AAPT UNITED STATES PHYSICS TEAM
AIP 2014

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts.
- Part A has four questions and is allowed 90 minutes.
- Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 4 parts: the cover sheet (page 2), Part A (pages 3-5), Part B (pages 7-8), and several answer sheets for two of the questions in part A (pages 10-13). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after April 15, 2014.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

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INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all four problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  AAPT ID #

  Doe, Jamie

  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 15, 2014.**

Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8\ \mathrm{N/kg}$	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8\ \mathrm{m/s}$	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J/K}$
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$1\mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

Inspired by: http://www.wired.com/wiredscience/2012/04/a-leaning-motorcycle-on-a-vertical-wall/

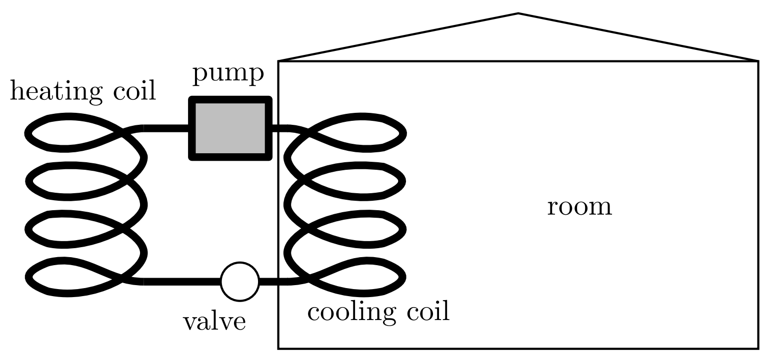
A unicyclist of total height h goes around a circular track of radius R while leaning inward at an angle θ to the vertical. The acceleration due to gravity is g.

a. Suppose $h \ll R$. What angular velocity ω must the unicyclist sustain?

b. Now model the unicyclist as a uniform rod of length h, where h is less than R but not negligible. This refined model introduces a correction to the previous result. What is the new expression for the angular velocity ω? Assume that the rod remains in the plane formed by the vertical and radial directions, and that R is measured from the center of the circle to the point of contact at the ground.

Question A2

A room air conditioner is modeled as a heat engine run in reverse: an amount of heat Q_L is absorbed from the room at a temperature T_L into cooling coils containing a working gas; this gas is compressed adiabatically to a temperature T_H; the gas is compressed isothermally in a coil *outside* the house, giving off an amount of heat Q_H; the gas expands adiabatically back to a temperature T_L; and the cycle repeats. An amount of energy W is input into the system every cycle through an electric pump. This model describes the air conditioner with the best possible efficiency.



Assume that the outside air temperature is T_H and the inside air temperature is T_L. The air-conditioner unit consumes electric power P. Assume that the air is sufficiently dry so that no condensation of water occurs in the cooling coils of the air conditioner. Water boils at 373 K and freezes at 273 K at normal atmospheric pressure.

a. Derive an expression for the maximum rate at which heat is removed from the room in terms of the air temperatures T_H, T_L, and the power consumed by the air conditioner P. Your derivation must refer to the entropy changes that occur in a Carnot cycle in order to receive full marks for this part.

b. The room is insulated, but heat still passes into the room at a rate $R = k\Delta T$, where ΔT is the temperature difference between the inside and the outside of the room and k is a constant. Find the coldest possible temperature of the room in terms of T_H, k, and P.

c. A typical room has a value of $k = 173$ W/°C. If the outside temperature is 40°C, what minimum power should the air conditioner have to get the inside temperature down to 25°C?

Question A3

When studying problems in special relativity it is often the invariant distance Δs between two events that is most important, where Δs is defined by

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]$$

where $c = 3 \times 10^8$ m/s is the speed of light.[1]

a. Consider the motion of a projectile launched with initial speed v_0 at angle of θ_0 above the horizontal. Assume that g, the acceleration of free fall, is constant for the motion of the projectile.

   i. Derive an expression for the invariant distance of the projectile as a function of time t as measured from the launch, assuming that it is launched at $t = 0$. Express your answer as a function of any or all of θ_0, v_0, c, g, and t.

   ii. The radius of curvature of an object's trajectory can be estimated by assuming that the trajectory is part of a circle, determining the distance between the end points, and measuring the maximum height above the straight line that connects the endpoints. Assuming that we mean "invariant distance" as defined above, find the radius of curvature of the projectile's trajectory as a function of any or all of θ_0, v_0, c, and g. Assume that the projectile lands at the same level from which it was launched, and assume that the motion is *not* relativistic, so $v_0 \ll c$, and you can neglect terms with v/c compared to terms without.

b. A rocket ship far from any gravitational mass is accelerating in the positive x direction at a constant rate g, as measured by someone *inside* the ship. Spaceman Fred at the right end of the rocket aims a laser pointer toward an alien at the left end of the rocket. The two are separated by a distance d such that $dg \ll c^2$; you can safely ignore terms of the form $(dg/c^2)^2$.

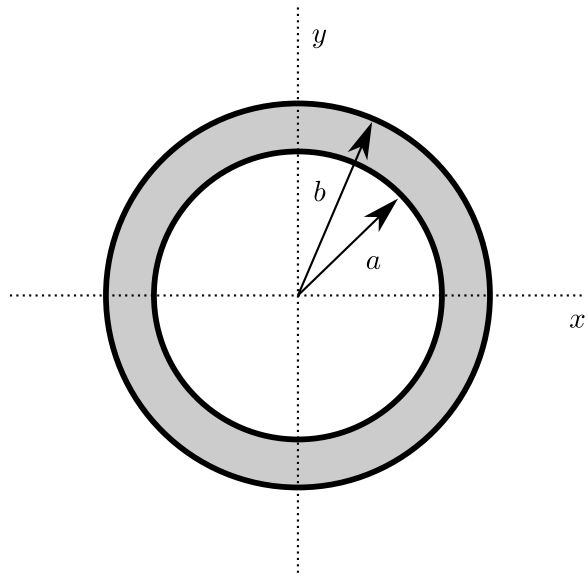
   i. Sketch a graph of the motion of both Fred and the alien on the space-time diagram provided in the answer sheet. The graph is *not* meant to be drawn to scale. Note that t and x are reversed from a traditional graph. Assume that the rocket has velocity $v = 0$ at time $t = 0$ and is located at position $x = 0$. Clearly indicate any asymptotes, and the slopes of these asymptotes.

   ii. If the frequency of the laser pointer as measured by Fred is f_1, determine the frequency of the laser pointer as observed by the alien. It is reasonable to assume that $f_1 \gg c/d$.

[1] We are using the convention used by Einstein

Question A4

A positive point charge q is located inside a neutral hollow spherical conducting shell. The shell has inner radius a and outer radius b; $b - a$ is not negligible. The shell is centered on the origin.



a. Assume that the point charge q is located at the origin in the very center of the shell.

   i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

   ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

   iii. Determine the electric potential at $x = a$.

   iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

b. Assume that the point charge q is now located on the x axis at a point $x = 2a/3$.

   i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

   ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

   iii. Determine the electric potential at $x = a$.

   iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

   v. Sketch a figure showing the electric field lines (if any) inside, within, and outside the conducting shell on the answer sheet provided. You should show at least eight field lines in any distinct region that has a non-zero field.

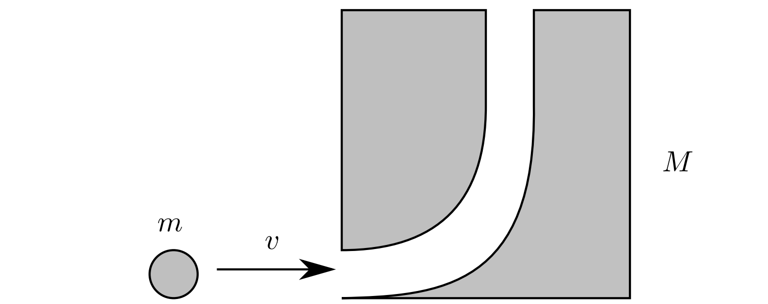
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

A block of mass M has a hole drilled through it so that a ball of mass m can enter horizontally and then pass through the block and exit vertically upward. The ball and block are located on a frictionless surface; the block is originally at rest.



frictionless horizontal surface

a. Consider the scenario where the ball is traveling horizontally with a speed v_0. The ball enters the block and is ejected out the top of the block. Assume there are no frictional losses as the ball passes through the block, and the ball rises to a height much higher than the dimensions of the block. The ball then returns to the level of the block, where it enters the top hole and then is ejected from the side hole. Determine the time t for the ball to return to the position where the original collision occurs in terms of the mass ratio $\beta = M/m$, speed v_0, and acceleration of free fall g.

b. Now consider friction. The ball has moment of inertia $I = \frac{2}{5}mr^2$ and is originally not rotating. When it enters the hole in the block it rubs against one surface so that when it is ejected upwards the ball is rolling without slipping. To what height does the ball rise above the block?

Question B2

In parts a and b of this problem assume that velocities v are much less than the speed of light c, and therefore ignore relativistic contraction of lengths or time dilation.

a. An infinite uniform sheet has a surface charge density σ and has an infinitesimal thickness. The sheet lies in the xy plane.

  i. Assuming the sheet is at rest, determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

  ii. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{x}}$ (parallel to the sheet), determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

  iii. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{x}}$, determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

  iv. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$ (perpendicular to the sheet), determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

  v. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$, determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

b. In a certain region there exists only an electric field $\vec{\mathbf{E}} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ (and no magnetic field) as measured by an observer at rest. The electric and magnetic fields $\vec{\mathbf{E}}'$ and $\vec{\mathbf{B}}'$ as measured by observers in motion can be determined entirely from the local value of $\vec{\mathbf{E}}$, regardless of the charge configuration that may have produced it.

  i. What would be the observed electric field $\vec{\mathbf{E}}'$ as measured by an observer moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$?

  ii. What would be the observed magnetic field $\vec{\mathbf{B}}'$ as measured by an observer moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$?

c. An infinitely long wire wire on the z axis is composed of positive charges with linear charge density λ which are at rest, and negative charges with linear charge density $-\lambda$ moving with speed v in the z direction.

  i. Determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) at points outside the wire.

  ii. Determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) at points outside the wire.

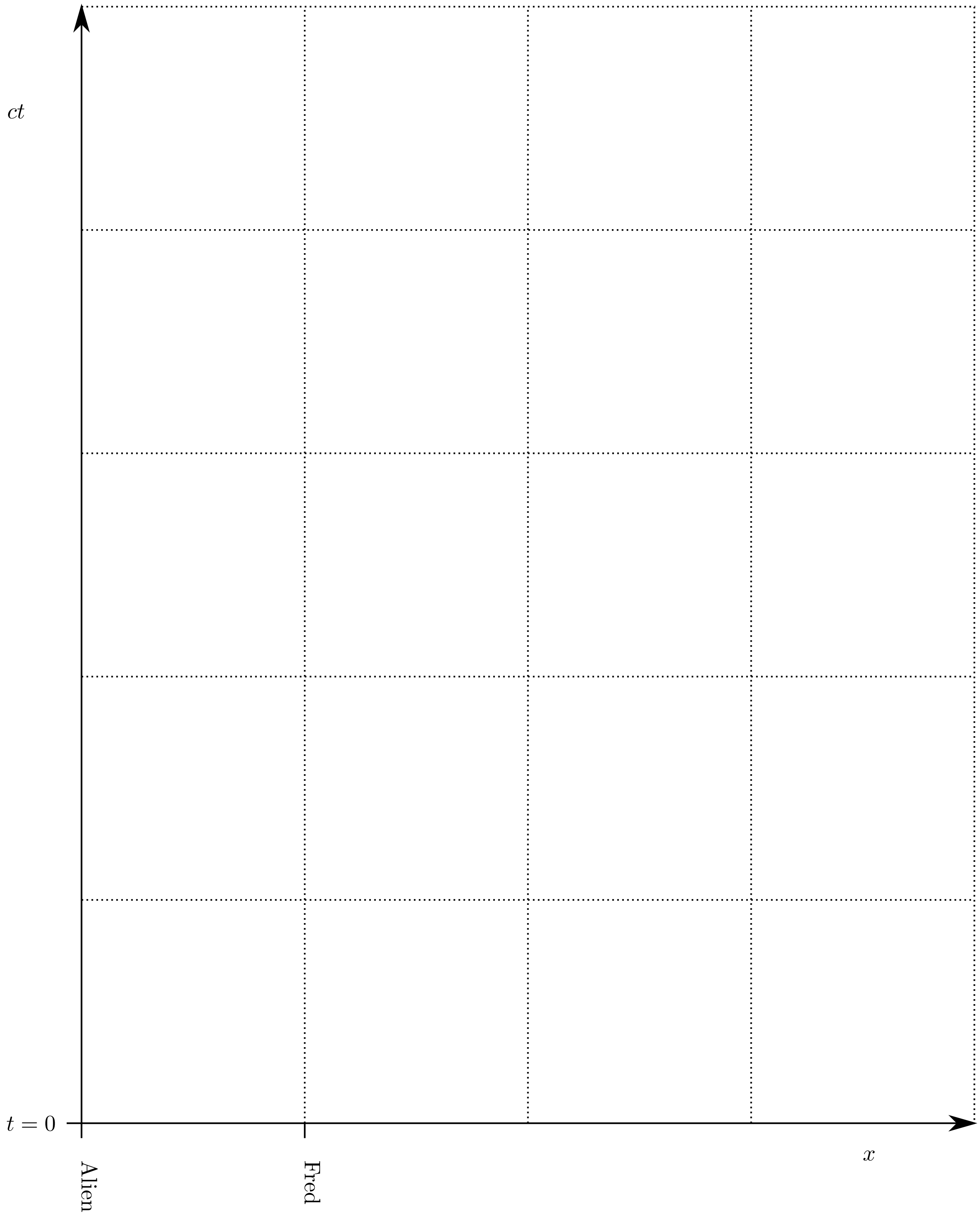
  iii. Now consider an observer moving with speed v parallel to the z axis so that the negative charges appear to be at rest. There is a symmetry between the electric and magnetic fields such that a variation to your answer to part b can be applied to the magnetic field in this part. You will need to change the multiplicative constant to something dimensionally correct and reverse the sign. Use this fact to find and describe the electric field measured by the moving observer, and comment on your result. (Some familiarity with special relativity can help you verify the direction of your result, but is not necessary to obtain the correct answer.)

Answer Sheets

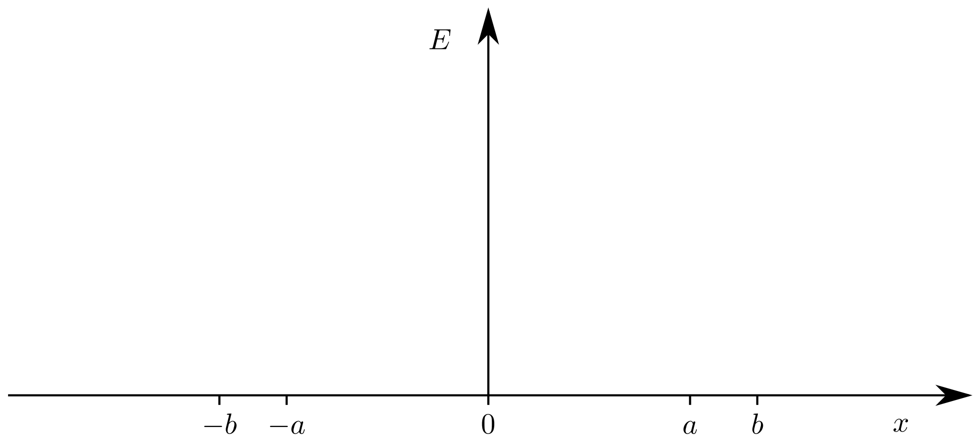
Following are answer sheets for some of the graphical portions of the test.

Answer for Part A, Question 3

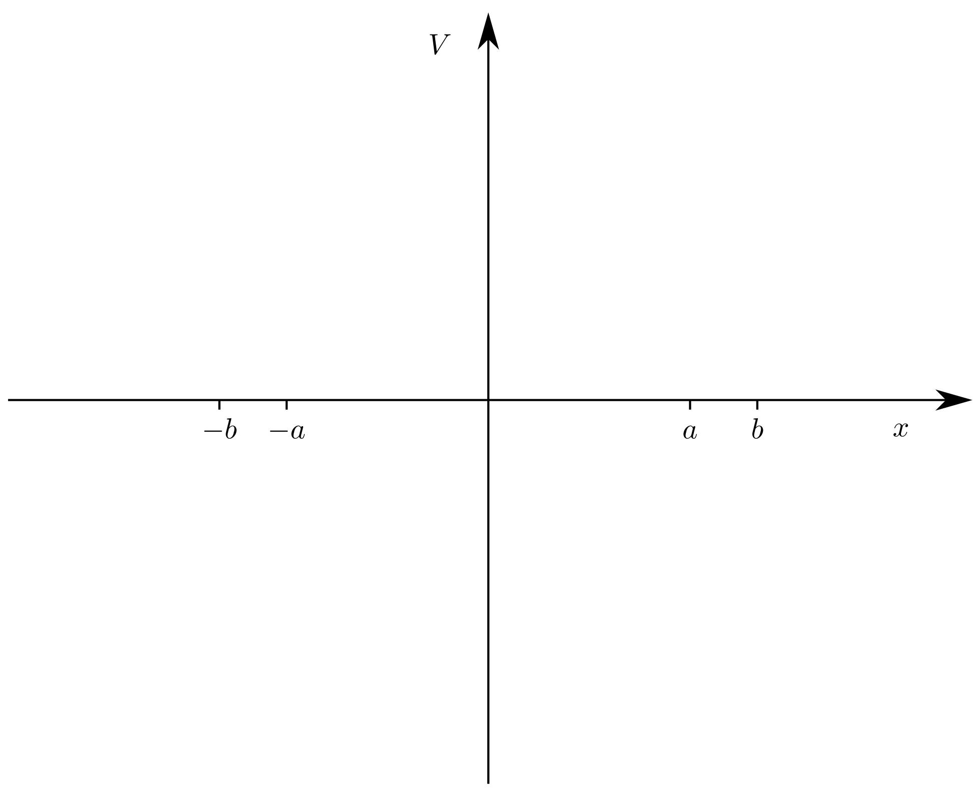
Space-time graph for accelerated rocket. The positions of Fred and the Alien at $t = 0$ are shown.



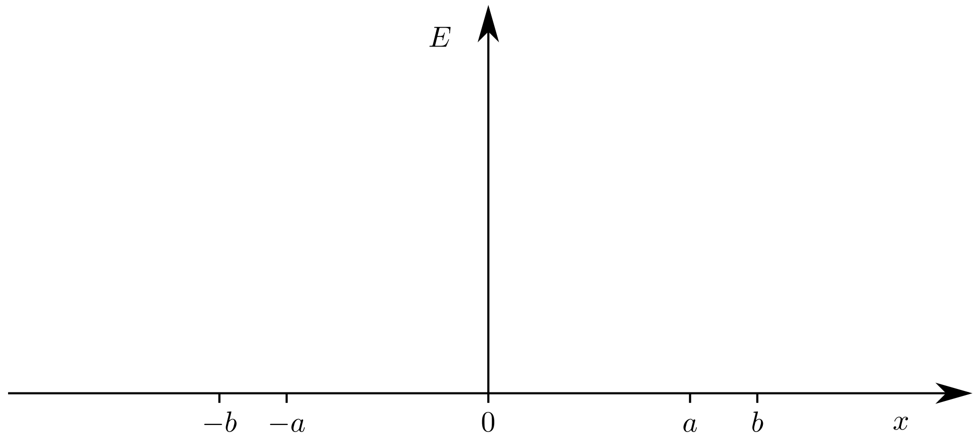
Answer for Part A, Question 4



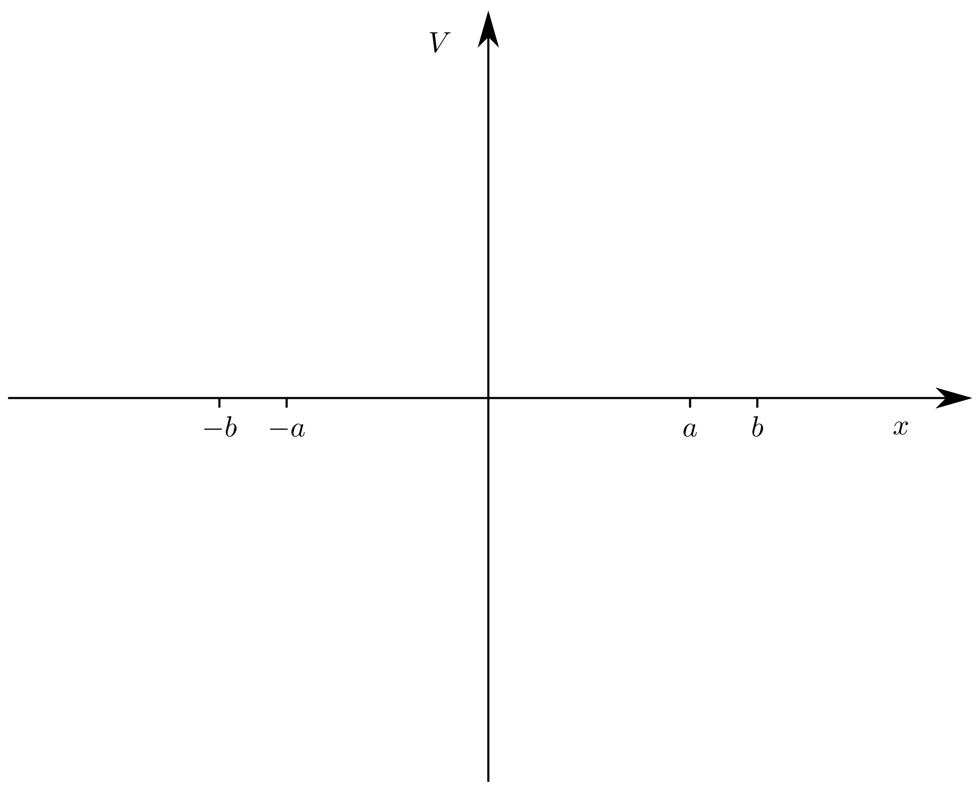
Answer for Part A, Question 4



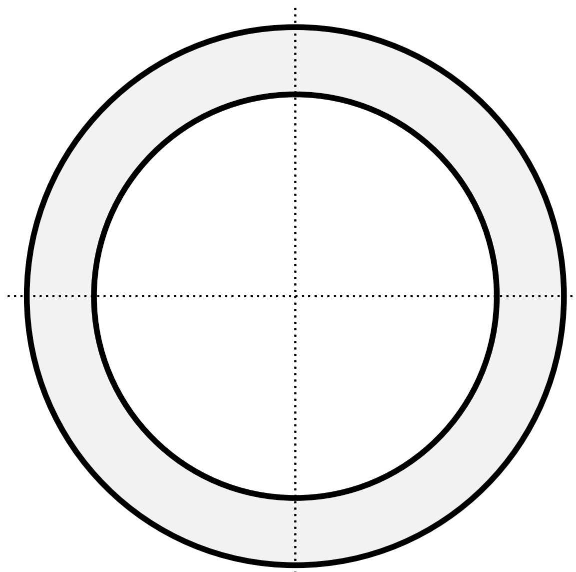
Answer for Part A, Question 4



Answer for Part A, Question 4



Answer for Part A, Question 4



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  A1 - 1/3

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$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
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$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x\rvert \ll 1$
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Part A

Question A1

Inspired by: http://www.wired.com/wiredscience/2012/04/a-leaning-motorcycle-on-a-vertical-wall/

A unicyclist of total height h goes around a circular track of radius R while leaning inward at an angle θ to the vertical. The acceleration due to gravity is g.

a. Suppose $h \ll R$. What angular velocity ω must the unicyclist sustain?

