = \int_a^b S_z 2\pi r \,\mathrm{d}r = \int_a^b \frac{VI}{\ln(b/a)} \frac{\mathrm{d}r}{r} \tag{5}$$

where we have substituted S_z from (4).

But $\displaystyle\int_a^b \frac{\mathrm{d}r}{r} = \ln(b/a)$ (6)

$$\therefore \quad P = VI \tag{7}$$

This is the entire power transmitted by the cable. It follows that the Poynting theorem indicates that the entire flow of energy resides in the space between the conductors.

If the resistance of the cable cannot be neglected then V is no longer constant. An axial component of $\boldsymbol{E}$ is necessary to maintain the flow of current to compensate for the Ohmic energy loss.

14.71 The current density in the wire is $(I/\pi a^2)\hat{e}_z$. Therefore the electric field in the wire, including on the surface of the wire, will be $(I/\pi a^2\sigma_{\mathrm{E}})\hat{e}_z$.
The magnetic field intensity by Ampere's theorem is $(I/2\pi a^2)\hat{e}_\phi$.
The Poynting vector at the surface is given by

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \left(\frac{I}{\pi a^2 \sigma_{\mathrm{E}}}\hat{e}_z\right) \times \left(\frac{I}{2\pi a}\hat{e}_\phi\right) = -\frac{I^2}{2\pi^2 a^3 \sigma_{\mathrm{E}}}\hat{e}_r \tag{1}$$

Now Poynting's theorem is

$$-\int_s (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{s} = \frac{\partial}{\partial t}\int_V \frac{\varepsilon\,|\boldsymbol{E}|^2}{2}\mathrm{d}V + \frac{\partial}{\partial t}\int_V \frac{\mu\,|\boldsymbol{H}|^2}{2}\mathrm{d}V + \int_V \boldsymbol{E} \cdot \boldsymbol{J}\,\mathrm{d}V \tag{2}$$

Since the fields are constant in time, the first two terms on the right of (2) which contain time derivative $\partial/\partial t$ vanish. The power dissipated in the wire is then

$$P = \int \boldsymbol{E} \cdot \boldsymbol{J}\mathrm{d}V = -\int (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{s} = -\frac{I^2}{2\pi^2 \sigma_{\mathrm{E}} a^3}(2\pi a L) = \frac{I^2 L}{\pi \sigma_{\mathrm{E}} a^2} = I^2 R$$

14.72 Given $\quad \boldsymbol{B} = -\dfrac{m_{\mathrm{e}}}{ne^2}\nabla \times \boldsymbol{J} \qquad (1)$

Use the vector identity

$$\nabla \times (\nabla \times \boldsymbol{B}) = -\nabla^2 \boldsymbol{B} + \nabla(\nabla \cdot \boldsymbol{B}) \tag{2}$$

Use Maxwell's equations

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3}$$

$$\nabla \times \frac{\boldsymbol{B}}{\mu_0} = \boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{4}$$

Here $\quad \dfrac{\partial \boldsymbol{D}}{\partial t} = \varepsilon_0 \dfrac{\partial \boldsymbol{E}}{\partial t} = 0 \qquad (5)$

so that (4) becomes

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{6}$$

Using (3) and (6) in (2)

$$\mu_0 \nabla \times \boldsymbol{J} = -\nabla^2 \boldsymbol{B} \tag{7}$$

Using (1) in (7)

$$\nabla^2 \boldsymbol{B} = \frac{\mu_0 n e^2 \boldsymbol{B}}{m_e}$$

14.73 High-frequency resistance

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} \tag{1}$$

Direct current resistance per metre

$$R = \frac{1}{\sigma A} = \frac{1}{\sigma \pi r^2} \tag{2}$$

Further the skin thickness

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{3}$$

For metals assume $\mu = \mu_0$. Combining (1), (2) and (3)

$$\frac{R_s}{R} = \frac{\pi r^2}{\delta}\sqrt{\frac{\mu_0}{2}} = \frac{\pi(10^{-3})^2}{6.6 \times 10^{-5}}\sqrt{\frac{4\pi \times 10^{-7}}{2}} = 3.77 \times 10^{-5}$$

14.74 When a charge q moves in a magnetic field, it experiences a magnetic force

$$F_m = q\boldsymbol{v} \times \boldsymbol{B} \tag{1}$$

When an electric conductor is physically moved across a magnetic field, the free electrons in the conductor will experience a force on them in the direction of the force. The flow of electrons implies the existence of a potential difference between the ends of the conductor. The situation is the same as if an electric field had been set up in the conductor which is expressed by the relation

$$E_m = F_m/q = \boldsymbol{v} \times \boldsymbol{B}\,\mathrm{V/m} \tag{2}$$

Equation (2) implies that every moving magnetic field is accompanied by an electric field.
From (2) and the definition of the emf ξ of a source, the instant emf of the source is

$$\xi = \int \boldsymbol{E}_{\mathrm{m}} \cdot \mathrm{d}\boldsymbol{l} = \int \boldsymbol{v} \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}\,\mathrm{V} \tag{3}$$

This is the general expression for motional emf. Now Faraday's law states that

$$\oint \boldsymbol{E}_{\mathrm{m}} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{4}$$

This equation states that every time-changing magnetic field has an electric field associated with it. Now the total flux through the surface is

$$\phi = \int_s \boldsymbol{B} \cdot \hat{n}\mathrm{d}s \tag{5}$$

Therefore

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{B} \cdot \hat{n}\,\mathrm{d}s \tag{6}$$

If the source and only the induction are changing, $\mathrm{d}/\mathrm{d}t$ outside the integral may be replaced by $\partial/\partial t$ inside the integral. The expression becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{n}\,\mathrm{d}s \tag{7}$$

Combining (7) with (4)

$$\oint_c \boldsymbol{E}_{\mathrm{m}} \cdot \mathrm{d}\boldsymbol{l} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{n}\,\mathrm{d}s \tag{8}$$

Transforming the line integral in (8) into surface integral by the use of Stokes' theorem

$$\oint_c \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_s \hat{n} \cdot (\nabla \times \boldsymbol{E})\,\mathrm{d}s \tag{9}$$

Combining (8) and (9)

$$\int_s n \cdot (\nabla \times \boldsymbol{E})\,\mathrm{d}s = -\int \hat{n} \cdot \frac{\partial \boldsymbol{B}}{\partial t}\,\mathrm{d}s \tag{10}$$

Since this expression is true for any surface, the two integrals in (10) can be equated to yield

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{11}$$

This is known as Faraday's law in point form or differential form.

14.75 Ampere's law is expressed by

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 i \tag{1}$$

where B is known as magnetic induction or magnetic field. Its unit is weber per metre square or tesla. If magnetic materials are placed in the field of induction, the elementary magnetic dipoles, permanent or induced, will set up its own field that will modify the original field. A large value of B in an iron core is explained by a subsidiary vector, the magnetization M which is the magnetic moment per unit volume of the core material. A hypothetical current i_M is introduced and Ampere's law, (1), is modified accordingly:

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 (i + i_\mathrm{M}) \tag{2}$$

Writing

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 i + \mu_0 \oint M \cdot \mathrm{d}\boldsymbol{l} \tag{3}$$

we find

$$\oint \left(\frac{\boldsymbol{B} - \mu_0 \boldsymbol{M}}{\mu_0}\right) \cdot \mathrm{d}\boldsymbol{l} = i \tag{4}$$

or $$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = i \tag{5}$$

where $$H = \frac{B - \mu_0 M}{\mu_0} \tag{6}$$

is known as the magnetic field strength.

$$\therefore \quad B = \mu_0 (H + M) \tag{7}$$

The unit of H is henry/metre.

(**a**) For paramagnetic material $\boldsymbol{B}$ is directly proportional to $\boldsymbol{H}$, the relation being $B = k_m \ \mu_0 H$, where k_m is the permeability of the magnetic medium, which is a constant for a given temperature and density of the material.

(**b**) In ferromagnetic materials the relationship between B and H is far from linear. The B–H curve is known as the familiar hysteresis curve. k_m is a function not only of the value of H but also because of hysteresis and is a function of the magnetic and thermal history of the specimen.

14.76 Consider Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon \tag{1}$$

and $\quad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t}$ (2)

Define a new electric potential function $\phi(r, t)$ such that

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{3}$$

The reason for redefining the scalar potential in this fashion is that (3) is consistent with Faraday's law, $\nabla \times \boldsymbol{E} = -(\partial \boldsymbol{B}/\partial t)$, as can be verified by substitution. On the other hand the electrostatic definition, $\boldsymbol{E} = -\nabla V$ is inconsistent with Faraday's law and therefore cannot be used in electrodynamics. However, the relation $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, continues to be correct. Substituting (3) in (1), we obtain

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{\rho}{\varepsilon} \tag{4}$$

Substituting $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ in (2), we get

$$\nabla^2\boldsymbol{A} - \nabla(\nabla \cdot \boldsymbol{A}) = -\mu\boldsymbol{J} - \mu\varepsilon\frac{\partial \boldsymbol{E}}{\partial t} \tag{5}$$

It is convenient to choose a Lorentz gauge given by

$$\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t} \quad \text{(Lorentz condition)} \tag{6}$$

With the use of (6), (4) and (5) are simplified to

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{7}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J} \tag{8}$$

14.77 $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ (Faraday's law) (1)

Writing the curl in rectangular form gives

$$\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

Integrating in time and choosing the constant of integration as zero, we obtain

$$B_x = \frac{1}{v}f(z - vt) \tag{2}$$

Notice that the variation of B is exactly the same as the variation of E, except that E_{y} and B_{x} are at right angles to each other and perpendicular to the direction of propagation. From (2) and the relation $H_x = B_x/\mu_0$ we find

$$H_x = \frac{1}{\mu_0 v} f(x - vt) \tag{3}$$

so that $E_{\mathrm{y}} = \mu_0 v H_{\mathrm{x}}$ (4)

Using the relation $v = \sqrt{1/\mu_0 \varepsilon_0}$, we can write (4) in the form

$$E_y = Z_0 H_x$$

with $\quad Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 376.6\,\Omega.$

14.78 **(a)** $\boldsymbol{E}(z,t) = E_0[\hat{x}\sin(kz - \omega t) + \hat{y}\cos(kz - \omega t)]$. This is a plane polarized wave polarized in the xy-plane and propagating in the positive z-direction.

(b) The magnetic lines will suffer refraction in passing from one magnetic medium to another.

(i) The continuity of B lines is first specified as a necessary condition. Figure 14.8 shows a bundle of B lines in passing through the interface between two magnetic media characterized by μ_1 and μ_2.

(ii) Since $di\mathrm{v}\ B = 0$, it is required that the magnetic flux associated with the flux lines be constant in passing through the interface.

$$\phi = B_1\,\mathrm{d}s_1 = B_2\,\mathrm{d}s_2$$

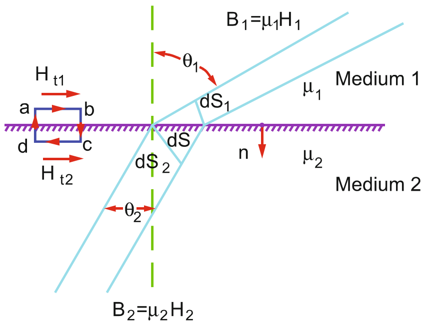


Fig. 14.8 The refraction of magnetic lines

where $\mathrm{d}s_1$ and $\mathrm{d}s_2$ are the cross-section of the flux lines in medium 1 and 2, respectively. Dividing by $\mathrm{d}s$, the corresponding area on the interface, we get

$$B_1 \frac{\mathrm{d}s_1}{\mathrm{d}s} = B_2 \frac{\mathrm{d}s_2}{\mathrm{d}s}$$

which from Fig. 14.8 may be written as

$$B_1 \cos\theta_1 = B_2 \cos\theta_2 \tag{1}$$

which may be written as

$$\boldsymbol{B}_1 \cdot \hat{n} = \boldsymbol{B}_2 \cdot \hat{n}$$

which shows that the normal component of the B vector is the same on both sides of the boundary.

(iii) Next we apply Ampere's circuital law to the path across the interface, Fig. 14.8. Assuming that no current exists in the interface, for the path considered

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Breaking the integral into individual parts of the path

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_a^b \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_b^c \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_c^d \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_d^a \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

In the limit the path shrinks approaching the interface

$$\int_b^c \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_d^a \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

$$\therefore \quad \int_a^b \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_c^d \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Thus $H_{t1} = H_{t2}$

i.e. $\boldsymbol{H}_1 \times \hat{n} = \boldsymbol{H}_2 \times \hat{n}$ (2)

This implies that the tangential component of the $\boldsymbol{H}$ vector is the same on both sides of the boundary.

(c) Dividing (2) by (1)

$$\frac{H_1 \sin\theta_1}{B_1 \cos\theta_2} = \frac{H_2 \sin\theta_2}{B_2 \cos\theta_2}$$

$$\therefore \quad \frac{1}{\mu_1} \tan\theta_1 = \frac{1}{\mu_2} \tan\theta_2$$

$$\therefore \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} \quad \text{(law of refraction)}$$

14.79 Consider a plane wave normally incident on a dielectric discontinuity, as in Fig. 14.9. In the region $z < 0$, $\varepsilon = \varepsilon_1$, and for $z > 0$, $\varepsilon = \varepsilon_2$. The boundary condition on E is that its tangential component is continuous, the boundary condition on H is that its tangential component is also continuous.

$$E_\mathrm{i}(z = 0) + E_\mathrm{r}(z = 0) = \boldsymbol{E}_\mathrm{t}(z = 0) \tag{1}$$

$$H_\mathrm{i}(z = 0) + H_\mathrm{r}(z = 0) = \boldsymbol{H}_\mathrm{t}(z = 0)$$

Letting $E_\mathrm{i} = E_0 \mathrm{e}^{-\mathrm{j}kz} e_x,\ E_\mathrm{r} = E_1 \mathrm{e}^{\mathrm{j}kz} e_x,\ E_\mathrm{t} = E_2 \mathrm{e}^{-\mathrm{j}kz} e_x,$

$$H_\mathrm{i} = \frac{E_0}{\eta} \mathrm{e}^{-\mathrm{j}kz} e_y \text{ and } H_\mathrm{r} = -\frac{E_1}{\eta} \mathrm{e}^{-\mathrm{j}kz} e_y \tag{2}$$

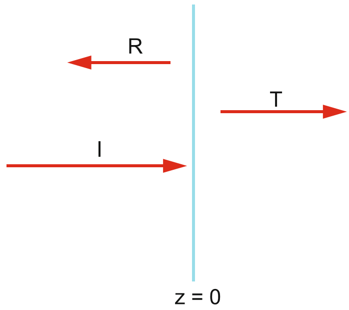


Fig. 14.9 Reflection of plane waves normally incident on the interface between two dielectrics

and substituting in (1), we obtain

$$E_0 + E_1 = E_2 \tag{3}$$

$$\frac{\mathrm{E}_0}{\eta_1} - \frac{\mathrm{E}_1}{\eta_1} = \frac{\mathrm{E}_2}{\eta_2} \tag{4}$$

where $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$ and $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$. Solving, we find

$$\frac{E_1}{E_0} = \rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{5}$$

$$\frac{E_2}{E_0} = \tau = \frac{2\eta_2}{\eta_2 + \eta_1} \tag{6}$$

Note the minus sign in the last relation for H_r in (2) arises because the Poynting vector $S = E \times H$ must be in the direction of propagation (right-hand rule).

Substituting $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$ and $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$ in (5) and (6) and setting $\mu_1 = \mu_2 = \mu_0$ for non-magnetic substances, and putting $\sqrt{\varepsilon_1/\varepsilon_2} = n_1/n_2$ for the refractive index, we obtain

$$R = \rho^2 = \frac{(n_2 - n_1)^2}{(n_2 + n_1)^2} \tag{7}$$

$$T = \tau^2 \frac{n_1}{n_2} = \left(\frac{2n_2}{n_2 + n_1}\right)^2 \frac{n_1}{n_2} = \frac{4n_1 n_2}{(n_1 + n_2)^2} \tag{8}$$

Adding (7) and (8) it follows that $R + T = 1$. This is simply the consequence of conservation of energy.

14.80 Refer to Fig. 14.10. Consider a plane perpendicular to the propagation direction through the origin. Let the distance from this plane measured in the direction of propagation be called l. If the coordinates of a point are x, z, then

$$l_I = x \sin\theta + z \cos\theta \tag{1}$$

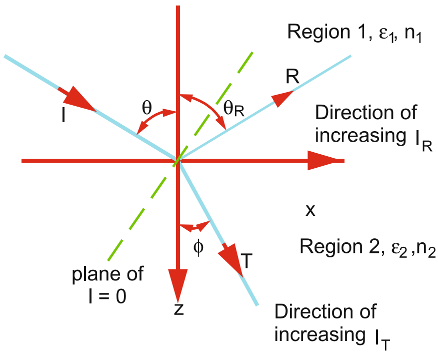


Fig. 14.10 Reflection and Refraction of electromagnetic wave

As the electric field is in the plane of incidence, for the incident wave

$$\begin{cases} E_{\mathrm{I}} = E_0 \mathrm{e}^{\mathrm{j}k_{\mathrm{I}} l_{\mathrm{I}}} (\cos\theta \, e_x - \sin e_z) \\ H_{\mathrm{I}} = \frac{E_0}{\eta_1} \mathrm{e}^{-\mathrm{j}k_{\mathrm{I}} l_{\mathrm{I}}} e_y \end{cases} \tag{2}$$

where $k_1 = \omega\sqrt{\nu\varepsilon_1}$ and $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$ are the values of the propagation constant and characteristic impedance in region 1.

For the reflected wave $l_{\mathrm{R}} = x \sin\theta' - z \cos\theta'$ (3)

$$\begin{cases} E_{\mathrm{R}} = E_{\mathrm{R}} \mathrm{e}^{-\mathrm{j}k_{\mathrm{I}} l_{\mathrm{R}}} (\cos\theta_{\mathrm{R}} \, e_x + \sin\theta_{\mathrm{R}} e_z) \\ H_{\mathrm{R}} = -\dfrac{E_{\mathrm{R}}}{\eta_1} \mathrm{e}^{-\mathrm{j}k_{\mathrm{I}} l_{\mathrm{R}}} e_y \end{cases} \tag{4}$$

For the transmitted wave the relations are

$$l_T = x \sin\phi + z \cos\phi \tag{5}$$

$$\begin{cases} E_\mathrm{T} = E_\mathrm{T} e^{-\mathrm{j}k_\mathrm{T} l_\mathrm{T}} (\cos\phi\, e_x - \sin\phi e_z) \\ H_\mathrm{T} = \dfrac{E_\mathrm{T}}{\eta_2} e^{-\mathrm{j}k_\mathrm{T} l_\mathrm{T}} e_y \end{cases} \tag{6}$$

The boundary conditions at $z = 0$ require that the tangential E that is E_x and tangential H_y be continuous. Setting $E_x(z = 0)$ in region 1 equal to $E_x(z = 0)$ in region 2,

$$E_0 \cos\theta\, e^{-\mathrm{j}k_\mathrm{I} x \sin\theta} + E_\mathrm{R} \cos\theta_\mathrm{R}\, e^{-\mathrm{j}k_\mathrm{I} x \sin\theta_\mathrm{R}} = E_\mathrm{T} \cos\phi e^{-\mathrm{j}k_\mathrm{T} x \sin\phi} \tag{7}$$

If this equation is to hold for all values of x then

$$k_\mathrm{I} \sin\theta = k_\mathrm{I} \sin\theta_\mathrm{R} = k_\mathrm{T} \sin\phi \tag{8}$$

It follows that $\theta_\mathrm{R} = \theta$, that is, the angle of incidence is equal to angle of reflection as in a plane mirror.
Further

$$\frac{\sin\phi}{\sin\theta} = \frac{k_I}{k_T} = \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{n_1}{n_2} \tag{9}$$

where n is the index of refraction of the material. Equation (9) then gives Snell's law ($n_1 \sin\theta = n_2 \sin\phi$) which holds irrespective of the nature of polarization. Using (9) in (7) and cancelling the exponential terms we have

$$E_0 \cos\theta + E_\mathrm{R} \cos\theta = E_\mathrm{T} \cos\phi \tag{10}$$

Using (8), the boundary conditions on H_y yield

$$\frac{E_0}{\eta_1} - \frac{E_\mathrm{R}}{\eta_1} = \frac{E_\mathrm{T}}{\eta_2} \tag{11}$$

Solving (10) and (11) we get

$$\frac{E_\mathrm{R}}{E_0} = \frac{\eta_2 \cos\phi - \eta_1 \cos\theta}{\eta_2 \cos\phi + \eta_1 \cos\theta} \tag{12}$$

$$\frac{E_\mathrm{T}}{E_0} = \frac{2\eta_2 \cos\theta}{\eta_2 \cos\phi + \eta_1 \cos\theta}. \tag{13}$$

Substituting $\eta_1 = \sqrt{\mu_1/\varepsilon_1}, \eta_2 = \sqrt{\mu_2/\varepsilon_2}$, $\mu_1 = \mu_2 = \mu_0$ for non-magnetic dielectric and using (9) in (12), we obtain

$$R = \left(\frac{E_1}{E_0}\right)^2 = \left(\frac{n_2 \cos\theta - n_1 \cos\phi}{n_2 \cos\theta + n_1 \cos\phi}\right)^2$$

14.81 **(a)** Reflectance $R = 0$ if $n_2 \cos\theta - n_1 \cos\phi = 0$ (1)

Given $\tan\theta = \frac{n_2}{n_1}$ (Brewster's law of polarization) (2)

$\frac{\sin\theta}{\sin\phi} = \frac{n_2}{n_1}$ (Snell's law of refraction) (3)

Combining (2) and (3), we have (Fig. 14.11)

$\sin\phi = \cos\theta$ (4)

or $\phi = 90° - \theta$ (5)

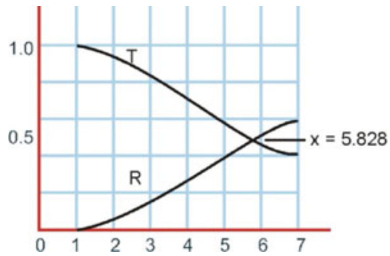


Fig. 14.11 R and T against n_1/n_2

Eliminating n_2 between (1) and (2)

$n_2 \cos\theta - n_1 \cos\phi = n_1(\sin\theta - \cos\phi) = n_1(\sin\theta - \sin\theta) = 0$ where we have used (5).

(b) $\tan\theta = \frac{n_2}{n_1} = \frac{1.5}{1} = 1.5$

$\therefore \quad \theta = 56.31°$

14.82 **(a)** $R = \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2} = \frac{\left(\frac{n_1}{n_2} - 1\right)^2}{\left(\frac{n_1}{n_2} + 1\right)^2} = \frac{(x-1)^2}{(x+1)^2}$ (1)

where $x = \frac{n_1}{n_2}$

$T = 1 - R = \frac{4x}{(x+1)^2}$ (2)

(b) Setting $R = T$ yields the quadratic equation $x^2 - 6x + 1 = 0$, whose solution is $x = 3 + 2\sqrt{2}$ or 5.828. Thus for $n_1/n_2 = 5.828$, we get $R = T = 0.5$.

14.83 Using Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = 0 \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

(a) $$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}.\boldsymbol{r} + \phi)} \tag{4}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 e^{i(\omega t - \boldsymbol{k}.\boldsymbol{r} + \phi)} \tag{5}$$

Let the wave be propagated in the z-direction. Then

$$\nabla \cdot \boldsymbol{E} = -i\,\boldsymbol{k} \cdot \boldsymbol{E} = 0$$

and $\nabla \cdot \boldsymbol{B} = -i\,\boldsymbol{k} \cdot \boldsymbol{B} = 0$

This shows that both $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to $\boldsymbol{k}$, which is the direction of propagation. Therefore both $\boldsymbol{E}$ and $\boldsymbol{B}$ are transverse oscillations.

(b) $\boldsymbol{B}$ and $\boldsymbol{E}$ are in phase

(c) Using (3)

$$\nabla \times \boldsymbol{E} = -i\,\boldsymbol{k} \times \boldsymbol{E} = -i\omega \boldsymbol{B}$$

$$\therefore \quad \boldsymbol{B} = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega} = \frac{1}{c}\frac{\boldsymbol{k} \times \boldsymbol{E}}{k}$$

$$\therefore \quad \boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{s}} \times \boldsymbol{E} \tag{6}$$

where $\hat{s} = \boldsymbol{k}/k$ is a unit vector in the direction of propagation. The three vectors $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ form a right-handed rectangular coordinate system. From (6) we obtain $B = E/c$.

14.84 $E = \dfrac{\sigma}{2\varepsilon_r \varepsilon_0}$

$$\therefore \quad \sigma = 2\varepsilon_0 \varepsilon_r E = 2 \times 8.85 \times 10^{-12} \times 6 \times 2 \times 10^3$$

$$= 2.124 \times 10^{-7}\,\mathrm{C/m^2}$$

14.3.3 Phase Velocity and Group Velocity

14.85 Using the result of prob. 14.93

$$\frac{1}{V_g} - \frac{1}{V_p} = \frac{V_p - V_g}{V_g V_p} = \frac{V_p - V_g}{c^2} - = \frac{\omega}{c}\frac{dn}{d\omega}$$

$$\therefore \quad \frac{V_\mathrm{p} - V_\mathrm{g}}{V_\mathrm{p}} = \frac{\omega c}{V_\mathrm{P}} \frac{\mathrm{d}n}{\mathrm{d}\omega} \tag{1}$$

But $\omega = 2\pi\nu = 2\pi \mathrm{c}/\lambda$ (2)

$$V_\mathrm{p} = c/n \tag{3}$$

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} = \frac{\mathrm{d}n}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\omega} = -\frac{\lambda^2}{2\pi c}\frac{\mathrm{d}n}{\mathrm{d}\lambda} \tag{4}$$

Substituting (2), (3) and (4) in (1)

$$\frac{V_\mathrm{p} - V_\mathrm{g}}{V_\mathrm{p}} = -n\lambda \frac{\mathrm{d}n}{\mathrm{d}\lambda} \tag{5}$$

$n_1 - 1$	$n_2 - 1$	$n = \frac{n_1+n_2}{2}$	Δn	$\lambda_1(A)$	$\lambda_2(A)$	$\lambda = \frac{\lambda_1+\lambda_2}{2}$	$\Delta\lambda = \lambda_1 - \lambda_2$
2.786×10^{-4}	2.781×10^{-4}	$1 + 2.784 \times 10^{-4}$	5×10^{-7}	4800	5000	4900 A	−200 A
2.781×10^{-4}	2.777×10^{-4}	$1 + 2.779 \times 10^{-4}$	4×10^{-7}	5000	5200	5100 A	−200 A

Using formula (5), the first set of data gives $(V_\mathrm{p} - V_\mathrm{g})/V_\mathrm{p} = 1.22 \times 10^{-5}$ and the second set 1.02×10^{-5}.

14.86 $\omega = ak^2$

(a) $\upsilon_\mathrm{p} = \dfrac{\omega}{k} = ak$

(b) $\upsilon_\mathrm{g} = \dfrac{\mathrm{d}\omega}{\mathrm{d}k} = 2ak = 2\upsilon_\mathrm{p}$

14.87 **(a)** $\upsilon_\mathrm{p} = \dfrac{1}{\sqrt{\varepsilon\mu}}$ (1)

(b) $n = \sqrt{\varepsilon_\mathrm{r}}$

$$\varepsilon_\mathrm{r} = n^2 = 1 - \frac{D^2}{\omega^2}$$

$$\therefore \quad n = \sqrt{1 - \frac{D^2}{\omega^2}} \tag{2}$$

$$\upsilon_\mathrm{p} = \frac{c}{n} = \frac{c}{\sqrt{1 - \frac{D^2}{\omega^2}}} \tag{3}$$

Squaring (3) and re-arranging

$$\omega^2 = D^2 + \frac{\omega^2 c^2}{\upsilon_\mathrm{p}^2} = D^2 + k^2c^2 \tag{4}$$

$$\therefore \quad \upsilon_\mathrm{p} = \omega/k$$

(c) Differentiating with D with c constant, $\omega \dfrac{\mathrm{d}\omega}{\mathrm{d}k} = c^2 k$

$$\therefore \quad v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c^2 k}{\omega} = \frac{c^2}{v_\mathrm{p}} = c\sqrt{1 - \frac{D^2}{\omega^2}}$$

$$\therefore \quad v_\mathrm{p} v_\mathrm{g} = c^2 \tag{5}$$

(d) Substituting $D = 1.2 \times 10^{11}/\mathrm{s}$, $\omega = 2\pi \times 20 \times 10^9$ Hz and $c = 3 \times 10^8$ m/s in (3), we find $v_\mathrm{p} = 1.016 \times 10^9$ m/s
Substituting $v_\mathrm{p} = 1.016 \times 10^9$ m/s in (5) we find $v_\mathrm{g} = 8.858 \times 10^7$ m/s.
It is observed that while $v_\mathrm{p} > \mathrm{c}$, $v_\mathrm{g} < \mathrm{c}$.

14.88 $$V_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{1}$$

$$\omega = vk \tag{2}$$

$$\therefore \quad V_\mathrm{g} = \frac{\mathrm{d}}{\mathrm{d}k}(vk) = v + k\frac{\mathrm{d}v}{\mathrm{d}k} \tag{3}$$

14.89 $$V_\mathrm{g} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} \quad \text{(by prob. 14.88)} \tag{1}$$

$$V = c/n \tag{2}$$

Substituting (2) in (1)

$$V_\mathrm{g} = c/n + ck\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{n}\right) = \frac{c}{n} - \frac{ck}{n^2}\frac{\mathrm{d}n}{\mathrm{d}k} \tag{3}$$

$$\text{Now} \quad \frac{\mathrm{d}n}{\mathrm{d}k} = \frac{\mathrm{d}n}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}k} = \frac{\mathrm{d}n}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{2\pi}{k}\right) = -\frac{2\pi}{k^2}\frac{\mathrm{d}n}{\mathrm{d}\lambda} \tag{4}$$

Using (4) in (3)

$$V_\mathrm{g} = \frac{c}{n} + \frac{\lambda c}{n^2}\frac{\mathrm{d}n}{\mathrm{d}\lambda}$$

14.90 $$v \propto \frac{1}{\lambda} \quad \text{(by problem)}$$

$$\therefore \quad v = Ak \quad (\text{where } A = \text{constant})$$

$$v_\mathrm{g} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} = Ak + k\frac{\mathrm{d}}{\mathrm{d}k}(Ak) = Ak + Ak = 2Ak = 2v$$

14.91 $$v_\mathrm{g} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} = v + k\frac{\mathrm{d}v}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{d}k} = v + kv_\mathrm{g}\frac{\mathrm{d}v}{\mathrm{d}\omega} \tag{1}$$

Now $\quad v = c/n \qquad (2)$

$$\therefore \quad \frac{dv}{d\omega} = \frac{dv}{dn}\frac{dn}{d\omega} = -\frac{c}{n^2}\frac{dn}{d\omega} \qquad (3)$$

Substituting (2) and (3) in (1) and using $\omega = kv$ and rearranging we get

$$v_g = \frac{c}{n + \omega(dn/d\omega)}$$

14.92 $V_g = \dfrac{\text{distance}}{\text{time}} = \dfrac{50}{1 \times 10^{-6}} = 5 \times 10^7$ m/s

$$V_g = c\sqrt{1 - (\lambda/2a)^2}$$

$$\therefore \quad 5 \times 10^7 = 3 \times 10^8 \sqrt{1 - (\lambda/5)^2}$$

Solving for λ, we find the free-space wavelength $\lambda = 4.93$ cm.

$$V_p = \frac{c^2}{V_g} = \frac{(3 \times 10^8)^2}{5 \times 10^7} = 1.8 \times 10^9 \text{ m/s}$$

14.93 $V_g = d\omega/dk \qquad (1)$

Rewriting (1), $1/V_g = dk/d\omega$

$$\therefore \quad \frac{1}{V_g} = \frac{d}{d\omega}\left(\frac{\omega}{V_p}\right) = \frac{1}{V_p} - \frac{\omega}{V_p^2}\frac{dV_p}{d\omega} \qquad (2)$$

Substituting $V_p = c/n$ in (2)

$$\frac{1}{V_g} = \frac{1}{V_p} - \frac{\omega n^2 c}{c^2}\left(-\frac{1}{n^2}\frac{dn}{d\omega}\right) = \frac{1}{V_p} + \frac{\omega}{c}\frac{dn}{d\omega} \qquad (3)$$

14.94 $V_g = \dfrac{\partial \omega}{\partial k}$

$$\therefore \quad \frac{1}{V_g} = \frac{\partial k}{\partial \omega} = \frac{\partial\,(2\pi/\lambda)}{\partial(2\pi\nu)} = \frac{\partial\,(1/\lambda)}{\partial \nu}$$

But $n = \dfrac{c}{v_p} = \dfrac{c}{\nu\lambda} \rightarrow \dfrac{1}{\lambda} = \dfrac{n\nu}{c}$

$$\therefore \quad V_g = \frac{\partial \nu}{\partial\left(\frac{1}{\lambda}\right)} = \frac{\partial \nu}{\partial\,(n\nu/c)} = \frac{c\partial \nu}{\partial(n\,\nu)}$$

14.95 $v_g = \frac{d\omega}{dk} = \frac{d(\omega\hbar)}{d(k\hbar)} = \frac{dE}{dp} = \frac{d\left(p^2/2m\right)}{dp} = \frac{1}{2m}\frac{dp^2}{dp} = \frac{2p}{2m} = \frac{mv}{m} = v$

14.96 By prob. (14.85)

$$\frac{V_p - V_g}{V_p} = -n\lambda\frac{dn}{d\lambda} \tag{1}$$

where $V_p = \frac{c}{n}$ (2)

Re-arranging (1) with the aid of (2) and writing μ for n we find

$$V_g = c\left[\frac{1}{\mu} + \lambda\frac{d\mu}{d\lambda}\right] \tag{3}$$

$$\mu = 1.420 + \frac{3.60 \times 10^{-14}}{\lambda^2} \quad \text{(by problem)} \tag{4}$$

Substituting $\lambda = 500\,\text{nm} = 5 \times 10^{-7}$ m in (4)

$$\mu = 1.564 \tag{5}$$

Differentiating μ with respect to λ in (4)

$$\frac{d\mu}{d\lambda} = -\frac{7.2 \times 10^{-14}}{\lambda^3}$$

or $\lambda\frac{d\mu}{d\lambda} = -\frac{7.2 \times 10^{-14}}{\lambda^2} = -0.288$ (6)

Substituting (5) and (6) in (3), we find $V_g = 0.35$ c.

14.3.4 Waveguides

14.97 **(a)** Assuming the dominant mode, for $a = 2.5$ cm, and $b = 2.5$ cm, for the rectangular waveguide, and $c = 3 \times 10^8$ m/s, the velocity of electromagnetic waves in free space, the phase velocity is given by

$$V_p = \frac{c}{\sqrt{1 - (\lambda/2a)^2}} = \frac{3 \times 10^8}{\sqrt{1 - (4/5)^2}} = 5 \times 10^8\ \text{m/s}$$

(b) $V_g = c\sqrt{1 - (\lambda/2a)^2} = 3 \times 10^8\sqrt{1 - (4/5)^2} = 1.8 \times 10^8$ m/s

(c) $\lambda_g = \frac{\lambda}{\sqrt{1 - (\lambda/2a)^2}} = \frac{4}{\sqrt{1 - (4/5)^2}} = 6.67$ cm

14.98 $$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} \tag{1}$$

$$\lambda_g = 3\lambda \quad \text{(by problem)} \tag{2}$$

Combining (1) and (2) and solving for λ with $a = 3$ cm we find the free-space wavelength $\lambda = 4\sqrt{2}$ cm.

14.99 **(a)** $\lambda_g = \dfrac{\lambda}{\sqrt{1-(\lambda/2a)^2}} = \dfrac{8}{\sqrt{1-(8/10)^2}} = 13.33\,\text{cm}$

(b) The cut-off wavelength $\lambda_c = 2a = 2 \times 5 = 10$ cm.

14.100 $$N = \frac{8\pi\nu^2 \mathrm{d}\nu V}{c^3} = \frac{8\pi \mathrm{d}\lambda \cdot V}{\lambda^4}$$

Mean $\lambda = 5500\text{Å} = 5.5 \times 10^{-7}$ cm

$$\mathrm{d}\lambda = (6000 - 5000)\mathring{A} = 10^{-7}\,\text{cm}$$

$$V = (0.5)^3\,\text{cm}^3$$

$$N = \frac{8\pi \times 10^{-7} \times (0.5)^3}{(5.5 \times 10^{-7})^4} = 3.43 \times 10^{18}$$

14.101 The cut-off frequency of the TM_{mn} or TE_{mn} mode is

$$\omega_{mn} = c\left[\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2\right]^{1/2}$$

The lowest value we can have for ν_{mn} is for the choice $m = 1$, $a = 15$ and $n = 0$, that is, for TM_{10} wave.

$$\therefore \quad \nu_{10} = \frac{c}{2} \times \frac{1}{15} = \frac{3 \times 10^8}{30} = 10^7\,\text{Hz} = 10^4\,\text{kHz}$$

This is much above the range of AM waves (530–1600 kHz). Hence AM waves cannot propagate in the tunnel.

14.102 The cut-off frequency will be least for TE_{10} waves. Of course TM_{10} waves do not exist.

$$\nu_{10} = \frac{c}{2\pi}\frac{\pi}{a} = \frac{c}{2a} = \frac{3 \times 10^8}{2 \times 0.05} = 3 \times 10^9\,\text{Hz} = 3\,\text{GHz}$$

Note that we take the higher dimension (5 cm) for the lower value of cut-off frequency.

14.103 The relation between ω and k in a rectangular waveguide is

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \tag{1}$$

For TE_{01} waves $m = 0$, $n = 1$, $a = 1\,\text{cm}$ and $b = 2\,\text{cm}$. Equation (1) is then reduced to

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{\pi}{0.02}\right)^2 \tag{2}$$

The phase velocity is given by

$$v_p = \frac{\omega}{k} \tag{3}$$

The group velocity v_g is given by

$$v_g = \frac{d\omega}{dk} \tag{4}$$

$$v_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} = \frac{c^2}{v_p}$$

$$\text{or } v_p v_g = c^2 \tag{5}$$

The cut-off frequency is given by

$$\omega_{01} = \frac{\pi c}{b} \tag{6}$$

The $\omega - k$ plot for $m = 0$, $n = 1$, $b = 0.02m$ is shown in Fig. 14.12. For convenience the variables are chosen as dimensionless.

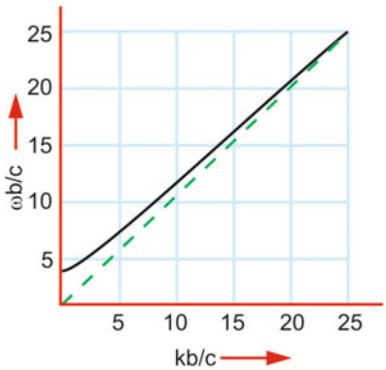


Fig. 14.12 Dispersion diagram for the TE_{01} mode of rectangular waveguide for $b = 2\,\text{cm}$

At high frequencies, the curve is asymptotic to the line $\omega - kc$. Thus at high frequencies, both the phase and group velocities approach c. However, the

$\omega - k$ plot is always above the $\omega = kc$ line. This implies that the phase velocity $\nu_p = \omega/k$ is always larger than c.

The group velocity $v_g = d\omega/dk$ is determined by the slope of the curve which is less than the slope of the line $\omega = kc$, and so v_g is always less than c.

There is a minimum frequency, known as the cut-off frequency, below which k becomes imaginary and the wave ceases to exist. As the frequency approaches the cut-off frequency the phase velocity becomes infinite and the group velocity becomes zero.

14.104 **(a)** $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z = (k^2 - \omega^2 \mu \varepsilon) E_z \tag{1}$$

For TM waves H_z vanishes, and E_z must be a solution of (1). Let

$$E_z = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)\, e^{-jkz} \tag{2}$$

where A, B, C, D are arbitrary constants and k_x and k_y are constants to be determined by boundary conditions. Substitution of (2) in (1) yields

$$\left(k_x^2 + k_y^2\right) = \omega^2 \mu \varepsilon - k^2 \tag{3}$$

The form of (2) is further constrained by the boundary conditions. Assuming that the walls of the waveguide are perfect conductors, E_z which is tangential to the walls must vanish at $x = 0$, $x = a$, $y = 0$ and $y = b$, Fig. 14.13. In (2) E_z will not vanish at $x = 0$ unless $A = 0$. Similarly the boundary condition at $y = 0$ is satisfied if $C = 0$. The boundary conditions at $x = a$ and $y = b$ are satisfied by putting.

$$k_x = \frac{m\pi}{a}$$

$$k_y = \frac{m\pi}{b} \tag{4}$$

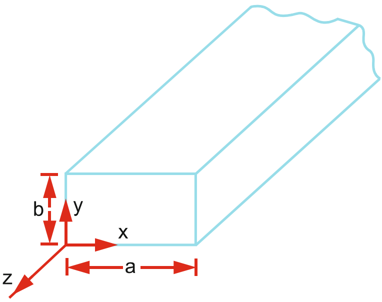


Fig. 14.13 Rectangular hollow metal waveguide

where m and n are positive integers. Finally, we get the result

$$E_z = K \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \mathrm{e}^{-\mathrm{j}kz} \tag{5}$$

where $K = (B)(D)$ is another constant. Note that E_z vanishes if either $m = 0$ or $n = 0$. In that case $\mathrm{TM_{m0}}$ or $\mathrm{TM_{0n}}$ wave does not exist. The other field components are given from equations which are obtained by manipulating Maxwell's equations.

$$E_x = -\frac{j}{\omega^2\mu\varepsilon - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) \tag{6}$$

$$E_y = -\frac{j}{\omega^2\mu\varepsilon - k^2}\left(-k\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) \tag{7}$$

Using (5) in (6) and (7) and putting $H_z = 0$

$$E_x = -\frac{jK\,km\pi}{(\omega^2\mu\varepsilon - k^2)a} \cos\frac{m\pi x}{a} \sin\frac{n\pi x}{b} \mathrm{e}^{-\mathrm{j}kz} \tag{8}$$

$$E_y = -\frac{jK\,kn\pi}{(\omega^2\mu\varepsilon - k^2)b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \mathrm{e}^{-\mathrm{j}kz} \tag{9}$$

The solutions (5), (8) and (9) represent an infinitely large family of waves, characterized by different values of the integers m and n. They differ from one another by the values of the integers m and n. They also differ in their velocity as well as field configuration.

(b) Combining (3) and (4) we get

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \tag{10}$$

where we have substituted $\mu\varepsilon = 1/c^2$. The cut-off frequency is obtained by setting $k = 0$.

$$\omega_{mn} = \pi c\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2} \tag{11}$$

14.105 **(a)** For TE waves there is no E_z. Here, we must find boundary conditions on H_z that cause the tangential component of E_z to vanish. The given equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_z = (k^2 - \omega^2\mu\varepsilon)\, H_z \tag{1}$$

The general solution is

$$H_z = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)\mathrm{e}^{-\mathrm{j}kz} \tag{2}$$

$$\therefore \quad \frac{\partial H_z}{\partial x} = (-k_2 A\sin k_x x + Bk_2\cos k_x x)$$
$$(C\cos k_y y + D\sin k_y y)\mathrm{e}^{-\mathrm{j}kz}$$

For $\dfrac{\partial H_z}{\partial x} = 0$ at $x = 0$, it is necessary that $B = 0$.

Similarly

$$\frac{\partial H_z}{\partial y} = (A\cos k_x x + B\sin k_x x)(-Ck_y\sin k_y y + D\cos k_y y)\mathrm{e}^{-\mathrm{j}kz}$$

For $\dfrac{\partial H_z}{\partial y} = 0$, $D = 0$. Therefore (2) becomes

$$H_z = K\cos(k_x x)\cos(k_y y)\ \mathrm{e}^{-\mathrm{jkz}} \tag{3}$$

where K is the product of A and B is another constant. Imposing boundary conditions at $x = a$ and $y = b$, we have

$$\frac{\partial H_z}{\partial x} = -K\,k_x\sin k_x x\cos k_y y = 0$$

yielding

$$k_x a = m\pi \tag{4}$$

and $\dfrac{\partial H}{\partial y} = -K\,k_y\cos k_x x\sin k_y y = 0$

yielding

$$k_y b = n\pi \tag{5}$$

$$\therefore \quad H_z = K\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{6}$$

Substituting (2) in (1) we obtain

$$k_x^2 + k_y^2 = \omega^2\mu\varepsilon - k^2 \tag{7}$$

which is identical with (3) of prob. (14.104).

(b) Substituting the values of k_x and k_y from (4) and (5) in (7) and setting $k = 0$ gives the cut-off frequency.

$$\omega_{mn} = \pi c \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]^{1/2} \tag{8}$$

which is identical with (11) of prob. (14.104).

(c) Thus the features which are identical for the TM and TE modes are

(i) $\omega - k$ plot
(ii) Phase
(iii) Group velocity
(iv) Cut-off frequency

However the important difference is that when either $m = 0$ or $n = 0$, the TM mode fails to exist. On the other hand H_z does not vanish for $m = 0$ or $n = 0$ (see (6)). Because of this fact the TE_{10} is the mode which has the lowest cut-off frequency. Here we have assumed that $a > b$. The cut-off frequency for TE_{10} mode is given by

$$\omega_{10} = \frac{\pi c}{a} \tag{9}$$

the free space wavelength being $2a$.

The small dimension (b) has no bearing on the cut-off frequency for this mode. The other advantage is that single mode operation is feasible over a wide range of frequencies.

Chapter 15
Optics

Abstract *Chapter 15* deals with geometric optics. Problems are solved under internal reflection in slabs and prisms, fibre optics, matrix methods, Fraunhofer diffraction by single slit, double slit and grating, missing orders, resolving power, Rayleigh's criterion, interference, colours in thin films, Newton's rings, polarization, Malu's law and Brewster's law.

15.1 Basic Concepts and Formulae

Geometrical Optics

Fermat's principle: A ray of light traverses from one point to another by a route which takes least time.

$$\textit{Momentum of photon} \qquad p = h\nu/c \tag{15.1}$$

Fraction (f) of light escaping from an isotropic point source in a medium of refractive index n through a flat surface is given by

$$f = \frac{1}{2}\left[1 - \frac{1}{n}\sqrt{n^2 - 1}\right] \tag{15.2}$$

Intensity of light (I) at distance r from a point source of power W is related to pressure P by

$$P = \frac{I}{c} = \frac{W}{4\pi r^2 c} \tag{15.3}$$

Mirage is a type of illusion formed by light rays coming from the low region of the sky in front of the observer on a sunny day.

$$\textit{Optical path length (O.P.L.)} = \sum\nolimits_{i=1}^{N} n_i s_i \tag{15.4}$$

where s_i is the path length of the ray in the ith medium.

Fibre optics: The maximum acceptance angle θ_{max} outside which entering rays will not be totally reflected within the fibre is given by

$$n_0 \ \sin\theta_{max} = \sqrt{n_f^2 - n_c^2} \tag{15.5}$$

where n_0 is the refractive index of the medium outside the fibre, n_f that of the fibre and n_c that of the cladding material.

Snell's law of refraction

$$n_1 \sin_i = n_2 \sin r \tag{15.5a}$$

where i = angle of incidence and r = angle of refraction.

Prisms (Fig. 15.1)

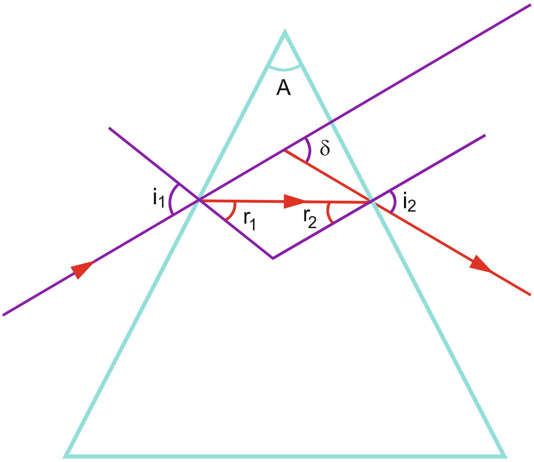


Fig. 15.1

$$r_1 + r_2 = A \tag{15.6}$$

$$\delta = (i_1 - r_1) + (i_2 - r_2) = i_1 + i_2 - A \tag{15.7}$$

For minimum angle of deviation

$$\delta = D, i_1 = i_2, r_1 = r_2 \tag{15.8}$$

$$n = \sin\frac{1}{2}(A + D)/\sin\frac{1}{2}(A) \tag{15.9}$$

If the prism of index n_1 is placed in a medium of index n_2 then n should be replaced by n_1/n_2.

Lenses: Object distance (u), image distance (v) and focal length (f)

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \tag{15.10}$$

Lens Maker's Formula

Sign Convention

$$\frac{1}{F} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad \text{(lens maker's formula)} \tag{15.11}$$

r is positive if the refracting surface facing the object is convex and r is negative if the refracting surface facing the object is concave.

If the lens of refractive index n_1 is immersed in a medium of index n_2, then $n = n_1/n_2$.

Combination of Two Thin Lenses: Let lense L_1 of focal length f_1 facing the object at distance u_1 and L_2 of focal length f_2 be located at a distance d behind L_1. Then the final image is located from L_2 at a distance v_2 given by

$$v_2 = \frac{f_2[f_1u_1 - d(u_1 - f_1)]}{f_1u_1 + (u_1 - f_1)(f_2 - d)} \tag{15.12}$$

$$\text{Magnification} = \frac{\text{Height of the final image}}{\text{Height of the object}} \tag{15.13}$$

If v_2 is positive then the image is real, if v_2 is negative, the image is virtual.

$$\textit{System matrix}: \qquad I_2 = R_{21}\ T_2\ R_{12}\ I_1 \tag{15.14}$$

where the initial image I_1 in medium 1 is transformed into the final image I_2 in medium 2. R_{12} is the refraction matrix at the first surface (air to glass), T_2 is the translation matrix in the second medium (glass) and R_{21} is the refraction matrix at the second surface (glass to air).

$$\text{The matrix } S = R_{21}\ T_2\ R_{12} \tag{15.15}$$

is known as the system matrix.

Interference

Conditions: Light sources are coherent, i.e. their phase difference remains constant and that the distance between the sources is reasonably small (of the order of a few Angstroms for visible light).

Young's Double-Slit Experiment

$$x_m = m\lambda\frac{D}{d} \qquad \text{(bright fringes)}, m = 0, 1, 2, \ldots \tag{15.16}$$

$$x_m = \left(m + \frac{1}{2}\right)\lambda\frac{D}{d} \qquad \text{(dark fringes)}, m = 0, 1, 2, \ldots \tag{15.17}$$

where x_m is the distance of the mth fringe from the central fringe, D is the source–screen distance, d is the separation of slits and λ is the wavelength of monochromatic light used.

If white light is used, the central fringe will be white, flanked by coloured fringes on either side with the system of violet fringes closer to the central fringe and red one farther. In three dimensions the shape of the fringes is that of hyperboloids, and on the screening the shape would be that of a set of hyperbolas. However, because of limited field of view the fringes appear as a set of equidistant straight lines. The separation of fringes is known as bandwidth (β):

$$\beta = \frac{\lambda D}{d} \qquad \text{(bandwidth)} \tag{15.18}$$

Shift of fringes (Δ): When a thin film of thickness t is introduced in the path of one of the rays

$$(n-1)t = m\lambda \tag{15.19a}$$

$$\Delta = \frac{D}{d}(n-1)t \tag{15.19b}$$

and the entire system of fringes undergoes a lateral shift.

The intensity distribution of the fringes is given by

$$I = 4A^2\cos^2\left(\frac{\pi dx}{\lambda D}\right) \tag{15.20}$$

The principle of *optical reversibility* states that if there is no absorption of light, then a light ray that is reflected or refracted will retrace its original path if its direction is reversed.

Interference by Reflection from Thin Films

When reflection occurs from an interface beyond which the medium has a lower index of refraction, the reflected ray does not undergo a phase change; when the medium beyond the interface has a higher index, there is a phase change of π. The transmitted wave does not undergo a change of phase in either case.

Thus for air–glass–air media (Fig. 15.2) for normal incidence

$$2tn = (m + ½)\lambda, \qquad m = 0, 1, 2, \ldots \text{(maxima)} \tag{15.21}$$

The term ½ λ is introduced because of the change of phase of 180° which is equivalent to half a wavelength

$$2tn = m\lambda, \qquad m = 0, 1, 2, \ldots \text{ (minima)} \tag{15.22}$$

Fig. 15.2

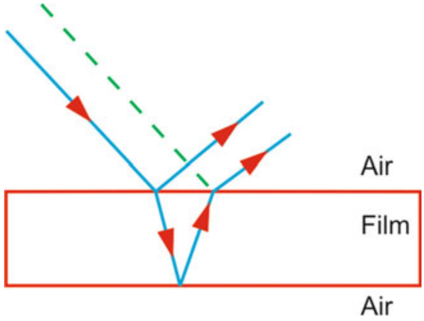
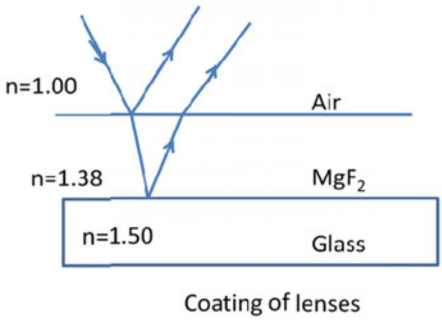


Fig. 15.3



These equations are valid if n of the film is higher (for example, air–glass–air) or lower (for example, glass–water–glass) than the indices of media on each side of the film.

For oblique incidence, with refracting angle r, the left-hand side of (15.21) and (15.22) must be multiplied by $\cos r$.

For coating lenses, Fig. 15.3 shows a typical arrangement in which the indices of the media are in the ascending order. Here the conditions are reversed:

$$2tn = m\lambda, m = 0, 1, 2 \ldots \text{(maxima)} \tag{15.23}$$

$$2tn = (m + ½)\lambda, m = 0, 1, 2 \ldots \text{(minima)} \tag{15.24}$$

Wedge film

$$\beta = \frac{\lambda}{2n\theta} \tag{15.25}$$

where θ is the angle of the wedge.

$$\textit{Biprism} \qquad \beta = \frac{\lambda D}{2d} \qquad \text{(bandwidth)} \tag{15.26}$$

Newton's Rings (in Reflected Light)

Radius of the mth ring (r_m) for the convex lens of radius R is given by

$$r_m = \sqrt{m\lambda R} \quad \text{(dark rings, } m = 0, 1, 2, \ldots) \tag{15.27a}$$

$$2t = m\lambda \quad \text{(dark rings)} \tag{15.27b}$$

$$r_m = \sqrt{\left(m + \frac{1}{2}\right)\lambda} \quad \text{(bright rings, } m = 0, 1, 2, \ldots) \tag{15.28a}$$

$$2t = (m + ½)\lambda \quad \text{(bright rings)} \tag{15.28b}$$

where t is the thickness of air gap.

Michelson's Interferometer

M_1 and M_2 are two plane mirrors mounted (Fig. 15.4), M_1 being movable and M_2 fixed. The plate P_2 compensates for the extra pathlength in P_1. The interference fringes form from the superposition of the two beams and are viewed at E (see prob. 15.38).