Solution

We work in a frame rotating with angular velocity ω, where the unicyclist is static. Four forces act on the unicyclist: a normal and frictional force at the point of contact, gravity downwards at the center of mass, and a fictitious centrifugal force.

If $h \ll R$, all parts of the unicyclist are at a distance of approximately R from the center of the circle, so the centripetal acceleration of every part of the unicyclist is $\omega^2 R$. The centrifugal force can then be taken to act at the center of mass for purposes of computing the torque. If the center of mass is a distance l from the point of contact, the torque about the point of contact is

$$\tau = m\omega^2 Rl\cos\theta - mgl\sin\theta$$

Since the unicyclist is stationary in this frame, $\tau = 0$, and solving for ω gives

$$\omega = \sqrt{\frac{g}{R}\tan\theta}.$$

b. Now model the unicyclist as a uniform rod of length h, where h is less than R but not negligible. This refined model introduces a correction to the previous result. What is the new expression for the angular velocity ω? Assume that the rod remains in the plane formed by the vertical and radial directions, and that R is measured from the center of the circle to the point of contact at the ground.

Solution

The centripetal acceleration now varies along the length of the unicyclist. In the rotating frame, the torque about the point of contact is given by

$$\tau_c = \int \omega^2 rz\, dm$$

where r is the distance from the center of the circle, z is the height above the ground, and dm is a mass element. Because the mass of the unicyclist is uniformly distributed along a length h,

$$dm = \frac{m}{h}\, ds$$

where s is the length along the unicyclist. Then

$$\tau_c = \int_0^h \omega^2 (R - s\sin\theta)(s\cos\theta)\,\frac{m}{h}\,ds = m\omega^2 h\cos\theta\left(\frac{R}{2} - \frac{h}{3}\sin\theta\right).$$

Gravity continues to act at the center of mass, a distance $h/2$ from the point of contact, and in the opposite direction,

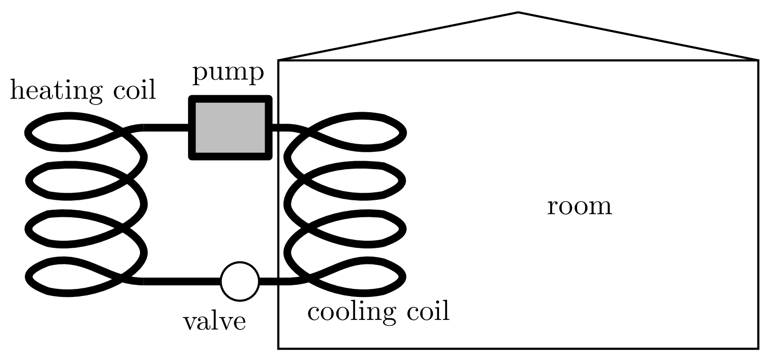
$$\tau_g = -mg\frac{h}{2}\sin\theta.$$

Again, the total torque is zero, so $\tau_c + \tau_g = 0$. Solving for ω gives

$$\omega = \sqrt{\left(\frac{g}{R}\tan\theta\right)\left(1 - \frac{2}{3}\frac{h}{R}\sin\theta\right)^{-1}}.$$

Question A2

A room air conditioner is modeled as a heat engine run in reverse: an amount of heat Q_L is absorbed from the room at a temperature T_L into cooling coils containing a working gas; this gas is compressed adiabatically to a temperature T_H; the gas is compressed isothermally in a coil *outside* the house, giving off an amount of heat Q_H; the gas expands adiabatically back to a temperature T_L; and the cycle repeats. An amount of energy W is input into the system every cycle through an electric pump. This model describes the air conditioner with the best possible efficiency.



Assume that the outside air temperature is T_H and the inside air temperature is T_L. The air-conditioner unit consumes electric power P. Assume that the air is sufficiently dry so that no condensation of water occurs in the cooling coils of the air conditioner. Water boils at 373 K and freezes at 273 K at normal atmospheric pressure.

a. Derive an expression for the maximum rate at which heat is removed from the room in terms of the air temperatures T_H, T_L, and the power consumed by the air conditioner P. Your derivation must refer to the entropy changes that occur in a Carnot cycle in order to receive full marks for this part.

Solution

The optimal performance is attained by a Carnot cycle running in reverse. Since a Carnot cycle is reversible, it keeps the total entropy of the heat reservoirs constant. The change in entropy for a reservoir of temperature T absorbing heat Q is $\Delta S = Q/T$, so

$$\frac{Q_H}{T_H} = \frac{Q_L}{T_L}.$$

Energy conservation states $Q_H = Q_L + W$. Eliminating Q_H and solving for Q_L,

$$Q_L = W\left(\frac{T_L}{T_H - T_L}\right).$$

Finally, the rate of heat removal is Q_L/t, so dividing both sides by t,

$$\frac{Q_L}{t} = P\left(\frac{T_L}{T_H - T_L}\right).$$

b. The room is insulated, but heat still passes into the room at a rate $R = k\Delta T$, where ΔT is the temperature difference between the inside and the outside of the room and k is a constant. Find the coldest possible temperature of the room in terms of T_H, k, and P.

Solution

We equate $k\Delta T$ with the cooling rate Q_L/t found in the previous section. Writing the equation in terms of T_H and $\Delta T = T_H - T_L$,

$$k\Delta T = P\frac{T_L}{\Delta T} = P\frac{T_H - \Delta T}{\Delta T}.$$

Rearranging, we have

$$(\Delta T)^2 = \frac{P}{k}(T_H - \Delta T)$$

which is a quadratic in ΔT. Letting $x = P/k$, we have

$$\Delta T = \frac{x}{2}\left(-1 \pm \sqrt{1 + 4T_H/x}\right)$$

but only the positive root has physical significance. Therefore,

$$T_L = T_H - \frac{x}{2}\left(\sqrt{1 + 4T_H/x} - 1\right).$$

c. A typical room has a value of $k = 173$ W/°C. If the outside temperature is 40°C, what minimum power should the air conditioner have to get the inside temperature down to 25°C?

Solution

From our work above,

$$P = \frac{k(\Delta T)^2}{T_L} = 130 \text{ W}.$$

A common mistake is to forget to convert Celsius to Kelvin.

Question A3

When studying problems in special relativity it is often the invariant distance Δs between two events that is most important, where Δs is defined by

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]$$

where $c = 3 \times 10^8$ m/s is the speed of light.[1]

a. Consider the motion of a projectile launched with initial speed v_0 at angle of θ_0 above the horizontal. Assume that g, the acceleration of free fall, is constant for the motion of the projectile.

   i. Derive an expression for the invariant distance of the projectile as a function of time t as measured from the launch, assuming that it is launched at $t = 0$. Express your answer as a function of any or all of θ_0, v_0, c, g, and t.

Solution

Let the particle start at the origin. Then its path satisfies

$$x = v_0 t \cos\theta_0, \quad z = v_0 t \sin\theta_0 - \frac{1}{2}gt^2$$

by ordinary kinematics. Then

$$s^2 = (ct)^2 - (v_0 t\cos\theta_0)^2 - (v_0 t \sin\theta_0 - \frac{1}{2}gt^2)^2$$

which can be simplified to

$$s^2 = (c^2 - v_0^2)t^2 + gv_0 \sin\theta_0\, t^3 - \frac{1}{4}g^2t^4.$$

   ii. The radius of curvature of an object's trajectory can be estimated by assuming that the trajectory is part of a circle, determining the distance between the end points, and measuring the maximum height above the straight line that connects the endpoints. Assuming that we mean "invariant distance" as defined above, find the radius of curvature of the projectile's trajectory as a function of any or all of θ_0, v_0, c, and g. Assume that the projectile lands at the same level from which it was launched, and assume that the motion is *not* relativistic, so $v_0 \ll c$, and you can neglect terms with v/c compared to terms without.

Solution

Plugging in

$$t_f = \frac{2v_0 \sin\theta}{g}$$

[1] We are using the convention used by Einstein

the invariant distance between the endpoints is approximately

$$s^2 \approx (ct_f)^2 \quad \Rightarrow \quad s \approx 2c\frac{v_0 \sin\theta}{g}.$$

The maximum height above the ground is

$$z_{\max} = \frac{(v_0 \sin\theta)^2}{2g}.$$

Suppose this path subtends an angle θ of a circle of radius R in spacetime. Then

$$s \approx R\theta, \quad z_{\max} \approx R\left(1 - \cos\frac{\theta}{2}\right) \approx \frac{R\theta^2}{8}$$

and eliminating θ yields

$$R \approx \frac{1}{8}\frac{s^2}{z_{\max}} = \frac{c^2}{g}.$$

Indeed, if we solved the problem exactly in relativity, we would find that the path of the projectile is a hyperbola in spacetime with semimajor axis c^2/g. Here we just computed the radius of curvature near the vertex.

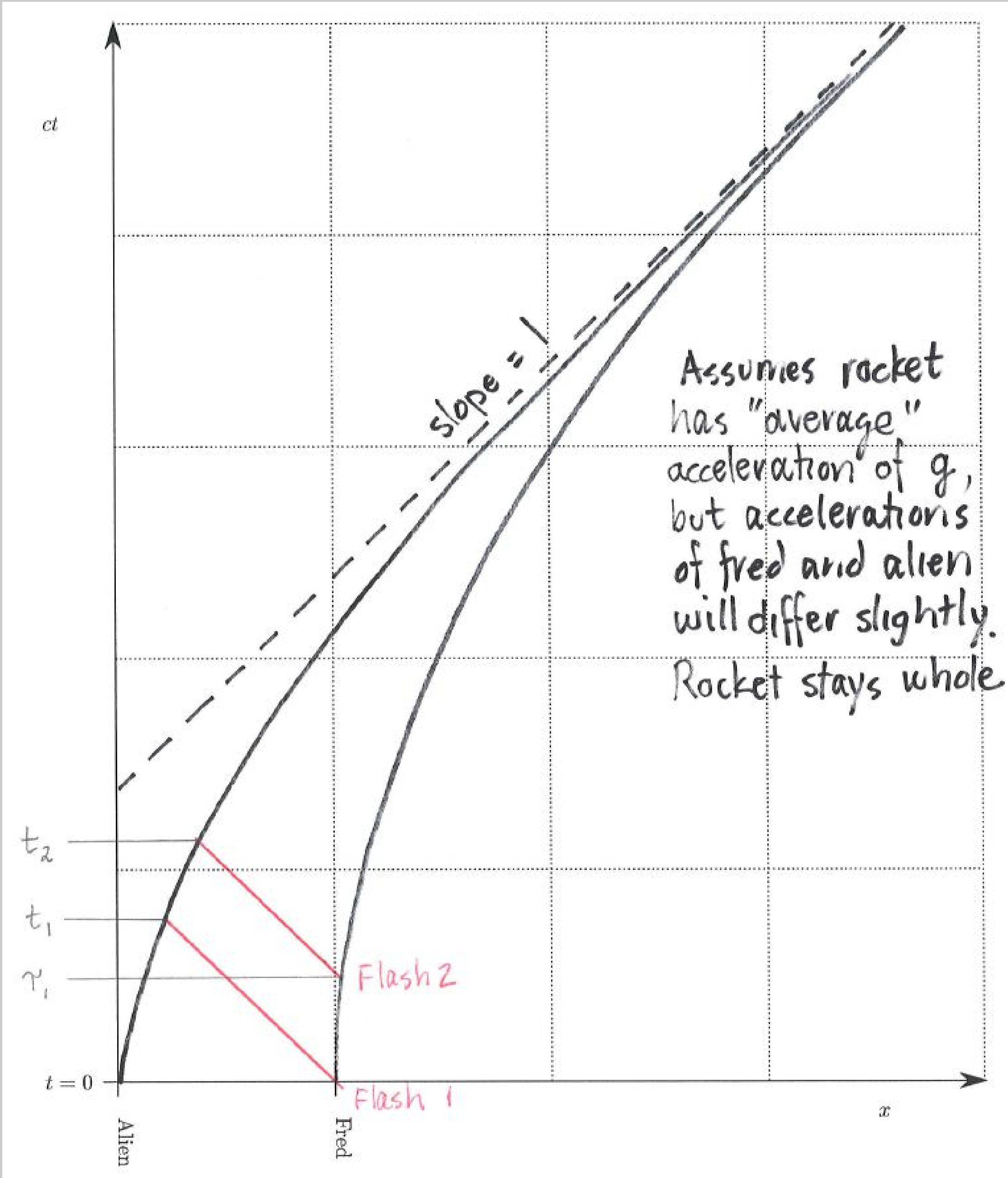
b. A rocket ship far from any gravitational mass is accelerating in the positive x direction at a constant rate g, as measured by someone *inside* the ship. Spaceman Fred at the right end of the rocket aims a laser pointer toward an alien at the left end of the rocket. The two are separated by a distance d such that $dg \ll c^2$; you can safely ignore terms of the form $(dg/c^2)^2$.

   i. Sketch a graph of the motion of both Fred and the alien on the space-time diagram provided in the answer sheet. The graph is *not* meant to be drawn to scale. Note that t and x are reversed from a traditional graph. Assume that the rocket has velocity $v = 0$ at time $t = 0$ and is located at position $x = 0$. Clearly indicate any asymptotes, and the slopes of these asymptotes.

Solution

Since the rocket is constantly accelerating but cannot exceed the speed of light, the curves must asymptote with a slope of one; in an exact analysis we would find they are hyperbolas. However, there is a slight challenge to consider: do Fred and the alien approach the same asymptote, or two different asymptotes?

Since the rocket ship is solid, it maintains the same proper length. Since moving objects are length contracted, it must length contract in our diagram, approaching a length of zero. Indeed, if there were no length contraction, then in the instantaneous rest frame of the ship, the ship would be getting longer and longer, eventually breaking apart.



ii. If the frequency of the laser pointer as measured by Fred is f_1, determine the frequency of the laser pointer as observed by the alien. It is reasonable to assume that $f_1 \gg c/d$.

Solution

To solve this problem, we replace the light with a series of discrete flashes, then find how the frequencies of these flashes are seen by Fred and the alien. Let Fred emit a flash of light at time $t = 0$ and a second flash of light at time $t = \tau$, where τ is very small. Let the alien see the flashes at times t_1 and t_2. Then by ordinary kinematics,

$$ct_1 = d - \frac{1}{2}gt_1^2, \quad c(t_2 - \tau) = \frac{1}{2}g\tau^2 + d - \frac{1}{2}gt_2^2.$$

Note that we are ignoring time dilation effects because they are second order in the velocity, and hence second order in g.

Now subtracting these equations, we have

$$c(t_2 - t_1 - \tau) = \frac{g}{2}(\tau^2 + t_1^2 - t_2^2).$$

Defining $\Delta t = t_2 - t_1$ and simplifying, we have

$$\Delta t \left(1 + \frac{g}{2c}(t_1 + t_2)\right) = \tau \left(1 + \frac{g\tau}{2c}\right) \approx \tau$$

since τ is extremely small, so

$$\frac{\tau}{\Delta t} = 1 + \frac{g}{2c}(t_1 + t_2).$$

Since we are working to first order in g, we may use $t_1 \approx t_2 \approx h/c$ on the right-hand side, so

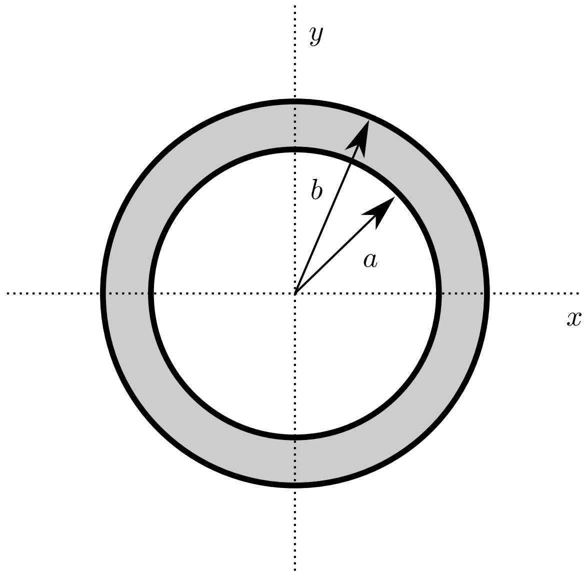
$$\frac{f_{\text{Alien}}}{f_{\text{Fred}}} = \frac{\tau}{\Delta t} \approx 1 + \frac{gh}{c^2}.$$

By the equivalence principle, this is a derivation of gravitational redshift.

The problem can also be solved by thinking in terms of the ordinary Doppler effect. Consider Fred's frame at time $t = 0$. In this frame, the alien is also stationary. The light takes a time h/c to reach the alien; at this point the alien has picked up a velocity of gh/c. Then using the ordinary Doppler shift formula gives a frequency shift of $1 + gh/c^2$ as seen above. This is valid, as all the effects we implicitly ignored were higher order in g, but it harder to see this.

Question A4

A positive point charge q is located inside a neutral hollow spherical conducting shell. The shell has inner radius a and outer radius b; $b - a$ is not negligible. The shell is centered on the origin.



a. Assume that the point charge q is located at the origin in the very center of the shell.

i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

Solution

We apply Gauss's law for a sphere with radius $r > b$ centered about the origin. Since the shell is neutral, the enclosed charge is q, so by spherical symmetry

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2}$$

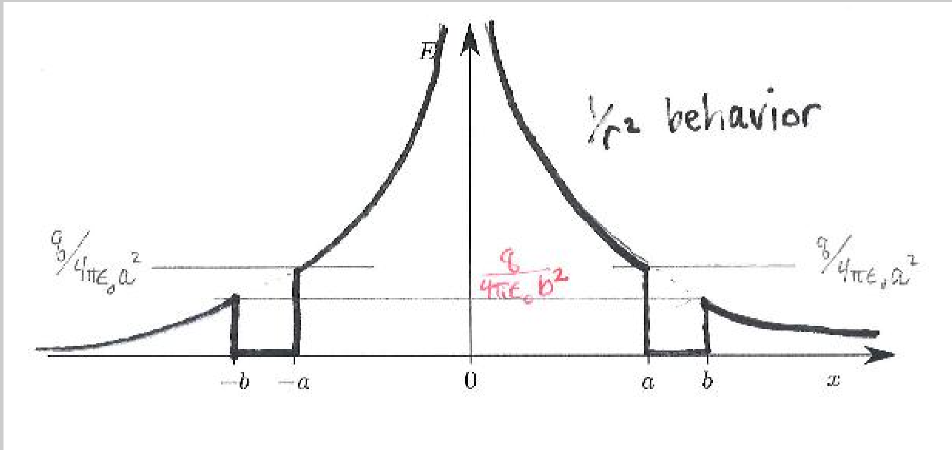
outside the shell. Just outside the shell, the field is $q/4\pi\epsilon_0 b^2$.

ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

Solution

Since the shell is conducting, the electrostatic field is zero inside it. By Gauss's law, this is achieved by having a charge of $-q$ on the inner surface $r = a$ and a charge of q on the outer surface $r = b$, both uniformly distributed.

For $r < a$, we can apply Gauss's law again to conclude $E(r) = \frac{q}{4\pi\epsilon_0 r^2}$, just as it is outside the shell.



iii. Determine the electric potential at $x = a$.

Solution

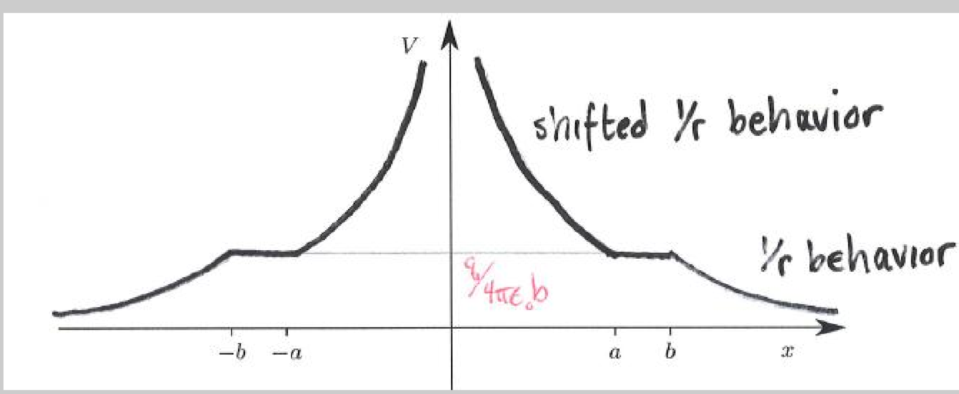
The shell is a conductor, so it is an equipotential surface. Then the potential at $r = a$ is same as the potential at $r = b$. However, outside the shell the field looks just like that of a point charge q at the origin, so

$$V(a) = V(b) = \frac{q}{4\pi\epsilon_0 b}.$$

iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

Solution

As we've shown above, the potential is proportional to $1/r$ outside $r = b$, and is constant between $r = a$ and $r = b$. Then the potential for $r < a$ is *not* proportional to $1/r$. Instead, it is a $1/r$ curve shifted by a constant, so that the potential is continuous at $r = a$.



b. Assume that the point charge q is now located on the x axis at a point $x = 2a/3$.

i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

Solution

The conducting shell acts like a Faraday cage. As in the previous part, by Gauss's law, we must have a charge of $-q$ on the inner surface, so a charge of q on the outer surface. The charge on the inner surface is distributed non-uniformly to perfectly cancel out the asymmetric field of the point charge; these two contributions sum to exactly zero everywhere outside $r = a$. Then by spherical symmetry, the charges on the outer surface are uniformly distributed.

One might wonder why the charges on the outer and inner surfaces can't *both* be non-uniformly distributed. A more rigorous argument would appeal to the uniqueness theorems for electrostatics: given the setup we've given, there is only one way to satisfy all the boundary conditions, so the configuration we gave above must be it. The general principle in that in electrostatics, the only information that can be seen across a shielding conducting shell is the total charge.

In any case, by the same logic as in part (a),

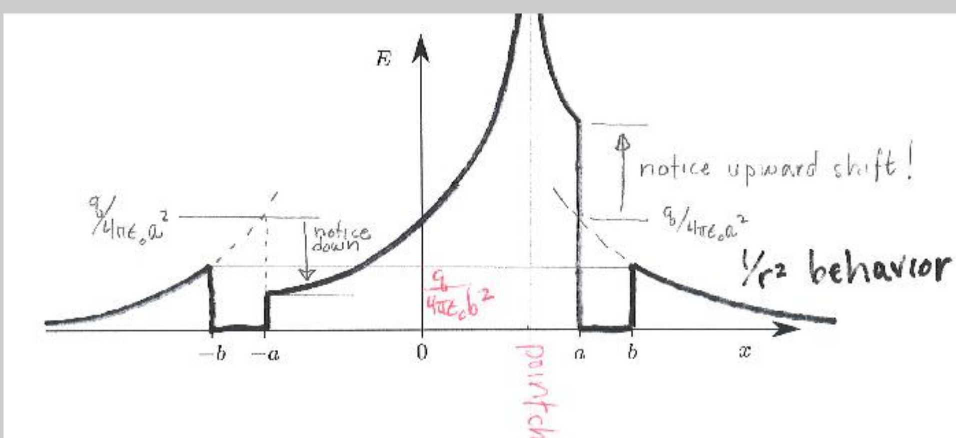
$$E(r) = \frac{q}{4\pi\epsilon_0 r^2}$$

outside the shell. Just outside the shell, the field is $q/4\pi\epsilon_0 b^2$.

ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

Solution

For $r < b$, the field is just that of a point charge at the origin, by the shell theorem. The field inside is more complicated because it depends on the distribution of charge on the inner surface; all that is required is that it diverges at $x = 2a/3$ and is higher at $x = a$ than $x = -a$.



Incidentally, one can find the field for $r < a$ exactly using the method of image charges: for $r < a$, the shielding charges on the inner surface produce the exact same field as a single point charge would. The location of this "image" charge can be found by inverting the original point charge about the circle $r = a$.

iii. Determine the electric potential at $x = a$.

Solution

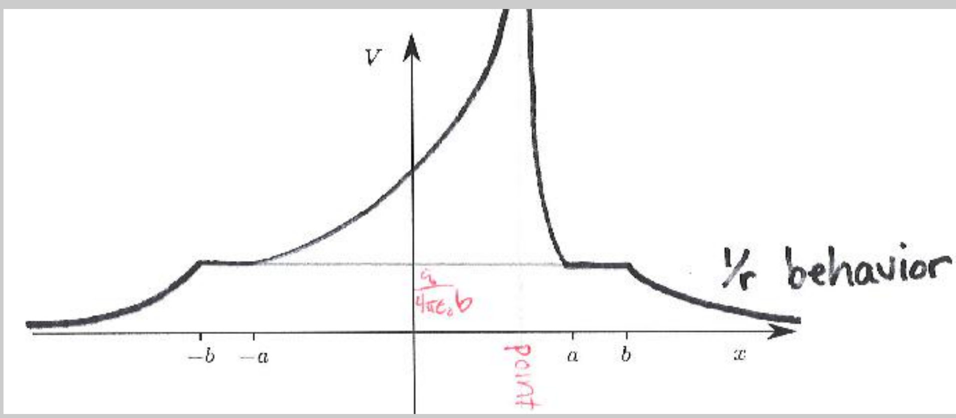
By the same logic as in part (a),

$$V(a) = \frac{q}{4\pi\epsilon_0 b}.$$

iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

Solution

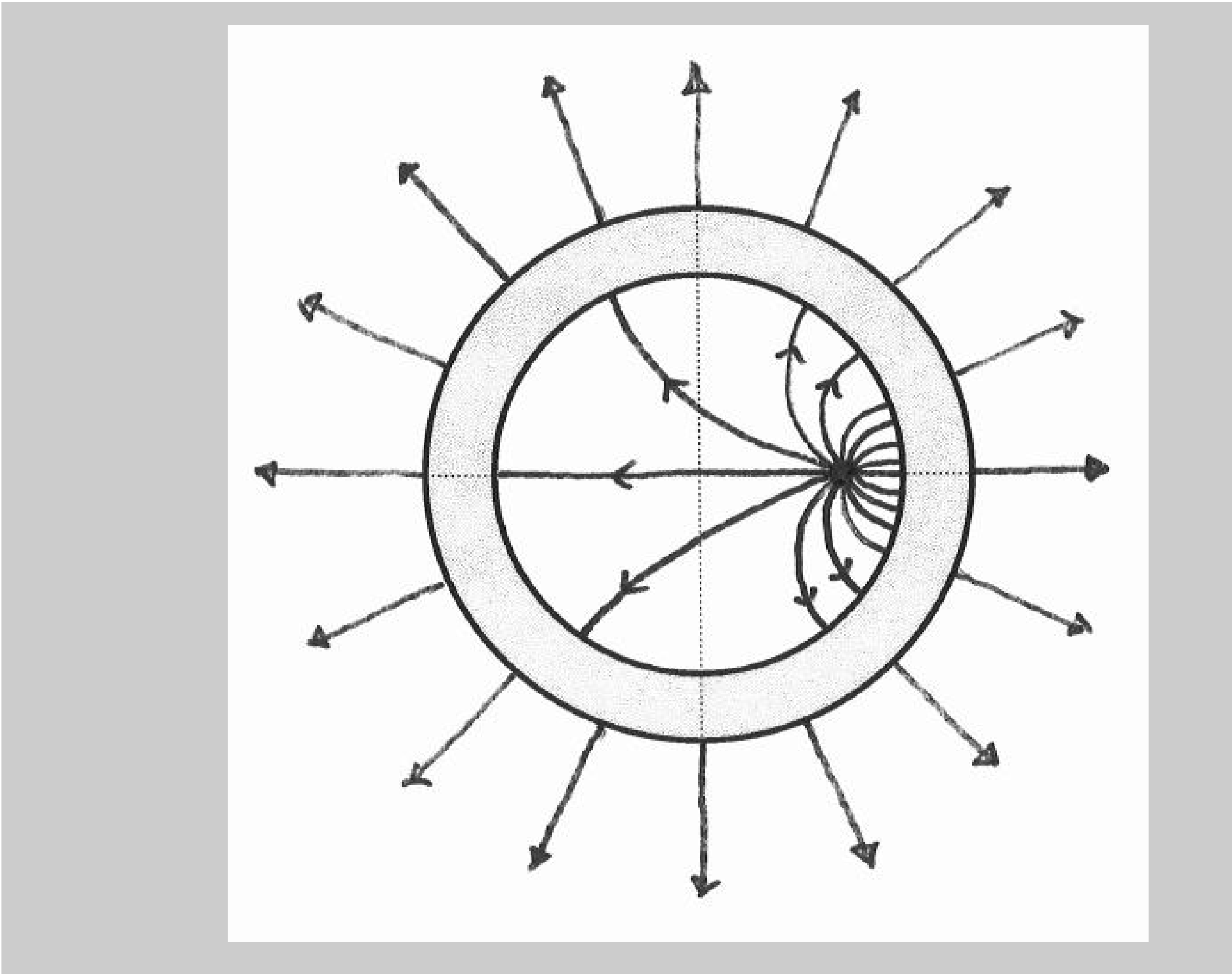
Again, the potential is proportional to $1/r$ outside $r = b$, and is constant between $r = a$ and $r = b$. The potential inside is more complicated, diverging at $x = 2a/3$.



v. Sketch a figure showing the electric field lines (if any) inside, within, and outside the conducting shell on the answer sheet provided. You should show at least eight field lines in any distinct region that has a non-zero field.

Solution

The field should be spherically symmetric outside the shell, zero within the shell, and nonuniform inside. The field lines should terminate perpendicular to the conductor.



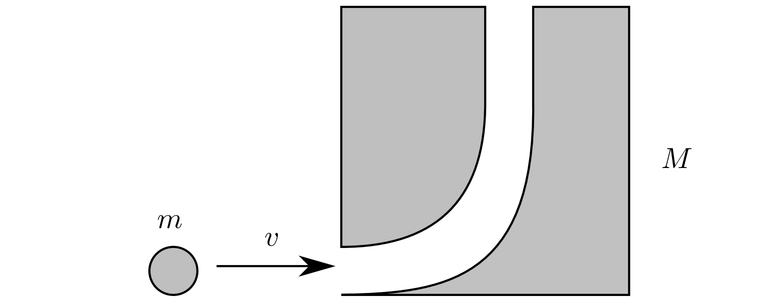
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

A block of mass M has a hole drilled through it so that a ball of mass m can enter horizontally and then pass through the block and exit vertically upward. The ball and block are located on a frictionless surface; the block is originally at rest.



frictionless horizontal surface

a. Consider the scenario where the ball is traveling horizontally with a speed v_0. The ball enters the block and is ejected out the top of the block. Assume there are no frictional losses as the ball passes through the block, and the ball rises to a height much higher than the dimensions of the block. The ball then returns to the level of the block, where it enters the top hole and then is ejected from the side hole. Determine the time t for the ball to return to the position where the original collision occurs in terms of the mass ratio $\beta = M/m$, speed v_0, and acceleration of free fall g.

Solution

After the collision, the ball and block have the same horizontal velocity v_1. Since the horizontal momentum is conserved,

$$v_1 = \frac{m}{m+M} v_0.$$

Let v_2 be the vertical component of the velocity of the ball immediately after the collision. Since there are no frictional losses, conservation of energy yields

$$\frac{1}{2} m v_0^2 = \frac{1}{2} M v_1^2 + \frac{1}{2} m (v_1^2 + v_2^2).$$

Since the ball rises to a height much higher than the height of the block, the gravitational potential energy is negligible, so we have ignored it. This assumption also means we can ignore the duration of the collision itself in our calculations below.

Plugging in our result for v_1,

$$m v_0^2 - (M+m) \left(\frac{m}{m+M} \right)^2 v_0^2 = m v_2^2 \quad \Rightarrow \quad v_2 = \sqrt{\frac{M}{m+M}} v_0.$$

The time spent by the ball in the air is

$$t_1 = 2 v_2 / g.$$

The distance traveled horizontally by the ball while it is in the air is

$$x = v_1 t_1 = \frac{2v_1 v_2}{g} = \frac{2v_0^2}{g}\sqrt{\frac{m^2 M}{(m+M)^3}}.$$

The ball then falls back into the block and is ejected horizontally.

Since energy and momentum are conserved from before the first collision and after the second, the final horizontal velocity v_3 of the ball is given by the result for a perfectly elastic collision,

$$v_3 = v_0 \frac{m-M}{m+M}$$

as can be derived from the conservation laws. Note that v_3 is positive for $m > M$. Thus if $\beta \leq 1$, the time t is infinite; the ball never returns to its starting position.

Assuming that $\beta > 1$, the ball must move a distance x towards its original collision point, where we have neglected the time taken for the collisions themselves. Thus the time for the ball to return to its original position horizontally is

$$t_2 = -\frac{x}{v_3} = \frac{2v_0}{g}\sqrt{\frac{m^2 M}{(M+m)(M-m)^2}}.$$

The total time since the first collision is

$$t = t_1 + t_2 = \frac{2v_0}{g}\sqrt{\frac{M}{m+M}}\left(\frac{M}{M-m}\right) = \frac{2v_0}{g}\sqrt{\frac{\beta}{1+\beta}}\left(\frac{\beta}{\beta-1}\right)$$

where $\beta > 1$.

b. Now consider friction. The ball has moment of inertia $I = \frac{2}{5}mr^2$ and is originally not rotating. When it enters the hole in the block it rubs against one surface so that when it is ejected upwards the ball is rolling without slipping. To what height does the ball rise above the block?

Solution

As before, we have

$$v_1 = \frac{m}{m+M} v_0.$$

Let v_4 be the vertical velocity of the ball after the collision; it is less than v_2 due to the friction force f. Friction slows the ball down with an impulse given by

$$\Delta p = f\Delta t = m(v_2 - v_4)$$

while increasing the angular momentum by

$$\Delta L = \tau \Delta t = r f \Delta t = r \Delta p.$$

We also know that $\Delta L = I\omega = Iv_4/r$, so

$$Iv_4/r = mr(v_2 - v_4) \quad \Rightarrow \quad v_4 = \frac{v_2}{1 + 2/5}.$$

The vertical velocity of the ball will take it to a height

$$h = \frac{v_4^2}{2g} = \frac{v_0^2}{2g} \frac{1}{(1 + 2/5)^2} \frac{M}{m + M} = \frac{v_0^2}{2g} \frac{25}{49} \frac{\beta}{1 + \beta}.$$

Question B2

In parts a and b of this problem assume that velocities v are much less than the speed of light c, and therefore ignore relativistic contraction of lengths or time dilation.

a. An infinite uniform sheet has a surface charge density σ and has an infinitesimal thickness. The sheet lies in the xy plane.

i. Assuming the sheet is at rest, determine the electric field $\tilde{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

Solution

By symmetry, the fields above and below the sheet are equal in magnitude and directed away from the sheet. By Gauss's Law, using a cylinder of base area A,

$$2EA = \frac{\sigma A}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{2\epsilon_0}$$

pointing directly away from the sheet in the z direction, or

$$\mathbf{E} = \frac{\sigma}{2\epsilon} \times \begin{cases} \hat{\mathbf{z}} & \text{above the sheet,} \\ -\hat{\mathbf{z}} & \text{below the sheet.} \end{cases}$$

ii. Assuming the sheet is moving with velocity $\tilde{\mathbf{v}} = v\hat{\mathbf{x}}$ (parallel to the sheet), determine the electric field $\tilde{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

Solution

The motion does not affect the electric field, so the answer is the same as that of part (i).

iii. Assuming the sheet is moving with velocity $\tilde{\mathbf{v}} = v\hat{\mathbf{x}}$, determine the magnetic field $\tilde{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

Solution

Assuming $v > 0$, the right-hand rule indicates there is a magnetic field in the $-\tilde{\mathbf{y}}$ direction for $z > 0$ and in the $+\tilde{\mathbf{y}}$ direction for $z < 0$. From Ampere's law applied to a loop of length l normal to the $\tilde{\mathbf{x}}$ direction,

$$2Bl = \mu_0 \sigma v l.$$

To get the right-hand side, note that in time t, an area vtl moves through the loop, so a charge σvtl moves through. Then the current through the loop is σvl.
Applying symmetry, we have

$$\mathbf{B} = \frac{\mu_0 \sigma v}{2} \times \begin{cases} -\hat{\mathbf{y}} & \text{above the sheet,} \\ \hat{\mathbf{y}} & \text{below the sheet.} \end{cases}$$

iv. Assuming the sheet is moving with velocity $\tilde{\mathbf{v}} = v\hat{\mathbf{z}}$ (perpendicular to the sheet), determine the electric field $\tilde{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

Solution

Again the motion does not affect the electric field, so the answer is the same as that of part (i).

v. Assuming the sheet is moving with velocity $\tilde{\mathbf{v}} = v\hat{\mathbf{z}}$, determine the magnetic field $\tilde{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

Solution

Applying Ampere's law and symmetry, there is no magnetic field above and below the sheet.

Interestingly, there's no magnetic field at the sheet either. Consider an Amperian loop of area A in the xy plane as the sheet passes through. The loop experiences a current of the form

$$A\sigma\delta(t).$$

But the loop also experiences an oppositely directed change in flux of the form

$$A\frac{\sigma}{\epsilon_0}\delta(t),$$

so the right-hand side of Ampere's law, including the displacement current term, remains zero.

b. In a certain region there exists only an electric field $\tilde{\mathbf{E}} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ (and no magnetic field) as measured by an observer at rest. The electric and magnetic fields $\tilde{\mathbf{E}}'$ and $\tilde{\mathbf{B}}'$ as measured by observers in motion can be determined entirely from the local value of $\tilde{\mathbf{E}}$, regardless of the charge configuration that may have produced it.

i. What would be the observed electric field $\tilde{\mathbf{E}}'$ as measured by an observer moving with velocity $\tilde{\mathbf{v}} = v\hat{\mathbf{z}}$?

Solution

In part (a), we showed that if the electric field was produced by a sheet of charge, then it was unaffected by the motion of an observer. Thus, in general,

$$\mathbf{E}' = \mathbf{E}.$$

ii. What would be the observed magnetic field $\tilde{\mathbf{B}}'$ as measured by an observer moving with velocity $\tilde{\mathbf{v}} = v\hat{\mathbf{z}}$?

Solution

No magnetic field was created by the motion of the sheet of charge in the direction of the electric field, so the magnetic field in the frame of reference of the moving observer should likewise not depend on the component of the electric field in the direction of motion. When the sheet of charge was moving in the $+\hat{\mathbf{x}}$ direction, a magnetic field was created in the $-\hat{\mathbf{y}}$ direction; the observer moving in the $+\hat{\mathbf{x}}$ direction is equivalent to the sheet of charge moving in the $-\hat{\mathbf{x}}$ direction, creating a magnetic field in the $+\hat{\mathbf{y}}$ direction. That is, an electric field in the $\hat{\mathbf{z}}$ direction causes an observer moving in the $\hat{\mathbf{x}}$ direction to observe a magnetic field in the $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ direction.

Furthermore, the magnitudes of the fields satisfied

$$B = \mu_0 \epsilon_0 v E = \frac{1}{c^2} v E.$$

Combining this with the previous equation,

$$\mathbf{B}' = -\frac{1}{c^2}\mathbf{v} \times \mathbf{E} = \frac{v}{c^2}\left(E_y \hat{\mathbf{x}} - E_x \hat{\mathbf{y}}\right).$$

c. An infinitely long wire wire on the z axis is composed of positive charges with linear charge density λ which are at rest, and negative charges with linear charge density $-\lambda$ moving with speed v in the z direction.

i. Determine the electric field $\tilde{\mathbf{E}}$ (magnitude and direction) at points outside the wire.

Solution

The wire as a whole is neutral, so there is no electric field outside the wire.

ii. Determine the magnetic field $\tilde{\mathbf{B}}$ (magnitude and direction) at points outside the wire.

Solution

The current in the wire is λv, so Ampere's law yields

$$B = \mu_0 \frac{\lambda v}{2\pi r}$$

in the tangential direction. The current is in the $-\hat{\mathbf{z}}$ direction, so by the right-hand rule, the circular $\mathbf{B}$ field lines would point clockwise looking in that direction.

iii. Now consider an observer moving with speed v parallel to the z axis so that the negative charges appear to be at rest. There is a symmetry between the electric and magnetic fields such that a variation to your answer to part b can be applied to the magnetic field in this part. You will need to change the multiplicative constant to something dimensionally correct and reverse the sign. Use this fact to find and describe the electric field measured by the moving observer, and comment on your result. (Some familiarity with special relativity can help you verify the direction of your result, but is not necessary to obtain the correct answer.)

Solution

The result of part (b) was

$$\mathbf{B}' = -\frac{1}{c^2}\mathbf{v} \times \mathbf{E}.$$

Exchanging the electric and magnetic fields, reversing the sign, and fixing the dimensions,

$$\mathbf{E}' = \mathbf{v} \times \mathbf{B}.$$

Taking the cross product yields an electric field vector that points outward, with magnitude

$$E' = v\mu_0 \frac{\lambda v}{2\pi r} = \frac{\lambda}{2\pi\epsilon_0 r}\frac{v^2}{c^2}.$$

Physically, this can be explained by length contraction of the positive charges and inverse length contraction of the negative charges, which are now stationary. That is, we have derived a relativistic effect, second-order in v/c, from the first-order field transformations!

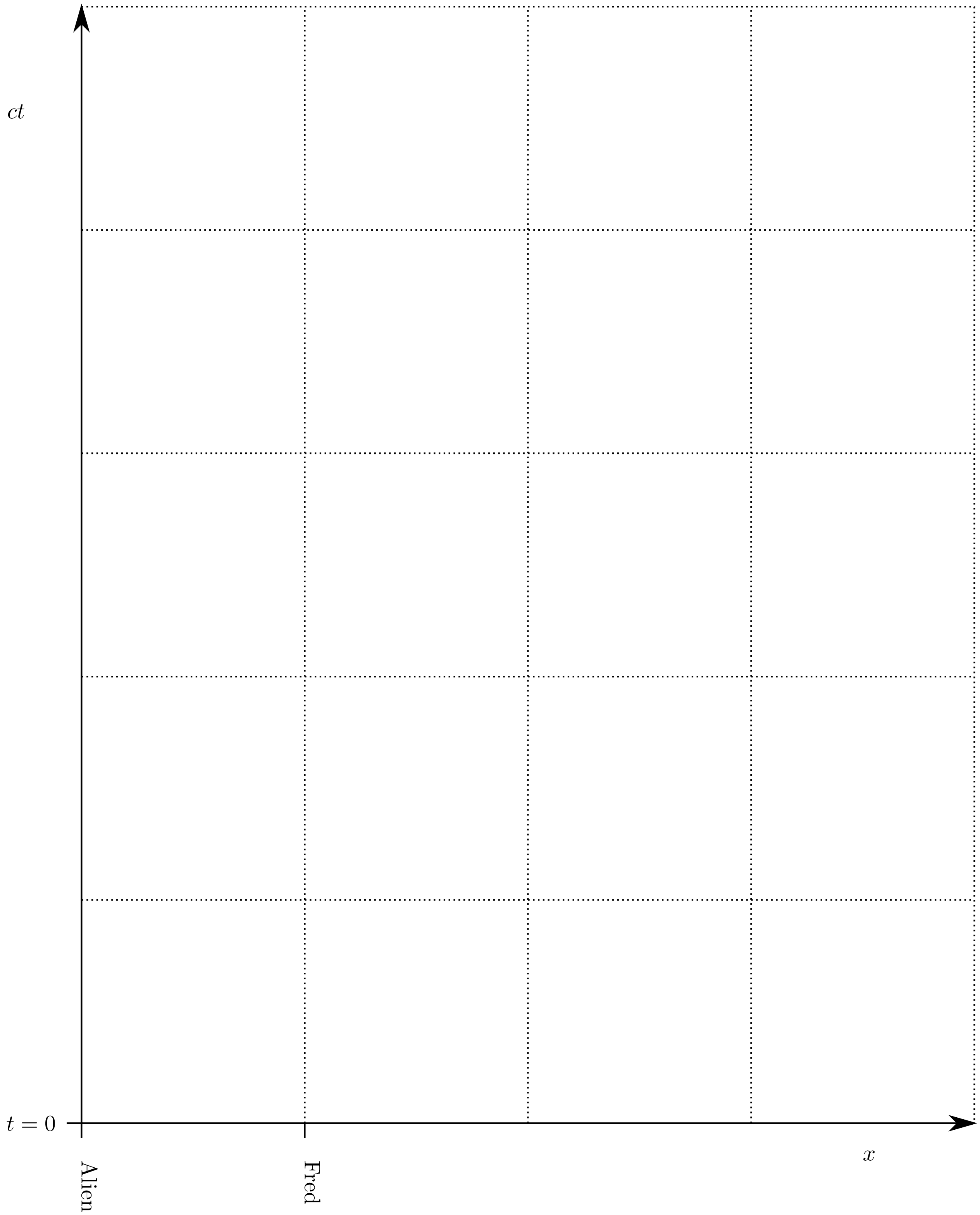
Why wasn't this derivation used to discover relativity the moment Maxwell's equations were written down? We have implicitly assumed that Maxwell's equations are the same in all reference frames, but historically it was thought they were only valid in one frame, the reference frame of the ether. Assuming that Maxwell's equations are indeed the same in all frames yields an invariant speed, the speed of light, leaving inevitably to all of special relativity. Here we've taken one of many possible paths.

Answer Sheets

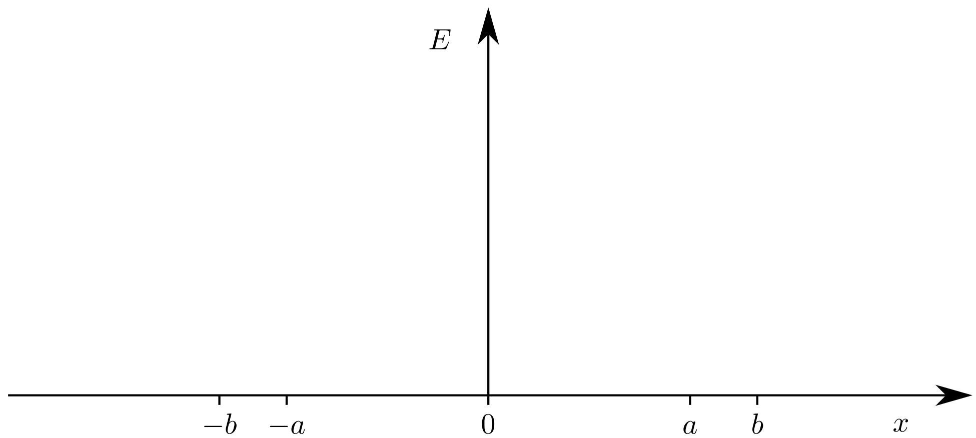
Following are answer sheets for some of the graphical portions of the test.