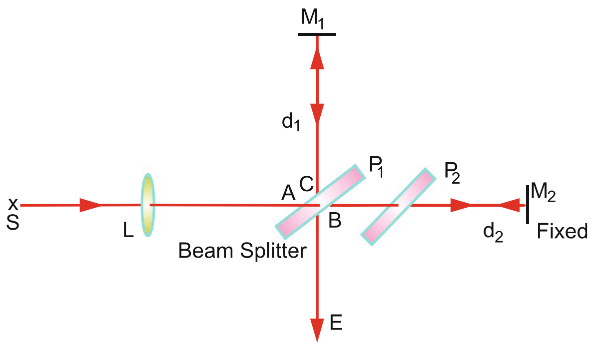


Fig. 15.4

The mirror displacement L when n fringes cross the field of view is given by

$$L = \frac{n\lambda}{2} \tag{15.29}$$

Diffraction (Fraunhofer)

Single slit $\quad a \sin\theta = m\lambda \quad$ (minima, $m = 1, 2, 3, \ldots$) $\quad$ (15.30)

where a is the slit width and θ is the diffraction angle.

Relative intensities of the secondary maxima

$$I_\theta = I_{\max}\left(\frac{\sin\alpha}{\alpha}\right)^2 \tag{15.31}$$

$$\text{where } \alpha = \frac{\pi a}{\lambda}\sin\theta \tag{15.32}$$

$$\frac{I_\theta}{I_{\max}} = \left(\frac{\sin\left(m+\frac{1}{2}\right)\pi}{\left(m+\frac{1}{2}\right)\pi}\right)^2 \tag{15.33}$$

The secondary maxima lie approximately halfway between the minima.

Double-Slit and *N*-Slits (Grating)

$$I = I_{\max}\left(\frac{\sin\alpha}{\alpha}\right)^2\frac{\sin^2 N\beta}{\sin^2\beta} \tag{15.34}$$

$$\text{where } \alpha = \frac{\pi a\sin\theta}{\lambda} \qquad \text{and} \qquad \beta = \frac{\pi d\sin\theta}{\lambda}$$

a is the slit width and d is the slit spacing. In (15.32) the first factor arises due to diffraction from a single slit, the second one is due to interference of light waves from different slits.

For $N = 1$ we obtain the single-slit pattern and for large N we are dealing with a grating.

$$\textit{Maximum number of order} \qquad m_{\max} = a/\lambda \tag{15.35}$$

Condition for *overlapping* of spectral lines:

$$m_1\lambda_1 = m_2\lambda_2 = m_3\lambda_3 = \ldots \tag{15.36}$$

Missing orders: The missing orders occur when the condition for a maximum of the interference and for a minimum of the diffraction are both fulfilled for the same value of θ:

$$\begin{aligned} &d\sin\theta = m\lambda, m = 0, 1, 2\ldots \\ &a\sin\theta = p\lambda, p = 1, 2, 3, \ldots \\ &\frac{d}{a} = \frac{m}{p} \end{aligned} \tag{15.37}$$

Since both m and p are integers, d/a must be in the ratio of two integers. Thus if order 3 is missing then $d/a = 3$. Other interference fringes which are missing are $6, 9, \ldots$.

If the ratio d/a is not exactly equal to the ratio of two integers, then the intensity of a particular order will not be zero but would be quite small.

Grating: Total number of lines (N) on the grating

$$N = N'W \tag{15.38}$$

where N' is the number of lines per unit length and W is the grating width.

Resolving Power (R.P)

$$R = \lambda/\mathrm{d}\lambda = Nm \tag{15.39}$$

Dispersive Power of a Prism (*D*)

$$D = \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{\mathrm{d}\mu}{\mu - 1} \tag{15.40}$$

Resolving power (R) for a prism of baselength B is given by

$$R = \frac{\lambda}{\mathrm{d}\lambda} = B\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \tag{15.41}$$

where $\mathrm{d}\mu/\mathrm{d}\lambda$ is the variation of refractive index of the prism with wavelength, λ is the mean wavelength and $d\lambda$ is the difference in wavelengths to be resolved.

Diffraction from a Disc of Radius *a*

$$I = I_0\left[\frac{2J_1(\rho)}{\rho}\right]^2 \tag{15.42}$$

where $\rho = 2\pi\frac{a}{\lambda}\sin\theta$ and $J_1(\rho)$ is the Bessel function of the first kind.

Rayleigh's Criterion

The minimum angular resolution of a telescope of diameter D is

$$\theta = 1.22\frac{\lambda}{D} \tag{15.43}$$

where D is the diameter of the telescope.

Zone Plate

$$\frac{1}{a} + \frac{1}{b} = \frac{n\lambda}{r_n^2} = \frac{1}{f_n} \tag{15.44}$$

where a and b are the distances of the object and image from the zone plate, r_n is the radius of the nth zone and f the focal distance.

Polarization

Let the x- and y-components of the electric vector be given by

$$E_x = a_1 \sin \omega t \tag{15.45}$$

$$E_y = a_2 \sin(\omega t + \delta) \tag{15.46}$$

$$\frac{E_x^2}{a_1^2} + \frac{E_y^2}{a_2^2} - 2\frac{E_x E_y}{a_1 a_2} \cos \delta = \sin^2 \delta \tag{15.47}$$

(i) *Plane-polarized* (linearly polarized) light:

If $\delta = 2n\pi$ where $n = 0, 1, 2, \ldots$ for which

$$E_y = \frac{a_1}{a_2} E_x \tag{15.48}$$

or $\delta = (2n + 1)\pi$, for which

$$E_y = -\frac{a_1}{a_2} E_x \tag{15.49}$$

In either case the electric field oscillates on a straight line.
(ii) *Elliptically polarized* light:
If $\delta = (n + ½)\pi$, where $n = 0, 1, 2, \ldots$ and $a_1 \neq a_2$

$$\frac{E_x^2}{a_1^2} + \frac{E_y^2}{a_2^2} = 1 \tag{15.50}$$

If $\delta = \pi/2, 5\pi/2, 9\pi/2, \ldots$ the tip of the electric vector rotates clockwise for an observer towards whom light approaches, and the light is said to be left-handed elliptically polarized.

If $\delta = 3\pi/2, 7\pi/2, 11\pi/2, \ldots$ the ellipse is described in the counterclockwise direction and the light is said to be right-handed elliptically polarized.
(iii) *Circularly polarized* light

If $\delta = (n + ½)\pi$ where $n = 0, 1, 2, \ldots$ and $a_1 = a_2 = a$

$$E_x^2 + E_y^2 = a^2 \tag{15.51}$$

Right-handed and left-handed circularly polarized light are described as in case (ii).

(iv) *Unpolarized light*:
If the phase difference δ is random between two linearly polarized waves at right angles to each other, light is said to be unpolarized.

Malus' Law

Consider two polarizing sheets P_1 and P_2 parallel to each other. Let the polarized light of intensity I_m from P_1 be incident on P_2 whose polarizing axis is oriented at angle θ with that of P_1. Then the transmitted intensity I from P_2 is given by

$$I = I_m \cos^2\theta \qquad \text{(Malus' law)} \tag{15.52}$$

Brewster's Law

When an unpolarized light beam is incident on a dielectric surface then for a particular angle of incidence, called polarizing angle θ_p, the reflected light is completely plane polarized with its plane of vibration at right angles to the plane of incidence (σ-component).

In this situation the reflected light and the refracted light beams are at right angles

$$\tan\theta_p = n = n_2/n_1 \text{ (Brewster's law)} \tag{15.53}$$

where n is the refractive index of medium 2 with respect to medium 1.

At the polarizing angle the π-component of the beam (plane of vibration parallel to the incident plane) is entirely refracted with an admixture of σ-component.

Brewster windows are used in laser technology to produce plane polarized light.

Birefringence: Optically isotropic substances exhibit optical properties such as refractive indices independent of the direction of propagation of the electromagnetic wave and the state of polarization of the wave. However, there are crystalline solids such as calcite and quartz which exhibit optically anisotropic properties. Such substances are called birefringent. In this case there are two refractive indices. If the polarization is parallel to the optic axis, light will travel with one velocity, if the polarization is perpendicular to the axis, light will travel with a different velocity.

Let a linearly polarized light be incident at the polarizing angle θ with the optical axis. The polarization can be resolved into x- and y-components. Since the x- and y-components travel with different speeds, their phases change at a different rate as light travels down the material.

Full Wave Plate, Half-Wave Plate and Quarter Wave Plate

Full wave plate is the one in which the vibrations of two components which were in phase initially remain in phase after light emerges from a thickness t of the material:

$$(n_{\text{slow}} - n_{\text{fast}})t = m\lambda, \qquad m = 1, 2, 3 \ldots \tag{15.54}$$

where λ is the wavelength in vacuum.

For half-wave plate

$$(n_{\text{slow}} - n_{\text{fast}})t = (2m + 1)\lambda/2 \tag{15.55}$$

This implies that if two components enter the slab in phase, they emerge from the slab 180° out of phase. The half-wave plate rotates the polarization direction but otherwise leaves the polarization unaffected.

For quarter wave plate

$$(n_{\text{slow}} - n_{\text{fast}})t = (2m + 1)\lambda/4 \tag{15.56}$$

Here, the two components have the phase difference of 90° as they emerge from the slab of thickness t.

If two quarter wave plates are put together the combination acts as a half-wave plate.

When an unpolarized beam is incident on a birefringent material, it is split up into an ordinary ray which travels in a normal way, obeying Snell's law, and an extraordinary ray which is displaced, the two emerging rays being linearly polarized at right angles to each other.

Polarimeter: When a polarized beam of light is passed through an optically active liquid such as sugar solution then the polarizing plane rotates through an angle θ:

$$\theta = \alpha L D \tag{15.57}$$

where α is the specific rotation, L is the length of the tube in decimetres and D is the amount of solvent in grams per 100 c.c.

15.2 Problems

15.2.1 Geometrical Optics

General

15.1 Show that the fraction F of light that escapes from a point source within a medium across a flat surface is given by

$$F = \frac{1}{2}\left[1 - \frac{1}{\mu}\sqrt{\mu^2 - 1}\right]$$

where μ is the refractive index.

15.2 Assuming that a 1000 W light bulb radiates equally in all directions, calculate the radiation pressure on a perfectly absorbing surface at a distance of 2 m.

15.3 Define optical path and state Fermat's principle. Using *Fermat's principle*, derive Snell's law of refraction at the plate interface between two materials of refractive index n and n'.

15.4 Use the concept of optical path to briefly explain why a mirage occurs.
Early in the morning, on a sunny day, the heat of the sun produces a thin layer of warm air above the surface of a long straight road. Consider a possible light ray path such as that illustrated in Fig. 15.5. This connects an eye-level point on the tree with an observer of height $h = 2\,\text{m}$. If the layer of hot air has refractive index $n_2 = 1.00020$, while the cold air has refractive index $n_1 = 1.00030$.

(a) Show that the *optical path length* along ABCD is approximately

$$n_2 x + n_1\sqrt{(d-x)^2 + 4h^2}$$

(b) By using Fermat's principle, determine the actual distance that the ray travels in the layer of hot air when $d = 500\,\text{m}$.

(c) As the observer walks towards the tree, she finds that the mirage disappears. At what distance from the tree does this occur?

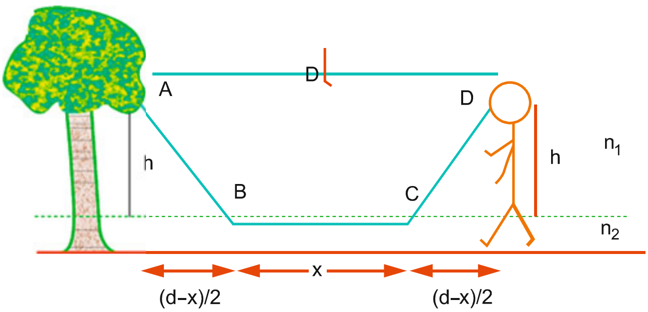


Fig. 15.5

15.5 An optical fibre consists of an inner material (the fibre) with refractive index n_f and an outer material of lower refractive index n_c, known as cladding, as in Fig. 15.6.

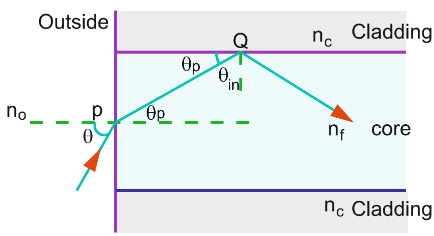


Fig. 15.6

(a) What is the purpose of cladding?

(b) Show that the maximum acceptance angle $\theta_{\max}$ is given by

$$n_0 \sin\theta_{\max} = \sqrt{n_f^2 - n_c^2}$$

(c) Discuss two main fibre loss mechanisms.

15.2.2 Prisms and Lenses

15.6 A triangular glass prism ($n = 1.6$) is immersed in a liquid ($n = 1.1$) as shown in Fig. 15.7. A thin ray of light is incident as shown on face AB making an angle of 20° with the normal. Calculate the angle that the ray emerging from AC makes with the ground when it leaves AC and strikes the ground.

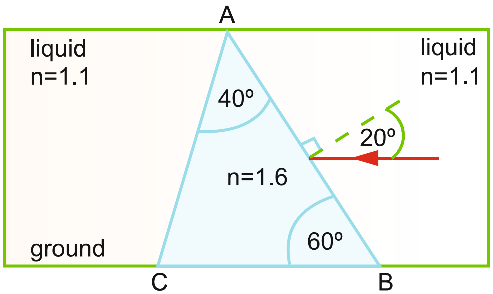


Fig. 15.7

15.7 **(a)** What is the critical angle for a block of glass, with refractive index $n_g = 1.45$, in air?

(b) Two narrow beams of microwave radiation are incident normal to one surface of a large wedge of chocolate as shown in Fig. 15.8. If the index of refraction for chocolate relative to air for these microwaves is 1.2, calculate the angle between the two emerging beams, shown as α on the

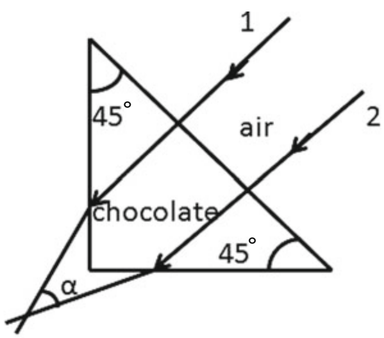


Fig. 15.8

diagram. (You can assume that the chocolate does *not* melt because of the microwaves.)

[University of Aberystwyth, Wales]

15.8 Each of the angles of a prism is 60° and the refraction index for sodium light is 1.5. Sodium light is incident at the correct angle for minimum deviation. Calculate the deviation of that portion of the light, which finally emerges from the prism, after having suffered one internal reflection.

[University of Durham]

15.9 Write down the lens maker's formula relating the focal length of a lens to the object and image distances. Explain the sign convention used for the distances involved.

Show that as two lenses are brought into contact, the focal length of the combined system, f, can be expressed as

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

where f_1 and f_2 are the focal lengths of the two separate lenses.

[University of Durham 2001]

15.10 An object is placed at a fixed distance D from the screen. Real images of the object are formed on the screen for two positions of a lens, separated by a distance d. Show that

(a) the ratio between the sizes of the two images will be $\dfrac{(D-d)^2}{(D+d)^2}$

(b) The object size $= \sqrt{I_1 I_2}$, where I_1 and I_2 are the sizes of the images

(c) $f = \dfrac{D^2 - d^2}{4D}$

(d) $D > 4f$

15.11 **(a)** Derive the lens maker's formula

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

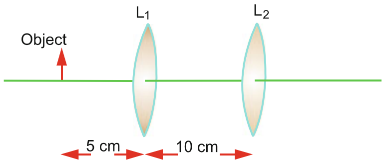
for a thin lens.

(b) A biconvex lens of plastic of refractive index $n = 1.2$ is immersed in water ($n = 1.33$). Would the lens act as a converging lens or diverging lens?

15.12 A combination of two thin convex lenses are placed as in Fig. 15.9. An object is placed 5 cm in front of the first lens which has a focal length of 10 cm. The second lens is 10 cm behind the first lens and has a focal length of 12 cm.

(a) Locate the image of the first lens with the aid of a ray diagram.
(b) Is the image real or virtual? Erect or inverted?
(c) Locate the final image for the lens combination.
(d) Is the final image real or virtual? erect or inverted?

Fig. 15.9



15.13 Let a glass sphere of radius r lie with its centre on the x-axis. A ray of light parallel to the x-axis will form an image on the other side of the sphere. Show that the distance of the image from the centre of the sphere will be equal to $\dfrac{\mu r}{2(\mu - 1)}$, where μ is the refractive index of the glass.

15.14 The light from a 100 W bulb uniformly spreads out in all directions. Find the intensity I of the electromagnetic waves and the amplitude E_0 at a distance of 5 m from the bulb.

15.15 An astronomical telescope has the focal lengths of objective and eyepiece in the ratio 8:1. Both the lenses are convex. A tower 100 m tall is at a distance of 10,000 m: (a) locate the image and (b) find the height of the image.

15.16 A 1000 W laser beam is concentrated by a lens of a cross-sectional area of 10^{-5} cm^2. Find the corresponding (a) intensity and (b) the amplitude of the electric field.

15.2.3 Matrix Methods

15.17 Derive expressions for the refraction matrix and translation matrix for a single lens.

15.18 Obtain the matrix equation for a pair of surfaces of radii r_1 and r_2 and refractive index n, separated by distance d and placed in air.

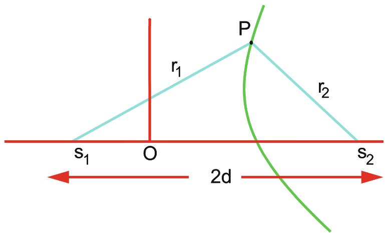
15.19 Using the results of prob. (15.18) show that for a thin lens,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

15.2.4 Interference

15.20 Consider two point sources s_1 and s_2, Fig. 15.10, which emit coherent waves. Show that curves such as that traced by p, for which the phase difference for rays r_1 and r_2 is a constant, are hyperboloids in three dimensions.

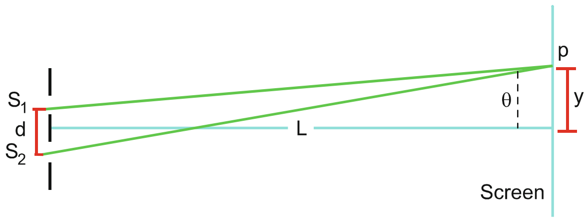
Fig. 15.10



15.21 A beam of monochromatic light with wavelength λ is incident upon two slits S_1 and S_2 a distance d apart, as shown in Fig. 15.11. Derive an expression, in terms of λ, L, d, and m for the distance y_m from the central point to the mth bright fringe of the interference pattern on a screen a distance L away when $L >> y_m$.

Two narrow slits separated by 2 mm are illuminated with a helium–neon laser of wavelength 612 nm. Calculate the spacing of the fringes observed on a screen 4 m.

Fig. 15.11



15.22 When a thin transparent plate of thickness t and refractive index μ is introduced in the path of one of the two interfering monochromatic beams of wavelength λ in Young's double-slit experiment, then fringes are shifted. Show that $(\mu - 1)t = n\lambda$.

15.23 In Young's double-slit experiment the bandwidth β is given by the expression: $\beta = \dfrac{\lambda L}{d}$, where L is the slit-screen distance, d is the slit separation and λ is the wavelength of light used:

(a) Obtain an expression for the intensity distribution in the fringe system of Young's double-slit experiment.
(b) Hence show that the average intensity of the fringes is equal to $2I$, where I is the intensity of each beam.

15.24 In a Fresnel's biprism experiment, the bandwidth of 0.195 mm is observed at a distance of 1 m from the slit. The image of the coherent sources is then produced at the same distance from the slit by placing a convex lens a 30 cm from the slit. Two images are found to be separated by 0.7 cm. Calculate the wavelength of light used.

[Kakatiya University 2002]

15.25 The inclined faces of a biprism of refractive index 1.50 make an angle of 2° with the base. A slit illuminated by monochromatic light is placed at a distance of 10 cm from the biprism. If the distance between two dark fringes observed at a distance of 1 m from the prism is 0.18 mm, find the wavelength of light used.

[University of Delhi]

15.26 A beam of monochromatic light of wavelength 5.82×10^{-7} m falls normally on a glass wedge with the wedge angle of 20 s of an arc. If the refractive index of glass is 1.5, find the number of interference fringes per centimetre of the wedge length.

[Indian Administration Services]

15.27 Light of wavelength 6000 Å falls normally on a thin wedge film of refractive index 1.4, forming fringes that are 2 mm apart. Find the angle of the wedge.

[Delhi University]

15.28 In Newton's rings apparatus, the radii of the nth and $(n + 20)$th dark rings are found to be 0.162 and 0.368 cm, respectively, when light of wavelength 546 nm is used. Calculate the radius of curvature, R, of the lower surface of the lens.

[University of Manchester 2007]

15.29 The radius of the 10th dark ring in Newton's rings apparatus changes from 60 to 50 mm when a liquid is introduced between the lens and the plate. Calculate the refraction index of the liquid.

[Nagarjuna University 2003]

15.30 Newton's rings may be formed in the reflective light by two curved surfaces as in Fig. 15.12a,b with the monochromatic light of wavelength λ incident from the top.

Show that the radius of the nth ring is given by the expression for the two situations:

$$r_n^2 \left(\frac{1}{R_1} \pm \frac{1}{R_2} \right) = n\lambda \qquad \text{(dark rings)}$$

$$= (n + \frac{1}{2})\lambda \qquad \text{(bright rings)}$$

Minus sign in the bracket of left side for situation (a) and plus sign for situation (b).

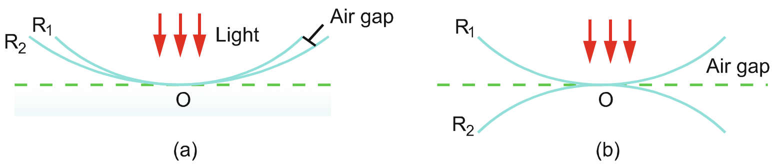


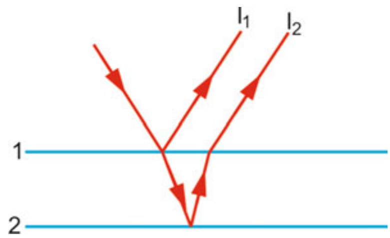
Fig. 15.12

15.31 In Young's experiment for what order does the band of wavelength of red light ($\lambda = 780\,\text{nm}$) coincide with $(m + 1)$th order in the band of blue light ($\lambda = 520\,\text{nm}$)?

15.32 Each of the parallel glass plates 1 and 2 reflects 25% of narrow monochromatic beam of light incident on it and transmits the remainder. Find the ratio of the minimum and maximum intensities in the interference pattern formed by the two beams I_1 and I_2 (Fig. 15.13).

[adapted from Hyderabad Central University 1991]

Fig. 15.13



15.33 A thin 4×10^{-5} cm thick film of refractive index 1.5 is illuminated by white light normal to its surface. Which colour will be intensified in the visible spectrum?

15.34 A parallel beam of light ($\lambda = 5890\,\text{A}^\circ$) is incident on a thin glass plate of refractive index 1.5 such that the angle of refraction in the plate is 60°. Calculate the smallest thickness of the plate which will appear dark by reflection.

[Srivenkateswara University 2000]

15.35 A beam of waves of wavelength ranging from 5800 to 3500 Å is allowed to fall normaly on a thin air film of thickness 0.2945 μm. What is the colour shown in reflection by the film?

[Osmania University]

15.36 Show that the minimum thickness of non-reflecting film is $\lambda/4\mu$.

[Kakatiya University 2001]

15.37 A beam of parallel rays is incident at an angle of 30° with the normal on a plane-paralleled film of thickness 4×10^{-5} cm and refractive index 1.50. Show that the reflected light whose wavelength is 7542 Å will reinforce.

[Mumbai University]

15.38 **(a)** Explain with necessary theory how a Michelson interferometer may be employed to find the difference in wavelength of D_1 and D_2 lines in the sodium spectrum.

(b) The distance through which the mirror of the Michelson interferometer has to be displaced between two consecutive positions of maximum distinctness of D_1 and D_2 lines of sodium is 2.89×10^{-5} cm. Calculate $\Delta\lambda$ assuming that $\lambda_1 \approx \lambda_2 = 5.89 \times 10^{-5}$ cm.

15.39 In Michelson's interferometer 100 fringes cross the field of view when the movable mirror is displaced through 0.02948 mm. Calculate the wavelength of the monochromatic light used.

[Delhi University]

15.40 The plates of Fabry – Perot interferometer have a reflectance amplitude of $r = 0.90$. Calculate the resolving power of wavelengths near 600 nm when the plates are separated by 2 mm.

[University of Wales, Aberystwyth 2005]

15.2.5 Diffraction

15.41 In Fraunhofer diffraction due to a narrow slit a screen is placed 2 m away from the lens to obtain the pattern. If the slit width is 0.2 mm and the first minima are 5 mm on either side of the central maximum, find the wavelength of light.

[Delhi University]

15.42 The intensity I_θ for the single-slit diffraction pattern is given by $I_\theta = I_m (\sin\alpha/\alpha)^2$, where $\alpha = \dfrac{\pi a}{\lambda}\sin\theta$, and I_m is the intensity of the central maximum. Show that the intensity maxima can be found out from the condition, $\tan\alpha = \alpha$.

15.43 **(a)** Obtain the expression for $\Delta\theta$, the half-width at half central maximum of single-slit Fraunhofer diffraction.

(b) Calculate $\Delta\theta$ for $\dfrac{a}{\lambda} = 4$.

[Osmania University]

15.44 A beam of light contains a mixture of wavelengths λ_1 and λ_2. When the light is incident on a single slit the first diffraction minimum of λ_1 coincides with the second minimum of λ_2. How are the two wavelengths related?

15.45 A single slit is illuminated normally by a monochromatic light of wavelength of 5600 Å and diffraction bands are observed on a screen 2 m away. If the centre of the second dark band is 1.6 cm from the central bright band, deduce the slit width.

15.46 **(a)** What are missing orders in the double-slit diffraction pattern? Explain.

(b) Deduce the missing order for a double-slit diffraction pattern, if the slit widths are 0.16 mm and they are 0.8 mm apart.

[Brahampur University]

15.47 What conditions must be satisfied for the central maximum of the envelope of the double-slit diffraction pattern to contain exactly n interference fringes? Find n given $d = 0.20$ mm and $a = 0.0120$ mm.

15.48 In a grating spectrum which spectral line in the fourth order will overlap with the third order of 5400 Å?

[Osmania University]

15.49 What is the highest order spectrum which may be seen with monochromatic light of wavelength 6000 Å by means of a diffraction grating with 5000 lines/cm.

[Delhi University]

15.50 A grating has slits that are each 0.1 mm wide. The distance between the centres of any two adjacent slits is 0.3 mm. Which of the higher order maxima are missing?

[Andhra University 1999]

15.51 A grating has 5×10^3 lines/cm. The opaque spaces are twice the transparent spaces. Find the orders of the spectrum that will be absent.

[Osmania University 2004]

15.52 How many orders will be observed by a grating having 4000 lines/cm, if a visible light in the range 4000–7000 Å is incident normally?

[Kanpur University]

15.53 Show that in a grating if the opaque and the transparent strips are of equal width then all the even orders, except $m = 0$, will be missing.

15.54 Show that the intensity of the first secondary maxima relative to that of central maxima in the single-slit diffraction is about 4.5%.

15.55 A plane diffraction grating in the first order shows an angle of minimum deviation of 20° at the mercury blue line of wavelength 4358 Å. Calculate the number of lines per centimetre.

[Andhra University 2003]

15.56 A diffraction grating used at normal incidence gives a green line, $\lambda = 5400$ Å in a certain order superimposed on the violet line, $\lambda = 4050$ Å of the next higher order. If the angle of diffraction is 30°, how many lines are there per centimetre in the grating?

[Delhi University]

15.57 Calculate the least width that a grating must have to resolve the components of D lines (5890 and 5896 Å) in the second order. The grating has 800 lines/cm.

[Osmania University 2002]

15.58 A grating of width $3''$ is ruled with 10,000 lines/in. Find the smallest wavelength separation that can be resolved in the first-order spectrum at a mean wavelength of 6000 Å.

[Kakatiya University2002]

15.59 Examine two spectral lines of wavelengths 5890 and 5896 Å which can be clearly resolved in the (i) first order and (ii) second order by diffraction grating 2 cm wide and having 425 lines/cm.

[Delhi University]

15.60 The refractive indices of a glass prism for the C and F lines are 1.6545 and 1.6635, respectively. The wavelength of these two lines in the solar spectrum are 6563 and 5270 Å, respectively. Calculate the length of the base of 60° prism which is capable of resolving sodium lines of wavelengths 5890 and 5896 Å.

[Vikram University]

15.61 Find the separation of two points on the moon that can be resolved by a 500 cm telescope. The distance of the moon is 3.8×10^5 km from the earth. The eye is most sensitive to light of wavelength 5500 Å.

[Nagpur University]

15.62 Lycopodium particles that have an average diameter of 30 μm are dusted on a glass plate. If a parallel beam of light of wavelength 589 nm is passed through the plate, what is the angular radius of the first diffraction maximum?

[Kakatiya University 2004]

15.63 The intensity distribution for Fraunhofer diffraction of a circular spectrum of radius R is of the form

$$I = I_0 \left[\frac{2J_1(\rho)}{\rho} \right]^2$$

where $\rho = \dfrac{2\pi a}{\lambda} \sin\theta$ and $J_1(x)$ is the Bessel function of the first kind. Show that by Rayleigh's criterion the minimum angular resolution of a telescope is given by $\theta \approx 1.22\dfrac{\lambda}{D}$, where D is the diameter of the circular aperture.

15.64 Calculate the radii of the 1st and 25th circles on a zone plate behaving like a convex lens of focal length 50 cm for $\lambda = 5000$ Å.

[Kakatiya University]

15.2.6 *Polarization*

15.65 The values of refractive indices for ordinary and extraordinary rays n_o and n_e for calcite are 1.642 and 1.478, respectively. Calculate the phase retardation for $\lambda = 6000$ Å with the plate thickness 0.04 mm.

[Kakatiya University 2003]

15.66 Two polarizing sheets have their polarizing directions parallel. Determine the angle by which either sheet must be turned so that the intensity falls to half of its value?

15.67 Sun rays incident obliquely on a pond are completely polarized by reflection. Find the elevation of the sun (in degrees) above the horizon.

15.68 Light is incident from water ($\mu = 1.33$) on the glass ($\mu = 1.5$). Find the polarizing angle for the boundary separating water and glass.

15.69 What is the minimum thickness of a quarter wave plate if the material has $\mu_0 = 1.553$ and $\mu_e = 1.544$ at a wavelength of 6000 Å.

[Andhra University 2003]

15.70 A tube 20 cm long containing sugar solution rotates the plane of polarization through an angle of 13.2° If the specific rotation is 66°, find the amount of sugar present in a litre of the solution.

[Osmania University 2003]

15.71 A system of three polarizing sheets intercept a beam of initially unpolarized light. The polarizing direction of the first sheet is parallel to the y-axis, that of the second sheet is at an angle of θ counterclockwise from the y-axis and that of the third sheet is parallel to the x-axis. The intensity of light emerging from the three-sheet system is 11.52% of the original intensity I_0. Determine the angle θ. In which direction is the emerging light polarized?

15.72 Describe briefly how a linear polarizer produces polarized light from an incident unpolarized beam. Is the transmission axis of a pair of polaroid sunglasses usually oriented horizontally or vertically for an observer standing upright and why is this?

What is a Brewster window? Calculate the inclination of a Brewster window with refractive index $n = 1.5$ in a laser cavity in which the gaseous medium has a refractive index $n = 1.0$.

[Durham University]

15.3 Solutions

15.3.1 Geometrical Optics

General

15.1 Let the point source be located at the centre O of a sphere of radius R, located within the medium. Light proceeding within a cone of semi-angle equal to the critical angle C can alone escape from a plane surface on the top, Fig. 15.14.

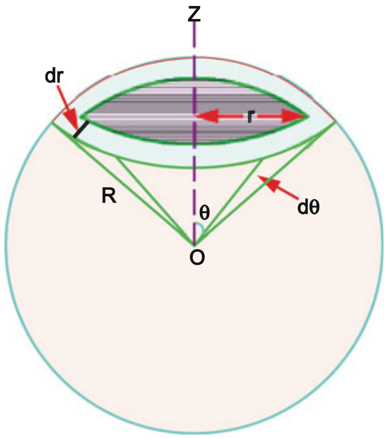


Fig. 15.14 Light from a point source escaping through a plane surface on the top

Consider a circular strip of radius r and width dr, symmetrical over the sphere's surface. The angle θ is measured with respect to the z-axis.

Area of the circular strip $= 2\pi r \mathrm{d}r$

Surface area of the sphere $= 4\pi R^2$

$$\text{Fraction } \mathrm{d}f = \frac{\text{area of the strip}}{\text{area of the sphere}} = \frac{2\pi r \mathrm{d}r}{4\pi R^2}$$

$$= \frac{1}{2}\frac{(R\sin\theta)(R\mathrm{d}\theta)}{R^2} = \frac{1}{2}\sin\theta\,\mathrm{d}\theta$$

Fraction of light escaping within the semi-angle θ is then given by

$$f = \int \mathrm{d}f = \int_0^\theta \frac{1}{2}\sin\theta \mathrm{d}\theta = \frac{1}{2}(1-\cos\theta)$$

Substituting $\theta = C$, the critical angle

$$\cos\theta = \cos C = \sqrt{1-\sin^2 C} = \sqrt{1-\frac{1}{\mu^2}} = \frac{1}{\mu}\sqrt{\mu^2-1}$$

$$\therefore \quad f = \frac{1}{2}\left[1 - \frac{1}{\mu}\sqrt{\mu^2-1}\right]$$

15.2 The momentum carried by each photon is $h\nu/c$. If it is incident normally on a black surface, it exerts an impulse of $h\nu/c$. The total pressure exerted would be $\frac{1}{c}\sum h\nu$, where the summation extends over photons of all frequencies incident on the surface per unit area per second. If W is the power of the source, then at distance r, the intensity $I = W/4\pi r^2$. Then pressure

$$P = \frac{I}{c} = \frac{W}{4\pi r^2 c} = \frac{1000}{(4\pi)(2^2)(3\times 10^8)} = 6.63\times 10^{-8}\,\text{Pa}$$

15.3 Suppose a ray in going from A to B traverses distance, $s_1, s_2, s_3, \ldots, s_p$ in media of indices $n_1, n_2, n_3, \ldots, n_p$, respectively. The total time of flight is then (Fig. 15.15)

$$t = \sum_{i=1}^{p}\frac{s_i}{v_i} = \frac{1}{c}\sum_{i=1}^{p} n_i s_i \tag{1}$$

The last summation is known as the optical path length (O.P.L).

Fermat's principle states that a ray of light traverses from one point to another by a route which takes least time.
A more, stringent formulation of Fermat's principle is as follows. A ray of light in traversing from one point to another, regardless of the media, adopts such a route which corresponds to stationary value of the optical path length:

$$\text{O.P.L} = \int_{\text{A}}^{\text{B}} n(s)\,\mathrm{d}s \tag{2}$$

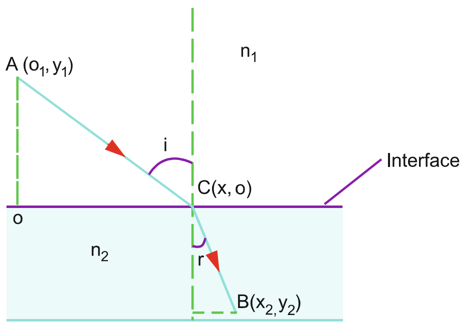


Fig. 15.15

A function $f(x)$ is said to have a stationary value at $x = x_0$, if its derivation $\mathrm{d}f/\mathrm{d}x$ vanishes at $x = x_0$. A stationary value could correspond to a maximum or minimum.

Let a ray of light proceed from $A(0, y_1)$ in medium of index n_1 and be incident at $C(x, 0)$ on the interface, get refracted and reach $B(x_2, y_2)$ in the medium of index n_2:

$$\text{O.P.L.} = n_1(\text{AC}) + n_2(\text{CB}) = n_1\sqrt{x^2 + y_1^2} + n_2\sqrt{(x_2 - x)^2 + y_2^2}$$

$$\frac{\mathrm{d(O.P.L)}}{\mathrm{d}x} = 0$$

$$\therefore \quad \frac{n_1 x}{\sqrt{x^2 + y_1^2}} - \frac{n_2(x_2 - x_1)}{\sqrt{(x_2 - x)^2 + y_2^2}} = 0$$

$$\therefore \quad n_1 \sin i - n_2 \sin r = 0$$

$$\therefore \quad \frac{\sin i}{\sin r} = \frac{n_2}{n_1} \qquad \text{(Snell's law)} \qquad (3)$$

15.4 A mirage is a type of illusion formed by light rays coming from the low region of the sky in front of the observer. On a sunny day, a road gets heated and a temperature gradient is established in the vertical direction with the upper air layer being slightly less warmer and the corresponding indices of refraction being slightly larger. As the rays penetrate the atmospheric depth they start bending due to refraction, becoming horizontal to the road and then bending upwards. The blue of the sky in the background produces a virtual image of a water pool, and the turbulence of air close to the road enhances the effect of a water pool with waves on the surface. This phenomenon is known as mirage.

(a) Referring to Fig. 15.5, optical path length (O.P.L) $= \sum_{i=1}^{2} n_i s_i = n_1\text{AB} + n_2\text{BC} + n_1\text{CD}$

$$= n_1\sqrt{(d - x)^2/4 + h^2} + n_2 x + n_1\sqrt{(d - x)^2/4 + h^2}$$

$$\text{O.P.L} = n_2 x + n_1\sqrt{(d - x)^2 + 4h^2} \qquad (1)$$

(b) $\dfrac{\mathrm{d(O.P.L)}}{\mathrm{d}x} = 0$ (Fermat's principle)

Using (1), $\quad n_2 - \dfrac{n_1(d - x)}{\sqrt{(d - x)^2 + 4h^2}} = 0$

Substituting $n_1 = 1.00030$, $n_2 = 1.00020$ and $h = 2\,\mathrm{m}$ and solving

$$d - x = 282.8 \qquad (2)$$

or $\quad x = 500 - 282.8 = 217.2\,\mathrm{m}$

(c) From (2) it is observed that as d decreases, x also decreases. The smallest value for x is zero, in which case the distance of the observer from the tree would be $d = 282.8\,\text{m}$ when the mirage disappears.

15.5 **(a)** The purpose of the cladding is to improve the transmission efficiency of the optical fibre. If cladding is not used then the signal is attenuated dramatically.

(b) Let a ray be incident at an angle θ, Fig. 15.6, the angle of refraction at P being θ_p. Let C be the critical angle at Q, interface of core and cladding:

$$\sin C = \frac{n_1}{n_2}$$

where n_1 and n_2 are the indices of the cladding and core, respectively:

$$\theta_\text{p} = 90 - \theta_\text{in}$$

where θ_in is the angle of incidence at Q:

$$n_0 \sin\theta = n_2 \sin\theta_\text{p} = n_2 \sin(90 - \theta_\text{in}) = n_2 \cos\theta_\text{in}$$

For internal reflection $\theta_\text{in} > C$ or $\cos\theta_\text{r} < \cos C$

$$\therefore \quad n_0 \sin\theta \leq n_2 \cos C$$

But $$n_2 \cos C = n_2\sqrt{1 - \sin^2 C} = \sqrt{n_2^2 - n_1^2}$$

$$\therefore \quad n_0 \sin\theta \leq \sqrt{n_2^2 - n_1^2}$$

This shows that there is a maximum angle of acceptance cone outside of which entering rays will not be totally reflected within the fibre. For the largest acceptance cone, it is desirable to choose the index of refraction of the cladding to be as small as possible. This is achieved if there is no cladding at all. However, this leads to other problems associated with the loss of intensity.

(c) The transmission is reduced due to multiple reflections and the absorption of the fibre core material due to impurities.

15.3.2 Prisms and Lenses

15.6 With reference to Fig. 15.6, using Snell's law

$$n_2 \sin r_1 = n_1 \sin 20°$$

$$\therefore \quad \sin r_1 = \frac{n_1}{n_2}\sin 20^\circ = \frac{1.1}{1.6}\times 0.342 = 0.2351$$

$$\therefore \quad r_1 = 13.6^\circ$$

From the geometry of Fig. 15.16

$$r_2 = 40 - 13.6^\circ = 26.4^\circ \quad (\because\ r_1 + r_2 = A,\ \text{the apex angle})$$

$$n_1 \sin i_2 = n_2 \sin r_2$$

$$\sin i_2 = \frac{n_2}{n_1}\sin r_2 = \frac{1.6}{1.1}\sin 26.4^\circ = 0.6467$$

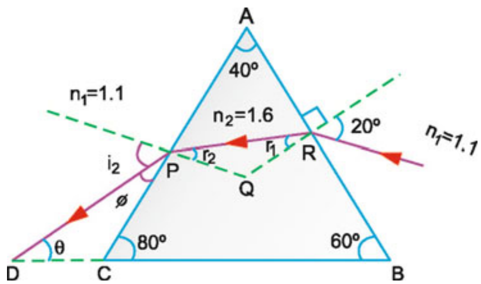
$$\therefore \quad i_2 = 40.3^\circ$$

$$\therefore \quad \phi = 90^\circ - 40.3^\circ = 49.7^\circ$$

$$\theta = 80 - 49.7^\circ = 30.3^\circ$$

($\because$ The exterior angle is equal to the sum of the interior angles).

Fig. 15.16 Triangular glass prism



15.7 **(a)** $\sin C = \dfrac{1}{n_g} = \dfrac{1}{1.45} = 0.6896$

$\therefore\ C = 43.6^\circ$

(b) In applying Snell's law instead of measuring angles with respect to the normal, we will measure them with respect to the interface, for convenience. Then

$$1.2\cos 45^\circ = 1.0\cos\theta$$

whence $\theta = 31.9^\circ$.

From Fig. 15.17 AC is parallel to EF and so $\mathrm{B\hat{A}C} = \mathrm{B\hat{C}A} = 45^\circ$. Thus in the triangle, ADC,

$$\alpha = 180^\circ - 2(\theta + 45^\circ) = 26.2^\circ$$

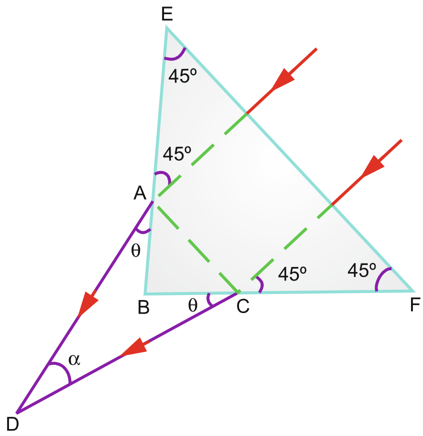


Fig. 15.17 Refraction in a prism

15.8 The net deviation of the ray

$$\delta = (i_1 - r_1) + (360^\circ - 2r_2) + (i_3 - r_3) \tag{1}$$

In the minimum angle position, $r_1 = r_2 = A/2 = 60/2 = 30^\circ$
From the geometry of Fig. 15.18, $r_3 = 30^\circ$.

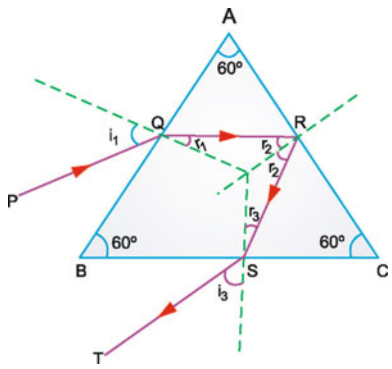


Fig. 15.18 Deviation of a ray incident in the minimum deviation position of the prism after suffering one internal reflection

By Snell's law

$\mathrm{Sin}\, i_1 = n \sin r_1 = 1.5 \times \sin 30^\circ = 0.75$
$\therefore \quad i_1 = 48.59^\circ$

Similarly, $i_3 = 48.59^\circ$

$$\therefore \qquad \delta = (48.59 - 30^\circ) + (360 - 2 \times 30^\circ) + (48.59 - 30^\circ) = 337.18^\circ$$

15.9 $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$

where u, the object distance, and v, the image distance, are measured from the centre of the lens. For real object and image u and v are positive, for virtual object or image u and v are negative. f is positive for convex lens and negative for concave lens.

Let the object o be placed at distance u_1 from lens L_1 of focal length f_1. The second tens L_2 is placed at distance d behind L_1, Fig. 15.19. The image I_1 is formed at distance v_1 from the lens L_1, alone. Then from the lens equation

$$\frac{1}{u_1}+\frac{1}{v_1}=\frac{1}{f_1} \text{ or } v_1=\frac{f_1u_1}{u_1-f_1} \tag{1}$$

The image acts as an object for the second lens L_2 (real or virtual)

$$-\frac{1}{v_1-d}+\frac{1}{v_2}=\frac{1}{f_2}$$

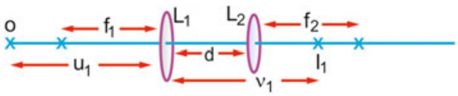


Fig. 15.19 Image due to combination of two lenses, a distance d apart

whence $v_2=\frac{f_2(v_1-d)}{f_2+v_1-d}$

Substituting v_1 from (1) and simplifying

$$v_2=f_2\frac{[f_1u_1-d(u_1-f_1)]}{f_1u_1+(u_1-f_1)(f_2-d)} \tag{2}$$

In the limit $d \to 0$, (2) reduces to

$$v_2=\frac{f_1f_2u_1}{f_1u_1+f_2u_1-f_1f_2}$$

$$\text{or} \quad \frac{1}{u_1}+\frac{1}{v_2}=\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}$$

15.10 The location of the object, the screen and the positions of the lens are indicated in Fig. 15.20

(a) $u_1+v_1=D \quad (1)$

$u_1-u_2=d \quad (2)$

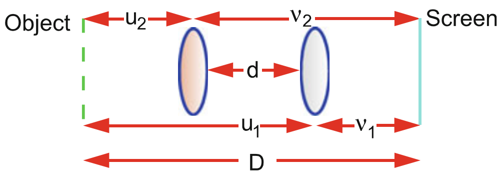


Fig. 15.20 Real images formed by a convex lens in two positions

By symmetry $\quad v_2 = u_1 \qquad (3)$

$$v_1 = u_2 \qquad (4)$$

Then (2) becomes

$$u_1 - v_1 = d \qquad (5)$$

$$\frac{\text{size of image}}{\text{size of object}} = \frac{I_1}{O} = \frac{v_1}{u_1} \qquad (6)$$

Similarly, $\quad \dfrac{I_2}{O} = \dfrac{v_2}{u_2} \qquad (7)$

$$\therefore \quad \frac{I_1}{I_2} = \frac{v_1}{v_2}\frac{u_2}{u_1} = \frac{v_1^2}{v_2^2} \qquad (8)$$

From (1) and (5), $v_2 = \dfrac{D+d}{2}$ and $v_1 = \dfrac{D-d}{2}$

Therefore, $\dfrac{I_1}{I_2} = \dfrac{(D-d)^2}{(D+d)^2}$

(b) Multiplying (6) and (7)

$$\frac{I_1}{O}\frac{I_2}{O} = \frac{v_1}{u_1}\frac{v_2}{u_2} = \frac{u_2}{u_1}\frac{u_1}{u_2} = 1$$

$$\therefore \quad O = \sqrt{I_1 I_2}$$

(c) $\dfrac{1}{u_1} + \dfrac{1}{v_1} = \dfrac{1}{f}$

$$\therefore \quad \frac{2}{D+d} + \frac{2}{D-d} = \frac{1}{f}$$

where we have used (1), (2) and (4).

$$\therefore \quad f = \frac{D^2 - d^2}{4D}$$

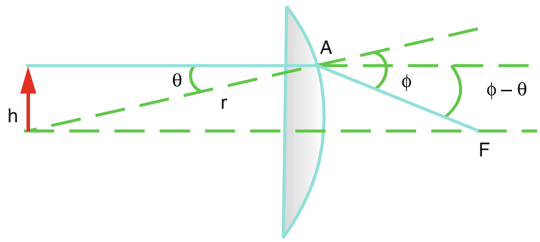
(d) From the result of **(c)**, we have

$$d^2 = \frac{D}{f}(D - 4f)$$

Since d^2 must be positive, it follows that $D > 4f$.

15.11 **(a)** Consider a plano-convex lens of focal length f. Let a paraxial ray be incident on the lens from a small object of height h. After striking the plano-convex lens normally, it gets refracted at the convex surface and passes through the principle focus F, behind the lens, Fig. 15.21. Let θ be the angle of incidence at A, the angle being measured with the radius of curvature of the curved surface r. Let ϕ be the angle of refraction:

Fig. 15.21



$$\sin\phi = n\sin\theta (\text{Snell's law})$$

$$\phi = n\theta (\because \text{ angles are small}) \tag{1}$$

$$\text{Also} \quad \theta = h/r, \phi - \theta = h/f \tag{2}$$

Combining (1) and (2)

$$\frac{1}{f} = \frac{n-1}{r} \tag{3}$$

We use the convention that r is positive if the refracting surface facing the object is convex and r is negative if the refracting index facing the object is concave.

A thin lens may be considered as two plano-convex or two plano-concave lenses in contact. Thus a thin biconvex lens whose faces have radii of curvature r_1 and r_2 are considered as two plano-convex lenses with their plane surfaces cemented together:

$$\frac{1}{f_1} = \frac{n-1}{r_1}, \quad \frac{1}{f_2} = \frac{n-1}{-r_2} \tag{4}$$

$$\therefore \quad \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{F} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \text{(lens maker's formula)} \tag{5}$$

For a biconvex lens r_1 is positive and r_2 negative. For double-concave lens r_1 is negative and r_2 positive.

(b) If the lens of refractive index n_1 is immersed in a medium of index n_2, then $n = n_1/n_2$, $n = 1.2/1.33 = 0.9$, and the first bracket in (5) becomes $n - 1 = -0.1$ resulting in a negative value of F. Thus the plastic lens immersed in water acts as a diverging lens.

15.12 **(a)** The first image due to L_1 alone is formed at I_1 at a distance v_1 given by

$$\frac{1}{u_1} + \frac{1}{v_1} = \frac{1}{f_1}$$

$$\therefore \quad v_1 = \frac{u_1 f_1}{u_1 - f_1} = \frac{5 \times 10}{5 - 10} = -10\,\text{cm}$$

The image is formed at 10 cm in front of the lens (Fig. 15.22).

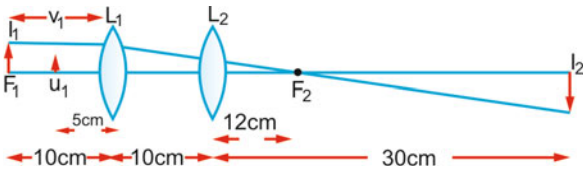


Fig. 15.22 Image due to combination of two convex lenses

(b) The image is virtual ($\because$ v is negative) and erect.

(c) The final image I_2 is located at distance v_2 from lens L_2 given by the result of prob. (15.9),

$$v_2 = \frac{f_2[f_1 u_1 - d(u_1 - f_1)]}{f_1 u_1 + (u_1 - f_1)(f_2 - d)} \tag{1}$$

Here $u_1 = 5$ cm, $d = 10$ cm, $f_1 = 10$ and $f_2 = 12$ cm.
Substituting these values in (1) we find $v_2 = 30$ cm behind the lens L_2.

(d) The final image is real ($\because$ v_2 is positive) and inverted.

15.13 Let a ray PA enter the sphere at A, refract at A and B and intersect the axis at F (Fig. 15.23). Let ϕ be the angle of incidence and θ the angle of refraction at A:

$$\text{Sin}\,\phi = \mu \sin\theta \qquad \text{(Snell's law)} \tag{1}$$

$$\phi = \mu\theta \qquad (\because \text{angles are small}) \tag{2}$$

In ΔBFC,

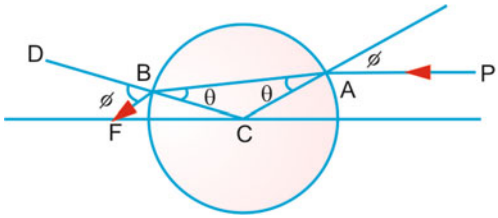
$$\frac{\text{FC}}{\text{BC}} = \frac{\sin \text{F}\hat{\text{B}}\text{C}}{\sin \text{B}\hat{\text{F}}\text{C}} = \frac{\sin(\pi - \phi)}{\sin 2(\phi - \theta)} = \frac{\sin \phi}{\sin 2(\phi - \theta)} = \frac{\phi}{2(\phi - \theta)} \tag{3}$$

since the angles are small. Hence the equivalent focal length

$$F_{\text{eq}} = \text{FC} = \frac{\text{BC}\phi}{2(\phi - \theta)} = \frac{\mu r}{2(\mu - 1)} \tag{4}$$

where we have used (2), (3) and (4).

Fig. 15.23



15.14 The intensity at a distance r is given by

$$I = \frac{W}{4\pi r^2} = \frac{100}{4\pi \times 5^2} = 0.318\,\text{W/m}^2$$

$$I = \frac{1}{2}\varepsilon_0 c E_0^2$$

$$\therefore \quad E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}} = \sqrt{\frac{2 \times 0.318}{8.85 \times 10^{-12} \times 3 \times 10^8}} = 15.48\ \text{V/m}$$

15.15 **(a)** If the object distance is u then the distance v_1 at which the image is formed by objective is given by

$$\frac{1}{u} + \frac{1}{v_1} = \frac{1}{f_\text{o}}$$

$$\text{or} \quad v_1 = -\frac{u f_\text{o}}{u - f_\text{o}} \tag{1}$$

The distance of this image from the eyepiece, u_1 is given by

$$u_1 = f_\text{o} + f_\text{e} - \frac{u f_\text{o}}{u - f_\text{o}} = \frac{u f_\text{e} - f_\text{o}^2 - f_\text{o} f_\text{e}}{u - f_\text{o}} \tag{2}$$

where $f_\text{o} + f_\text{e}$ is the distance between the objective and eyepiece.

If the final image is formed at a distance v from the eyepiece then

$$\frac{1}{v} + \frac{1}{u_1} = \frac{1}{f_e}$$

$$\text{or} \quad \frac{1}{v} = \frac{1}{f_e} - \frac{u - f_o}{u f_e - f_e^2 - f_o f_e}$$

$$\text{or} \quad v = -\frac{f_e(u f_e - f_o^2 - f_o f_e)}{f_o^2} \tag{3}$$

Now $u >> f_e, f_o$

$$\therefore \qquad v = -u\left(\frac{f_e}{f_o}\right)^2 = 10^4\left(\frac{1}{8}\right)^2 = 156.25\,\text{m}$$

(b) $$M = \frac{\text{Height of the final image}}{\text{Height of the object}} = \frac{v_1}{u} \times \frac{v}{u_1}$$

$$\simeq -\frac{f_e}{f_o} = -\frac{1}{8} = -0.125$$

where we have used (1), (2) and (3).

$\therefore$ Height of final image = (height of the object).M

$= -100 \times 0.125 = -12.5\,\text{m}$

The negative sign indicates that the final image is virtual.

Note that the height of the image is only one-eighth of that of the object, but the image is closer than the object by a factor of 64, so it subtends an angle eight times large, that is, the image appears eight times larger.

15.16 **(a)** The Intensity is given by

$$I = \frac{\text{power}}{\text{cross-section}} = \frac{W}{A}$$
$$= \frac{1000}{10^{-5} \times 10^{-4}} = 10^{12}\,\text{W/m}^2$$

(b) $$I = \frac{1}{2}\varepsilon_0 c E_0^2$$

$$E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}} = \sqrt{\frac{2 \times 10^{12}}{8.85 \times 10^{-12} \times 3 \times 10^8}}$$
$$= 2.74 \times 10^7\,\text{V/m}$$

15.3.3 Matrix Methods

15.17 Assume that light is always incident from the left, then the refracting surface is convex if r is positive (from V to C) and concave if r is negative (from V to C'), Fig. 15.24.