Answer for Part A, Question 3

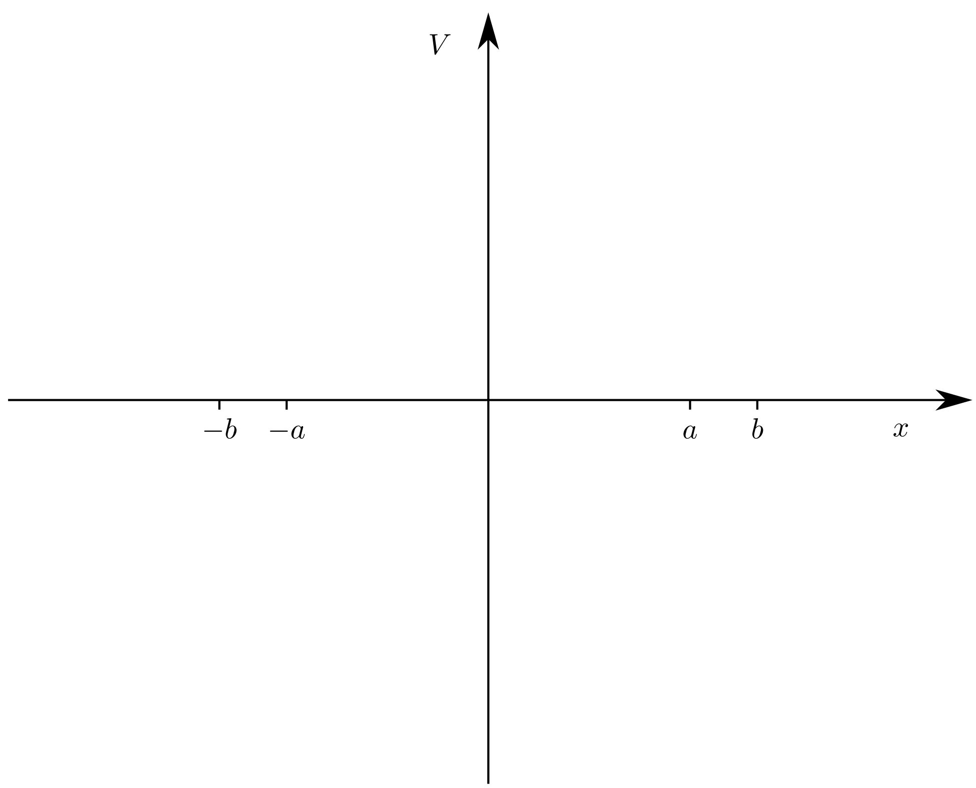
Space-time graph for accelerated rocket. The positions of Fred and the Alien at $t = 0$ are shown.



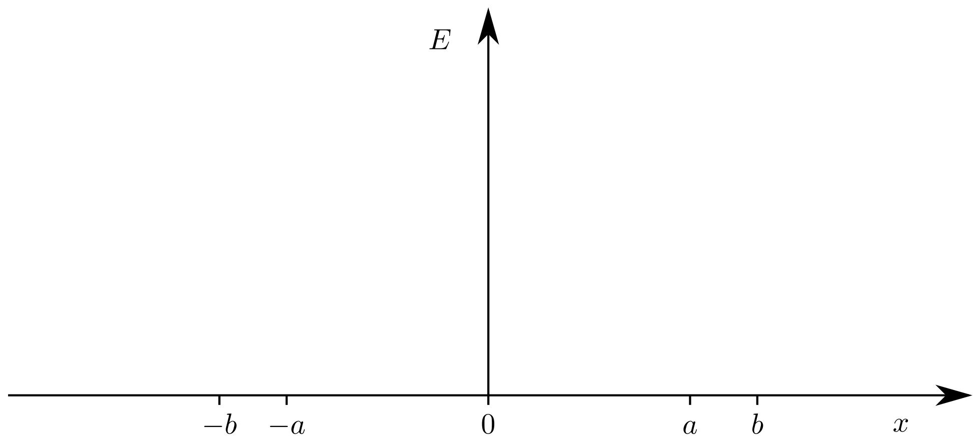
Answer for Part A, Question 4



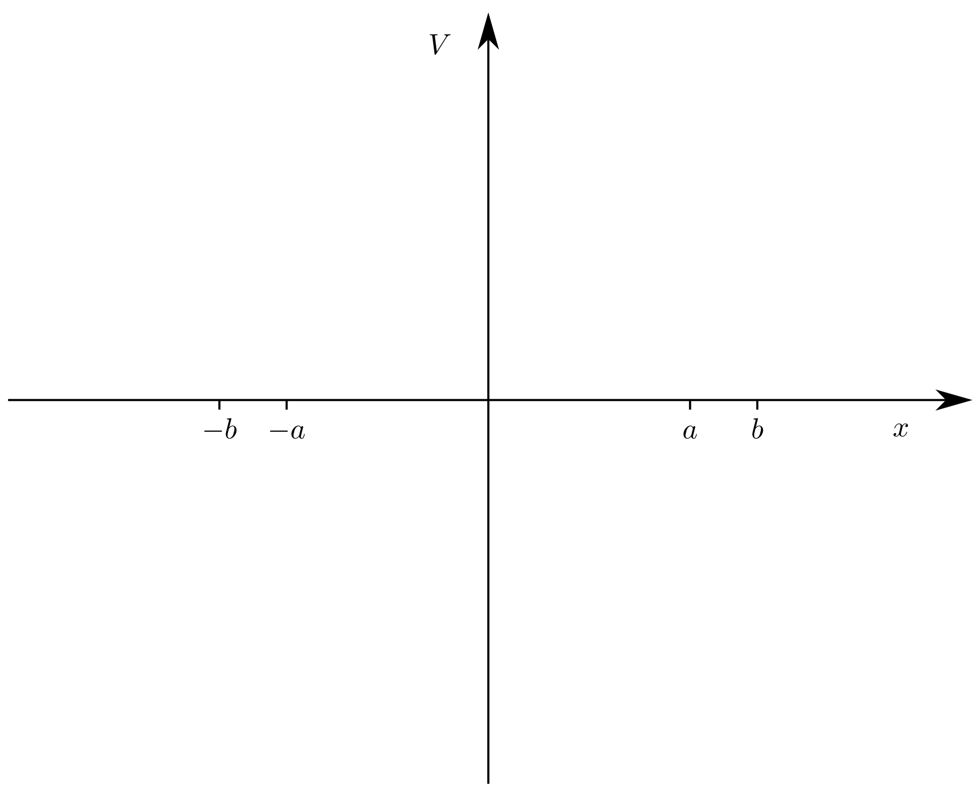
Answer for Part A, Question 4



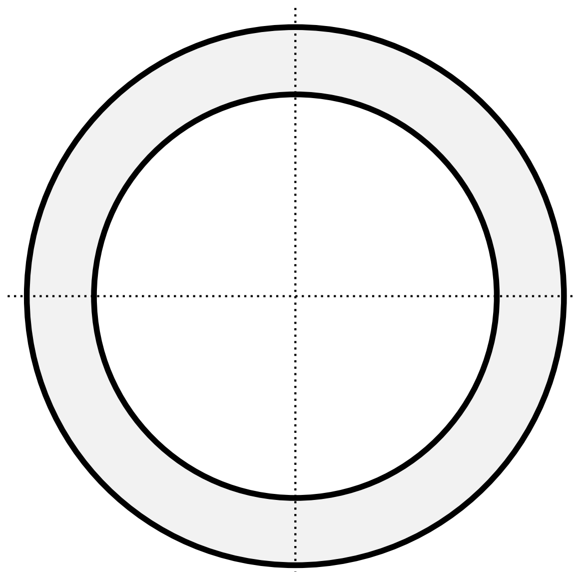
Answer for Part A, Question 4



Answer for Part A, Question 4



Answer for Part A, Question 4



AAPT UNITED STATES PHYSICS TEAM
AIP 2015

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts.
- Part A has four questions and is allowed 90 minutes.
- Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 4 parts: the cover sheet (page 2), Part A (pages 3-7), Part B (pages 9-11), and several answer sheets for one of the questions in Part A (pages 13-13). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
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- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

AAPT UNITED STATES PHYSICS TEAM
AIP 2015

USA Physics Olympiad Exam

INSTRUCTIONS

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- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  AAPT ID #
  Doe, Jamie
  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
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Possibly Useful Information. You may use this sheet for both parts of the exam.

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$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}$ C
$1\mathrm{eV} = 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x\rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

Consider a particle of mass m that elastically bounces off of an infinitely hard horizontal surface under the influence of gravity. The total mechanical energy of the particle is E and the acceleration of free fall is g. Treat the particle as a point mass and assume the motion is non-relativistic.

a. An estimate for the regime where quantum effects become important can be found by simply considering when the deBroglie wavelength of the particle is on the same order as the height of a bounce. Assuming that the deBroglie wavelength is defined by the maximum momentum of the bouncing particle, determine the value of the energy E_q where quantum effects become important. Write your answer in terms of some or all of g, m, and Planck's constant h.

b. A second approach allows us to develop an estimate for the actual allowed energy levels of a bouncing particle. Assuming that the particle rises to a height H, we can write

$$2\int_0^H p\,dx = \left(n + \frac{1}{2}\right) h$$

where p is the momentum as a function of height x above the ground, n is a non-negative integer, and h is Planck's constant.

i. Determine the allowed energies E_n as a function of the integer n, and some or all of g, m, and Planck's constant h.

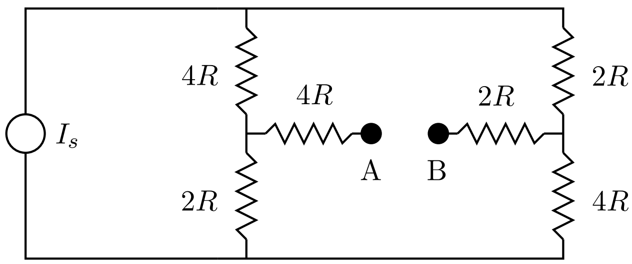
ii. Numerically determine the minimum energy of a bouncing neutron. The mass of a neutron is $m_n = 1.675 \times 10^{-27}$ kg $= 940$ MeV/c^2; you may express your answer in either Joules or eV.

iii. Determine the bounce height of one of these minimum energy neutrons.

c. Let E_0 be the minimum energy of the bouncing neutron and f be the frequency of the bounce. Determine an order of magnitude estimate for the ratio E/f. It only needs to be accurate to within an order of magnitude or so, but you do need to show work!

Question A2

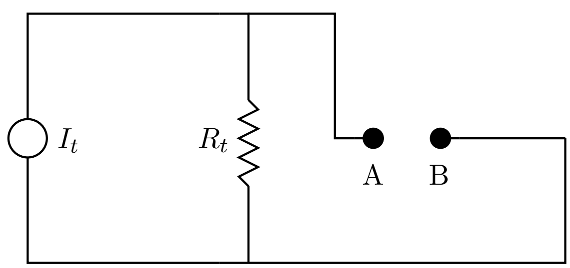
Consider the circuit shown below. I_s is a constant current source, meaning that no matter what device is connected between points A and B, the current provided by the constant current source is the same.



a. Connect an ideal voltmeter between A and B. Determine the voltage reading in terms of any or all of R and I_s.

b. Connect instead an ideal ammeter between A and B. Determine the current in terms of any or all of R and I_s.

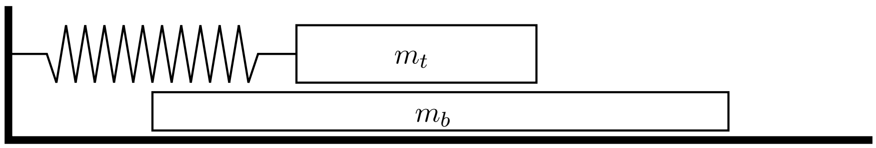
c. It turns out that it is possible to replace the above circuit with a new circuit as follows:



From the point of view of *any* passive resistance that is connected between A and B the circuits are identical. You don't need to prove this statement, but you do need to find I_t and R_t in terms of any or all of R and I_s.

Question A3

A large block of mass m_b is located on a horizontal frictionless surface. A second block of mass m_t is located on top of the first block; the coefficient of friction (both static and kinetic) between the two blocks is given by μ. All surfaces are horizontal; all motion is effectively one dimensional. A spring with spring constant k is connected to the top block only; the spring obeys Hooke's Law equally in both extension and compression. Assume that the top block never falls off of the bottom block; you may assume that the bottom block is very, very long. The top block is moved a distance A away from the equilibrium position and then released from rest.



a. Depending on the value of A, the motion can be divided into two types: motion that experiences no frictional energy losses and motion that does. Find the value A_c that divides the two motion types. Write your answer in terms of any or all of μ, the acceleration of gravity g, the masses m_t and m_b, and the spring constant k.

b. Consider now the scenario $A \gg A_c$. In this scenario the amplitude of the oscillation of the top block as measured against the original equilibrium position will change with time. Determine the magnitude of the change in amplitude, ΔA, after one complete oscillation, as a function of any or all of A, μ, g, and the angular frequency of oscillation of the top block ω_t.

c. Assume still that $A \gg A_c$. What is the maximum speed of the bottom block during the first complete oscillation cycle of the upper block?

Question A4

A heat engine consists of a moveable piston in a vertical cylinder. The piston is held in place by a removable weight placed on top of the piston, but piston stops prevent the piston from sinking below a certain point. The mass of the piston is $m = 40.0$ kg, the cross sectional area of the piston is $A = 100\ \text{cm}^2$, and the weight placed on the piston has a mass of $m = 120.0$ kg.

Assume that the region around the cylinder and piston is a vacuum, so you don't need to worry about external atmospheric pressure.

- At point **A** the cylinder volume V_0 is completely filled with liquid water at a temperature $T_0 = 320$ K and a pressure P_{min} that would be just sufficient to lift the piston alone, except the piston has the additional weight placed on top.
- Heat energy is added to the water by placing the entire cylinder in a hot bath.
- At point **B** the piston and weight begins to rise.
- At point **C** the volume of the cylinder reaches V_{max} and the temperature reaches T_{max}. The heat source is removed; the piston stops rising and is locked in place.
- Heat energy is now removed from the water by placing the entire cylinder in a cold bath.
- At point **D** the pressure in the cylinder returns to P_{min}. The added weight is removed; the piston is unlocked and begins to move down.
- The cylinder volume returns to V_0. The cylinder is removed from the cold bath, the weight is placed back on top of the piston, and the cycle repeats.

Because the liquid water can change to gas, there are several important events that take place

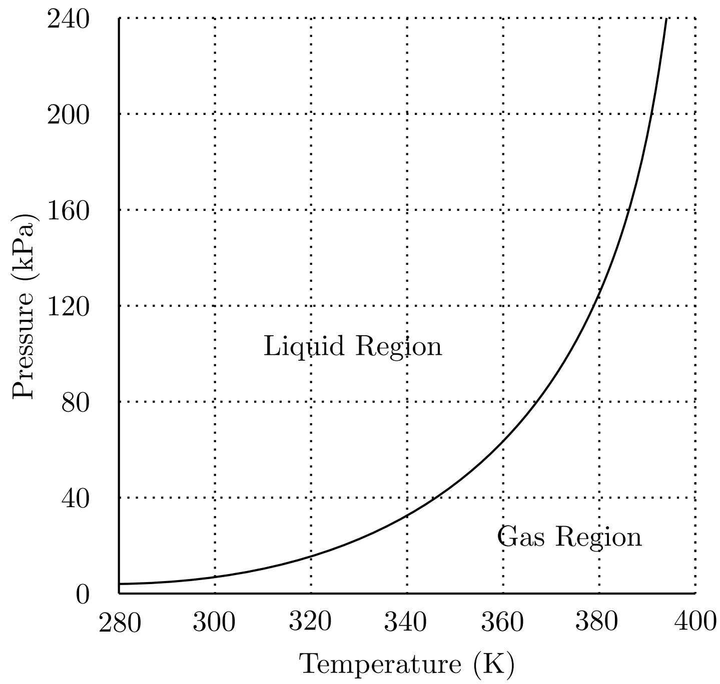
- At point **W** the liquid begins changing to gas.
- At point **X** all of the liquid has changed to gas. This occurs at the same point as point **C** described above.
- At point **Y** the gas begins to change back into liquid.
- At point **Z** all of the gas has changed back into liquid.

When in the liquid state you need to know that for water kept at constant volume, a change in temperature ΔT is related to a change in pressure ΔP according to

$$\Delta P \approx (10^6\ \text{Pa/K})\Delta T$$

When in the gas state you should assume that water behaves like an ideal gas.

Of relevance to this question is the pressure/temperature phase plot for water, showing the regions where water exists in liquid form or gaseous form. The curve shows the coexistence condition, where water can exist simultaneously as gas or liquid.



The following graphs should be drawn on the answer sheet provided.

a. Sketch a PT diagram for this cycle on the answer sheet. The coexistence curve for the liquid/gas state is shown. Clearly and accurately label the locations of points **B** through **D** and **W** through **Z** on this cycle.

b. Sketch a PV diagram for this cycle on the answer sheet. You should estimate a reasonable value for V_{max}, note the scale is logarithmic. Clearly and accurately label the locations of points **B** through **D** on this cycle. Provide reasonable approximate locations for points **W** through **Z** on this cycle.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

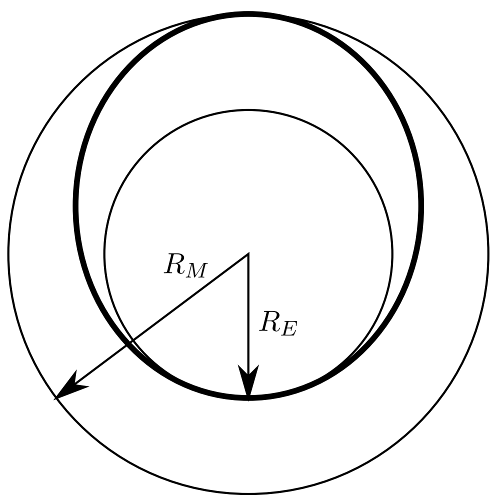
This problem is divided into three parts. It is possible to solve these three parts independently, but they are not equally weighted.

a. An ideal rocket when empty of fuel has a mass m_r and will carry a mass of fuel m_f. The fuel burns and is ejected with an exhaust speed of v_e relative to the rocket. The fuel burns at a constant mass rate for a total time T_b. Ignore gravity; assume the rocket is far from any other body.

   i. Determine an equation for the acceleration of the rocket as a function of time t in terms of any or all of t, m_f, m_r, v_e, T_b, and any relevant fundamental constants.

   ii. Assuming that the rocket starts from rest, determine the final speed of the rocket in terms of any or all of m_r, m_f, v_e, T_b, and any relevant fundamental constants.

b. The ship starts out in a circular orbit around the sun very near the Earth and has a goal of moving to a circular orbit around the Sun that is very close to Mars. It will make this transfer in an elliptical orbit as shown in bold in the diagram below. This is accomplished with an initial velocity boost near the Earth Δv_1 and then a second velocity boost near Mars Δv_2. Assume that both of these boosts are from instantaneous impulses, and ignore mass changes in the rocket as well as gravitational attraction to either Earth or Mars. Don't ignore the Sun! Assume that the Earth and Mars are both in circular orbits around the Sun of radii R_E and $R_M = R_E/\alpha$ respectively. The orbital speeds are v_E and v_M respectively.



   i. Derive an expression for the velocity boost Δv_1 to change the orbit from circular to elliptical. Express your answer in terms of v_E and α.

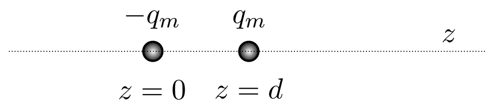
   ii. Derive an expression for the velocity boost Δv_2 to change the orbit from elliptical to circular. Express your answer in terms of v_E and α.

   iii. What is the angular separation between Earth and Mars, as measured from the Sun, at the time of launch so that the rocket will start from Earth and arrive at Mars when it reaches the orbit of Mars? Express your answer in terms of α.

Question B2

The nature of magnetic dipoles.

a. A "Gilbert" dipole consists of a pair of magnetic monopoles each with a magnitude q_m but opposite magnetic charges separated by a distance d, where d is small. In this case, assume that $-q_m$ is located at $z = 0$ and $+q_m$ is located at $z = d$.



Assume that magnetic monopoles behave like electric monopoles according to a coulomb-like force

$$F = \frac{\mu_0}{4\pi}\frac{q_{m1}q_{m2}}{r^2}$$

and the magnetic field obeys

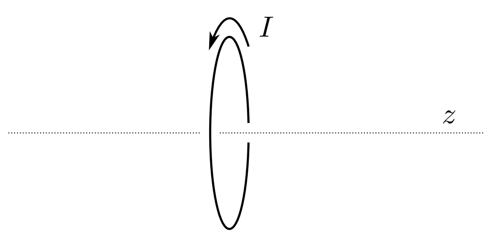
$$B = F/q_m.$$

i. What are the dimensions of the quantity q_m?

ii. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > d$. Write your answer in terms of q_m, d, z, and any necessary fundamental constants.

iii. Evaluate this expression in the limit as $d \to 0$, assuming that the product $q_m d = p_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of p_m, z, and any necessary fundamental constants.

b. An "Ampère" dipole is a magnetic dipole produced by a current loop I around a circle of radius r, where r is small. Assume the that the z axis is the axis of rotational symmetry for the circular loop, and the loop lies in the xy plane at $z = 0$.



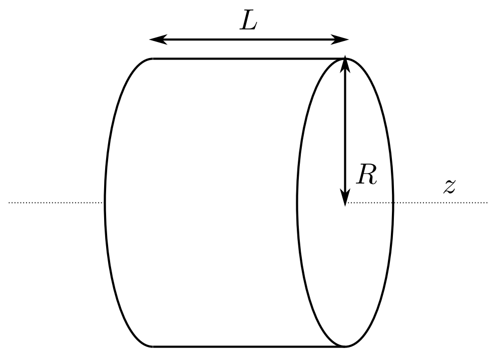
i. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > 0$. Write your answer in terms of I, r, z, and any necessary fundamental constants.

ii. Let kIr^γ have dimensions equal to that of the quantity p_m defined above in Part aiii, where k and γ are dimensionless constants. Determine the value of γ.

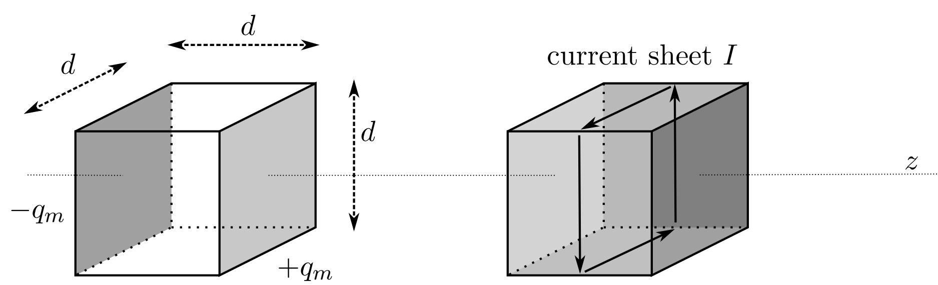
iii. Evaluate the expression in Part bi in the limit as $r \to 0$, assuming that the product $kIr^\gamma = p'_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of k, p'_m, z, and any necessary fundamental constants.

iv. Assuming that the two approaches are equivalent, $p_m = p'_m$. Determine the constant k in Part bii.

c. Now we try to compare the two approaches if we model a physical magnet as being composed of densely packed microscopic dipoles.



A cylinder of this uniform magnetic material has a radius R and a length L. It is composed of N magnetic dipoles that could be either all Ampère type or all Gilbert type. N is a *very* large number. The axis of rotation of the cylinder and all of the dipoles are all aligned with the z axis and all point in the same direction as defined above so that the magnetic field *outside* the cylinder is the same in either dipole case as you previously determined. Below is a picture of the two dipole models; they are cubes of side $d << R$ and $d << L$ with volume $v_m = d^3$.



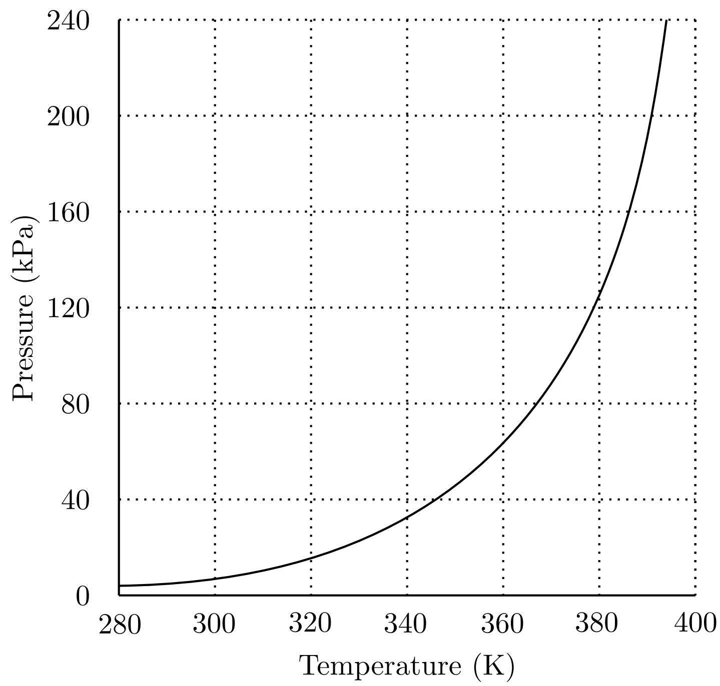
Gilbert Dipole Ampère Dipole

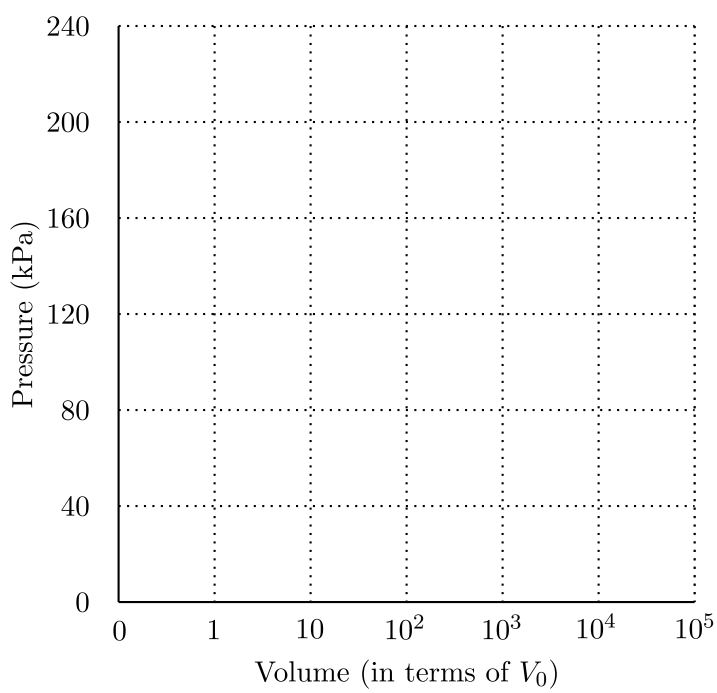
i. Assume that $R \gg L$ and only Gilbert type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.

ii. Assume that $R \ll L$ and only Ampère type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.

Answer Sheets

Following are answer sheets for some of the graphical portions of the test.


240
200
160
120
80
40
0
Pressure (kPa)
280
300
320
340
360
380
400
Temperature (K)


240
200
160
120
80
40
0
Pressure (kPa)
0
1
10
10^2
10^3
10^4
10^5
Volume (in terms of V_0)

AAPT UNITED STATES PHYSICS TEAM
AIP 2015

USA Physics Olympiad Exam

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AAPT AIP UNITED STATES PHYSICS TEAM 2015

USA Physics Olympiad Exam

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  Doe, Jamie

  A1 - 1/3
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$c = 3.00 \times 10^8$ m/s	$k_\text{B} = 1.38 \times 10^{-23}$ J/K
$N_\text{A} = 6.02 \times 10^{23}$ (mol)$^{-1}$	$R = N_\text{A} k_\text{B} = 8.31$ J/(mol $\cdot$ K)
$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)	$e = 1.602 \times 10^{-19}$ C
1eV $= 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s
$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
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Part A

Question A1

Consider a particle of mass m that elastically bounces off of an infinitely hard horizontal surface under the influence of gravity. The total mechanical energy of the particle is E and the acceleration of free fall is g. Treat the particle as a point mass and assume the motion is non-relativistic.

a. An estimate for the regime where quantum effects become important can be found by simply considering when the deBroglie wavelength of the particle is on the same order as the height of a bounce. Assuming that the deBroglie wavelength is defined by the maximum momentum of the bouncing particle, determine the value of the energy E_q where quantum effects become important. Write your answer in terms of some or all of g, m, and Planck's constant h.

Solution

The de Broglie wavelength is $p = h/\lambda$, so if the height H of the bounce is given by

$$E = mgH = \frac{p^2}{2m}$$

and $\lambda = H$, then

$$mgH = \frac{h^2}{2mH^2} \quad \Rightarrow \quad H^3 = \frac{h^2}{2m^2 g}$$

or

$$E_q = \sqrt[3]{\frac{1}{2}mg^2h^2}.$$

One can also use dimensional analysis to find $E_q \propto \sqrt[3]{mg^2h^2}$, though this will receive only partial credit.

b. A second approach allows us to develop an estimate for the actual allowed energy levels of a bouncing particle. Assuming that the particle rises to a height H, we can write

$$2\int_0^H p\,dx = \left(n + \frac{1}{2}\right) h$$

where p is the momentum as a function of height x above the ground, n is a non-negative integer, and h is Planck's constant.

i. Determine the allowed energies E_n as a function of the integer n, and some or all of g, m, and Planck's constant h.

ii. Numerically determine the minimum energy of a bouncing neutron. The mass of a neutron is $m_n = 1.675 \times 10^{-27}$ kg $= 940$ MeV/c^2; you may express your answer in either Joules or eV.

iii. Determine the bounce height of one of these minimum energy neutrons.

Solution

We simply evaluate the given integral,

$$\begin{aligned}\left(n+\frac{1}{2}\right)h &= 2\int_0^H p\,dx\\ &= 2\sqrt{2m}\int_0^H \sqrt{E-mgx}\,dx,\\ &= 2\sqrt{2mE}\int_0^H \sqrt{1-mgx/E}\,dx,\\ &= 2\sqrt{2mE}\frac{E}{mg}\int_0^1 \sqrt{1-u}\,du,\\ &= 2\sqrt{2mE}\frac{E}{mg}\int_0^1 \sqrt{v}\,dv,\\ &= 2\sqrt{2}\frac{E^{3/2}}{\sqrt{mg}}\frac{2}{3}\end{aligned}$$

so

$$E_n = \sqrt[3]{\frac{9mg^2h^2}{32}}\left(n+\frac{1}{2}\right)^{2/3}.$$

Solving for the minimum energy we get

$$E_0 = \sqrt[3]{\frac{9mg^2h^2}{128}} = \sqrt[3]{\frac{9(mc^2)g^2h^2}{128c^2}} = 1.1\times 10^{-12}\,\text{eV}.$$

The bounce height is given by

$$H = \frac{E_0}{mg} = 10\ \mu\text{m}.$$

This is a very measurable distance!

c. Let E_0 be the minimum energy of the bouncing neutron and f be the frequency of the bounce. Determine an order of magnitude estimate for the ratio E/f. It only needs to be accurate to within an order of magnitude or so, but you do need to show work!

Solution

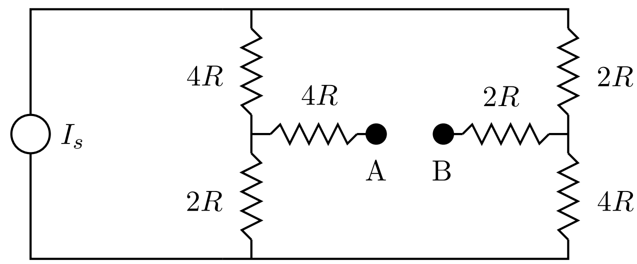
One can simply use the results found above. A quicker method is dimensional analysis: the only quantity with units of energy times time is h itself, so we must have

$$E/f \sim h.$$

A mnemonic to remember the units is the energy-time uncertainty principle $\Delta E\Delta t \sim h$.

Question A2

Consider the circuit shown below. I_s is a constant current source, meaning that no matter what device is connected between points A and B, the current provided by the constant current source is the same.



a. Connect an ideal voltmeter between A and B. Determine the voltage reading in terms of any or all of R and I_s.

Solution

An ideal voltmeter has infinite resistance, so no current flows between A and B. By symmetry, the same current must flow down each leg, so the current in each leg is $I_s/2$.

Assume the potential at the bottom is zero. The potential at A is the same as the junction to the left of A, so

$$V_A = \frac{I_s}{2} 2R = I_s R.$$

The potential at B is found the same way,

$$V_B = \frac{I_s}{2} 4R = 2 I_s R.$$

The difference is

$$V_A - V_B = -I_s R.$$

The sign is not important for scoring purposes.

b. Connect instead an ideal ammeter between A and B. Determine the current in terms of any or all of R and I_s.

Solution

An ideal ammeter has zero resistance, so we just need to find the current through the effective $6R$ resistor that connects the two vertical branches. This current will flow to the left.

By symmetry, the current through each vertical resistance of $2R$ must be the same, as well as the currents through each vertical resistance of $4R$. This gives the system of equations

$$\begin{aligned} I_s &= I_2 + I_4, \\ I_2 &= I_6 + I_4, \\ I_4(4R) &= I_2(2R) + I_6(6R). \end{aligned}$$

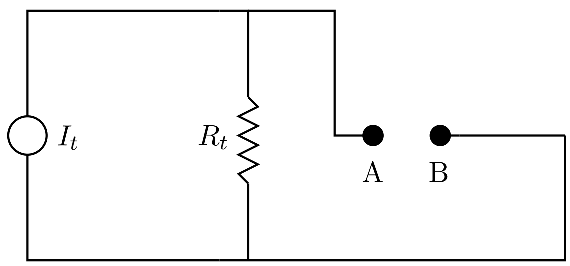
Eliminating I_2 gives

$$\begin{aligned} I_s &= I_6 + 2I_4, \\ 4I_4 &= 2\left(I_6 + I_4\right) + 6I_6. \end{aligned}$$

Finally, eliminating I_4 gives $I_4 = 4I_6$ and

$$I_6 = \frac{1}{9} I_s.$$

c. It turns out that it is possible to replace the above circuit with a new circuit as follows:



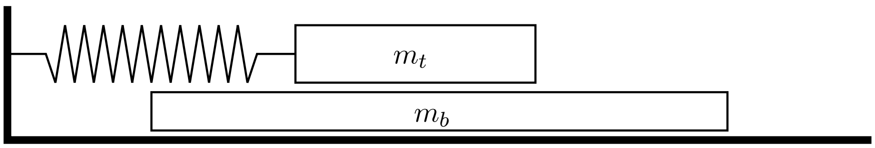
From the point of view of *any* passive resistance that is connected between A and B the circuits are identical. You don't need to prove this statement, but you do need to find I_t and R_t in terms of any or all of R and I_s.

Solution

We can simply use the previous results. If A and B are shorted, all of the current will flow through AB, so $I_t = I_6 = I_s/9$. If the resistance between A and B is infinite, the potential across AB will be $I_s R$, so $R_t = 9R$. The statement we made above is called Norton's theorem.

Question A3

A large block of mass m_b is located on a horizontal frictionless surface. A second block of mass m_t is located on top of the first block; the coefficient of friction (both static and kinetic) between the two blocks is given by μ. All surfaces are horizontal; all motion is effectively one dimensional. A spring with spring constant k is connected to the top block only; the spring obeys Hooke's Law equally in both extension and compression. Assume that the top block never falls off of the bottom block; you may assume that the bottom block is very, very long. The top block is moved a distance A away from the equilibrium position and then released from rest.



a. Depending on the value of A, the motion can be divided into two types: motion that experiences no frictional energy losses and motion that does. Find the value A_c that divides the two motion types. Write your answer in terms of any or all of μ, the acceleration of gravity g, the masses m_t and m_b, and the spring constant k.

Solution

The maximum possible acceleration of the top block without slipping is $m_b a_{\max} = \mu m_t g$. If the top block is *not* slipping then the angular frequency is given by

$$\omega_2 = \sqrt{\frac{k}{m_t + m_b}},$$

so

$$a_{\max} \geq A{\omega_2}^2$$

or

$$A_c = \mu g \frac{m_t}{k}\left(1 + \frac{m_t}{m_b}\right).$$

b. Consider now the scenario $A \gg A_c$. In this scenario the amplitude of the oscillation of the top block as measured against the original equilibrium position will change with time. Determine the magnitude of the change in amplitude, ΔA, after one complete oscillation, as a function of any or all of A, μ, g, and the angular frequency of oscillation of the top block ω_t.

Solution

The energy of an oscillation is approximately equal to

$$E = \frac{1}{2}kA^2.$$

Taking the differential gives the energy loss due to friction,

$$\Delta E = kA\Delta A.$$

If $A \gg A_c$, then the top block has almost completed a complete half cycle before the bottom block catches up with it, so the energy lost in half a cycle is approximately

$$\frac{1}{2}\Delta E = 2Af = 2A\mu m_t g$$

where f is the friction force. Combining,

$$4\mu m_t g = k\Delta A \quad \Rightarrow \quad \Delta A = 4\frac{\mu m_t g}{k} = 4\frac{\mu g}{{\omega_t}^2}.$$

c. Assume still that $A \gg A_c$. What is the maximum speed of the bottom block during the first complete oscillation cycle of the upper block?

Solution

In the limit $A \gg A_c$, the bottom block has little influence on the motion of the top block. The top block oscillates with its usual angular frequency $\omega_t = \sqrt{k/m_t}$, with a negligibly shifted equilibrium point. Thus, the bottom block experiences a leftward acceleration due to friction of $a = \mu g m_t/m_b$ during roughly the first half of the period, and a rightward acceleration during the second half. Its maximum speed occurs halfway through the oscillation cycle,

$$v_b = a\frac{\pi}{\omega_t} = \pi\mu g\frac{m_t}{m_b}\sqrt{\frac{m_t}{k}}.$$

Question A4

A heat engine consists of a moveable piston in a vertical cylinder. The piston is held in place by a removable weight placed on top of the piston, but piston stops prevent the piston from sinking below a certain point. The mass of the piston is $m = 40.0$ kg, the cross sectional area of the piston is $A = 100$ cm^2, and the weight placed on the piston has a mass of $m = 120.0$ kg.

Assume that the region around the cylinder and piston is a vacuum, so you don't need to worry about external atmospheric pressure.

- At point **A** the cylinder volume V_0 is completely filled with liquid water at a temperature $T_0 = 320$ K and a pressure P_{min} that would be just sufficient to lift the piston alone, except the piston has the additional weight placed on top.
- Heat energy is added to the water by placing the entire cylinder in a hot bath.
- At point **B** the piston and weight begins to rise.
- At point **C** the volume of the cylinder reaches V_{max} and the temperature reaches T_{max}. The heat source is removed; the piston stops rising and is locked in place.
- Heat energy is now removed from the water by placing the entire cylinder in a cold bath.
- At point **D** the pressure in the cylinder returns to P_{min}. The added weight is removed; the piston is unlocked and begins to move down.
- The cylinder volume returns to V_0. The cylinder is removed from the cold bath, the weight is placed back on top of the piston, and the cycle repeats.

Because the liquid water can change to gas, there are several important events that take place

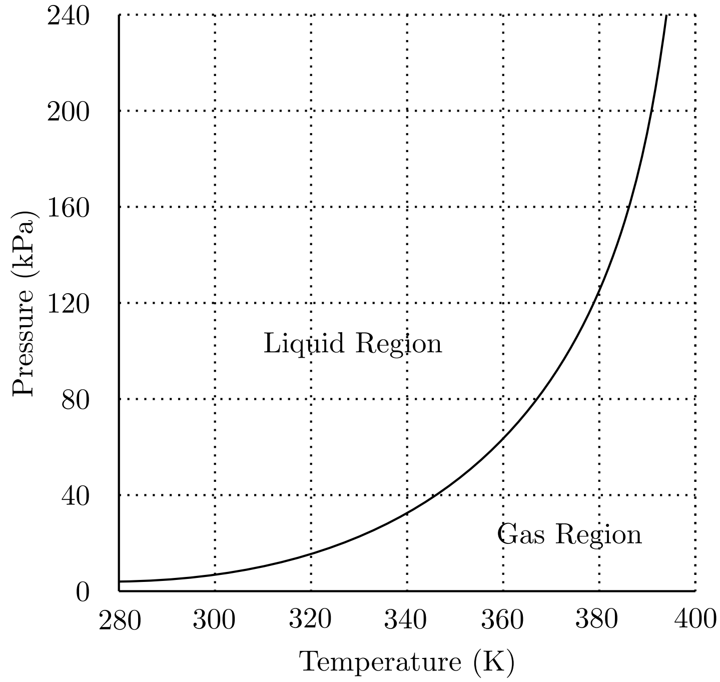
- At point **W** the liquid begins changing to gas.
- At point **X** all of the liquid has changed to gas. This occurs at the same point as point **C** described above.
- At point **Y** the gas begins to change back into liquid.
- At point **Z** all of the gas has changed back into liquid.

When in the liquid state you need to know that for water kept at constant volume, a change in temperature ΔT is related to a change in pressure ΔP according to

$$\Delta P \approx (10^6 \text{ Pa/K})\Delta T$$

When in the gas state you should assume that water behaves like an ideal gas.

Of relevance to this question is the pressure/temperature phase plot for water, showing the regions where water exists in liquid form or gaseous form. The curve shows the coexistence condition, where water can exist simultaneously as gas or liquid.



The following graphs should be drawn on the answer sheet provided.

Solution

Before we get started solving the problem, let's say a bit about where the data in the problem comes from. We are using the Magnus form to approximate the coexistence curve,

$$P = (610.94\text{ Pa})e^{17.625/(1+243.04/T)}$$

where T is measured in centigrade. This is closely related to the result that can be derived from the Clausius-Clapeyron equation for ideal gases,

$$P = P_0 e^{\frac{L}{R}\left(\frac{T-T_0}{TT_0}\right)}$$

where we assume the temperature is low compared to the critical temperature and the latent heat L is a constant.

To get $\Delta P/\Delta T$, we used the cyclic chain rule

$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial T}{\partial V}\right)_P = -1$$

where subscripts indicate what is being held constant. Dropping those for convenience,

$$\frac{\partial P}{\partial T} = \left(-V\frac{\partial P}{\partial V}\right)\left(\frac{1}{V}\frac{\partial V}{\partial T}\right) = \frac{\beta_{VT}}{\beta_{PV}} \approx \frac{(6\times 10^{-4}\text{ K}^{-1})}{(5\times 10^{-10}\text{ Pa}^{-1})} \approx 10^6\text{ Pa/K}.$$

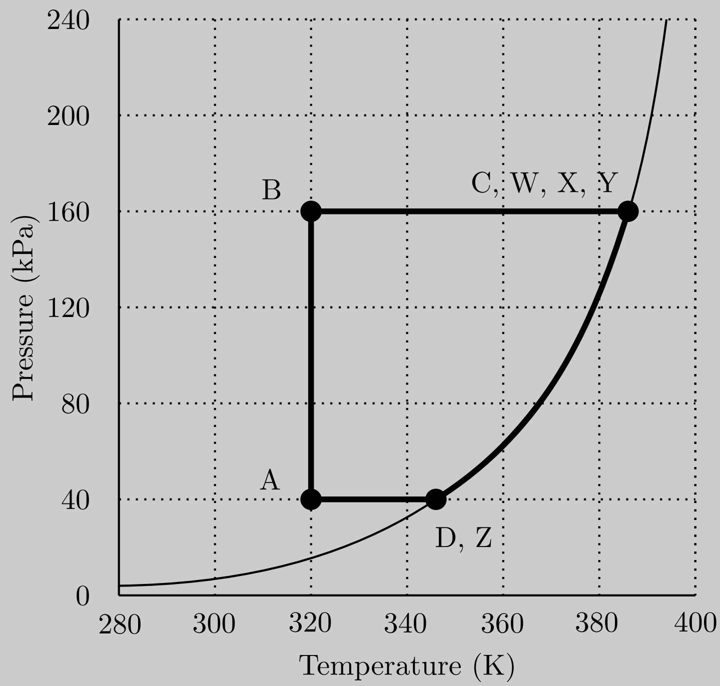
The specific value is not important; the point is that a very small change in the temperature of the liquid in a fixed volume will result in a very large change in the pressure.

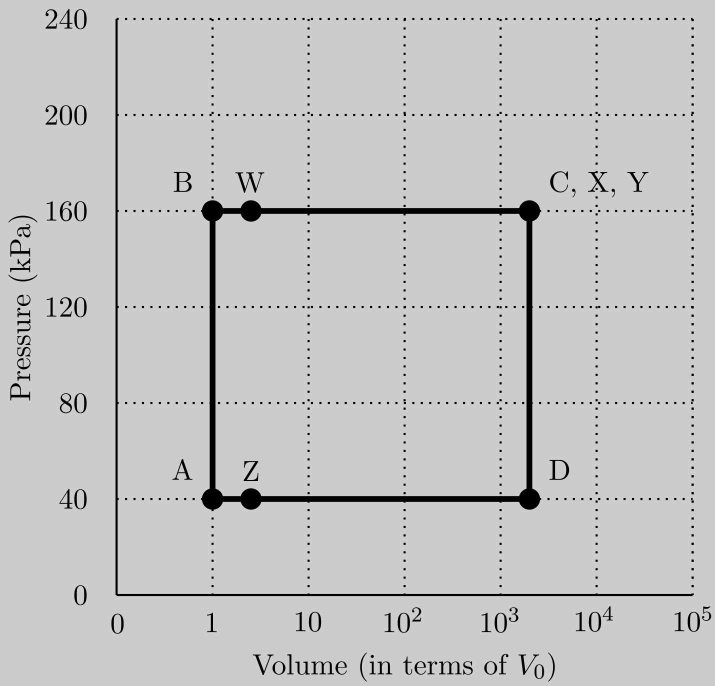
a. Sketch a PT diagram for this cycle on the answer sheet. The coexistence curve for the liquid/gas state is shown. Clearly and accurately label the locations of points **B** through **D** and **W** through **Z** on this cycle.

b. Sketch a PV diagram for this cycle on the answer sheet. You should estimate a reasonable value for V_{max}, note the scale is logarithmic. Clearly and accurately label the locations of points **B** through **D** on this cycle. Provide reasonable approximate locations for points **W** through **Z** on this cycle.