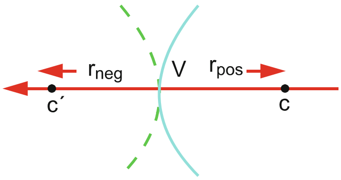


Fig. 15.24 Light is incident from left. Sign convention for refraction at a curved optical surface

The symbols used are object distance x_0, object size y_0, image distance x_i, image size y_i and radius of curvature, r. The object–image equation can be written as

$$\frac{n_1}{-x_0} + \frac{n_2}{x_\mathrm{i}} = \frac{n_2 - n_1}{r} \tag{1}$$

The convention results in negative value for the object distance. Paraxial rays are considered so that α_1 and α_2 are small. Each ray is given a height and an angle, Fig. 15.25. The distance ε on the axis, called sagitta, is nearly zero. Considering counterclockwise angles as positive and clockwise angles as negative (1) becomes

$$\frac{n_1\alpha_1}{l_1} + n_2\left(-\frac{\alpha_2}{l_2}\right) = \frac{n_2 - n_1}{r} \tag{2}$$

As $l_2 = l_1$, two simultaneous equations can be written in the variables α_j and $l_j (j = 1, 2)$:

$$l_2 = l_1 \tag{3}$$

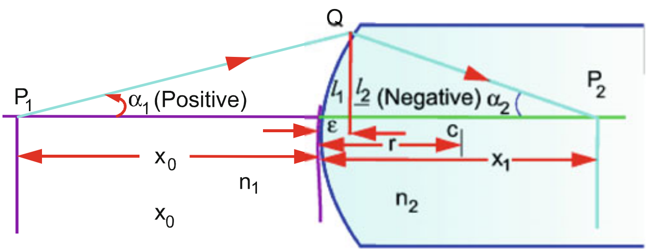


Fig. 15.25 Passage of a light ray P_1 in medium of refractive index n_1 into medium of refractive index n_2 onto P_2

$$\alpha_2 = \frac{n_1}{n_2}\alpha_1 - \left(\frac{n_2 - n_1}{n_2 r}\right) l_1 \tag{4}$$

In the matrix form (3) and (4) can be written as

$$\begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{r}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix} \tag{5}$$

The initial image in medium 1 described by the column vector $I_1 = \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$ is transformed into the final image in medium 2 described by the column vector $I_2 = \begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix}$. The transformation is accomplished by the refraction matrix

$$R_{12} = \begin{pmatrix} 1 & 0 \\ \frac{1}{r}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix} \tag{6}$$

In the matrix notation (5) is written as

$$I_2 = R_{12}\, I_1 \tag{7}$$

Next consider a parallel translation of a ray through a distance d in some homogeneous medium, Fig. 15.26.

Since $\alpha_1 = \alpha_2$, for small angles

$$l_2 = l_1 + \alpha_1 d \tag{8}$$

In matrix form (8) can be written as

$$\begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix} \tag{9}$$

or $\quad I_2 = T_2 I_1 \qquad (10)$

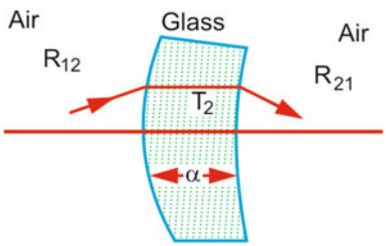


Fig. 15.26 Sequences of refraction, translation and refraction for a thick lens placed in air

where the translation matrix is

$$T_2 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{11}$$

In Fig. 15.26, the overall transformation can be written as

$$I_2 = R_{21}\, T_2 R_{12} I_1 \tag{12}$$

where R_{12} is the refraction matrix at the first surface (air to glass), T_2 is the translation matrix in the second medium (glass) and R_{21} is the refraction matrix at the second surface (glass to air). The matrix $R_{21}T_2R_{12}$ is known as the system matrix.

15.18 Consider two curved surfaces of positive radii of curvature, r_1 and r_2. The final image in Fig. 15.27 is obtained from the equation:

$$\begin{pmatrix} l_3 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{n-1}{r_2} & n \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{1}{r_1}\left(\dfrac{1-n}{n}\right) & \dfrac{1}{n} \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$$

Symbolically,

$$I_3 = R_{23}\, T_2\, R_{12}\, I_1$$

Let $P_{12} = \dfrac{1}{r_1}\dfrac{(1-n)}{n}$ and $P_{23} = \dfrac{n-1}{r_2}$

Then $$\begin{pmatrix} l_3 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ P_{23} & n \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ P_{12} & \dfrac{1}{n} \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + dP_{12} & \dfrac{d}{n} \\ P_{23} + dP_{23}P_{12} + nP_{12} & \dfrac{dP_{23}}{n} + 1 \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$$

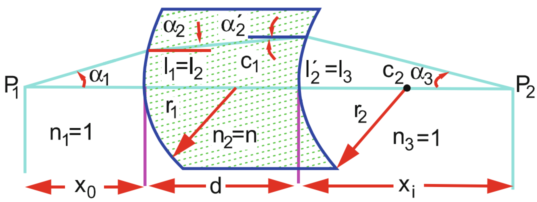


Fig. 15.27 Refraction in a double convex lens of thickness d placed in air

15.19 For a thin lens $d = 0$. The transformation matrix then becomes (see prob. 15.18)

$$\begin{pmatrix} 1 & 0 \\ P_{23} + nP_{12} & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 0 \end{pmatrix}$$

Thus $-\frac{1}{f_1} - \frac{1}{f_2} = -\frac{1}{f}$ or $\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f}$

15.3.4 Interference

15.20 A constant phase difference implies a constant difference in length between r_1 and r_2, Fig. 15.28:

$$r_1 - r_2 = 2a \tag{1}$$

$$\sqrt{(x+d)^2 + y^2} - \sqrt{(d-x)^2 + y^2} = 2a \tag{2}$$

Transposing the second radical

$$\sqrt{(x+d)^2 + y^2} = 2a + \sqrt{(d-x)^2 + y^2} \tag{3}$$

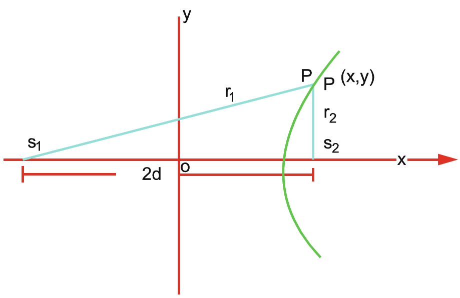


Fig. 15.28 Locus of a point with constant phase difference from two coherent point sources s_1 and s_2

Squaring and simplifying

$$x^2(d^2 - a^2) - y^2a^2 = a^2(d^2 - a^2) \tag{4}$$

Dividing by $a^2(d^2 - a^2)$

$$\frac{x^2}{a^2} - \frac{y^2}{d^2 - a^2} = 1 \tag{5}$$

Since $2d > 2a$ or $d > a$, $d^2 - a^2$ will be positive. Writing $d^2 - a^2 = b^2$, (5) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{6}$$

This is the equation of a hyperbola with the centre at the origin and the foci on the x-axis. In three dimensions, the locus of P would be a hyperboloid, the figure of revolution of the hyperbola.

In an actual Young's experiment on the observation of interference fringes one looks at a limited field of view and consequently the central portions of hyperbolas appear as straight lines as in Fig. 15.29 (within the dotted lines).

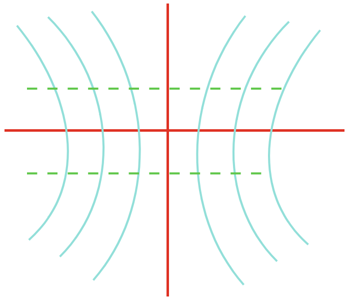


Fig. 15.29 Interference fringes in Young's experiment

15.21 At any point P on the screen at a distance y from the axis, the phase difference due to the waves coming from S_1 and S_2 will be due to the optical path difference (S_2P − S_1P), Fig. 15.11:

$$\delta = \frac{2\pi}{\lambda}(S_2\mathrm{P} - S_1\mathrm{P}) \tag{1}$$

If $(S_2\mathrm{P} - S_1\mathrm{P}) = n\lambda (n = 0, 1, 2, \ldots)$, the phase difference $\delta = 2n\pi$, and the intensity is maximum at P:

If $(S_2\mathrm{P} - S_1\mathrm{P}) = \left(m + \frac{1}{2}\right)\lambda$, where $m = 0, 1, 2, \ldots$, the phase difference, $\delta = (2m + 1)\pi$ and the intensity will be zero at P. With reference to Fig. 15.11,

$$(S_2\mathrm{P})^2 = L^2 + (x + d/2)^2$$
$$(S_1\mathrm{P})^2 = L^2 + (x - d/2)^2$$
$$\therefore \quad (S_2\mathrm{P})^2 - (S_1\mathrm{P})^2 = (S_2\mathrm{P} + S_1\mathrm{P})(S_2\mathrm{P} - S_1\mathrm{P}) = 2xd$$

In practice d is quite small, typically 0.5 mm, compared to L, typically 1.0 m, and P is close to the axis, $y << L$, so that S_1P as well as S_2P are only slightly greater than L. We can then set $S_2\mathrm{P} + S_1\mathrm{P} = 2L$. Therefore

$$S_2\mathrm{P} - S_1\mathrm{P} = \frac{2xd}{2L} = \frac{xd}{L} \tag{2}$$

Therefore, the condition for maximum intensity at P is

$$\frac{xd}{L} = m\lambda \qquad \text{(bright fringes)} \tag{3}$$

and the condition for zero intensity at P is

$$\frac{xd}{L} = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(dark fringes)} \tag{4}$$

m is called the order of fringe system. On the axis, at $y = 0$, $m = 0$ and we have the intensity maximum. The central bright band on the screen is flanked on either side by a series of b bright and dark bands corresponding to $m = 1, 2, 3, \ldots$, the mth bright fringe being at a distance y_m from the axis:

$$y_m = \frac{m\lambda L}{d} \tag{5}$$

and the $(m + 1)$th bright fringe being at a distance

$$y_{(m+1)} = \frac{(m+1)\ \lambda L}{d} \tag{6}$$

The separation of the fringes β called the bandwidth is given by

$$\Delta y = y_{(m+1)} - y_m = \beta = \frac{\lambda L}{d} \tag{7}$$

$$\beta = \frac{\lambda L}{d} = \frac{612 \times 10^{-9} \times 4}{2 \times 10^{-3}} = 1.124 \times 10^{-3}\,\text{m} = 1.124\ \text{mm}$$

15.22 Let a transparent plate of thickness t and refractive index μ be introduced in the path of one of the two interfering beams of monochromatic light, Fig. 15.30. A ray travelling from S_1 to O covers a distance t in the plate while the rest of the distance $(S_1\text{O} - t)$ is covered in air. The effective optical path length would be

$$\mu t + (S_1\text{O} - t) \text{ or } S_1\text{O} + (\mu - 1)t$$

The optical path length for the ray emanating from S_2 would be S_2O. Clearly

$$S_1\text{O} + (\mu - 1)t > S_2\text{O}$$

Consequently, the central fringe corresponding to zero path difference is not formed at O, the normal position of the central fringe in the absence of the plate. The new position of the central fringe would be at O′ such that

$$S_1\text{O}' + (\mu - 1)t = S_2\text{O}'$$

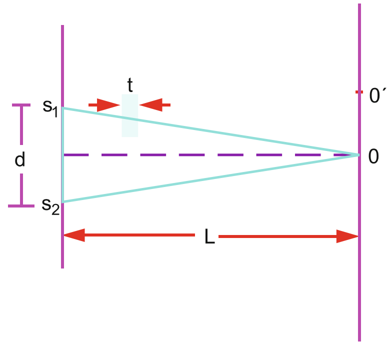


Fig. 15.30 Shift of fringes when a transparent plate is introduced in the path of one of the rays in Young's double-slit experiment

But $\quad S_2\mathrm{O}' - S_1\mathrm{O}' = \dfrac{d}{L} \cdot \mathrm{OO}'$

Calling $\mathrm{OO}' = \Delta$, the distance through which the central fringe shifts,

$$\Delta = \frac{L}{d}(\mu - 1)\, t$$

Furthermore, $(\mu - 1)t = n\lambda$.
This shift is towards the side on which the plate is placed.
Note that the bandwidth of the fringe system is unaffected and the entire fringe system undergoes a lateral shift.
With the use of monochromatic light it is not possible to detect shift of fringes. However, if white light is used then the central fringe being white is easily distinguished from the coloured fringes and its shift can be easily measured. Thus, by the use of the above procedure the thickness of the plate can be accurately measured.

15.23 (**a**) Let two light waves of the same wavelength λ and amplitude A pass through a given point and be represented by

$$y_1 = A \sin \omega t \tag{1}$$
$$y_2 = A \sin(\omega t - \delta) \tag{2}$$

where $\omega = 2\pi\nu$ and $\delta =$ constant phase difference between the two waves. The resulting displacement is then given by

$$\begin{aligned} y &= y_1 + y_2 \\ &= A \sin \omega t + A \sin(\omega t - \delta) \\ &= 2A \cos \frac{\delta}{2} \sin\left[\omega t - \frac{\delta}{2}\right] \end{aligned} \tag{3}$$

where we have used the identity

$$\sin B + \sin C = 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \tag{4}$$

Equation (3) represents simple harmonic vibration of frequency $\omega/2\pi$ and amplitude $A' = 2\text{A}\cos(\delta/2)$. The amplitude of the resulting wave varies from $2A$ through 0 to $-2A$ according to the value of δ. The resulting intensity I at the given point is proportional to the square of the amplitude or A'^2:

$$I \propto 4A^2\cos^2\frac{\delta}{2} \tag{5}$$

If $\delta = (2n+1)\pi(n = 0, 1, 2, \ldots)$ then $A' = 0$, i.e. the crests of one wave coincide with the troughs of the other, the two waves interfere destructively to give zero intensity, i.e. darkness.

If $\delta = 2n\pi$, then $A' = 2A$, i.e. the two waves interfere constructively to produce maximum intensity of $4A^2$. Here the crests of one wave are an integral number of wavelengths ahead of crest of the other so that the waves are reinforced:

By eqn (1), prob. (15.21), $\delta = \left(\frac{2\pi}{\lambda}\right)(S_2P - S_1P) = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\text{d}x}{L}\right)$

$$\therefore \quad I = 4A^2\cos^2\left(\frac{\pi\,\text{d}x}{\lambda L}\right) \tag{6}$$

Figure 15.31 shows the intensity distribution of Young's fringes. Here x is measured on the screen. The bright central fringe occurs at $x = 0$, in the centre of the fringe system. The other bright fringes are separated by distance, $x = \lambda\,L/d,\ 2\ \lambda\,L/d,\ 3\ \lambda\,L/d\ \ldots$ on either side. Halfway between two neighbouring bright fringes, the centres of dark fringes occur. The light intensity does not drop off suddenly but varies as $\cos^2(\delta/2)$. At maxima the intensity reaches a value of $4A^2$, and at minima, it is equal to zero. At other points it is given by (6). Thus the intensity in the interference pattern varies between $4A^2$ and zero. In the absence of interference each beam would contribute A^2 so that from two incoherent sources, there would be a uniform intensity of $2A^2$, which is indicated by the horizontal dotted line.

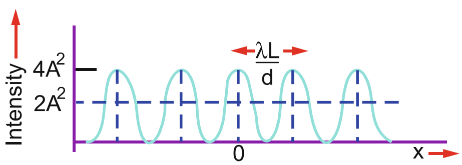


Fig. 15.31 Intensity distribution of fringes in young's double-slit experiment

(**b**) Now the average intensity of the interference pattern is obtained by averaging over the $\cos^2$ function in (6)

$$< I >= \frac{\int_0^{\lambda L/d} I \mathrm{d}x}{\int_0^{\lambda L/d} \mathrm{d}x} = \frac{4A^2 \int_0^{\lambda L/d} \cos^2(\pi \mathrm{d}x/\lambda L)\mathrm{d}x}{\lambda L/d} = 2A^2 \tag{7}$$

The average intensity is equal to $2A^2$ as expected. Although at maximum the intensity is double the average value, at minima, it becomes zero and on the whole it averages out to $2A^2$. Hence there is no violation of energy conservation.

15.24 $u + v = D = 100\,\text{cm} \to v = 100 - u = 100 - 30 = 70\,\text{cm}$

$$\frac{I}{O} = \frac{0.7}{O} = \frac{v}{u} = \frac{70}{30}$$

$\therefore \quad 2d = O = 0.30\,\text{cm}$ (distance between two coherent sources)

$$\beta = \frac{.\lambda D}{2d} \qquad \text{(bandwidth)}$$

$$\therefore \quad \lambda = \frac{2d\beta}{D} = \frac{(0.3)(0.0195)}{100} = 5.85 \times 10^{-5}\text{cm} = 5850\,\text{Å}$$

15.25 $D = y_1 + y_2 = 10 + 100 = 110\,\text{cm}$

$$2d = 2(\mu - 1)y_1\alpha = 2 \times (1.5 - 1) \times \left(\frac{2}{57.3}\right) \times 10$$
$$= 0.349\,\text{cm}$$
$$\lambda = \frac{(2d)\beta}{D} = \frac{0.349 \times 0.018}{110} = 5.711 \times 10^{-5}\text{cm}$$
$$= 5711\,\text{Å}.$$

15.26 $\theta = 20\,\text{s} = \left(\frac{20}{3600}\right) \times \frac{1}{57.3}\text{rad}$

$$= 9.695 \times 10^{-5}\,\text{rad}$$

$$\beta = \frac{\lambda}{2\mu\theta} = \frac{5.82 \times 10^{-5}\text{cm}}{2 \times 1.5 \times 9.695 \times 10^{-5}} = 0.2\,\text{cm}$$

Number of fringes per centimetre$= \frac{1}{\beta} = \frac{1}{0.2} = 5.$

15.27 $\theta = \frac{\lambda}{2\mu\beta} = \frac{6 \times 10^{-5}\text{cm}}{2 \times 1.4 \times (0.2\,\text{cm})} = 1.07 \times 10^{-4}\,\text{rad}$

$$= 1.07 \times 10^{-4} \times 57.3^\circ = 22'' \text{ of arc.}$$

15.28 The radius of the nth dark ring is given by

$$r_n = \sqrt{n\lambda R} \tag{1}$$

The radius of the $(n+m)$th dark ring is given by

$$r_{n+m} = \sqrt{(n+m)\lambda R} \tag{2}$$

Squaring (1) and (2), subtracting and solving for R, the radius of curvature of lower lens

$$R = \frac{r_{n+m}^2 - r_n^2}{m\lambda} = \frac{r_{n+20}^2 - r_n^2}{20\lambda}$$

$$= \frac{(0.368)^2 - (0.162)^2}{20 \times 5.46 \times 10^{-5}} = 100\,\text{cm}$$

15.29 $r_n = \sqrt{n\lambda R}$ (dark ring in air)

$r'_n = \sqrt{n\lambda R/\mu}$ (dark ring in liquid)

$$\therefore \quad \mu = \frac{r_n^2}{r_n'^2} = \frac{60^2}{50^2} = 1.44$$

15.30 First, we calculate the air thickness t of the air gap between the horizontal surface and the lower surface of the lens where Newton's ring is formed as in Fig. 15.32.
DE $= t$ is the thickness of air gap; CB $= R$, the radius of curvature of the lens; and DA $=$ DB $= r$ is the radius of the ring. From a theorem in geometry on intersecting chords

$$\text{DE} \times \text{DG} = \text{DA} \times \text{DB}$$

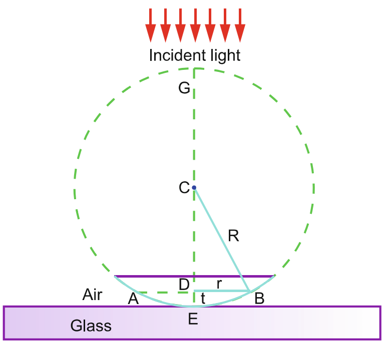


Fig. 15.32 Newton's rings with convex lens on a flat surface

or $\quad t \times (2R - t) = r^2$

$$\therefore \quad t = \frac{r^2}{2R} \qquad (\because\ t << 2R) \tag{1}$$

(a) Centres of curvature on the same side.
An air film BC is formed sandwiched between two curved surfaces of radii R_1 and R_2 in contact at O. The centres of curvature of lenses are on the same side, Fig. 15.33a. The thickness of air film

$$t = \mathrm{BC} = \mathrm{AC} - \mathrm{AB}$$

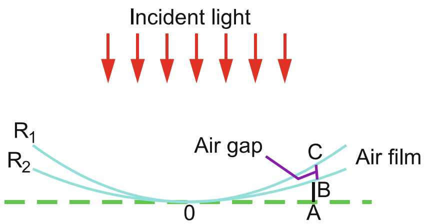


Fig. 15.33 a Newton's rings formed by two curved surfaces with the centres of curvature on the same side

Using (1)

$$\mathrm{AC} = \frac{r_m^2}{2R_1}; \qquad \mathrm{AB} = \frac{r_m^2}{2R_2}$$

$$t = \frac{r_m^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2}$$

where r_m is the radius of the mth ring:

$$2t = m\lambda \qquad \text{(dark rings)} \tag{3}$$

Eliminating t between (2) and (3)

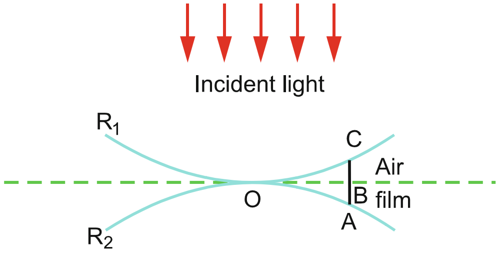
$$r_m^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = m\lambda \qquad (m = 0, 1, 2, \ldots \text{ dark rings}) \tag{4}$$

$$2t = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(bright rings)} \tag{5}$$

$$r_m^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \left(m + \frac{1}{2}\right)\lambda \qquad (m = 0, 1, 2, \ldots \text{ bright rings}) \tag{6}$$

(b) Centres of curvature on the opposite side
The surfaces in contact at O are as in Fig. 15.33b; the thickness of air film t is

Fig. 15.33 b Newton's rings formed by two curved surfaces with centres of curvature on the opposite side



$$\mathrm{AC} = \mathrm{BC} + \mathrm{AB}$$

$$t = \frac{r_m^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{7}$$

$$2t = m\lambda \qquad \text{(dark fringes)} \tag{8}$$

Eliminating t between (7) and (8)

$$r_m^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = m\lambda \qquad (m = 0, 1, 2\ldots,\ \text{dark fringes}) \tag{9}$$

$$2t = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(bright fringes)} \tag{10}$$

Eliminating t between (7) and (10)

$$r_m^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \left(m + \frac{1}{2}\right)\lambda \qquad (m = 0, 1, 2\ldots,\ \text{bright fringes}) \tag{11}$$

15.31 $x_m = \dfrac{m\,\lambda_1 D}{d} = (m+1)\lambda_2 \dfrac{D}{d}$

$$m \times 780 = (m+1) \times 520$$

$$\therefore \quad m = 2$$

15.32 Let the amplitude of the incident beam be a and intensity I:

$$I = a^2$$

The intensity of the reflected beam from the first face, Fig. 15.13

$$I_1 = \frac{1}{4}I \qquad \text{(by problem)}$$

$$\therefore \quad a_1 = \frac{a}{2}$$

The intensity of the transmitted beam at the first face will be

$$I_1' = \frac{3}{4}I$$

The corresponding amplitude will be

$$a_1' = \frac{\sqrt{3}}{2}a$$

The reflected beam at the second face will have amplitude

$$a_1'' = \frac{1}{2} \times \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a$$

The emerging beam from the first face will have amplitude

$$a_2 = \frac{\sqrt{3}}{2}a_1'' = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{4}a = \frac{3}{8}a$$

The two beams reflected from the first face will interfere:

$$\frac{I_{\min}}{I_{\max}} = \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} = \frac{\left(\frac{a}{2} - \frac{3}{8}a\right)^2}{\left(\frac{a}{2} + \frac{3}{8}a\right)^2} = \frac{1}{49}$$

15.33 For constructive interference

$$2\,\mu t = \left(m + \frac{1}{2}\right)\lambda$$

$$\therefore \quad \lambda = \frac{2\mu t}{m + \frac{1}{2}} = \frac{2 \times 1.5 \times 4 \times 10^{-5}}{m + \frac{1}{2}} = \frac{12 \times 10^{-5}}{m + \frac{1}{2}}\ \text{cm}$$

Only for $m = 2$, we get $\lambda = 4.8 \times 10^{-5}$ cm or 4800 Å, corresponding to blue colour in the visible region.

15.34 $2\mu t \cos r = m\lambda \quad$ (minima)

For smallest thickness, $m = 1$

$$t = \frac{m\lambda}{2\mu \cos r} = \frac{1 \times 5890}{2 \times 1.5 \times \cos 60^\circ} = 3927\ \text{Å}$$

15.35 $2\mu t\cos r = \left(m+\frac{1}{2}\right)\lambda \qquad \text{(maxima)}, m = 0, 1, 2\ldots$

$$r = 0, \mu = 1$$

$$\lambda = \frac{2t}{m+\frac{1}{2}} = \frac{2\times(2945\,\text{Å})}{m+\frac{1}{2}} = \frac{5890}{m+\frac{1}{2}}\text{Å}$$

For $m = 0, \quad \lambda = 11{,}980\,\text{Å}$

$m = 1, \quad \lambda = 3927\,\text{Å}$

$m = 2, \quad \lambda = 2356\,\text{Å}$

$\lambda = 3927\,\text{Å}$ falls within the range of visible spectrum. The colour is violet.

15.36 $2\mu t\cos r = \left(m+\frac{1}{2}\right)\lambda$

$r = 0, m = 0$

$t = \lambda/4\mu$

15.37 For constructive interference

$$2\mu t\cos r = \left(m+\frac{1}{2}\right)\lambda, \quad m = 0, 1, 2\ldots$$

$$\sin r = \frac{\sin i}{\mu} = \frac{\sin 30^\circ}{1.5} = 0.3333$$

$$\therefore \quad \cos\ r = 0.9428$$

$$\lambda = \frac{2\mu t\cos r}{m+\dfrac{1}{2}} = \frac{2\times 1.5\times 4\times 10^{-5}\times 0.9428}{m+\dfrac{1}{2}}\,\text{cm}$$

For $m = 1$, we find $\lambda = 7.542\times 10^{-5}\,\text{cm} = 7542\,\text{Å}$.