Solution

The correct graphs are shown below.





We start by computing pressures. The minimum pressure is attained when only the piston is to be lifted, so

$$P_{\text{min}} = \frac{F}{A} = \frac{mg}{A} = \frac{(40\text{ kg})(10\text{ m/s}^2)}{(0.01\text{ m}^2)} = 40\text{ kPa}.$$

The maximum pressure is attained when lifting the piston with extra weight,

$$P_{\text{max}} = \frac{F}{A} = \frac{mg}{A} = \frac{(160\text{ kg})(10\text{ m/s}^2)}{(0.01\text{ m}^2)} = 160\text{ kPa}.$$

Point A is clearly at (P_{min}, T_0) on the PT graph.

For liquid water a small temperature increase results in a large pressure increase, so point B is effectively at the same temperature as point A. Process A $\rightarrow$ B is therefore essentially isothermal, and it is also a constant volume process.

Afterwards, the pressure is sufficient to lift the piston and weight, so the volume expands at constant pressure for the process B $\rightarrow$ C. However, liquid cannot change to gas until we reach the coexistence curve. This defines the location of point W. On the PT graph we are "stuck" on the coexistence curve until all the liquid has changed into gas, so X and C are also at the same point.

Upon reaching C the piston is locked in place, fixing the volume, and the cylinder is allowed to cool. To figure out what happens, suppose water vapor obeyed the ideal gas law $PV = nRT$. For a constant volume process, $T \propto P$, so the path would be a straight line towards the origin of the PT diagram. This isn't what happens here, because we run into the coexistence curve, and the pressure is decreased by some of the gas condensing to liquid. Thus during the entire C $\rightarrow$ D process, we follow the coexistence curve downward; it is impossible to cross it until *all* of the vapor condenses.

The next process C $\rightarrow$ D is constant volume, but not isothermal. On the PT graph we follow the coexistence curve to the minimum pressure, at which time the piston is freed and allowed to lower at constant pressure. Since Z is the point where all of the gas has changed to liquid, it must be on the coexistence curve. On the PV diagram it is just to the right of point A.

There are a few numbers needed on the PV diagram that are not given in the problem. You should know from everyday experience that liquid water will only expand slightly when heated over this temperature range, so point W must be very close to B on the PV diagram. You might also know that the density of liquid water is about $2,000$ times the density of water vapor in these conditions. In terms of grading policy, the points C, X, Y, and D may have volumes in the range $[1000V_0, 5000V_0]$ on the PV diagram for full credit, while points W and Z may have volumes in the range $[V_0, 2V_0]$ for full credit and $[2V_0, 5V_0]$ for partial credit.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

This problem is divided into two parts. It is possible to solve these two parts independently, but they are not equally weighted.

a. An ideal rocket when empty of fuel has a mass m_r and will carry a mass of fuel m_f. The fuel burns and is ejected with an exhaust speed of v_e relative to the rocket. The fuel burns at a constant mass rate for a total time T_b. Ignore gravity; assume the rocket is far from any other body.

   i. Determine an equation for the acceleration of the rocket as a function of time t in terms of any or all of t, m_f, m_r, v_e, T_b, and any relevant fundamental constants.

Solution

Since there are no external forces on the system,

$$0 = \frac{dp}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt}$$

which means

$$a = -\frac{1}{m(t)}v_e\frac{dm}{dt} = \frac{v_e}{m_r + m_f(1 - t/T)}\frac{m_f}{T}.$$

   ii. Assuming that the rocket starts from rest, determine the final speed of the rocket in terms of any or all of m_r, m_f, v_e, T_b, and any relevant fundamental constants.

Solution

Rearrange the previous result for

$$\frac{1}{v_e}dv = -\frac{1}{m}dm.$$

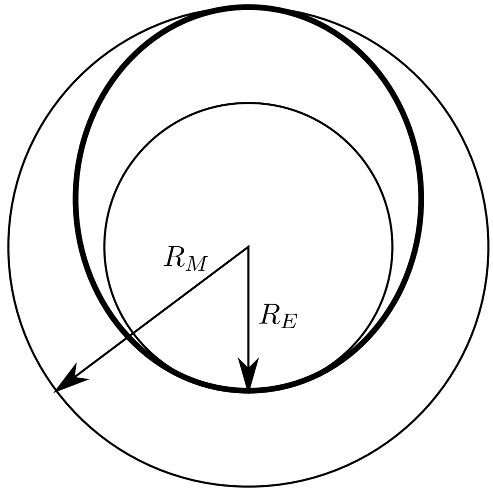
Integrating both sides gives

$$\frac{1}{v_e}v = \ln\left(\frac{m_r + m_f}{m_r}\right) \quad \Rightarrow \quad v = v_e \ln\left(\frac{m_r + m_f}{m_r}\right).$$

This result is called the ideal rocket equation.

b. The ship starts out in a circular orbit around the sun very near the Earth and has a goal of moving to a circular orbit around the Sun that is very close to Mars. It will make this transfer in an elliptical orbit as shown in bold in the diagram below. This is accomplished with an initial velocity boost near the Earth Δv_1 and then a second velocity boost near Mars Δv_2. Assume that both of these boosts are from instantaneous impulses, and ignore mass changes in the rocket as well as gravitational attraction to either Earth or Mars. Don't ignore the

Sun! Assume that the Earth and Mars are both in circular orbits around the Sun of radii R_E and $R_M = R_E/\alpha$ respectively. The orbital speeds are v_E and v_M respectively.



i. Derive an expression for the velocity boost Δv_1 to change the orbit from circular to elliptical. Express your answer in terms of v_E and α.

Solution

First off, for a circular orbit of radius R_c, we have

$$\frac{GM_S}{{R_c}^2} = \frac{{v_c}^2}{R_c}$$

where M_S is the mass of the sun, so

$$v_E = \sqrt{\frac{GM_S}{R_E}}, \quad v_M = \sqrt{\frac{GM_S}{R_M}}.$$

Now consider an elliptical orbit with minimum radius R_1 and maximum radius R_2. Energy and angular momentum give

$$\frac{1}{2}v^2 - \frac{GM_S}{r} = E, \quad v_1 R_1 = v_2 R_2.$$

Combining and eliminating v_2,

$$\frac{1}{2}{v_1}^2 - \frac{GM_S}{R_1} = \frac{1}{2}{v_1}^2\left(\frac{R_1}{R_2}\right)^2 - \frac{GM_S}{R_2}$$

which can be solved for v_1,

$$\frac{1}{2}{v_1}^2\left(1 - \left(\frac{R_1}{R_2}\right)^2\right) = GM_S\frac{R_2 - R_1}{R_1 R_2}.$$

Setting $R_1 = R_E$ and $R_2 = R_M$, we have $\alpha = R_1/R_2$, so

$$\frac{1}{2}{v_1}^2\left(1 - \alpha^2\right) = \frac{GM_S}{R_1}\left(1 - \alpha\right) \quad \Rightarrow \quad v_1 = v_E\sqrt{\frac{2}{1+\alpha}}.$$

As expected, this is greater than v_E, and the boost is

$$\Delta v_1 = v_E \left(\sqrt{\frac{2}{1+\alpha}} - 1 \right)$$

ii. Derive an expression for the velocity boost Δv_2 to change the orbit from elliptical to circular. Express your answer in terms of v_E and α.

Solution

This is similar to the previous part, except we now eliminate v_1,

$$\frac{1}{2} {v_2}^2 \left(1 - (1/\alpha)^2\right) = \frac{GM_S}{R_2} \left(1 - (1/\alpha)\right) \quad \Rightarrow \quad v_2 = v_M \sqrt{\frac{2}{1+1/\alpha}}.$$

This is *less* than v_M, so the rocket must receive a second positive boost,

$$\Delta v_2 = v_M \left(1 - \sqrt{\frac{2}{1+1/\alpha}}\right) = v_E \sqrt{\alpha} \left(1 - \sqrt{\frac{2}{1+1/\alpha}}\right).$$

where we used $v_M = v_E \sqrt{\alpha}$.

iii. What is the angular separation between Earth and Mars, as measured from the Sun, at the time of launch so that the rocket will start from Earth and arrive at Mars when it reaches the orbit of Mars? Express your answer in terms of α.

Solution

Kepler's third law gives the time for the orbital transfer,

$$\frac{T}{T_M} = \frac{1}{2} \left(\frac{(R_E + R_M)/2}{R_M} \right)^{3/2} = \frac{1}{2} \left(\frac{\alpha + 1}{2} \right)^{3/2}.$$

During this time Mars moves through an angle of

$$2\pi \frac{T}{T_M} = \pi \left(\frac{\alpha+1}{2} \right)^{3/2}$$

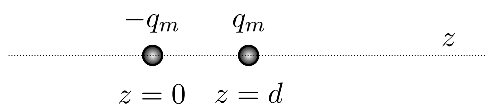
while the rocket moves through an angle of π, so the angular separation from Earth will be

$$\theta = \pi \left(1 - \left(\frac{\alpha+1}{2} \right)^{3/2} \right).$$

Question B2

The nature of magnetic dipoles.

a. A "Gilbert" dipole consists of a pair of magnetic monopoles each with a magnitude q_m but opposite magnetic charges separated by a distance d, where d is small. In this case, assume that $-q_m$ is located at $z = 0$ and $+q_m$ is located at $z = d$.



Assume that magnetic monopoles behave like electric monopoles according to a coulomb-like force

$$F = \frac{\mu_0}{4\pi}\frac{q_{m1}q_{m2}}{r^2}$$

and the magnetic field obeys

$$B = F/q_m.$$

i. What are the dimensions of the quantity q_m?

Solution

By the second expression, q_m must be measured in Newtons per Tesla. But since Tesla are also Newtons per Ampere per meter, then q_m is also measured in Ampere meters.

ii. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > d$. Write your answer in terms of q_m, d, z, and any necessary fundamental constants.

Solution

Adding the two terms,

$$B(z) = -\frac{\mu_0}{4\pi}\frac{q_m}{z^2} + \frac{\mu_0}{4\pi}\frac{q_m}{(z-d)^2} = \frac{\mu_0 q_m}{4\pi}\left(\frac{1}{(z-d)^2} - \frac{1}{z^2}\right).$$

iii. Evaluate this expression in the limit as $d \to 0$, assuming that the product $q_m d = p_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of p_m, z, and any necessary fundamental constants.

Solution

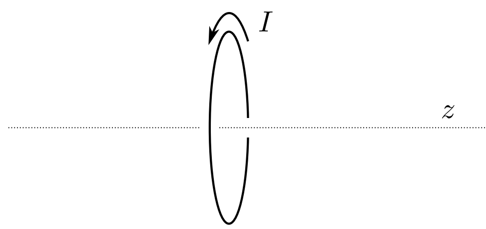
By combining the fractions,

$$B(z) = \frac{\mu_0 q_m}{4\pi}\frac{z^2 - (z-d)^2}{z^2(z-d)^2} = \frac{\mu_0 q_m}{4\pi}\frac{2zd - d^2}{z^2(z-d)^2}.$$

In the limit $d \to 0$, the d^2 term in the numerator can be neglected, and the denominator can be approximated as z^4, giving

$$B(z) = \frac{\mu_0 q_m d}{2\pi z^3} = \frac{\mu_0 p_m}{2\pi z^3}.$$

b. An "Ampère" dipole is a magnetic dipole produced by a current loop I around a circle of radius r, where r is small. Assume the that the z axis is the axis of rotational symmetry for the circular loop, and the loop lies in the xy plane at $z = 0$.



i. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > 0$. Write your answer in terms of I, r, z, and any necessary fundamental constants.

Solution

Applying the Biot-Savart law, with $\mathbf{s}$ the vector from the point on the loop to the point on the z axis,

$$B(z) = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l} \times \mathbf{s}}{s^3} = \frac{\mu_0 I}{4\pi} \frac{2\pi r}{r^2 + z^2} \sin\theta$$

where θ is the angle between the point on the loop and the center of the loop as measured by the point on the z axis, so

$$\sin\theta = \frac{r}{\sqrt{r^2 + z^2}}.$$

Then we have

$$B(z) = \frac{\mu_0 I}{4\pi} \frac{2\pi r^2}{(r^2 + z^2)^{3/2}}.$$

ii. Let kIr^γ have dimensions equal to that of the quantity p_m defined above in Part aiii, where k and γ are dimensionless constants. Determine the value of γ.

Solution

We know p_m must have dimensions of Amperes times meters squared, so $\gamma = 2$.

iii. Evaluate the expression in Part bi in the limit as $r \to 0$, assuming that the product $kIr^\gamma = p'_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of k, p'_m, z, and any necessary fundamental constants.

Solution

Using our previous result,

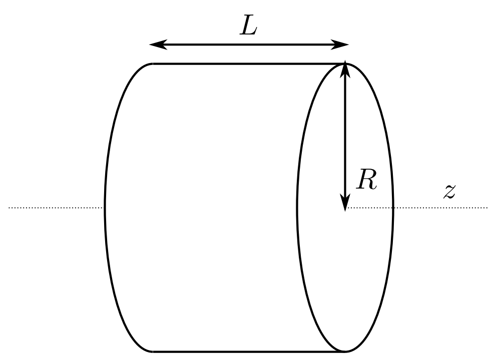
$$B(z) = \frac{\mu_0 I}{4\pi} \frac{2\pi r^2}{(r^2+z^2)^{3/2}} \approx \frac{\mu_0 I}{2\pi} \frac{\pi r^2}{z^3} = \frac{\mu_0}{2\pi} \frac{\pi}{k} \frac{p'_m}{z^3}$$

iv. Assuming that the two approaches are equivalent, $p_m = p'_m$. Determine the constant k in Part bii.

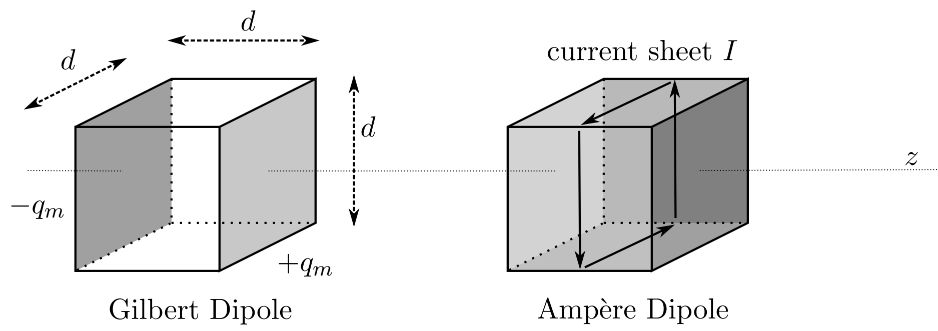
Solution

By inspection, $k = \pi$.

c. Now we try to compare the two approaches if we model a physical magnet as being composed of densely packed microscopic dipoles.



A cylinder of this uniform magnetic material has a radius R and a length L. It is composed of N magnetic dipoles that could be either all Ampère type or all Gilbert type. N is a *very* large number. The axis of rotation of the cylinder and all of the dipoles are all aligned with the z axis and all point in the same direction as defined above so that the magnetic field *outside* the cylinder is the same in either dipole case as you previously determined. Below is a picture of the two dipole models; they are cubes of side $d << R$ and $d << L$ with volume $v_m = d^3$.



i. Assume that $R \gg L$ and only Gilbert type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.

Solution

The monopoles that make up the dipoles cancel out except on the flat surfaces. Then the cylinder acts like a parallel plate capacitor.

If the size of a dipole is d, then the surface density of monopole charge is

$$\sigma_m = q_m/d^2.$$

Using the analogy with a parallel place capacitor, the magnitude of B is

$$B = \mu_0 \sigma_m = \mu_0 \frac{p_m}{d^3}$$

and the direction is to the left.

ii. Assume that $R \ll L$ and only Ampère type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.

Solution

The currents that make up the dipoles all cancel out except on the cylindrical surfaces. Then the cylinder acts like a solenoid, with

$$B = \frac{\mu_0 I}{d}$$

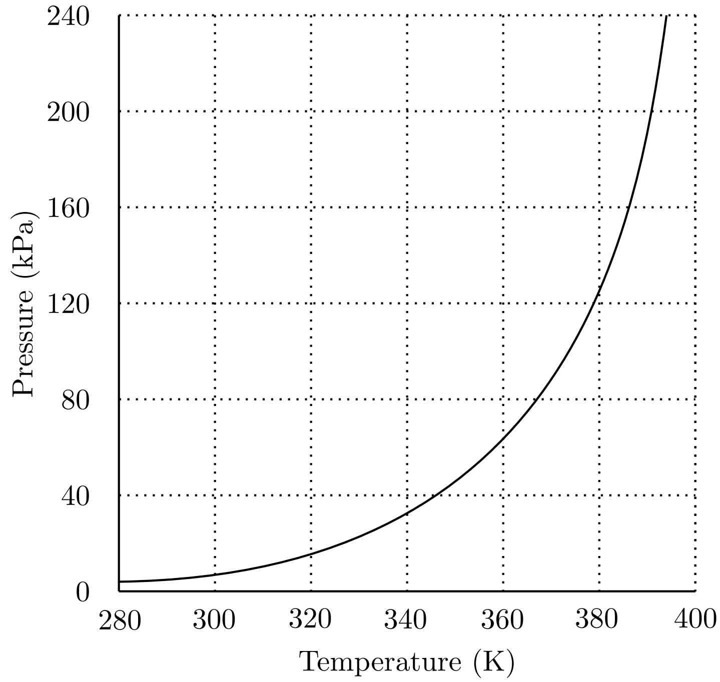
where I/d is the surface current density. The magnitude of B is

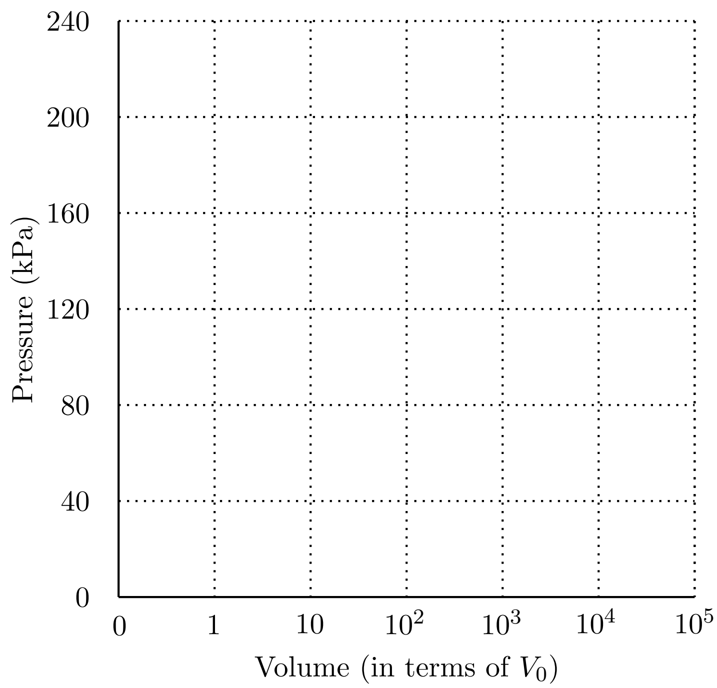
$$B = \frac{\mu_0 I}{d} = \mu_0 \frac{p_m}{d^3}$$

and the direction is to the right.

Answer Sheets

Following are answer sheets for some of the graphical portions of the test.


240
200
160
120
80
40
0
Pressure (kPa)
280
300
320
340
360
380
400
Temperature (K)


240
200
160
120
80
40
0
Pressure (kPa)
0
1
10
10^2
10^3
10^4
10^5
Volume (in terms of V_0)

AAPT UNITED STATES PHYSICS TEAM
AIP 2016

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has four questions and is allowed 90 minutes; Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 4 parts: the cover sheet (page 2), Part A (pages 3-6), Part B (pages 8-9), and several answer sheets for one of the questions in Part A (pages 11-12). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
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AAPT UNITED STATES PHYSICS TEAM
AIP 2016

USA Physics Olympiad Exam

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- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  AAPT ID #

  Doe, Jamie

  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
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Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}$ C
$1\mathrm{eV} = 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

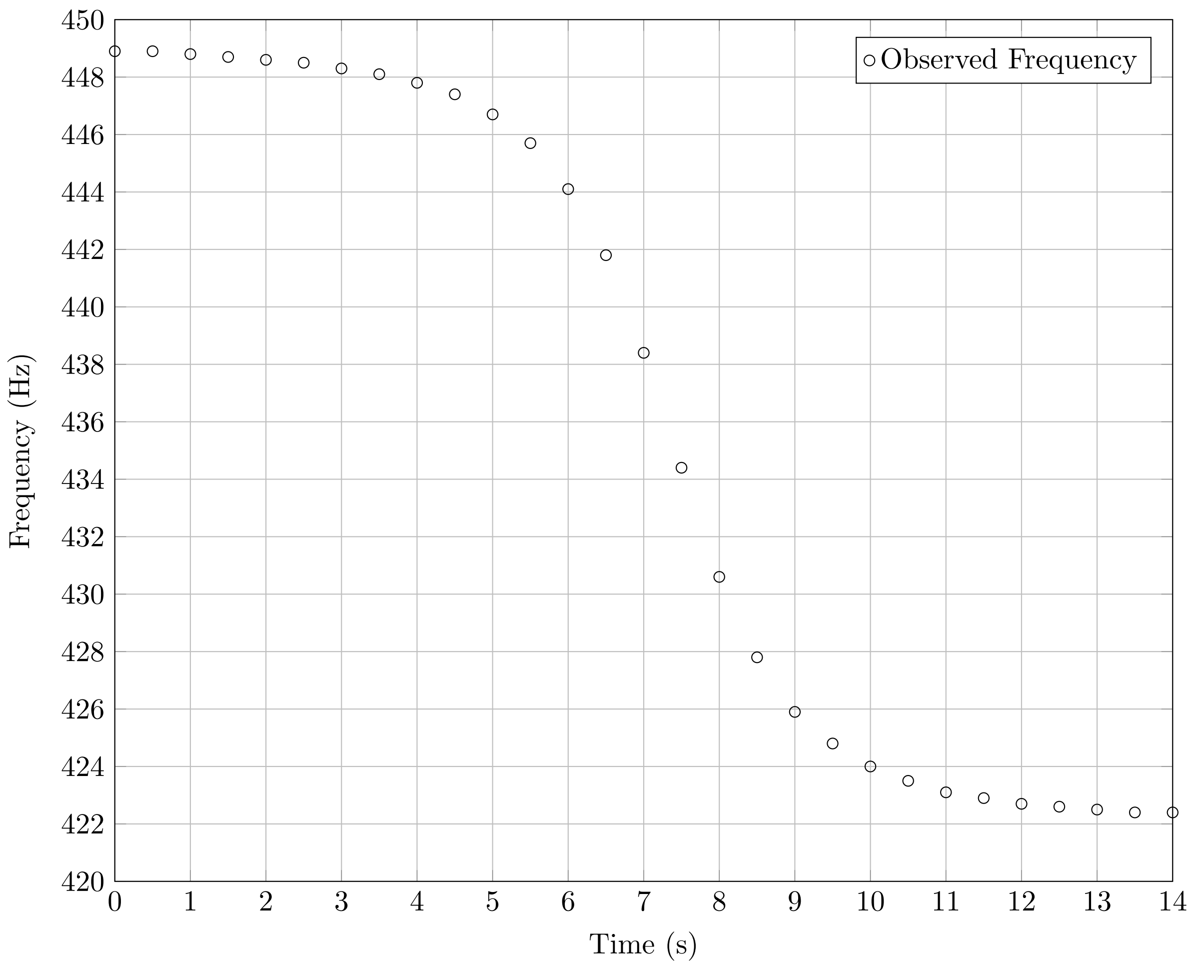
Question A1

The Doppler effect for a source moving relative to a stationary observer is described by

$$f = \frac{f_0}{1 - (v/c)\cos\theta}$$

where f is the frequency measured by the observer, f_0 is the frequency emitted by the source, v is the speed of the source, c is the wave speed, and θ is the angle between the source velocity and the line between the source and observer. (Thus $\theta = 0$ when the source is moving directly towards the observer and $\theta = \pi$ when moving directly away.)