15.38 **(a)** The two mirrors M_1 and M_2 are adjusted such that their distance from the beam splitter are approximately equal, Fig. 15.4. The wavelengths λ_1 and λ_2 for the D_1 and D_2 lines of sodium differ only by a few angstroms. Two different sets of fringes arise due to two wavelengths. If M_1 is slowly moved away there is a gradual separation of the two sets, and finally the bright band of the one lies over the dark band of the other, resulting in uniform illumination. Thus the fringes disappear and reappear periodically. Between two successive disappearances the mirror has to be moved by, say d cm. This corresponds to a path difference of $2d$ cm. Assuming that $\lambda_2 < \lambda_1$, this path difference must contain exactly one more wavelength of λ_2 than of λ_1. Thus expressing λ_1 and λ_2 in cm,

$$\frac{2d}{\lambda_2} - \frac{2d}{\lambda_1} = 1$$

$$\therefore \quad \lambda_1 - \lambda_2 = \Delta\lambda = \frac{\lambda_1\lambda_2}{2d}$$

(b) $\Delta\lambda = \dfrac{(5.89 \times 10^{-5})^2}{2 \times 2.89 \times 10^{-5}} = 6\,\text{Å}$

15.39 $\lambda = \dfrac{2d}{N} = \dfrac{2 \times 2.948 \times 10^{-3}\,\text{cm}}{100} = 5.896 \times 10^{-5}\,\text{cm} = 5896\,\text{Å}$

15.40 Resolving power $\dfrac{\lambda}{\Delta\lambda} = \dfrac{g\pi D}{\lambda}$

where D = distance between the plates = 2 mm = 0.2 cm:

$$\lambda = 600\,\text{nm} = 6 \times 10^{-5}\,\text{cm}$$

$$g = \frac{2r}{1-r^2} = \frac{2 \times 0.9}{1-(0.9)^2} = 9.4737$$

$$R.P = \frac{\lambda}{\Delta\lambda} = \frac{9.4737 \times 3.14 \times 0.2}{6 \times 10^{-5}} = 9.9 \times 10^4$$

15.3.5 Diffraction

15.41 $a \sin\theta = m\lambda$ (1)

$$\therefore \quad \sin\theta = \frac{m\lambda}{a} = 1 \times \frac{\lambda}{a} = \frac{\lambda}{a} \tag{2}$$

If D is the slit screen distance and x the distance of the first minimum from the central maximum, then

$$\tan\theta \simeq \sin\theta = x/D \tag{3}$$

Combining (2) and (3)

$$\lambda = \frac{ax}{D} = \frac{0.02 \times 0.5}{200} = 5 \times 10^{-5}\,\text{cm} = 5000\,\text{Å}$$

15.42 $I_\theta = I_\text{m}\left(\dfrac{\sin\alpha}{\alpha}\right)^2$

Differentiating I_θ with respect to α and setting $\dfrac{dI_\theta}{d\alpha} = 0$,

$$\frac{dI_\theta}{d\alpha} = \frac{I_\text{m}(2\alpha^2 \sin\alpha\cos\alpha - 2\alpha\sin^2\alpha)}{\alpha^4} = 0$$

$$\therefore \quad 2\alpha \sin\alpha(\alpha\cos\alpha - \sin\alpha) = 0$$

$$\therefore \quad \tan\alpha = \alpha$$

15.43 **(a)** The half-width is the angle between the two points in the pattern where the intensity is one-half the centre of the pattern:

$$\frac{I_\theta}{I_\mathrm{m}} = \frac{1}{2} = \left(\frac{\sin\alpha_x}{\alpha_x}\right)^2 \tag{1}$$

The solution of (1) found by numerical method is

$$\alpha_x = 1.40\,\mathrm{rad} \tag{2}$$

$$\alpha_x = \frac{\pi a}{\lambda}\sin\theta_x = 1.40 \tag{3}$$

where $\theta_x = \frac{1}{2}\Delta\theta.$

$$\sin\theta_x = \frac{1.4\lambda}{\pi a} \quad \text{or} \quad \theta_x = \sin^{-1}\left(\frac{1.4\lambda}{\pi a}\right)$$

$$\text{or} \quad \Delta\theta = 2\theta_x = 2\sin^{-1}\left(\frac{1.4\lambda}{\pi a}\right)$$

(b) $\Delta\theta = 2\sin^{-1}\left(\frac{1.4}{4\pi}\right) = 12.8°$

15.44 $a\ \sin\theta = m\lambda$

$$a\ \sin\theta_1 = 1\cdot\lambda_1$$

$$a\ \sin\theta_2 = 2\lambda_2$$

But $\theta_1 = \theta_2$

$$\therefore \quad \lambda_1 = 2\lambda_2$$

15.45 $\sin\theta = \frac{x}{D} = \frac{n\lambda}{a}$

$$\therefore \quad a = \frac{n\lambda D}{x} = \frac{2\times 5.6\times 10^{-5}\times 200}{1.6} = 0.014\,\mathrm{cm}$$

$$= 0.14\,\mathrm{mm}.$$

15.46 **(a)** Let m be the order of interference. If the slit width a is maintained constant and the separation of the slits d is varied, the scale of the interference pattern varies, but that of the diffraction pattern remains unchanged. If the diffraction angle corresponds to the minimum given by (1) then a particular order of interference maxima may be absent. This is called a missing order. Thus, a missing order is realized for an angle θ for which the following two equations are simultaneously satisfied:

$$d \sin\theta = m\lambda (m = 1, 2, 3 \ldots) \tag{1}$$
$$a \sin\theta = n\lambda (n = 1, 2, 3 \ldots) \tag{2}$$

Dividing (1) by (2)

$$\frac{d}{a} = \frac{m}{n} \tag{3}$$

Since m and n are both integers, missing orders will occur when d/a is in the ratio of two integers. Expressing d as the sum of the slit width a and the opaque space b between consecutive slits, that is

$$d = a + b \tag{4}$$

(3) an be written as

$$\frac{a+b}{a} = \frac{m}{n} \tag{5}$$

In particular, if $\frac{a+b}{a} = 1,\ b = 0$. In this case the first-order spectrum will be absent and the resultant diffraction pattern will be similar to that of a single slit.

If $\frac{a+b}{a} = 2,\ a = b$, that is the width of the slit is equal to the width of the opaque space. Here the second-order spectrum will be absent.

(b) $$\frac{a+b}{a} = \frac{m}{n}$$

$$\frac{0.16 + 0.8}{0.16} = 6 = \frac{m}{n}$$

The above relation is satisfied for

$$n = 1, 2, 3, \ldots$$
$$m = 6, 12, 18, \ldots$$

Thus the order 6, 12, 18, etc. of the interference maxima will be missing, in the diffraction pattern.

15.47 The central diffraction peak is limited by the first minima. The angular locations of these minima are given by

$$a \sin\theta = \lambda \qquad (\because\ m = 1) \tag{1}$$

The angular locations of the bright interference fringes are given by

$$d \sin\theta = m\lambda \qquad (m = 0, 1, 2 \ldots) \tag{2}$$

We can locate the first diffraction minimum within the double-slit fringe pattern by dividing (2) by (1)

$$n = \frac{d}{a}$$
$$n = \frac{0.2}{0.012} = 16.66$$

Therefore 16 interference fringes will lie within the central maximum.

15.48 $m_1\lambda_1 = m_2\lambda_2$

$$\lambda_1 = \frac{m_2\lambda_2}{m_1} = \frac{3}{4} \times 5460 = 4095\,\text{Å}$$

15.49 $d \sin\theta = m\lambda$

For $\quad m = m_{\max}, \theta = \theta_{\max} = 90^\circ,$ Given $\dfrac{1}{d} = 5000$

$$m_{\max} = \frac{d}{\lambda} = \frac{1}{5000} \times \frac{1}{6 \times 10^{-5}} = 3.33$$
$$\therefore \quad m_{\max} = 3$$

15.50 $\dfrac{d}{a} = \dfrac{m}{n}$ (condition for missing orders)

$$\frac{0.3}{0.1} = \frac{3}{1} = \frac{m}{n}$$

where m and n are integers. The above relation is satisfied for

$$m = 3, 6, 9$$
$$n = 1, 2, 3$$

Thus the maxima will be missing in the third, sixth, ninth, etc. orders.

15.51 $\dfrac{d}{a} = \dfrac{a+b}{a} = \dfrac{a+2a}{a} = 3 = \dfrac{m}{n}$

The above relation is satisfied for

$$m = 3, 6, 9, \ldots$$
$$n = 1, 2, 3, \ldots$$

Thus the interference maxima will be missing in the third, sixth, ninth, etc. orders.

15.52 $$n_1(\max) = \frac{d}{\lambda_1} = \frac{1}{N\lambda_1} = \frac{1}{4000 \times 4 \times 10^{-5}} = 6.25$$

$$n_2(\max) = \frac{d}{\lambda_2} = \frac{1}{N\lambda_2} = \frac{1}{4000 \times 7 \times 10^{-5}} = 3.57$$

The maximum order of spectrum varies between 3 (towards red) and 6 (towards violet).

15.53 $$\frac{d}{a} = \frac{a+b}{a} = \frac{a+a}{a} = 2 = \frac{m}{n} \qquad \text{(condition for missing orders)}$$

$$\therefore \quad m = 2n$$

The above condition is satisfied for

$$n = 1, 2, 3 \ldots$$

$$m = 2, 4, 6 \ldots$$

Thus all the even orders of interference fringes (2, 4, 6, ...), except $m = 0$, are missing.

15.54 The secondary maxima lie approximately halfway between the minima. Now, the intensity at an angle θ is given by

$$I_\theta = I_\mathrm{m} \left(\frac{\sin\alpha}{\alpha} \right)^2 \tag{1}$$

$$\text{where} \quad \alpha = \frac{\pi\, a}{\lambda} \sin\theta \tag{2}$$

Minima occur in (1) when

$$\alpha = m\pi \qquad (m = 1, 2, 3 \ldots) \tag{3}$$

Therefore the first secondary maximum would occur halfway between first minimum and second minimum. Therefore,

$$\alpha = \left(m + \frac{1}{2} \right) \pi = \frac{3\pi}{2} \tag{4}$$

Substituting (4) into (1)

$$\frac{I}{I_\mathrm{m}} = \left(\frac{\sin(3\pi/2)}{3\pi l2} \right)^2 = 0.045 \text{ or } 4.5\%$$

15.55 In Fig. 15.34 the grating space AD $= d$. In the ΔABD, B$\hat{\text{A}}$D $= i$, the angle of incidence. Also D$\hat{\text{A}}$C $= \theta$, the angle of diffraction. In the ΔABD, BD the path difference between the incident rays, YD and XA is

$$\text{BD} = d \sin i$$

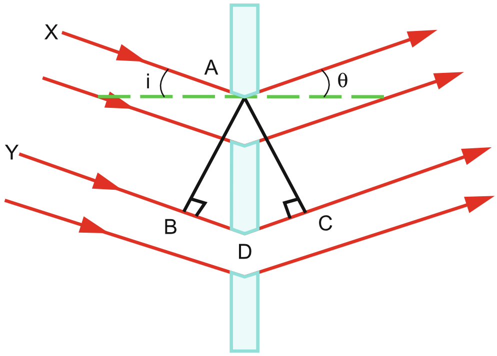


Fig. 15.34 Diffraction by a grating for a parallel beam of light which is obliquely incident

Similarly, the path difference between the diffracted rays is

$$DC = d \sin\theta$$

The total path difference

$$\text{BD} + \text{DC} = d(\sin i + \sin\theta) \tag{1}$$

For the mth primary maximum

$$d(\sin\theta_m + \sin i) = m\lambda \tag{2}$$

$$\text{or} \qquad \sin\left(\frac{\theta_m + i}{2}\right) = \frac{m\lambda}{2d\cos\left(\frac{\theta_m - i}{2}\right)} \tag{3}$$

The angle of deviation of the diffracted beam is

$$\delta_m = \theta_m + i \tag{4}$$

For δ_m to be minimum, $\cos\dfrac{\theta_m - i}{2}$ must be maximum, that is

$$(\theta_m - i)/2 = 0 \rightarrow \theta_m = i \tag{5}$$

Then δ_m will be minimum, say D_m, and is given by

$$D_m = \theta_m + i = 2i$$

$$\text{or} \qquad i = D_m/2 \tag{6}$$

Substituting (5) and (6) into (3)

$$2d \sin\left(\frac{D_m}{2}\right) = m\lambda \tag{7}$$

$$N = \frac{1}{d} = \frac{2\sin(20/2)}{1 \times 4.358 \times 10^{-5}} = 7969 \text{ lines/cm}$$

15.56 Condition for overlapping is

$$m\lambda_1 = (m+1)\lambda_2$$
$$5400\, m = 4050 \times (m+1)$$
$$\therefore \quad m = 3$$
$$d \sin\theta = m\lambda_1$$
$$N = \frac{1}{d} = \frac{\sin\theta}{m\lambda_1} = \frac{\sin 30°}{3 \times 5.4 \times 10^{-5}}$$
$$= 3086 \text{ lines/cm}$$

15.57 $\overline{\lambda} = 5893\,\text{Å} = 5.893 \times 10^{-5}\text{cm}$

$$\Delta\lambda = \lambda_1 - \lambda_2 = 5896 - 5890 = 6\,\text{Å} = 6 \times 10^{-8}\text{cm}$$

Resolving power

$$R = \frac{\overline{\lambda}}{\mathrm{d}\lambda} = \frac{5.893 \times 10^{-5}}{6 \times 10^{-8}} = 982$$
$$R = Nm$$
$$\therefore \quad N = \frac{R}{m} = \frac{982}{2} = 491 \qquad \text{(Total number of lines)}$$

If N' is the number of lines/cm, then the width of the grating

$$W = \frac{N}{N'} = \frac{491}{800} = 0.614\,\text{cm}$$

15.58 Total number of lines on the grating

$$N = N'W \tag{1}$$

where $N' =$ number of lines/in. and W is the grating width (in inch)

$$\therefore \quad N = 10,000 \times 3 = 30,000$$

$$R = \frac{\lambda}{\mathrm{d}\lambda} = Nm$$

$$\therefore \quad \mathrm{d}\lambda = \frac{\lambda}{Nm} = \frac{6 \times 10^{-5}}{3 \times 10^4 \times 1} = 2 \times 10^{-9}\,\mathrm{cm} = 0.2\,\text{Å}$$

15.59 Total number of lines

$$N = N'W = 2 \times 425 = 850$$

$$\overline{\lambda} = 5893\,\text{Å}, \ \mathrm{d}\lambda = 5896 - 5890 = 6\,\text{Å}$$

(**i**) First order:

$$R = \frac{\overline{\lambda}}{\mathrm{d}\lambda} = Nm$$

$$N = \frac{1}{m}\frac{\overline{\lambda}}{\mathrm{d}\lambda} = \frac{1}{1} \times \frac{5893}{6} = 982\,\text{lines}$$

As the required number of lines (982) exceeds the total number of lines (850) the lines are not resolved in the first order.

(**ii**) Second order

$$N = \frac{1}{m}\frac{\overline{\lambda}}{\mathrm{d}\lambda} = \frac{1}{2} \times \frac{5893}{6} = 491$$

As the required number of lines (491) is less than the total number of lines, the lines are resolved in the second order.

15.60 The resolving power for a prism of base length B is given by

$$R = \frac{\lambda}{\Delta\lambda} = \frac{B\mathrm{d}\mu}{\mathrm{d}\lambda} \tag{1}$$

where $\mathrm{d}\mu/\mathrm{d}\lambda$ is the variation of refraction index of the prism with wavelength, λ is the mean wavelength and $\Delta\lambda$ is the difference in wavelengths to be resolved:

$$\lambda = 5893\,\text{Å}, \Delta\lambda = 5896 - 5890 = 6\,\text{Å}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \frac{1.6635 - 1.6545}{(6563 - 5270) \times 10^{-8}\mathrm{cm}} = 696/\,\mathrm{cm}$$

Substituting the above values in (1) and solving for B, we find the length of the base of the prism, $B = 1.41\mathrm{cm}$.

15.61 The limit of resolution of a telescope is

$$d\theta = 1.22\frac{\lambda}{D} = \frac{1.22 \times 55 \times 10^{-5}}{500} = 1.342 \times 10^{-7}\,\text{rad}$$

If the distance between two points is x and the moon–earth distance r, then

$$x = r\,d\theta = 3.8 \times 10^{8} \times 1.342 \times 10^{-7}\,\text{m} = 51\,\text{m}$$

15.62 $\theta = \dfrac{1.22\lambda}{d} = \dfrac{1.22 \times 5.89 \times 10^{-7}}{30 \times 10^{-6}} = 0.024\,\text{rad}$

15.63 $I = I_0\left[\dfrac{2J_1(\rho)}{\rho}\right]^2$

where $\rho = \dfrac{2\pi}{\lambda}a\sin\theta$ and $J_1(\rho)$ is the Bessel function of the first kind. According to the Rayleigh criterion, the separation of the peaks is equal to the distance between the first minimum and the centre of the diffraction pattern, that is, the first minimum of the Bessel function is at $\rho = 3.83$, and we have

$$\rho = 3.83 = \frac{2\pi a}{\lambda}\left(\frac{R_1}{X}\right)$$

$$\text{or} \qquad \gamma = \frac{R_1}{X} = 0.61\frac{\lambda}{a} = 1.22\frac{\lambda}{2a} = 1.22\frac{\lambda}{D}$$

where $2a = D$ is the diameter of the lens, γ is the angle between the two stars, the distance between the observation screen and the lens is X (equal to focal length of the lens) and the position of the details of the diffraction pattern is R_1, Fig. 15.35a, b.

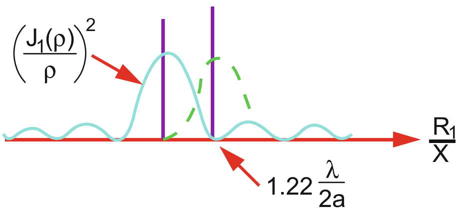


Fig. 15.35 a Intensity distribution for diffraction from a circular aperture described by Bessel function

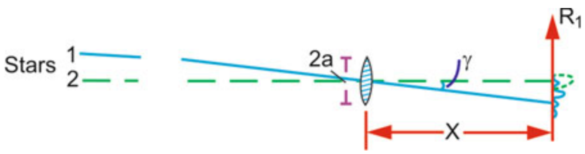


Fig. 15.35 b Resolution of diffraction patterns for two stars, the angle between them being γ. For the explanation of parameters X, R_1 and ρ, see the text

15.64 $r_n = \sqrt{f_n n\lambda}$

$$r_1 = \sqrt{50 \times 1 \times 5 \times 10^{-5}} = 0.05\,\text{cm}$$
$$r_{25} = \sqrt{50 \times 25 \times 5 \times 10^{-5}} = 0.25\,\text{cm}$$

15.3.6 Polarization

15.65 Path difference introduced by the plate

$$\Delta x = (n_0 - n_\text{e})t = (1.642 - 1.478) \times 4 \times 10^{-5}$$
$$= 6.56 \times 10^{-6}\,\text{m}$$

Phase difference

$$\delta = \frac{2\pi\,\Delta x}{\lambda} = \frac{(2\pi)(6.56 \times 10^{-6})}{6 \times 10^{-7}} = 68.66 \text{ rad}$$

15.66 $I_\theta = I_{\max} \cos^2\theta$ (Malus' law)

$$\cos\theta = \sqrt{\frac{I_\theta}{I_{\max}}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$\therefore \quad \theta = \pm 45, \pm 135°.$

15.67 $\tan\theta_\text{p} = \mu = 1.33$ (Brewster's law)

$\therefore \quad \theta_\text{p} = 53°$

Therefore the elevation of the sun is $90 - 53° = 37°$

15.68 $\tan Q_\text{p} = \mu_\text{wg} = \dfrac{\mu_\text{g}}{\mu_\text{w}} = \dfrac{1.5}{1.33} = 1.1278$

$\therefore Q_\text{p} = 48.4°$

15.69 $t = \dfrac{\lambda}{4(\mu_0 - \mu_\text{e})} = \dfrac{6 \times 10^{-5}}{4(1.553 - 1.544)} = 1.67 \times 10^{-3}\,\text{cm}$

15.70 $\theta = \alpha L D$

where θ is the angle of rotation of plane of polarization, α is the specific rotation, L is the length of the tube in decimetres and D is the amount of solvent in grams per 100 c.c:

$L = 20\,\text{cm} = 2\,\text{dm}$

$$D = \frac{\theta}{\alpha L} = \frac{13.2^\circ}{66^\circ \times 2} = 0.1\,\text{g}/100\,\text{c.c.} = 1.0\ \text{g/L}$$

15.71 As the light passing through the first sheet is unpolarized, the intensity I_1 of the light transmitted by the first sheet is given by the one-half rule:

$$I_1 = \frac{1}{2} I_0 \tag{1}$$

Because the polarizing direction of the first sheet is parallel to the y-axis, the polarization of the light transmitted by it is also along y-axis.

Because the light reaching the second sheet is polarized, the intensity I_2 of the light transmitted by that sheet is given by the cosine-squared rule. The angle θ in the rule is the angle between the polarization direction of the incoming light, parallel to the y-axis, and the polarizing direction of the second sheet, θ being counterclockwise from the y-axis. Thus

$$I_2 = I_1 \cos^2\theta \qquad \text{(Malus' law)} \tag{2}$$

Because the light entering the third sheet is polarized and the polarizing between the second and the third sheets is $90 - \theta$, the transmitted intensity is again given by Malus' law. Thus

$$I_3 = I_2\,\mathrm{Cos}^2(90 - \theta) = I_2 \sin^2\theta \tag{3}$$

From (3), (2) and (1)

$$I_3 = I_2 \sin^2\theta = I_1 \cos^2\theta \sin^2\theta = \frac{I_0}{2}\cos^2\theta\,(1 - \cos^2\theta) \tag{4}$$

$$\text{or}\quad \frac{I_3}{I_0} = 0.1152 = \frac{1}{2}\cos^2\theta\ (1 - \cos^2\theta)$$

$$\text{or}\quad \cos^4\theta - \cos^2\theta + 0.2304 = 0$$

$$\therefore\quad \cos^2\theta = 0.64 \text{ or } 0.36$$

$$\text{or}\quad \cos\theta = \pm 0.8 \text{ or } \pm 0.6$$

Taking only the positive value

$$\theta = 36.87^\circ \text{ or } 53.13^\circ$$

The polarization is along the x-axis.

15.72 Let an unpolarized light beam be incident from air on a dielectric such as glass. The E vector for the wave can be resolved into two components – one parallel to the plane of incidence, that is, the plane of paper and the second one perpendicular to the plane of incidence. The former one is represented by double arrow and is called π-component while the latter shown by dots is known as σ-component; Fig. 15.36.

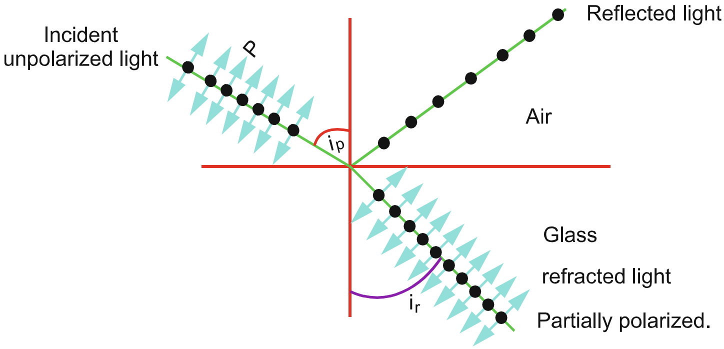


Fig. 15.36 Polarization by reflection • σ-component $\leftrightarrow$ π-component

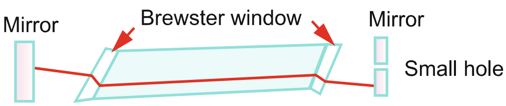
For a given dielectric, there is a particular angle of incidence i_p, called the polarizing angle or Brewster angle, at which the reflected beam is completely polarized with its plane of vibration perpendicular to the plane of incidence, that is, the reflected beam contains σ-component alone. Experiments show that at the polarizing angle, the reflected and refracted beams are at right angles. This leads to Brewster's law

$$\tan i_p = \mu$$

where μ is the refractive index. Note that at the polarizing angle the π-component is entirely refracted containing some amount of σ-component as well. Thus the transmitted light is only partially polarized. With the use of a stack of glass plates the proportion of σ-component can be increased in the reflected beam from multiple reflections and at the same time π-component can be made richer in the transmitted beam. Thus the transmitted beam is rendered plane polarized with the plane of vibration in the plane of incidence.

Dielectrics such as glass, water can partially or fully polarize light by reflection. If the surface is horizontal, the light is partially or fully polarized horizontally resulting in a bright spot (the glare) on the surface where reflection takes place. Such a glare from horizontal surfaces is eliminated by mounting the lenses in polarizing glasses with their polarizing direction vertical.

Fig. 15.37 Brewster windows mounted on a laser tube



Brewster windows are used in laser technology (Fig. 15.37). The tubes of gas lasers are sealed by mounting them in such a way that laser light is incident at Brewster's angle. Mirrors mounted outside the tube reflect the light back and forth through the tube. A small hole in one mirror permits the laser light to leave. Because the windows are tilted to the axis such that light is incident at Brewster's angle, the parallel component can traverse back and forth in the tube with minimum attenuation. The perpendicular component after several traversals gets substantially attenuated. Thus the light leaving the laser is polarized in the parallel direction

$$\tan\theta_{\mathrm{B}} = \mu = 1.5$$

$$\therefore \quad \theta_{\mathrm{B}} = 56.3^\circ, \text{ which is also the inclination of the Brewster window.}$$

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2.40 Half of kinetic energy is lost in completely inelastic collision of one body with one identical body at rest.
2.41 Speed of a bullet from its collision with a wooden block resting on a table.
2.42 The ratio M/m, V_c, KE in CMS in elastic collision.
2.43 In an elastic collision between M and m at rest ($M > m$), $\sin\theta_m = m/M$.
2.44 The ballistic pendulum.
2.45 Pressure exerted by fire engine jet (elastic collision with the wall).

2.46 Pressure exerted by fire engine jet (inelastic collision with the wall).
2.47 Symmetric elastic scattering of one ball with another identical ball at rest.
2.48 Total time for a ball dropped from a height h on a fixed plane to come to rest.
2.49 In prob. (2.48) total distance travelled.
2.50 In prob. (2.48) the height to which the ball goes up in the nth rebound.
2.51 For inelastic collision energy that is wasted is proportional to the square of the relative velocity of approach.
2.52 Magnitude of change in initial and final momentum in projectile's motion.
2.53 Explosion of a shell into three fragments at the highest point of trajectory.
2.54 Hovering of a helicopter.
2.55 Firing of a machine gun.
2.56 Scale reading of a balance pan when particles fall from a height and make elastic collisions with the pan.
2.57 In prob. (2.56) collisions are completely inelastic.
2.58 A partly elastic collision.
2.59 Acceleration of a car in a boat.
2.60 Minimum exhaust velocity for rocket to lift off immediately after firing.
2.61 Rate of fuel consumption to produce desired acceleration.
2.62 The rocket thrust, initial net acceleration, burn out velocity and time to reach burn-out velocity for Centaur rocket.
2.63 Rate of ejection of gas to provide necessary thrust.
2.64 Equation of motion for sliding of a rope over the edge of a table and its solution.
2.65 Variable mass problem applied to a running open car on horizontal rail under rain falling vertically down.
2.66 Pressure exerted by a falling chain on a table.
2.67 Velocity of rain drops.

Chapter 3

Rotational Kinematics

3.1 Given the parametric equations to find the path.
3.2 Given $a = f(r, t)$ in a circular orbit, to find power.
3.3 To find v where $r = 3\,\text{m}$, $r = 5\,\text{m/s}$ and $\omega = 4\,\text{rad/s}$.
3.4 A point moves along a circle of $r = 40\,\text{cm}$. Time needed for $a_N = a_t$.
3.5 A point moves along a circle of $r = 4\,\text{cm}$. Given $x = 0.3t^3$, to find a_N and a_t when $v = 0.4\,\text{m/s}$.
3.6 (a) Expression for $\boldsymbol{r}$ in polar form and (b) $\boldsymbol{a}$ is directed towards centre of circular motion.
3.7 $\propto$ of a wheel if a (total) of a point on the rim forms an angle of 30° with $\boldsymbol{v}$ in $t = 1.0\,\text{s}$.
3.8 A wheel rotates with constant acceleration $\alpha = 3\,\text{rad/s}^2$. $a(\text{total}) = 12\sqrt{10}\,\text{cm/s}^2$, to find R.

3.9 A car travels around a horizontal bend of $R = 150\,\text{m}$. To find v_{max} given $\mu_s = 0.85$.
3.10 Conical pendulum.
3.11 Difference in the level of the bob of conical pendulum when the frequency of revolutions is increased.
3.12 A rotating wheel plus a simple pendulum system.
3.13 Slipping of a coin placed on a rotating gramophone record.
3.14 Elongation of a spring with a particle attached to it.
3.15 Balancing a coin placed on the inside of a hollow rotating drum vertically.
3.16 A bead is located on a vertical circular wire frame so that its position vector makes an angle θ with the negative z-axis. If the frame is rotated, to find ω so that the bead does not slide.
3.17 A wire bent in a triangular form passes through a ring which revolves in a horizontal circle with a constant speed. To show that $v = \sqrt{gh}$ if the wires are to maintain the form.
3.18 A small cube placed on the inside of a funnel which rotates with constant frequency f. If μ is the coefficient of friction to find f_{max} for which the block will not move.
3.19 In prob. (3.18) to find f_{min} for which the block will not move.
3.20 A large mass M and a small mass m hang at two ends of a string that passes through a smooth tube. To find the frequency of rotation of mass m in a horizontal circle so that M may be stationary.
3.21 An object of weight W is being weighed on a spring balance going around a curve of known r and v. To find the weight registered.
3.22 Given the height of C.G of a carriage above the rails and the distance between the rails to find v_{max} on an unbanked curve of known radius.
3.23 v_{max} for given r, θ and μ on a curve on a highway.
3.24 Linear velocity of rotation of points on earth's surface at given latitude.
3.25 The speed of an aeroplane flying towards west such that the passenger may see the sun motionless.
3.26 The point at which a particle sliding from the highest point of smooth sphere would leave.
3.27 A sphere attached to a string is whirled in a vertical circle. To find the speed at the highest point, given that tension at bottom is equal to thrice the tension at the top.
3.28 A light rigid rod acting as simple pendulum when released from horizontal position has tension in the suspension equal to its weight when $\cos\theta = 1/3$.
3.29 Minimum speed of a motor cyclist in a circus stunt.
3.30 Minimum breaking strength of the string of a simple pendulum.
3.31 When the bob of a simple pendulum is deflected through a small arc s, and released, it would have velocity $v = \sqrt{g/L}$ at equilibrium position.
3.32 Tension in the string of a simple pendulum at $\theta = 45°$ when it swings with amplitude $\theta = 60°$.
3.33 When a simple pendulum is released from an angle θ, it has tension $T = 2\,\text{mg}$ at the lowest position. To determine θ.

3.34 The bob of a simple pendulum of $l = 1.0$ m has $v = 6$ m/s when at the bottom of the vertical circle. To find the point where it leaves the path.
3.35 A block released from height h on an incline enters the loop-the-loop track for $r = 12$ cm. To find h.
3.36 To find the point from where a particle leaves the loop-the-loop track.
3.37 To find the force exerted by a block on the given point of the loop-the-loop track.
3.38 In prob. (3.37), if $F = mg$ at the top of the loop then $h = 3R$.
3.39 It $v = 0.8944\sqrt{5gR}$ at the bottom of the loop-the-loop track then the particle would leave at 41.8° with the horizontal.
3.40 In the loop-the-loop track minimum height for completion of circular track is $2.5R$.
3.41 A nail is located at distance x vertically below the point of suspension of a simple pendulum of length 1 m. The pendulum bob is released from the position the string makes 60° with the vertical. To find x if the bob makes complete revolutions.
3.42 Complete revolutions made by a test tube when the cork flies out under pressure.
3.43 Minimum coefficient of friction between the tyres and road for a car to go round a level circular bend without skidding.

Chapter 4

Rotational Dynamics

4.1 Moment of inertia of a solid sphere.
4.2 Moment of inertia of a dumbbell.
4.3 Moment of inertia of a right circular cone.
4.4 Moment of inertia of a right circular cylinder.
4.5 Radius of gyration of a hollow sphere of radii a and b.
4.6 Moment of inertia of a thin rod.
4.7 Moment of inertia of a rectangular plate.
4.8 Moment of inertia of a triangular lamina.
4.9 A disc of known M.I is melted and converted into a solid sphere. To show that $I\,(\text{sphere}) = I\,(\text{disc})/5$.
4.10 Moment of inertia of a hollow sphere.
4.11 M.I. of a hollow sphere assuming that of a solid sphere.
4.12 Rolling of a solid cylinder down an incline.
4.13 Angular momentum and kinetic energy of a neutron star.
4.14 Rolling of a solid ball down an incline.
4.15 Least coefficient of friction that (a) solid cylinder and (b) loop, roll down an incline without slipping.
4.16 Tension in thread drawn through a hole in a table with constant velocity, the other end being attached to a small mass on the horizontal plane.

4.17 Spinning of an ice skater.
4.18 The distance to which a rolling sphere climbs up an incline.
4.19 A body attached to a string wound around a pulley is mounted on an axis. To determine linear acceleration and tension in the string.
4.20 A string is wound several times around a spool, the free end of the string being attached to a fixed point. To find acceleration of spool and tension in string.
4.21 Two unequal masses are suspended by a string over a heavy pulley. To find a, α and T_1/T_2.
4.22 Two wheels of M.I. I_1 and I_2 are set in motion with angular speed ω_1 and ω_2. When coupled face to face they rotate with common angular speed ω. To find ω and work done by frictional forces.
4.23 Toppling of a thin rod initially held vertically.
4.24 A circular disc of mass M rotates with angular velocity ω. Two particles each of mass m are attached at opposite end of diameter of disc. To find new angular velocity of disc.
4.25 Angular momentum from velocity and position vector.
4.26 Time and torque for a ball rolling down an incline.
4.27 A string is wrapped around a cylinder which is pulled vertically upward to prevent the centre of mass from falling. To find the tension in the string, the work done on the cylinder and length of string unwound.
4.28 Two cords are wrapped around a horizontal cylinder and vertically attached to the ceiling. To find a and T when cylinder is released.
4.29 A body rolling on level surface with speed u climbs up an incline to maximum height of $h = 3u^2/4g$. To figure out the geometrical shape of the body.
4.30 Four bodies of same mass and radius are released on an incline from the same height. To find the order in which three bodies reach the bottom of incline.
4.31 A tube filled with a liquid and closed of both ends is rotated horizontally about one end. To find the force by the liquid at the other end.
4.32 An inelastic collision of two point masses moving in opposite direction with a bar lying on a horizontal table.
4.33 Vertical oscillation of a system of rod carrying two point masses.
4.34 Duration of day if earth's radius suddenly decreases to half its present value.
4.35 a_{R} and a_{T} of a pole which cracks and falls over.
4.36 Magnitude of $\vec{\tau}$ and $\boldsymbol{J}$ from the expression for $\boldsymbol{J}$ and angle between $\boldsymbol{J}$ and $\boldsymbol{\tau}$.
4.37 Spinning of a disc about its axis on a horizontal surface.
4.38 Rolling of a solid sphere along the loop-the-loop track.
4.39 Motion of a particle along the interior of a smooth hemispherical bowl.
4.40 A spool with a thread wound on it is placed on an incline with the free end of thread attached to a nail and released, to find a.
4.41 Mean angular velocity of a flywheel.
4.42 Conical pendulum.
4.43 Sliding of a billiard hall.

Chapter 5

Gravitation

5.29 Separation of two components and mass of each component of a binary star.
5.30 A satellite fixed from the moon with velocity v_0 at 30° to vertical reaches maximum known height. To calculate v_0.
5.31 Semi-major axis of a satellite is given by $(T/2\pi)(v_{\max} \cdot v_{\min})^{1/2}$.
5.32 Mass of a planet from the radii of the satellite and planet, the shortest distance between the surfaces and period.
5.33 Angular momentum of the comet at a given point (r, θ) from the sun and speed of the comet at the closest distance of approach from given data.
5.34 The orbits of a comet and the earth are diagrammatically shown. To determine total energy component of velocity and angle at which comets orbit crosses that of the earth.
5.35 Verification of angular momentum conservation from data on the satellite 'Apple'.
5.36 Verification with which a satellite is to be fired to meet the subsequent motion.
5.37 Total energy and angular momentum of the fragments produced from internal explosion of a satellite.
5.38 When a particle approaches the nearer apse of an elliptic orbit the centre of force is transferred to the other focus. To determine the eccentricity of the new orbit.
5.39 Time average $<1/r>$ and $<v^2>$ for a satellite.
5.40 Changes in the major axis and time period of the earth when a small meteor falls into the sun and the earth is at the end of minor axis.
5.41 The velocity is doubled. To show that the new orbit will be a parabola or hyperbola accordingly as the apse is farther or nearer.
5.42 The axes of the new orbit when a particle is at the end of minor axis and force is increased by half.
5.43 In a circular orbit forces acting on the satellite, geosynchronous satellite.
5.44 Speed of a satellite at the apogee given its speed at the perigee.
5.45 Formula for the angle ϕ of encounter of a small body of velocity v with a massive body of mass M and impact parameter P.
5.46 Time required to describe an arc of a parabola under the force k/r^2 to the focus.
5.47 Maximum time comet remains inside earth's orbit.
5.48 Law of force for the orbit $r = a \sin n\theta$.
5.49 Law of force for the orbit $r = a(1 - \cos\theta)$.
5.50 In prob. (5.49) if Q be the force at the apse and v the velocity then $3V^2 = 4aQ$.
5.51 To determine orbit for inverse cube force.
5.52 Force necessary to describe the leminiscate is inversely proportional to the seventh power of r.
5.53 Force directed towards a point on the circular orbit of a particle is inversely proportional to the fifth power of r.
5.54 If the sun's mass suddenly decreased to half its value, then earth's orbit would become parabolic.

Chapter 6

Oscillations

Simple Harmonic Motion

6.1 Given E and T and x at t, to calculate A and m.
6.2 Given v_1 and v_2 at x_1 and x_2, to find A and T.
6.3 Given m, a and T for simple pendulum, to find, v_{max} and tension (max).
6.4 Given $T = 16$ s, at $t = 2$ s, $x = 0$ and at $t = 4$ s, $v = 4$ m/s, to find A.
6.5 To show that a floating body performs SHM vertically.
6.6 A box dropped is in a tunnel along earth's diameter. To show that it performs SHM and to find T.
6.7 Given the equation for SHM, to find A, T, f, ε, v, a at $t = 1$ s.
6.8 (a) Given KE = PE, to find x. (b) Given $x = A/2$, to show KE/PE $= 3/1$.
6.9 Mass M attached to a spring has $T = 2$ s. For $M + 2$, $T = 3$ s. To find M.
6.10 $a_{max} = 5\pi^2$ and at $x = 4$, $v = 3\pi$. To find A and T.
6.11 Given $a_{max} = \alpha$ and $v_{max} = \beta$, to show that $A = \beta^2/\alpha$ and $T = 2\pi\beta/\alpha$.
6.12 If tension in lowest position is 1.01 mg, angular amplitude of pendulum is 0.1 rad.
6.13 For SHM, $x = a$, b and c at t_0, $2t_0$ and $3t_0$. To find f.
6.14 For SHM a 4 kg mass has $T = 2$ s and $A = 2$ m. To find k and F_{max}.
6.15 Path of the particle from $x = a \sin \omega t$ and $y = b \cos \omega t$.
6.16 To show $\mathbf{F} = -kx\hat{i}$ is conservative and to find U.
6.17 K, A and f for vertical oscillations of a mass on a spring.
6.18 Amplitude of scale-pan oscillations when a mass is dropped from height h and sticks to the pan.
6.19 Probability of finding a particle between x and $x + \mathrm{d}x$ for SHM.
6.20 Mean K and mean U for SHM.
6.21 SHM for rolling of a cylinder plus spring system.
6.22 Time taken to gain a complete oscillation for two simple pendulums of given lengths.
6.23 Loss of time when a pendulum is taken from ground to top of a tower.
6.24 Loss of time for a pendulum when taken below earth's surface.
6.25 Oscillations of a liquid in U-tube.
6.26 SHM for cylinder–piston system.
6.27 Given equation to SHM and T, and $x = 4$ cm at $t = 0$ s. To find displacement at $t = 6$ s.
6.28 Determination of mass of spring from vertical oscillations of spring-mass system.
6.29 Work done on stiffer spring when stretched by the (a) same amount and (b) same force.
6.30 Oscillations of a solid cylinder rolling inside a large cylinder.
6.31 Physical pendulum.

6.32 Length of a simple pendulum that has the same T as a swinging rod.
6.33 g by a physical pendulum.
6.34 Frequency of oscillations of a semicircular disc pivoted freely.
6.35 Ratio of time periods of two modes of oscillations of a ring suspended on a nail.
6.36 Torsional oscillator.
6.37 Small oscillations of a pulley-spring-mass system.
6.38 Small oscillation of a cylinder attached to two springs.
6.39 Period of small oscillations of a particle in a one-dimensional potential field.
6.40 Effective spring constant of two springs in series.
6.41 Effective spring constant of two springs in parallel.
6.42 T for a block plus two spring system.
6.43 Oscillations of wire-spring-mass system.
6.44 Natural frequency of oscillation of a system consisting of a mass attached to one end of a rod which is connected to the centre of a cylinder.
6.45 Natural frequency of an oscillating semi-circular disc.
6.46 Eigen frequencies and normal modes of coupled pendula.
6.47 Equations of motion and energy in normal coordinates. Characteristics of normal coordinates.
6.48 To find frequency components and beat frequency from the resultant displacement.
6.49 Frequency of vibration of HCl molecule.
6.50 Resultant of three vibrations in the same straight line.
6.51 Logarithmic decrement for the vibrations of the mass-spring system.
6.52 Underdamped, overdamped and critically damped motion for the given equation.
6.53 Natural period, damping constant and logarithmic energy decrement.
6.54 Solution of equation of motion for a damped oscillator.
6.55 Position of a weight attached to a vertical spring and nature of oscillations.
6.56 Resonance frequency when periodic force is applied.
6.57 To find amplitude, phase lag, Q-factor and power dissipation from the equation for forced oscillations.
6.58 Q-factor for electric bell given frequency and time constant.
6.59 An oscillator has $T = 3$ s. Its amplitude decreases by 5% each cycle. To find energy decrease, time constant and Q-factor.
6.60 A damped oscillator loses 3% of energy in each cycle. Number of cycles required for half of energy to be dissipated and Q-factor.
6.61 Decrease of amplitude in each cycle when $\omega' = 9\omega_0/10$.
6.62 For small damping $\omega' \simeq (1 - r^2/8\,mk)\omega_0$.
6.63 Time elapsed between successive maximum displacements of a damped oscillator.
6.64 Period of oscillation from logarithmic decrement.
6.65 ω_0 and frequency of driving force from the given equation.
6.66 To show that $t_{1/2} = t_c \ln 2$.
6.67 Fraction of energy decrease in each cycle and Q-factor.

Chapter 7

Lagrangian and Hamiltonian Mechanics

7.1 Equations of motion for a particle under force $\mu m/r^2$.
7.2 Lagrangian for simple pendulum and proof for SHM.
7.3 Equations of motion for masses of Atwood machine by Lagrangian method.
7.4 Double Atwood machine.
7.5 Lagrangian and its equation of motion from given T and V.
7.6 Lagrangian for an SHO and time period.
7.7 Acceleration of a block on a fixed incline.
7.8 Acceleration of a block and inclined plane resting on smooth horizontal table.
7.9 Sliding of a bead on a straight wire which is constrained to rotate.
7.10 Spherical pendulum.
7.11 Equation of motion for a system of two blocks connected by spring on smooth horizontal table.
7.12 A double pendulum.
7.13 Hamilton's equations for spherical pendulum.
7.14 Hamilton's equation and solution from H for one-dimensional SHO.
7.15 Planetary motion using Hamilton's equations.
7.16 Natural frequencies of two coupled blocks.
7.17 Equations of motion of a simple pendulum pivoted to a block which slides on a smooth horizontal plane.
7.18 Equations of motion of an insect on a rod turning about one fixed end.
7.19 A rod attached at one end by a cord to a fixed end. To find inclination of string and rod when the system revolved about vertical through pivot.
7.20 Lagrangian in (r, θ) coordinates for a central potential and corresponding p_{r} and p_{θ} and H and conservation of E and J.
7.21 Coupled linear differential equations from Lagrangian equations and normal modes for a system of two masses plus three springs.
7.22 Lagrangian eigenfrequencies and normal modes for a system of two identical beads connected by two springs to a fixed wall.
7.23 Lagrangian and eigenfrequencies of a system of two beads of different masses connected to a wall by two springs.
7.24 Normal modes of oscillation for a system of three particles connected by springs.
7.25 Derivation of H for a single particle under conservative force.
7.26 Motion of a pendulum mounted on a block which can freely move on a horizontal surface.
7.27 Sliding of a particle down a smooth spherical bowl when it is (a) fixed and (b) free to move.
7.28 Hamilton and Hamiltonian equations from the given Lagrangian.
7.29 Lagrangian and equation of motion as in the given diagram.
7.30 Lagrangian and Lagrangian equation and equilibrium positions of a particle moving on an elliptic orbit in vertical plane.

7.31 Double pendulum with equal masses and lengths.
7.32 Coupled pendulums by Lagrangian method.
7.33 A bead sliding freely on a circular wire which rotates on a horizontal plane.
7.34 Equations of motion of a block-wedge spring system.
7.35 Rolling of a ball down a wedge which itself can slide on a horizontal table.

Chapter 8

Waves

8.1 Solution of a one-dimensional wave equation.
8.2 $y = 2A \sin(n\pi x/L) \cos 2\pi f t$ for standing wave is a solution of wave equation.
8.3 String plucked at the centre.
8.4 Superposition of the waves $y_1 = A \sin(kx - \omega t)$ and $y_2 = 3A \sin(kx - \omega t)$.
8.5 A sinusoidal wave has $v = 8$ m/s and $\lambda = 2$ m. To find K, f, ω and wave equation.
8.6 Equation of wave given k, ω and A.
8.7 When a standing wave is formed each point undergoes SHM transversely.
8.8 Frequencies of the first three harmonics for a plucked string.
8.9 Ratio of linear mass density of two strings.
8.10 A, f, v and λ of a transverse wave.
8.11 A and v of component waves of the given vibration. Distance between the nodes and the transverse velocity at a given point at a given time.
8.12 Phase velocity and phase difference.
8.13 Amplitude of resultant motion.
8.14 Suitable functions for one-dimensional wave equation.
8.15 Displacement of a cord when plucked at $x = L/3$.
8.16 Equation of a wave in the negative x-direction.
8.17 Superposition of the waves $y_1 = A \sin(kx - \omega t)$ and $y_2 = A \sin(kx + \omega t)$ is a standing wave.
8.18 Superposition of a harmonic wave with another wave travelling in the same direction but differing by phase difference δ of the same amplitude.
8.19 Average rate of energy transmission of a travelling wave.
8.20 Frequency of the fork by beat frequency with a monochord.
8.21 Velocity of a moving pulse.
8.22 Mass density of piano string frequencies of the first two harmonics. Length of a flute pipe.
8.23 (a) Sketch of first and second harmonic waves on a stretched string and (b) v and distance between nodes for given standing wave.
8.24 Wave function for the progressive wave from given data on A, ω and k.
8.25 Units for $(F/\mu)^{1/2}$.
8.26 For a sinusoidal wave on a string slope $\partial y/\partial x$ is equal to $\partial y/\partial t$ divided by v.
8.27 Wave equation for transverse waves.

8.28 Reflection of wave at the joint of two wires.
8.29 From the sketch to find f and λ if v is given and to find equation for the wave.
8.30 Given the linear density of the string to find energy sent down the string per second in prob. (8.29).
8.31 Energy of the string in the nth mode.
8.32 (a) Fundamental frequency of a steel bar for longitudinal vibrations and (b) comparison of frequencies for (i) free at both ends; (ii) damped at midpoint of bar (i); and (iii) clamped at both ends.
8.33 Fundamental frequency of vertical oscillations of a mass attached to a wire as (a) a simple oscillator and (b) system of vibrator fixed at one end and mass loaded at the other.
8.34 For a mass loaded bar the frequency condition $kL \tan kL = M/m$ reduces to that of SHO for $kL < 0.2$.
8.35 Velocity of long waves compared with those in deep liquid and canal waves.
8.36 Maximum depth of liquid for which the formula $v^2 = gh$ represents velocity of waves of length λ within 1%.
8.37 Surface tension of water by Ripple method.
8.38 Minimum velocities of surfaces waves for mercury and water.
8.39 v_p and v_g from dispersion relation for a piano wire.
8.40 v_p and v_g from dispersion relation for water waves of very short wavelength in deep water.
8.41 Given general dispersion relation for water wave to show that $v_\mathrm{p} = v_\mathrm{g} = \sqrt{gh}$ and for deep water $v_\mathrm{p} = (g/k + Sk/\rho)^{1/2}$ and to find v_g.
8.42 v_p and v_g in deep water for small ripples.
8.43 $v_p v_g = c^2$ for a relativistic particle.
8.44 Wavelength of surface waves on water.
8.45 $v_\mathrm{p}^2 = g/k$ for deep water waves. To show that $v_\mathrm{g} = v_\mathrm{p}/2$.
8.46 v_p and v_g from dispersion relation of sound in air.
8.47 Given $v_\mathrm{p}^2 = (g/k + Sk/\rho)$ for deep water waves, to show v_p is minimum for $\lambda = 2\pi(S/\rho g)^{1/2}$.
8.48 Relation between pressure amplitude and displacement amplitude and that they are out of phase by 90°.
8.49 Pressure amplitude corresponding to the threshold of hearing intensity.
8.50 Intensity of wave from intensity level of ordinary conversation.
8.51 A point source of sound radiates energy of 4 W. To find I and I.L at 25 m from source.
8.52 Maximum displacement from maximum pressure variation, f, ρ and v.
8.53 Ratio of intensities of two sound waves one in air and the other in water having equal pressure amplitude.
8.54 Pressure amplitude, f, ρ and v from the equation. $P = 2.4 \sin \pi(x - 330t)$ for progressive wave.
8.55 Amplitude of air vibrations by a note of given frequency and intensity.
8.56 To show a plane wave of effective acoustic pressure of a microbar has I.L of 74 dB.

8.57 Energy density and effective pressure of a plane wave in air of 70 dB I.L.
8.58 Pressure amplitude for $I = 1\,\mathrm{W/m^2}$ at pain threshold.
8.59 Theoretical speed of sound in H_2 at 0°C.
8.60 Given speed of sound in H_2 at 0°C, to calculate speed in O_2.
8.61 Two sound waves have $I_1 = 0.4$ and $10\,\mathrm{W/m^2}$. How many decibels is one louder than the other?
8.62 Ratio of intensities if one sound is 6.0 dB higher than the other.
8.63 Distance to which sound is audible if the source radiates at the rate of 0.009 W.
8.64 Calculation of displacement amplitude from the pressure amplitude.
8.65 Two sound waves of equal pressure amplitude and frequencies traverse in liquids with $\rho_1/\rho_2 = 3/4$ with $v_1/v_2 = 3/2$. To compare displacement amplitudes, intensities and energy densities.
8.66 One sound wave travels in air and the other in water, their intensities and frequencies being equal. To find ratio of wavelengths, pressure amplitude and particle amplitudes.
8.67 Characteristic impedance.
8.68 Intensity of a beam of plane waves, pressure amplitude displacement amplitude, acoustic particle velocity amplitude and condensation amplitude.
8.69 Laplace formula for sound velocity in a gas.
8.70 An empirical formula for sound velocity as a function of temperature.
8.71 The sound of whistle is reflected from the wall of the rock as the engine approaches a tunnel. To find the ratio of frequencies of reflected and direct sounds heard by the engine driver given the speed of the train.
8.72 Doppler effect when two train moves towards each other.
8.73 Doppler effect when two trains move away from each other.
8.74 Maximum and minimum frequencies heard by a listener from a rotating whistle.
8.75 The Kundt's tube experiment.
8.76 Wavelength of sound from a motion train when (i) train at rest; (ii) moving towards you; and (iii) moving away from you.
8.77 Shock wave, Mach number and angle of Mach cone.
8.78 Reverberation time for a room, Sabine's formula.
8.79 Echo of drum beating from a mountain.
8.80 Echo of rifle shot fired in the valley formed between two parallel mountains.
8.81 Beat frequency heard by a man who walks in line between two whistles sounding neighbouring frequencies.
8.82 Frequency of the unknown frequency by beat frequency in transferring load from one fork to another.
8.83 Beats produced by a fork sounding with an open organ pipe of an appropriate length.
8.84 Falling plate experiment in sound.
8.85 Effective length of a resonance tube closed at one end.
8.86 When an open pipe is suddenly closed so that f_3 of the closed pipe is higher by 100 Hz than f_2 of original pipe. To find f_1 of original pipe.

Chapter 9

Fluid Dynamics

9.1 Velocity of water in the narrower portion of pipe.
9.2 Verification of continuity equation from velocity components.
9.3 Work done in forcing water through pipe.
9.4 Lift on the wing of aeroplane.
9.5 Rate of flow of water using a venturi meter.
9.6 Speed of a plane using a Pitot tube.
9.7 Velocity of the spray in a sprinkler.
9.8 Test of steady incompressible flow.
9.9 Speed of flow past the lower and upper surface of the wing of an aeroplane.
9.10 Pressure drop and velocity in the throat of a venturimeter.
9.11 Application of Reynolds number to steady and turbulent flow.
9.12 A tube open at one end and closed at the other with small orifice is filled with liquid. To find the efflux velocity when rotated in a horizontal plane, about an axis through the open end.
9.13 Application of pitot tube.
9.14 Rate of flow of water in a horizontal pipe of varying cross-sections.
9.15 A cylinder with a small hole at the bottom is filled with water and fitted with a piston. To find the work done by a constant force when the system is rotated horizontally to squeeze all water from the cylinder.
9.16 A cylindrical vessel with water is rotated about its vertical axis. To find pressure distribution and shape of free space of water.
9.17 Speed of water flowing from a water tap.
9.18 Application of Torricelli's theorem.
9.19 Application of Torricelli's theorem.
9.20 Application of Torricelli's theorem.
9.21 Water leaks through a hole in the bottom of a tank. To find time for water level to decrease from h_1 and h_2.
9.22 Velocity of efflux through a hole in a tank filled with water.
9.23 Application of Torricelli's theorem to two tanks filled with a liquid and carrying hole of different areas and at different depths, the volume of flux being identical.
9.24 Efflux velocity of water in a bottom orifice in a tank filled with water + kerosene.
9.25 Two identical holes are punched on opposite sides at different height in a vessel filled with water. To calculate resultant force of reaction of water flow.
9.26 Pressure required to maintain water flow through a tube.
9.27 Pressure difference across a composite tube in Poiseuille's experiment.
9.28 Terminal velocity of rain drops.
9.29 Water flow through horizontal tubes connected in parallel.
9.30 Water flow through horizontal tubes connected in series.

Chapter 10

Heat and Matter

10.1 Mean free path of molecule given collision frequency and mean molecular speed.
10.2 rms of a molecule, mean free path and collision frequency.
10.3 Mean free path of gas molecules.
10.4 v_{mp} and v_{av} and T for two graphs for Maxwell–Boltzmann distribution.
10.5 Mort probable speed assuming Maxwell–Boltzmann distribution.
10.6 $< E >$, v_{rms}, v_{mp} and v_{av} for CO_2 gas.
10.7 Volume of a helium-filled weather balloon when it rises to high altitude.
10.8 Using van der Waal's equation to estimate the molar density of H_2S gas.
10.9 Thermal expansion of a bimetal bar.
10.10 Buckled rail due to thermal expansion.
10.11 Length of a steel and copper rod such that the difference is constant at any temperature.
10.12 Volume of mercury in a flask such that the volume of air inside the flask is constant at any temperature.
10.13 Tension in a wire with the decrease in temperature.
10.14 Specific gravity bottle experiment.
10.15 Determination of the gas constant.
10.16 Rising air bubble in a lake; Boyle's law.
10.17 Load carried by a balloon to a given height.
10.18 Two glass bulbs in communication kept at different temperatures. To find the pressure.
10.19 Conduction of heat through slabs in series.
10.20 Conduction of heat through slabs in parallel.
10.21 Temperature of interface of copper and iron bars in heat conduction.
10.22 Melting of ice kept at one end of a copper bar while the other end is heated.
10.23 Formula for heat conduction in a metal at low temperature given that thermal conductivity is proportional to absolute temperature.
10.24 Radial flow of heat between concentric spheres.
10.25 Radial flow of heat in a material across a coaxial cylinders.
10.26 Rate of addition of ice at the bottom of a layer of ice in a pond when air temperature drops to $-10°\mathrm{C}$.
10.27 Application of Newton's law of cooling.
10.28 Water equivalent of calorimeter from Newton's law of cooling.
10.29 Rate of cooling of steel balls of different radii.
10.30 Application of resistance thermometer.
10.31 Solar constant from the given data.
10.32 Temperature of sun's surface from given data.
10.33 Rate of heat loss by radiation at a given temperature.
10.34 Application of Wien's displacement law.