A sound source of constant frequency travels at a constant velocity past an observer, and the observed frequency is plotted as a function of time:



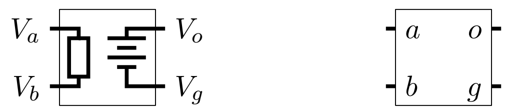
The experiment happens in room temperature air, so the speed of sound is 340 m/s.

a. What is the speed of the source?

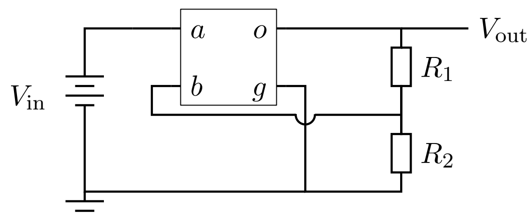
b. What is the smallest distance between the source and the observer?

Question A2

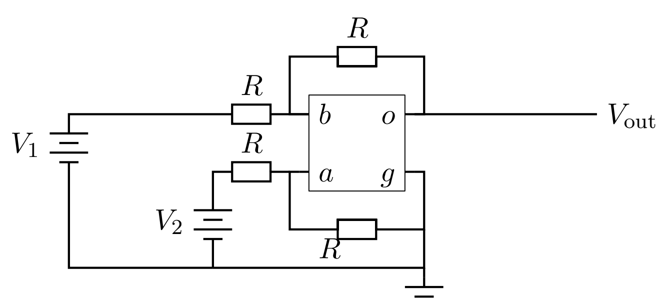
A simple integrated circuit device has two inputs, V_a and V_b, and two outputs, V_o and V_g. The inputs are effectively connected internally to a single resistor with effectively infinite resistance. The outputs are effectively connected internally to a perfect source of emf $\mathcal{E}$. The integrated circuit is configured so that $\mathcal{E} = G(V_a - V_b)$, where G is positive and very large. (In your answers below, you may neglect terms suppressed by $1/G$.) On the left is an internal schematic for the device; on the right is the symbol that is used in circuit diagrams.



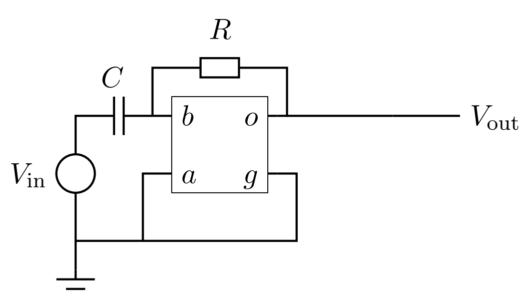
a. Consider the following circuit. $R_1 = 8.2\ k\Omega$ and $R_2 = 560\ \Omega$ are two resistors. Terminal g and the negative side of V_{in} are connected to ground, so both are at a potential of 0 volts. Determine the ratio $V_{\text{out}}/V_{\text{in}}$.



b. Consider the circuit below, where all four resistors have resistance R. Note that the positions of terminals a and b have been switched. Determine V_{out} in terms of V_1, V_2, and R.



c. Now consider a circuit with a capacitor C and a resistor R with time constant $RC = \tau$, where $V_{\text{in}}(t)$ smoothly varies in time. Determine V_{out} in terms of $V_{\text{in}}(t)$ and τ .



Question A3

Throughout this problem the inertial rest frame of the rod will be referred to as the rod's frame, while the inertial frame of the cylinder will be referred to as the cylinder's frame.

A rod is traveling at a constant speed of $v = \frac{4}{5}c$ to the right relative to a hollow cylinder. The rod passes through the cylinder, and then out the other side. The left end of the rod aligns with the left end of the cylinder at time $t = 0$ and $x = 0$ in the **cylinder's frame** and time $t' = 0$ and $x' = 0$ in the **rod's frame**.

The left end of the rod aligns with the left end of the cylinder at the same time as the right end of the rod aligns with the right end of the cylinder in the **cylinder's frame**; in this reference frame the length of the cylinder is 15 m.

For the following, sketch accurate, scale diagrams of the motions of the ends of the rod and the cylinder on the graphs provided. The horizontal axis corresponds to x, the vertical axis corresponds to ct, where c is the speed of light. Both the vertical and horizontal gridlines have 5.0 meter spacing.

a. Sketch the world lines of the left end of the rod (RL), left end of the cylinder (CL), right end of the rod (RR), and right end of the cylinder (CR) in the **cylinder's frame**.

b. Do the same in the **rod's frame**.

c. On **both** diagrams clearly indicate the following four events by the letters A, B, C, and D.

   A: The left end of the rod is at the same point as the left end of the cylinder

   B: The right end of the rod is at the same point as the right end of the cylinder

   C: The left end of the rod is at the same point as the right end of the cylinder

   D: The right end of the rod is at the same point as the left end of the cylinder

d. At event B a small particle P is emitted that travels to the left at a constant speed $v_P = \frac{4}{5}c$ in the **cylinder's frame**.

   i. Sketch the world line of P in the **cylinder's frame.**

   ii. Sketch the world line of P in the **rod's frame.**

e. Now consider the following in the **cylinder's frame**. The right end of the rod stops instantaneously at event B and emits a flash of light, and the left end of the rod stops instantaneously when the light reaches it. Determine the final length of the rod after it has stopped. You can assume the rod compresses uniformly with no other deformation.

Any computation that you do must be shown on a separate sheet of paper, and not on the graphs. Graphical work that does not have supporting computation might not receive full credit.

Question A4

The flow of heat through a material can be described via the *thermal conductivity* κ. If the two faces of a slab of material with thermal conductivity κ, area A, and thickness d are held at temperatures differing by ΔT, the thermal power P transferred through the slab is

$$P = \frac{\kappa A \Delta T}{d}$$

A large, flat lake in the upper Midwest has a uniform depth of 5.0 meters of water that is covered by a uniform layer of 1.0 cm of ice. Cold air has moved into the region so that the upper surface of the ice is now maintained at a constant temperature of $-10\ ^\circ\mathrm{C}$ by the cold air (an infinitely large constant temperature heat sink). The bottom of the lake remains at a fixed $4.0\ ^\circ\mathrm{C}$ because of contact with the earth (an infinitely large constant temperature heat source). It is reasonable to assume that heat flow is only in the vertical direction and that there is no convective motion in the water.

a. Determine the initial rate of change in ice thickness.

b. Assuming the air stays at the same temperature for a long time, find the equilibrium thickness of the ice.

c. Explain why convective motion can be ignored in the water.

Some important quantities for this problem:

Specific heat capacity of water	C_{water}	$4200\ \mathrm{J/(kg \cdot C^\circ)}$
Specific heat capacity of ice	C_{ice}	$2100\ \mathrm{J/(kg \cdot C^\circ)}$
Thermal conductivity of water	κ_{water}	$0.57\ \mathrm{W/(m \cdot C^\circ)}$
Thermal conductivity of ice	κ_{ice}	$2.2\ \mathrm{W/(m \cdot C^\circ)}$
Latent heat of fusion for water	L_{f}	$330,000\ \mathrm{J/kg}$
Density of water	ρ_{water}	$999\ \mathrm{kg/m^3}$
Density of ice	ρ_{ice}	$920\ \mathrm{kg/m^3}$

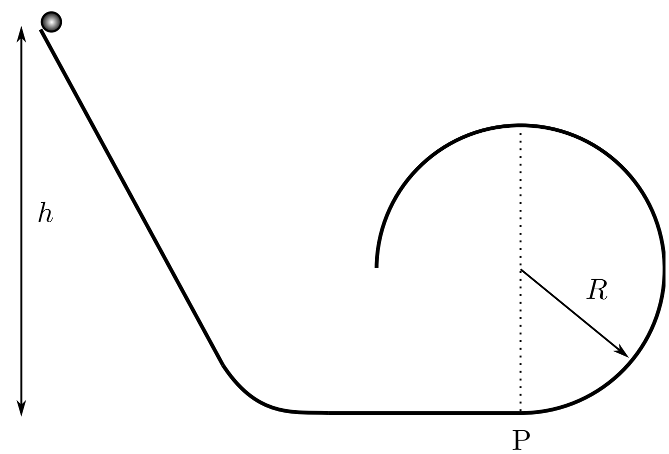
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

A uniform solid spherical ball starts from rest on a loop-the-loop track. It rolls without slipping along the track. However, it does not have enough speed to make it to the top of the loop. From what height h would the ball need to start in order to land at point P directly underneath the top of the loop? Express your answer in terms of R, the radius of the loop. Assume that the radius of the ball is very small compared to the radius of the loop, and that there are no energy losses due to friction.



Question B2

a. A spherical region of space of radius R has a uniform charge density and total charge $+Q$. An electron of charge $-e$ is free to move inside or outside the sphere, under the influence of the charge density alone. For this first part ignore radiation effects.

  i. Consider a circular orbit for the electron where $r < R$. Determine the period of the orbit T in terms of any or all of r, R, Q, e, and any necessary fundamental constants.

  ii. Consider a circular orbit for the electron where $r > R$. Determine the period of the orbit T in terms of any or all of r, R, Q, e, and any necessary fundamental constants.

  iii. Assume the electron starts at rest at $r = 2R$. Determine the speed of the electron when it passes through the center in terms of any or all of R, Q, e, and any necessary fundamental constants.

b. Accelerating charges radiate. The total power P radiated by charge q with acceleration a is given by

$$P = C\xi a^n$$

where C is a dimensionless numerical constant (which is equal to $1/6\pi$), ξ is a physical constant that is a function only of the charge q, the speed of light c, and the permittivity of free space ϵ_0, and n is a dimensionless constant. Determine ξ and n.

c. Consider the electron in the first part, except now take into account radiation. Assume that the orbit remains circular and the orbital radius r changes by an amount $|\Delta r| \ll r$.

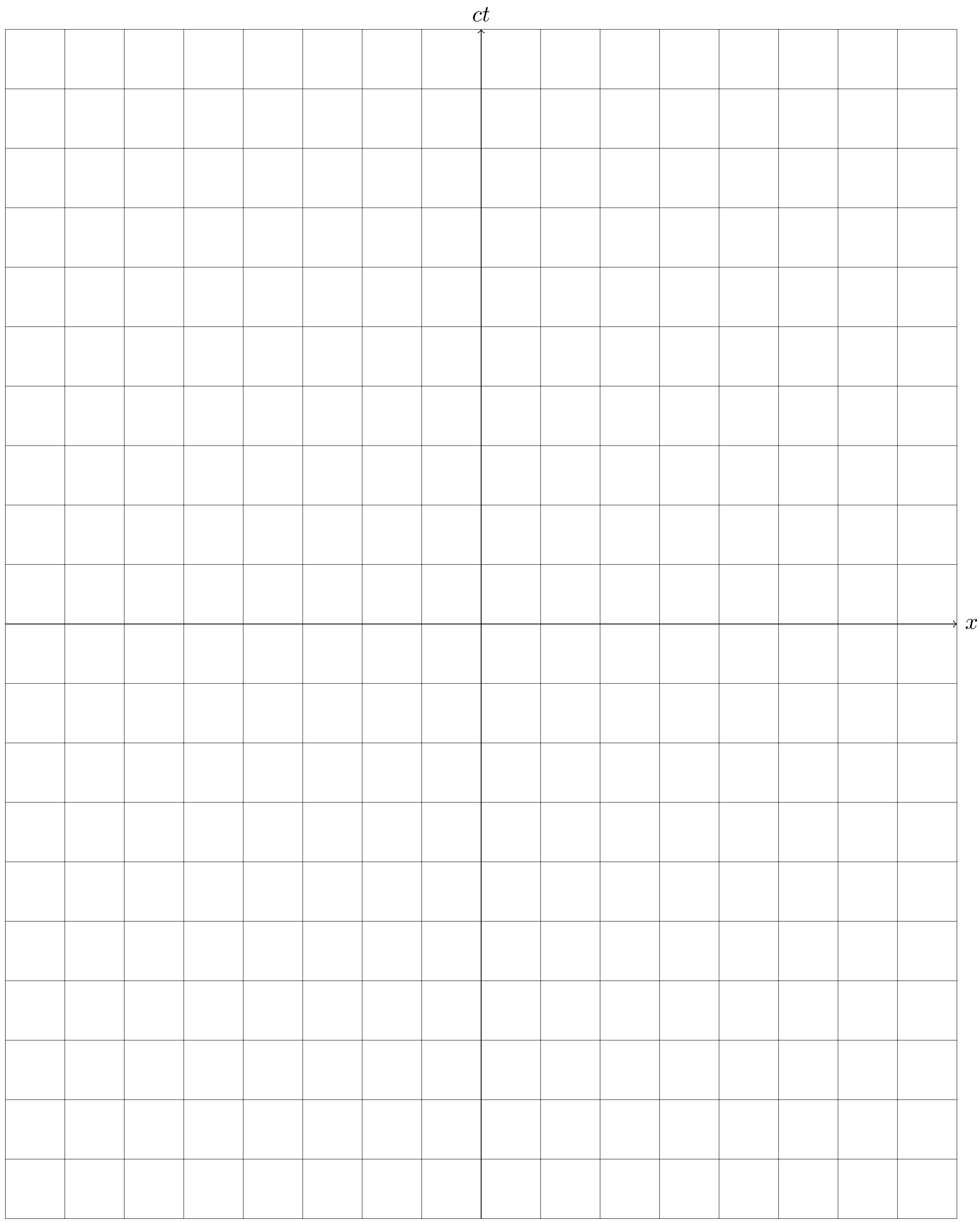
  i. Consider a circular orbit for the electron where $r < R$. Determine the change in the orbital radius Δr during one orbit in terms of any or all of r, R, Q, e, and any necessary fundamental constants. Be very specific about the sign of Δr.

  ii. Consider a circular orbit for the electron where $r > R$. Determine the change in the orbital radius Δr during one orbit in terms of any or all of r, R, Q, e, and any necessary fundamental constants. Be very specific about the sign of Δr.

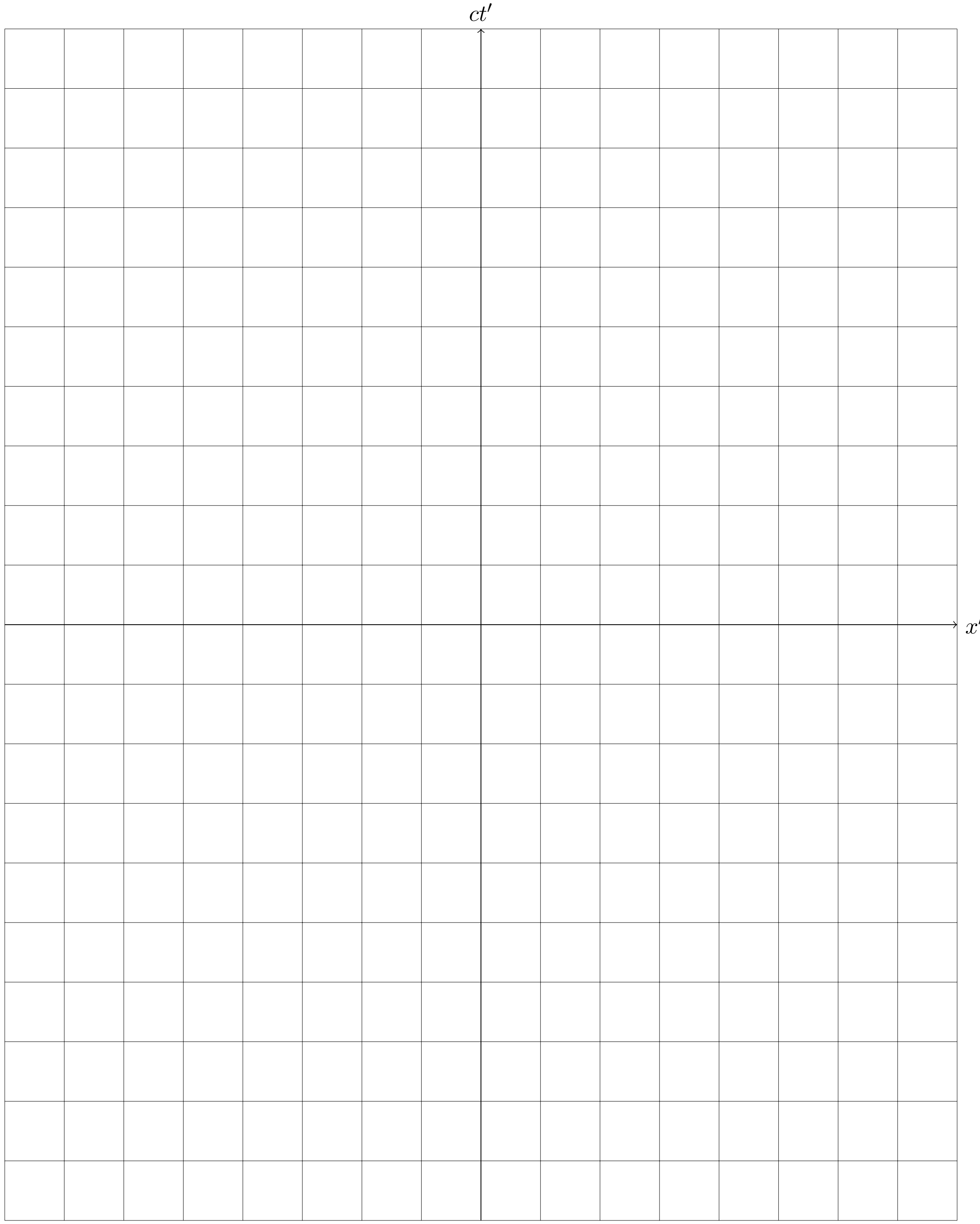
Answer Sheets

Following are answer sheets for some of the graphical portions of the test.

The Cylinder's Frame



The Rod's Frame



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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has four questions and is allowed 90 minutes; Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
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  Doe, Jamie

  A1 - 1/3

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$c = 3.00 \times 10^8\ \mathrm{m/s}$	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J/K}$
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$1\mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
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Part A

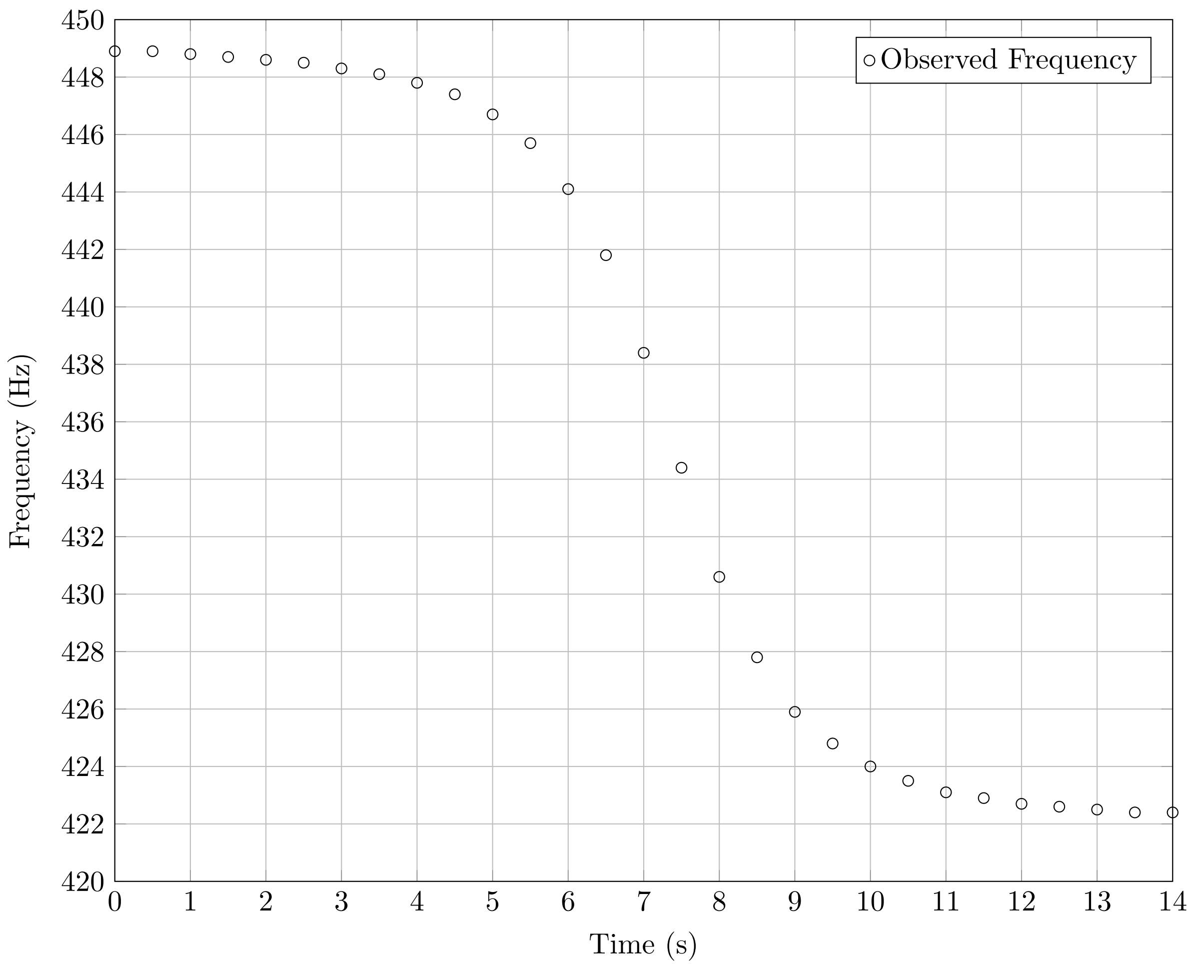
Question A1

The Doppler effect for a source moving relative to a stationary observer is described by

$$f = \frac{f_0}{1 - (v/c)\cos\theta}$$

where f is the frequency measured by the observer, f_0 is the frequency emitted by the source, v is the speed of the source, c is the wave speed, and θ is the angle between the source velocity and the line between the source and observer. (Thus $\theta = 0$ when the source is moving directly towards the observer and $\theta = \pi$ when moving directly away.)

A sound source of constant frequency travels at a constant velocity past an observer, and the observed frequency is plotted as a function of time:



The experiment happens in room temperature air, so the speed of sound is 340 m/s.

a. What is the speed of the source?

Solution

For $\theta = 0$ we have
$$f_a \approx f_0/(1 - v/c)$$
and for $\theta = \pi$,
$$f_b = f_0/(1 + v/c).$$
Read f_a and f_b off the early and late time portions of the graph and use
$$f_a/f_b = (1 + v/c)/(1 - v/c)$$
giving an answer of $v = 10.7$ m/s.