10.35 Latent heat of fusion determination.
10.36 Mean specific heat and specific heat at midpoint.
10.37 Equilibrium temperature from the mixture of liquid (A, B), (B, C) and (A, C).
10.38 Rise of temperature of the bullet in its collision with a block of wood (a) fixed and (b) free to move.
10.39 Rise in temperature due to water fall.
10.40 Rise in temperature due to fall of a lead piece from a given height on a non-conducting slab.
10.41 Work done on a gas during an adiabatic compression.
10.42 A thermodynamic cycle.
10.43 In an adiabatic process of a monatomic ideal gas $PV^{5/3} = \text{const.}$
10.44 Work done on a gas during an isothermal compression.
10.45 $P - V$ diagram for a sequence of thermodynamic processes.
10.46 Number of degrees of freedom for gas molecules.
10.47 Efficiency of a heat engine.
10.48 Temperatures of the source and sink of a heat engine.
10.49 Air pressure at a given attitude h, number density of gas molecules and p at $h/2$.
10.50 $P - V$ diagram for the Carnot cycle and the Sterling cycle.
10.51 Entropy change for isobaric and isochoric processes.
10.52 Change of values of thermodynamic parameters in reversible isothermal expansion.
10.53 Internal energy, heat, enthalpy, work and Gibbs function.
10.54 Shear modulus of a material.
10.55 Stress, strain and Young's modulus of a wire.
10.56 Isothermal elasticity and adiabatic elasticity.
10.57 Poisson's ratio from the ratio of Young's modulus and the rigidity modulus.
10.58 Young's modulus from the depression of a wire caused by attaching known load at midpoint of a horizontal wire.
10.59 Speed of an object released from a catapult.
10.60 Maximum angular speed of an object attached to the end of a wire whirled in a horizontal plane.
10.61 Maximum length of a wire which will not break under its own weight when it hangs freely.
10.62 Capillary rise.
10.63 Volume of air bubble at depth 100 m – gas equation.
10.64 Energy released in coalescing of droplets.
10.65 Work done in blowing a soap bubble to a larger size.
10.66 A hollow vessel with a small hole of radius r is immersed to a known depth when water just penetrates into vessel, to find r.
10.67 Depth of water column at which an air bubble is in equilibrium.
10.68 Inadequate capillary tube length.
10.69 Effect of charge on soap bubble.
10.70 Radius of bigger bubble when two bubbles coalesce.

Chapter 11

Electrostatics

11.1 (a) Force between two charges. Position of neutral point (b) E and F of proton at the position of electron in H atom.
11.2 (a) Tension in the thread when a charged ball hangs in an electric field and (b) $V_b - V_a$ positive or negative.
11.3 Electric potential along the axis of a charged circular loop.
11.4 Potential energy of four charges at the corners of a square and their stability.
11.5 V at the surface of a charged liquid drop, when two such drops coalesce, V at the surface of new drop.
11.6 A charged pendulum bob is in equilibrium in a horizontal electric field. Tension and angle of the thread with the vertical.
11.7 An infinite number of charges, each equal to q are placed at $x = 1, 2, 4, 8, \ldots$ units. V and E at $x = 0$.
11.8 In prob. (11.7) what will be V and E if consecutive charges have opposite sign.
11.9 A charged particle is released from rest on the axis of a fixed oppositely charged ring. To show that the particle has SHM and to find time period of oscillation.
11.10 Three charges each of value q are at the corners of an equilateral triangle and the fourth one at the centre. It $Q = -q$, charges move toward or away from the centre. Value of Q for the charge to be stationary.
11.11 Two identically charged spheres are suspended by strings of equal length. The strings make an angle of 30° with each other. When immersed in a liquid the angle remains the same. To find the dielectric constant of the liquid.
11.12 One charge is placed at one corner of a square and another at the center. Work done to move the charge from the centre to an empty corner.
11.13 A charged pith ball suspended by a thread is deflected by a known distance by a horizontal electric field. To find E.
11.14 Suspension of a charged oil drop under electric field and gravity.
11.15 Equal amount of charge on the earth and the moon to nullify gravitational attraction.
11.16 An energy output of 10^{-5} J results from a spark between insulated surfaces at P.D 5×10^6 V. To find q transferred and number of electrons flowed.
11.17 E at a given distance along the axis of a charged rod.
11.18 E along the axis of a charged disc.
11.19 Electronic charge by Millikan's oil drop method.
11.20 Total charge on a circular wire which has $\cos^2\theta$ charge density dependence.
11.21 Strength of electric force compared to gravitation force in H atom.
11.22 Charges on two spheres given their combined charge and force of repulsion at the given distance.

11.23 Four charges are placed at the four corners of a square as in the diagram. To find E and its direction.
11.24 E on the perpendicular bisector of a thin charged rod.
11.25 A thin non-conducting charged rod is bent to form an arc of circle and subtends an angle θ_0 at the centre of circle. To find E at the centre of circle.
11.26 In prob. (11.3) to find the distance at which E is maximum.
11.27 E and V at the centre of a charged non-conducting hemispherical cup.
11.28 E varies as $1/r^4$ for an electric quadrupole.
11.29 Electric and gravitational forces between two bodies each of mass m and charge q will be equal if $q/m = 8.6 \times 10^{-10}$/kg.
11.30 Two equally charged spheres are suspended from the same point by silk thread of the same length. To find the rate at which charge leaks out given the relative velocity of approach.
11.31 A long charged thread is placed on the axis of a charged ring with one end coinciding with the centre of the ring. To find force of interaction.
11.32 A very long wire with charge density λ is placed on the x-axis with one end coinciding with the origin. To calculate $\boldsymbol{E}$ from the y-axis.
11.33 E from the potential $\phi = cxy$.
11.34 To show that the locus of zero potential points for two fixed charges is a circle.
11.35 Two identical rings charged to Q_1 and Q_2 coaxially placed at a fixed distance. To find the work done in moving a charge q from the centre of one ring to that of the other.
11.36 To find x if the potentials at $(0, 2a)$ and $P_2(x, 0)$ are equal and to find the potential.
11.37 Value of Q if the interaction energy of three charges $+q_1$, $+q_2$ and Q placed at the vertices of a right-angled isosceles triangle is zero.
11.38 Work done in assembling the charges at the four corners of a square as in the diagram.
11.39 Total potential energy of a charged sphere.
11.40 A linear quadrupole.
11.41 Charge that can be placed on a sphere for the given field strength and the corresponding V.
11.42 Speed of electron when it approaches a positive charge.
11.43 For the linear quadrupole of prob. (11.40) for $r >> d$, $E(r) \propto 1/r^3$.
11.44 Force and acceleration of electron when it passes through a hole in a condenser plate.
11.45 Potential on the axis of a charged disc and the limiting case of $x >> R$.
11.46 Kepler's third law of motion is applicable to electron in H atom.
11.47 Equal charges are placed on four corners of a square. To calculate F on one of them due to other three.
11.48 E on the perpendicular bisector of a dipole. For $x >> d/2$, $E \propto 1/r^3$.
11.49 Application of Gauss' law to an infinite sheet of charge. σ of the sheet by deflection of a charged mass hanging by a silk thread.

11.76 Two capacitors are charged to a battery and connected in parallel. Find P.D of the combination if (a) positive ends are connected and (b) positive end is connected to negative terminal of the other.
11.77 Two capacitors are charged to P.D V_1 and V_2 and connected in parallel. ΔU when (a) positive ends are joined and (b) positive end of one is joined to negative end of the other.
11.78 Effect of dielectric on V, E, q, C and U, when the battery (a) remains connected and (b) is disconnected.
11.79 In prob. (11.78) dependence of the given quantities on the distance of separation of plates.
11.80 Force of attraction between the plates of a parallel plate capacitor.
11.81 n identical droplets each of radius r and charge q coalesce to form a large drop. To find relations of radius, C, V, σ and U for the large drop and droplet.
11.82 Half of the stored U_E of a cylindrical capacitor of radii a and b lies within a radius $\sqrt{ab}$ of the cylinder.
11.83 Capacitor of capacitance C_1 withstands maximum voltage V_1 and C_2 withstands maximum voltage V_2. Maximum voltage that the system of C_1 and C_2 can withstand when connected in series.
11.84 Application of Gauss' law to calculate the capacitance of Geiger–Muller counter.
11.85 Capacitance of a capacitor formed by two spherical metallic shells.
11.86 For two concentric shells the capacitance reduces to that of a parallel plate capacitor in the limit of large radii.
11.87 In the R – C circuit shown to find (i) time for charge to reach 90% of its final value; (ii) U stored in the capacitor at $t = \tau$; and (iii) Joule heating in R at $t = \tau$.
11.88 In prob. (11.87) number of time constants after which energy in capacitor will reach half of equilibrium value.
11.89 Capacitance of a parallel plate capacitor whose plates are slightly inclined.
11.90 and 11.91 In the given arrangements of capacitor to find P.D, q and U in the capacitors.
11.92 To find the effective capacitance between two points in the given arrangement of capacitors.
11.93 To obtain an expression for $q(t)$ for an $R - C$ circuit.
11.94 For the given $R - C$ circuit, to find battery current at $t = 0$ and $t = \infty$ when switch is closed. To find the current through R when switch is open after a long time.
11.95 A charged capacitor is discharged through a resistance. To find U, i, V_c at given time, τ and equation for t when q drops to half of its value.
11.96 Charge q is uniformly distributed in a sphere of radius R. (i) To find div $\boldsymbol{E}$ inside the sphere; (ii) electric force on a proton at $r < R$; and (iii) work done on proton to move it from infinity to a point at $r < R$.
11.97 The electric displacement D is uniform in a parallel plate capacitor. To obtain $E(x)$ for a non-uniform relative permittivity, using Gauss' law.
11.98 To obtain the differential Gauss' law for gravitation.

Chapter 12

Electric Circuits

12.1 Effective resistance between two points A and B in the given arrangement.
12.2 Change in resistance when a wire is stretched.
12.3 Equivalent resistance is p for two resistors in series and q for parallel, minimum value of n where $p = nq$.
12.4 Equivalent resistance for five resistors in the given arrangement.
12.5 Effective resistance of five resistors in the given arrangement.
12.6 Resistance between two terminals in the given network.
12.7 Equivalent resistance between two terminals in the given network.
12.8 Wheatstone bridge.
12.9 Five resistors are connected in the form of a square and a diagonal. To find R_{eq} across a side.
12.10 Effective resistance for a network of infinite number of resistors.
12.11 What equal length of an iron wire and a constantan wire of equal diameter must be joined in parallel to give equivalent resistance of 2 Ω?
12.12 Temperature coefficient of resistance.
12.13 A square ABCD is formed by bending a wire. B and D are joined by a similar wire and a battery of negligible internal resistance is included between A and C. To find R_{eq} and power dissipated.
12.14 Temperature dependence of resistance.
12.15 Equivalent resistance of a network in the form of a skeleton cube across the body diagonal.
12.16 Brightness of two bulbs in series.
12.17 Brightness of two bulbs in parallel.
12.18 Three resistors in parallel are connected with two in series. If a PD of 120 V is applied across the ends of circuit, to find PD drop across parallel arrangement.
12.19 Maximum power delivered to the external resistor.
12.20 Boiling of water by two heater coils when they are connected (a) in series and (b) in parallel.
12.21 Calculation of power expended in two resistors connected in series and in parallel.
12.22 Rate of energy loss in power transmission.
12.23 Power dissipated in three resistors in (a) series and (b) parallel.
12.24 The value of resistance in parallel with a heater so that the 1000 W heater operates at 62.5 W.
12.25 Number of cells wrongly connected in a battery.
12.26 Condition for maximum current in m external resistances connected to m rows of cells with each row containing n cells in series, each cell with internal resistance r.
12.27 Internal resistances of two cells in series and the external resistance.
12.28 Rate of heat production in resistors in series and in parallel.
12.29 Charging of a battery.

12.30 Comparison of power dissipated in resistance with the value for power supplied by battery.
12.31 Internal resistance of a cell by potentiometer.
12.32 Emf of a cell by potentiometer method.
12.33 Condition for null deflection and i_G/i in galvanometer of Wheatstone bridge.
12.34 To find current in an ammeter in a network.
12.35 Internal resistance of a battery.
12.36 Internal resistance of a cell by potentiometer method.
12.37 External resistance by potentiometer method.
12.38 Reading in the voltmeter.
12.39 Metre bridge.
12.40 A moving coil meter reading up to 1 mA to be converted into (a) 100 mA full scale and (b) 80 V full scale.
12.41 Pocket voltmeter.
12.42 Resistance of a galvanometer.
12.43 A battery of ξ_1 and r_1 with a second battery of ξ_2 and r_2 in parallel is joined to an external resistance R. To calculate i_1, i_2, P_1, P_2 and P.
12.44 Emf of batteries by applying the loop theorem.
12.45 Application of Kirchhoff's rules to determine currents in branches of a circuit.
12.46 Currents in various resistors and PD across the cells in a network.
12.47 Equivalent resistance of the circuit and power dissipated.
12.48 Given the internal resistance of one cell to calculate the internal resistance of the other cell.
12.49 Equivalent resistance, voltages, currents and power dissipated in a series parallel resistive circuit.
12.50 Current in the circuit and terminal voltage of battery under load conditions and power dissipated in R and r.
12.51 Equivalent resistance, currents, voltages and power dissipated in a series-parallel circuit.
12.52, 12.53, 12.54 Application of Kirchhoff's rules to the circuit to produce three equations with three unknown branch currents.
12.55 Application of Kirchhoff's rules to the given circuit.
12.56 Application of Kirchhoff's rules to the circuit to calculate currents in various branches.

Chapter 13

Electromagnetism I

13.1 Cyclotron frequency for alpha particles.
13.2 Energy and oscillator frequency for p and α
13.3 Mass spectrometer, velocity filter.
13.4 Identification of pion.
13.5 A particle of mass m and charge q travelling along x-axis is acted by $\boldsymbol{E}$ along y-axis. The trajectory.

13.40 Application of ampere's law to find B due to a current-carrying cylinder for $r < R$ and $r > R$.
13.41 To find B at the centre of two concentric arcs of radii r and $2r$ as in the diagram.
13.42 (a) To find B distance x from the midpoint of wire of length L. (b) To compare B at the centre of a loop when it is bent into (i) a square and (ii) an equilateral triangle.
13.43 To calculate B midway between Helmholtz coils, given the values of N, R and I.
13.44 $B(r)$ for the current-carrying long cylinder.
13.45 (a) L of a solenoid; (b) magnetic energy; and (c) application of Faraday's law.
13.46 (a) B at a given distance from a long straight wire carrying current and (b) $\mathrm{d}B$ at the given values of x, y and z from $I\mathrm{d}l$ at the origin.
13.47 B and H for a torus with and without magnetic material.
13.48 Current required to circulate the earth's core to produce the known dipolar magnetic field.
13.49 Emf generated by a revolving disc midway between Helmholtz coils.
13.50 Variation of voltage developed across a conductor moving with velocity v in a known field B.
13.51 A proton travelling with velocity $v = (\hat{i} + 3\hat{j})10^4$ m/s is located at $x = 2$ m and $y = 3$ m at some instant t. To find B.
13.52 To find F/l between two long straight wires separated by distance d carrying i_1 and i_2 in opposite directions.
13.53 Two parallel wires of distance d apart attract each other with a force F/l. If i_1 is given, to find i_2 and its direction.
13.54 Equilibrium between two current-carrying parallel wires separated by d vertically.
13.55 Three long parallel wires carrying 20 A are placed in the same plane with equal spacing of 10 cm. To find F/l for an outer wire and the central wire.
13.56 To calculate the force acting on a bent wire in a uniform magnetic field as in the diagram.
13.57 A rectangular coil of given dimensions is placed parallel to a long wire carrying current. To find force on each segment of rectangular coil and net force on it.
13.58 In a set-up similar to prob. (13.57) to derive an expression for the resultant force on the coil and find its value from the given data.
13.59 A loop is formed by two parallel rails, a resistor and a rod across the rails in a magnetic field. A force F drags the rod at velocity v. To find the current, total power and F.
13.60 In prob. (13.37) to calculate the magnetic moment of the disc.
13.61 To show that the magnetic dipole moment of the earth can be produced by wire carrying a current of 5×10^7 A around the magnetic equator.
13.62 Energy density at the centre of a current-carrying loop.
13.63 Magnetic energy density at the centre of H atom due to circulating electron.
13.64 Maximum torque of a circular coil in a magnetic field.

13.65 The ratio μ/L for a charged sphere rotating with constant angular velocity.
13.66 Potential energy of an electric dipole is an electric field.
13.67 Emf induced in an expanding flexible circular wire placed in a magnetic field.
13.68 Emf developed between the wing tips of an aeroplane flying over earth's magnetic field.
13.69 Potential difference between the centre and the outer edge of a spinning disc in the horizontal component of earth's magnetic field.
13.70 A coil in the given magnetic field is suddenly withdrawn from the field and a galvanometer in series with the coil records the charge passed around the circuit. To find the resistance of the coil and the galvanometer.
13.71 The induced emf and current in a wire loop when the magnetic field is reduced to zero in the given time.
13.72 Amplitude of the induced current when a square loop of wire rotates in a magnetic field.
13.73 A bar slides on parallel rods in a magnetic field when a current flows through the resistor. To find the speed of the bar.
13.74 A uniform magnetic field of induction fills a cylindrical volume. To calculate the emf produced at the end of a rod placed in it when B changes
13.75 A square wire of length l, mass m and resistance R slides on frictionless inclined rails. Magnetic field exists within the frame. To show that the wire frame acquires steady velocity.
13.76 A copper disc spins in a magnetic field and induced emf is recorded. To find B.
13.77 To verify that Faraday's law is dimensionally correct.
13.78 Given the equation of B waves, to find emf in a coil.
13.79 Ratio of $\xi_{\text{max}}(\text{television})/\xi_{\text{max}}(\text{radio})$ in a loop antenna.
13.80 Two rails are connected by a wire and are connected to a wire and a slider to form a loop. To find F on the slider to move it with velocity v.
13.81 Application of Faraday's law.
13.82 A wire carrying current is placed across a rectangular coil. To obtain the magnetic flux and emf induced.
13.83 To obtain magnetic flux in a betatron.
13.84 The magnetic and direction of the Hall field and concentration of free electrons.
13.85 Mobility of the electron given Hall coefficient and electrical conductivity.

Chapter 14

Electromagnetism II

14.1 Current, P.D and energy in an RC circuit.
14.2 P.D and phase difference in an LR circuit.
14.3 Given current and energy for an inductor, to find its value.
14.4 To find frequency and wavelength for an LC circuit.
14.5 Impedance and power dissipated in an RLC circuit.

14.6 L and C have equal X, at $f = 600\,\text{Hz}$, to find X_C/X_L at 60 Hz.
14.7 A capacitance has $X_C = 4\,\Omega$ at 250 Hz. To find C and X_C at 100 Hz and at 220 V, 50 Hz line.
14.8 In an LR series circuit across a 12 V–50 Hz supply, $i = 0.05$ A flows with $\phi = 60°$ with V. To find R and L, and to find C when connected in series to produce to $\phi = 0$.
14.9 i_rms and i_m when 0.6 H inductor is connected to 220 V–50 Hz AC line.
14.10 i_rms, Joule heat, V_rms in RL circuit for each component when $f = 50\,\text{Hz}$ and i_rms are available.
14.11 An ac applied to $R = 100\,\Omega$ given $V = 0.5\,V_\text{m}$ at $t = 1/300$ s. To find f.
14.12 To write an equation for the given LRC parallel circuit similar to that for standard equation.
14.13 To verify the formula for velocity of light.
14.14 Quantities RC, L/R and $\sqrt{LC}$ have units of time.
14.15 Fractional decrement of the resonance frequency in an RLC circuit.
14.16 Current in a damped LC circuit for low damping.
14.17 RLC circuit in series.
14.18 RLC circuit in parallel.
14.19 Differential equation for the charge of the given RLC circuit.
14.20 Solution of differential equation in prob. (14.19)
14.21 X_L, X_C, Z, I_T, ϕ, C_R, V_C, V_L and f_0 for the given RLC circuit.
14.22 A 40 Ω resistor and 50 μF capacitor in series and an AC of 5 V–300 Hz current in the circuit.
14.23 Z for the two networks involving L, R, C.
14.24 Definition of electric current, current density and quantization of charge, drift speed.
14.25 f_0 for two LC circuits in series is identical with individual circuits if $L_1C_1 = L_2C_2$.
14.26 Values of L and C if X_C at f_1 and X_L at f_2 are given.
14.27 A condenser of 0.01 μF is charged to 100 V. To find i_m when condenser is connected to an inductor of $L = 10$ mH.
14.28 An inductance of 1 mH has resistance of 5 Ω. Value of R and C to yield $f_0 = 500$ kHz in series and $Q = 150$.
14.29 f_0 and Q of a parallel RLC circuit with $L = 1$ mH, $R = 10\,\Omega$ and $C = 0.005$ μf.
14.30 In the discharge of a capacitor in an RC circuit to find error on $V(t)$, given error on R and C.
14.31 (a) Time for voltage on a condenser to fall to $1/e$ of initial value through a resister and (b) percentage error introduced if thermal effects are ignored.
14.32 Fractional half-width of resonance curve of an RLC circuit.
14.33 Dielectric constant from a plane em wave equation.
14.34 Speed of a Lorentz frame in which a pure magnetic field is observed.
14.35 Wave equation for B wave.
14.36 To show that $\nabla\left(\boldsymbol{j} + \frac{1}{\varepsilon_0}\frac{\partial \boldsymbol{E}}{\partial t}\right) = 0$.

14.37 Prediction that em waves propagate with velocity c.
14.38 Depth to which an em wave penetrates in aluminium.
14.39 To show that $E_0 = cB_0$
14.40 em wave equation for a conduction medium.
14.41 Capacitance and inductance per unit length for a coaxial cylinder.
14.42 **E, B** and **S** for a coaxial cable in the region between the central wire and tube.
14.43 To obtain the equation $u_\mathrm{B} = B^2/2\mu_0$
14.44 In em field $u_\mathrm{B} = u_\mathrm{E}$.
14.45 To obtain an expression for R if $u_\mathrm{B} = u_\mathrm{E}$ in a coaxial cable.
14.46 Radiation energy loss of a low energy proton in a cyclotron.
14.47 Amplitude of an E wave at a distance from a point source.
14.48 Intensity, E_0 and B_0 of a laser beam.
14.49 Calculation of irradiance of a laser beam.
14.50 Time-averaged power per unit area carried by a plane em wave.
14.51 Irradiance of a plane em wave.
14.52 $\mathbf{E} \times \mathbf{H}$ is in the propagation direction.
14.53 Given the E field equation to find f, λ, v, E_0 and polarization.
14.54 Equation for the B field associated with the E wave of prob. (14.53).
14.55 Speed of an approaching car by radar.
14.56 Beat frequency registered by a radar from a receding car.
14.57 Application of ampere's law of a coaxial cylinder to calculate magnetic field in four regions.
14.58 Energy stored in the magnetic field in the large Hadron collider's magnet.
14.59 Electric and magnetic field amplitude at the surface of the sun.
14.60 Amplitude of the electric field at the orbit of the earth.
14.61 Amplitude of B field at the surface of Mars and flux of radiation.
14.62 To show that $|E|/|H| = 377\,\Omega$.
14.63 Skin depth in copper.
14.64 Penetration of microwave in a copper screen.
14.65 Derivation of formula for skin depth.
14.66 Energy transported per square centimetre area for a light wave having $E_m = 10^{-3}$ V/m.
14.67 Proof and interpretation of Poynting's theorem.
14.68 Dispersion relation from Maxwell's equations in dielectric.
14.69 To show that the identity $\nabla \times (\nabla \times \boldsymbol{E}) = -\nabla^2 \boldsymbol{E} + \nabla(\nabla \cdot \boldsymbol{E})$ is true for the vector field $\boldsymbol{F} = \boldsymbol{x}^2 z^3\,\hat{i}$.
14.70 Use of Poynting vector to determine power flow in a coaxial cable.
14.71 Application of Poynting's theorem to show that power dissipated in a conducting wire is given by $i^2 R$.
14.72 Using Maxwell's equations to show that the equation for a super-conductor leads to the stated equation for B.
14.73 Ratio of high-frequency resistance to direct resistance.
14.74 Derivation of the expression curl $\boldsymbol{E} = -\partial B/\partial t$ using Stokes' theorem.
14.75 Derivation of the expression $B = \mu_0(H + M)$.

14.76 Gauss and Ampere's laws in free space subject to the Lorentz condition.
14.77 To show $E_y = Z_0 H_x$ for propagation of em wave where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$
14.78 Given E wave's direction of propagation and polarization of the wave. Boundary condition for media of different magnetic properties.
14.79 Use of boundary conditions to calculate the reflectance and transmittance of em waves at the dielectric discontinuity.
14.80 To obtain an expression for reflectance for an em wave in terms of angle of incidence and refraction and the refraction indices of two dielectrics.
14.81 Using the result in prob. (14.80), to show that reflectance is zero when tan $\theta = n_2/n_1$.
14.82 To sketch R and T against n_1/n_2 for normal incidence.
14.83 **(a)** **B** is perpendicular to **E**; **(b)** **B** is in phase with **E**; and **(c)** $B = E/c$.
14.84 Charge density induced on the surface of uncharged dielectric cube containing electric field.
14.85 Fractional difference between phase and group velocity at $\lambda = 5000$ Å.
14.86 Given the dispersion relation $\omega = ak^2$, to calculate v_{ph} and v_g.
14.87 To show that $v_\text{p} v_\text{g} = c^2$.
14.88 $v_\text{g} = v_\text{ph} + k\text{d}v_\text{p}/\text{d}k$.
14.89 $v_\text{g} = c/n + \left(\lambda c/n^2\right)\, \text{d}n/\text{d}\lambda$.
14.90 If $v_\text{ph} \propto 1/\lambda$ then $v_\text{g} = 2v_\text{ph}$.
14.91 $v_\text{g} = c/\left[n + \omega\,(\partial n/\partial\omega)\right]$.
14.92 Value of v_p and free space wavelength for radiation to traverse a length of a rectangular waveguide in given time.
14.93 $1/v_\text{g} = 1/v_\text{ph} + (\omega/c)\,\text{d}n/\text{d}\omega$.
14.94 $v_\text{g} = c\text{d}v/\text{d}(nv)$.
14.95 $v_g = v$ for a non-relativistic classical particle.
14.96 Given a relation between refractive index and λ, to calculate v_g.
14.97 v_ph, v_g and λ for rectangular wave guide of given dimensions.
14.98 In a rectangular guide of width $a = 3$ cm, value of λ if $\lambda_\text{g} = 3\lambda$.
14.99 To calculate λ_g and λ_c given a and λ
14.100 Number of states of em radiation between 5000 and 6000 Å in a cube of side 0.5 m.
14.101 Possibility of AM radio waves propagating in a tunnel of given dimensions.
14.102 Calculation of least cut-off frequency for TE_mn wave for guide of given dimension
14.103 Variation of v_ph and v_g of TE_{01} wave in a wave guide of given dimensions.
14.104 Given the wave equation for E_z, to find E_z and f_c.
14.105 Given the wave equation for H_z, to find H_z and f_c.

Chapter 15

Optics

15.1 Fraction of light from a point source in a medium escaping across a flat surface.