Alternatively, we can see that $v \ll c$ and approximate
$$f_a/f_b \approx 1 + 2v/c$$
which makes the calculation of v slightly faster. This is acceptable because the error terms are of order $(v/c)^2 \sim 0.1\%$.

b. What is the smallest distance between the source and the observer?

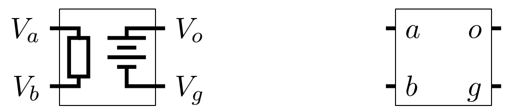
Solution

Let d be the (fixed) distance between the observer and the path of the source; let x be the displacement along the path, with $x = 0$ at closest approach. Then for $|x| \ll d$,
$$\cos\theta \approx \cot\theta = x/d$$
so we have
$$f = f_0/(1 - (v/c)(x/d)) \approx f_0(1 + (v/c)(x/d)).$$
Taking the time derivative, and noting that x' is simply v,
$$f' = f_0(v^2/c)d$$
Therefore we can read f' off the center region of the graph. We still need to find f_0, which we can do using our result from part (a) or simply by averaging f_a and f_b, since $v \ll c$, giving $f_0 = 435$ Hz and an answer of $d = 17.8$ m.

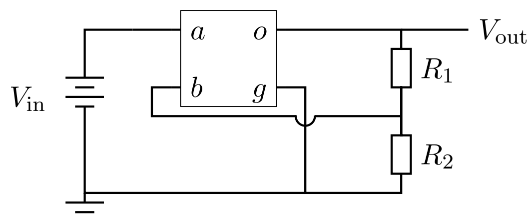
There's also a nice trick to speed up this computation. Draw lines at the asymptotic values and through the central data points. The two horizontal lines are $2f_0(v/c)$ apart in frequency, so the time between their intersections with the third line is simply $2d/v$.

Question A2

A simple integrated circuit device has two inputs, V_a and V_b, and two outputs, V_o and V_g. The inputs are effectively connected internally to a single resistor with effectively infinite resistance. The outputs are effectively connected internally to a perfect source of emf $\mathcal{E}$. The integrated circuit is configured so that $\mathcal{E} = G(V_a - V_b)$, where G is positive and very large. (In your answers below, you may neglect terms suppressed by $1/G$.) On the left is an internal schematic for the device; on the right is the symbol that is used in circuit diagrams.



a. Consider the following circuit. $R_1 = 8.2\ k\Omega$ and $R_2 = 560\ \Omega$ are two resistors. Terminal g and the negative side of V_{in} are connected to ground, so both are at a potential of 0 volts. Determine the ratio $V_{\text{out}}/V_{\text{in}}$.



Solution

Since terminal g is grounded, $V_g = 0$ and $V_a = V_{\text{in}}$, so $V_{\text{out}} = G(V_{\text{in}} - V_b)$. No current runs between a and b, so any current through R_1 also flows through R_2. Then Ohm's law gives

$$\frac{V_b}{R_2} = \frac{V_{\text{out}}}{R_1 + R_2} \quad \Rightarrow \quad V_{\text{out}} = G\left(V_{\text{in}} - V_{\text{out}}\frac{R_2}{R_1 + R_2}\right)$$

and solving for V_{out} gives

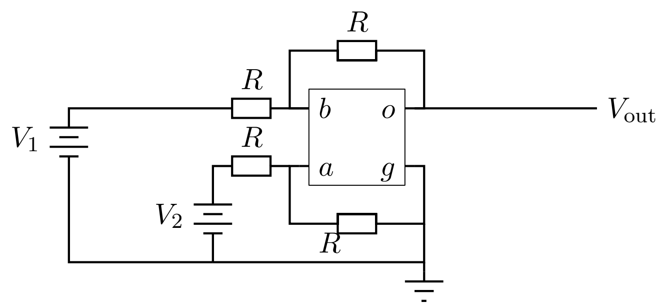
$$V_{\text{out}} = \frac{V_{\text{in}}}{\frac{1}{G} + \frac{R_2}{R_1+R_2}}.$$

Since $1/G$ is negligibly small, our final answer is

$$\frac{V_{\text{out}}}{V_{\text{in}}} \approx \frac{R_1 + R_2}{R_2}.$$

This circuit is an amplifier with feedback. In case you're wondering, the four-terminal circuit component used in this problem is called an operational amplifier, or "op amp".

b. Consider the circuit below, where all four resistors have resistance R. Note that the positions of terminals a and b have been switched. Determine V_{out} in terms of V_1, V_2, and R.



Solution

If we handled this part like part (a), accounting explicitly for G, then we would run into some complicated algebra. Instead, we notice that in the previous part, the solution for V_{out} sets $V_a \approx V_b$. The reason is that the circuit amplifies $V_a - V_b$ by the huge factor G, but its output *isn't* huge, which means the circuit has to drive $V_a - V_b$ to zero. Concretely, this occurs by negative feedback: if one slightly increases V_a, then one will dramatically increase V_o, which would then cause V_b to increase as well. Thus, $V_a \approx V_b$ is a stable equilibrium.

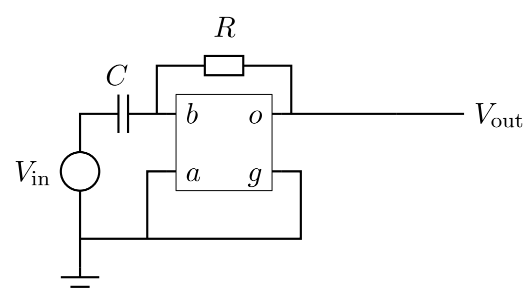
In general, such negative feedback will occur when the output is connected back to the input b. (If the output is instead connected back to a, then we get positive feedback instead, and the circuit is unstable.) And in this circuit, the output is indeed connected back to b, so we can simply assume $V_a \approx V_b$.

Now again $V_g = 0$, and if current I flows through the bottom resistor (below the a and g terminals) then $V_a = V_2/2$, since the voltage drop across the bottom two resistors is equal. Similarly, the voltage drop across the top two resistors is equal, so $V_1 + V_{\text{out}} = 2V_b$. Then

$$V_{\text{out}} = 2V_b - V_1 \approx 2V_a - V_1 = V_2 - V_1.$$

This circuit is a subtractor.

c. Now consider a circuit with a capacitor C and a resistor R with time constant $RC = \tau$, where $V_{\text{in}}(t)$ smoothly varies in time. Determine V_{out} in terms of $V_{\text{in}}(t)$ and τ .



Solution

Once again, this circuit is set up with negative feedback, which drives $V_b \approx V_a = 0$. Then the capacitor charge and current satisfy

$$Q = CV_{\text{in}}, \quad \dot{Q} = -\frac{V_{\text{out}}}{R}$$

where the second result follows from Ohm's law. Then

$$\frac{V_{\text{out}}}{R} = -C\frac{dV_{\text{in}}}{dt} \quad \Rightarrow \quad V_{\text{out}} = -\tau\frac{dV_{\text{in}}}{dt}.$$

This circuit is a differentiator.

Question A3

Throughout this problem the inertial rest frame of the rod will be referred to as the rod's frame, while the inertial frame of the cylinder will be referred to as the cylinder's frame.

A rod is traveling at a constant speed of $v = \frac{4}{5}c$ to the right relative to a hollow cylinder. The rod passes through the cylinder, and then out the other side. The left end of the rod aligns with the left end of the cylinder at time $t = 0$ and $x = 0$ in the **cylinder's frame** and time $t' = 0$ and $x' = 0$ in the **rod's frame**.

The left end of the rod aligns with the left end of the cylinder at the same time as the right end of the rod aligns with the right end of the cylinder in the **cylinder's frame**; in this reference frame the length of the cylinder is 15 m.

For the following, sketch accurate, scale diagrams of the motions of the ends of the rod and the cylinder on the graphs provided. The horizontal axis corresponds to x, the vertical axis corresponds to ct, where c is the speed of light. Both the vertical and horizontal gridlines have 5.0 meter spacing.

a. Sketch the world lines of the left end of the rod (RL), left end of the cylinder (CL), right end of the rod (RR), and right end of the cylinder (CR) in the **cylinder's frame**.

b. Do the same in the **rod's frame**.

c. On **both** diagrams clearly indicate the following four events by the letters A, B, C, and D.

A: The left end of the rod is at the same point as the left end of the cylinder

B: The right end of the rod is at the same point as the right end of the cylinder

C: The left end of the rod is at the same point as the right end of the cylinder

D: The right end of the rod is at the same point as the left end of the cylinder

d. At event B a small particle P is emitted that travels to the left at a constant speed $v_P = \frac{4}{5}c$ in the **cylinder's frame**.

i. Sketch the world line of P in the **cylinder's frame.**

ii. Sketch the world line of P in the **rod's frame.**

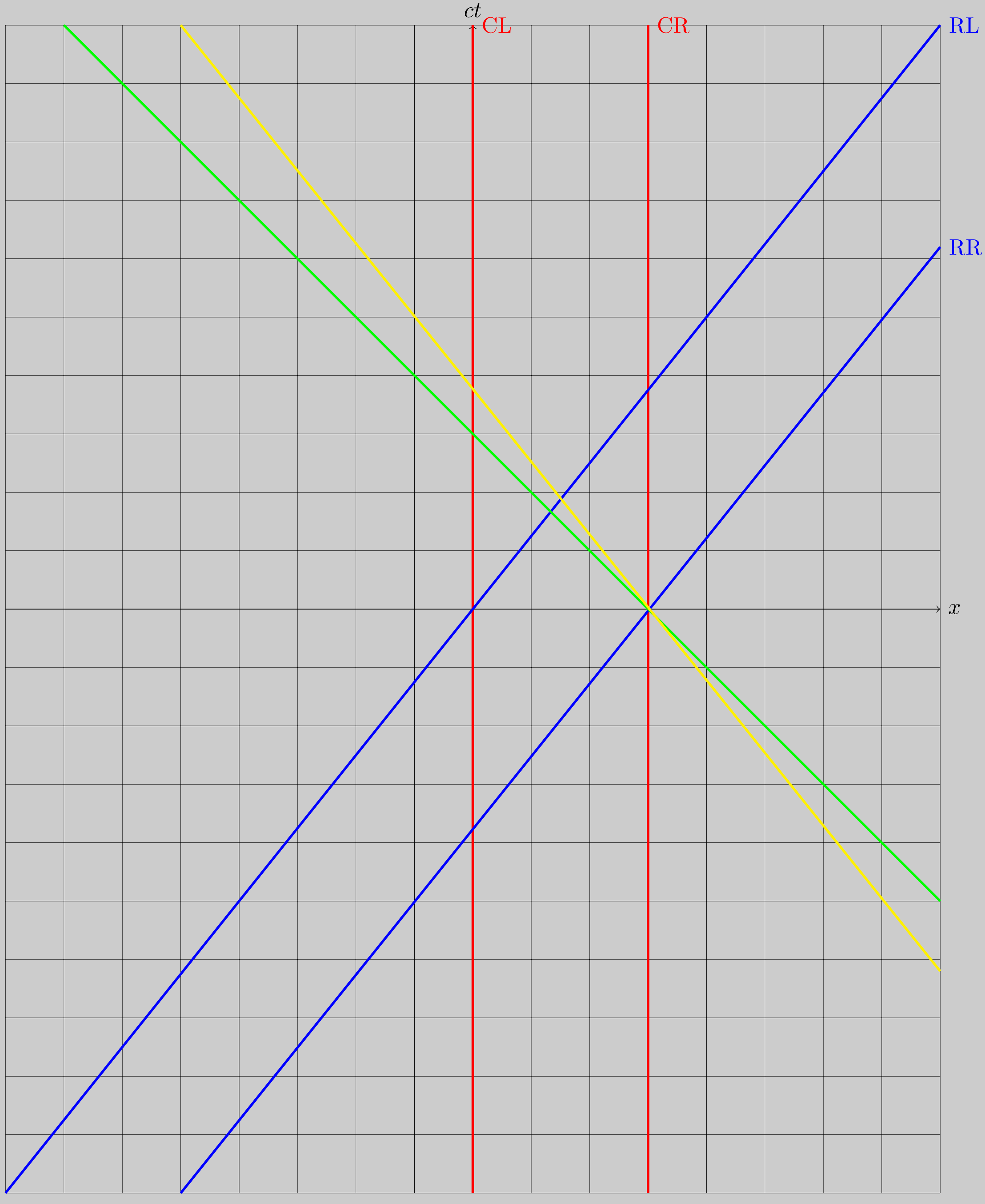
e. Now consider the following in the **cylinder's frame**. The right end of the rod stops instantaneously at event B and emits a flash of light, and the left end of the rod stops instantaneously when the light reaches it. Determine the final length of the rod after it has stopped. You can assume the rod compresses uniformly with no other deformation.

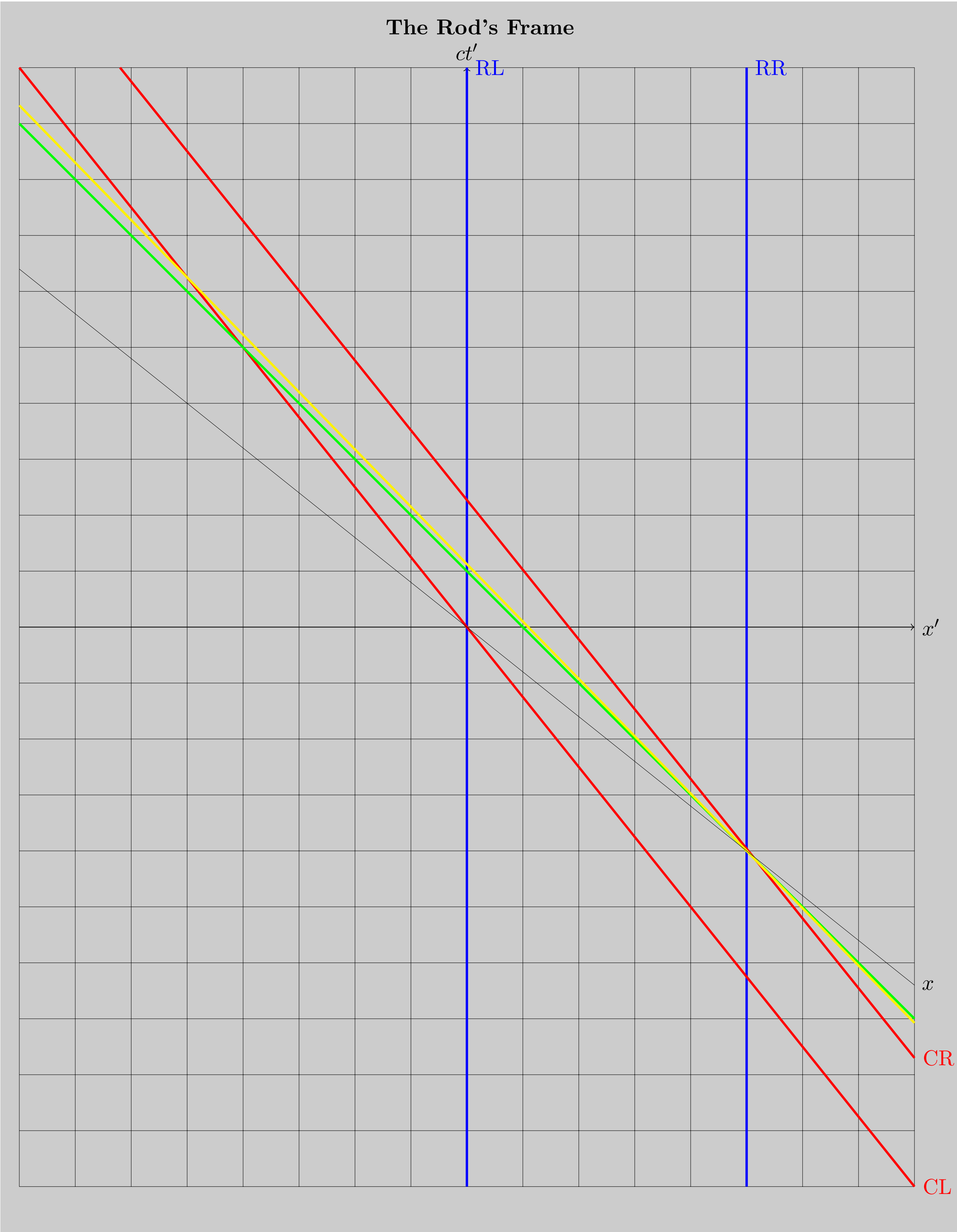
Any computation that you do must be shown on a separate sheet of paper, and not on the graphs. Graphical work that does not have supporting computation might not receive full credit.

Solution

The graphs are shown below, where the yellow line is the particle P and the green line is the flash of light. Solutions that used Galilean relativity received partial credit, as long as they were self-consistent. The final length of the rod is simply the distance between the line CR and the intersection of RL and the green line, i.e. 25/3 m. There's no need to apply length contraction, as we're already in the rest frame of the rod at this point. Nonetheless, the way we have chosen to stop the rod has squeezed it shorter.

The Cylinder's Frame





The Rod's Frame
ct'
RL
RR
x'
x
CR
CL

Question A4

The flow of heat through a material can be described via the *thermal conductivity* κ. If the two faces of a slab of material with thermal conductivity κ, area A, and thickness d are held at temperatures differing by ΔT, the thermal power P transferred through the slab is

$$P = \frac{\kappa A \Delta T}{d}$$

A large, flat lake in the upper Midwest has a uniform depth of 5.0 meters of water that is covered by a uniform layer of 1.0 cm of ice. Cold air has moved into the region so that the upper surface of the ice is now maintained at a constant temperature of -10 °C by the cold air (an infinitely large constant temperature heat sink). The bottom of the lake remains at a fixed 4.0 °C because of contact with the earth (an infinitely large constant temperature heat source). It is reasonable to assume that heat flow is only in the vertical direction and that there is no convective motion in the water.

a. Determine the initial rate of change in ice thickness.

Solution

The main effect is that the ice radiates heat into the air due to the temperature gradient through it, and this freezes the water next to the ice. However, there are many other effects that slightly change the answer.

i. There is another contribution to the thermal power from the temperature gradient in the water.
ii. As the water freezes, it lifts the ice above it.
iii. When a layer of water freezes into ice, all of the other water and ice becomes slightly colder.

The first point should be addressed for full credit. To do this, we will calculate both contributions. The water right at the bottom of the ice is at 0 C°. The temperature gradients in the water and ice are both uniform since the system is in quasi-equilibrium; physically, if the temperature gradient were not uniform, there would be a net flow of heat to or away from some regions, quickly making the gradient uniform again.

The temperature gradient in the water is 4 C°/5 m. Multiplying by the conductivity, we get a power of

$$P_w = \frac{4\ \mathrm{C}^\circ}{5\ \mathrm{m}} \frac{0.57\ \mathrm{W}}{\mathrm{mC}^\circ} = 0.456\ \mathrm{W/m}^2$$

delivered through the water. The same calculation for the ice gives power

$$P_i = \frac{10\ \mathrm{C}^\circ}{.01\ \mathrm{m}} \frac{2.2\ \mathrm{W}}{\mathrm{m} \cdot \mathrm{C}^\circ} = 2200\ \mathrm{W/m}^2$$

delivered through the ice. Thus P_w is negligible and can be ignored.

Now, each square meter of water directly underneath the ice loses 2200 J of energy per second. That is enough energy to freeze

$$2200\ \mathrm{W}/(330,000\ \mathrm{J/kg}) = 6.7 \times 10^{-3}\ \mathrm{kg/s}$$

of water into ice. Converting to volume, we have

$$(6.7 \times 10^{-3}\ \text{kg/s})/(920\ \text{kg/m}^3) = 7.2 \times 10^{-6}\ \text{m}^3/\text{s}$$

of ice formed for each square meter of ice, which means the ice is growing at a rate

$$r = 7.2 \times 10^{-6}\ \text{m/s} = 2.6\ \text{cm/hr}.$$

Next, we will account for the second and third points; these are not necessary for full credit. First consider the rising of the water. Each square meter of ice initially weighs 9.2 kg. A power of 2200 W is enough to lift this ice about 24 m/s against gravity. In reality, the ice is lifted at a much slower rate, so this accounts for a negligible portion of the energy.

The third point requires some more explanation. In an appropriate coordinate system, the temperature profile of the ice is

$$T(x, d) = \left(1 - \frac{x}{d}\right) \delta T, \quad x \in [0, d]$$

where d is the thickness and $\delta T = -10\ \text{C}^\circ$. As the thickness d increases, *all* of the ice must decrease slightly in temperature to maintain a linear temperature gradient,

$$\frac{\partial T}{\partial d} = \frac{x}{d^2} \delta T.$$

By drawing a graph, one can see this contribution is equal to the heat that would be needed to cool the new ice formed by 5 C°, which gives a 3% correction to the answer. There is also a similar contribution from cooling the water, which is negligible. Finally, we neglected the sublimation of the ice.

b. Assuming the air stays at the same temperature for a long time, find the equilibrium thickness of the ice.

Solution

This part is independent of the previous part. For convenience, define h_0 to be the depth of the lake if all the water were in liquid form. Accounting for the centimeter of ice, $h_0 = 5.01$ m to the number of significant digits we're using.

The ice will stop getting thicker when the energy flux through the water equals that through the ice,

$$\frac{\Delta T_w}{h_w} \kappa_w = \frac{\Delta T_i}{h_i} \kappa_i.$$

Since the thickness of the water is h_w, the amount of water that has frozen into ice had a thickness of $h_0 - h_w$. Setting the mass of water frozen equal to the mass of the ice,

$$h_i \rho_i = (h_0 - h_w) \rho_w \quad \Rightarrow \quad h_w = \frac{h_0 \rho_w - h_i \rho_i}{\rho_w}.$$

Plugging this into the previous expression gives

$$\frac{\Delta T_w \kappa_w \rho_w}{h_0 \rho_w - h_i \rho_i} = \frac{\Delta T_i \kappa_i}{h_i}.$$

Solving for h_i and plugging in numbers,

$$h_i = h_0 \frac{\Delta T_i \kappa_i \rho_w}{\Delta T_w \kappa_w \rho_w + \Delta T_i \kappa_i \rho_i} = 4.89 \text{ m}.$$

c. Explain why convective motion can be ignored in the water.

Solution

Convection occurs when boiling a pot of water because the hot water at the bottom of the pot has lower density than the colder water higher up. This means gravitational energy can be released when that hot, low-density water rises and cold, high-density water falls. When the hot water rises, it releases heat, cools, gets denser, and falls back down again, in a convection cycle. This phenomenon relies on the hotter water having lower density.

However, water reaches its maximum density at 4 C°, so the water at the bottom of the lake, though warmer, is more dense than the water above it. Convection does not occur because moving the water around vertically would not release any gravitational potential energy.

Some important quantities for this problem:

Specific heat capacity of water	C_{water}	$4200 \text{ J/(kg} \cdot \text{C}^\circ)$
Specific heat capacity of ice	C_{ice}	$2100 \text{ J/(kg} \cdot \text{C}^\circ)$
Thermal conductivity of water	κ_{water}	$0.57 \text{ W/(m} \cdot \text{C}^\circ)$
Thermal conductivity of ice	κ_{ice}	$2.2 \text{ W/(m} \cdot \text{C}^\circ)$
Latent heat of fusion for water	L_{f}	$330,000 \text{ J/kg}$
Density of water	ρ_{water}	999 kg/m^3
Density of ice	ρ_{ice}	920 kg/m^3

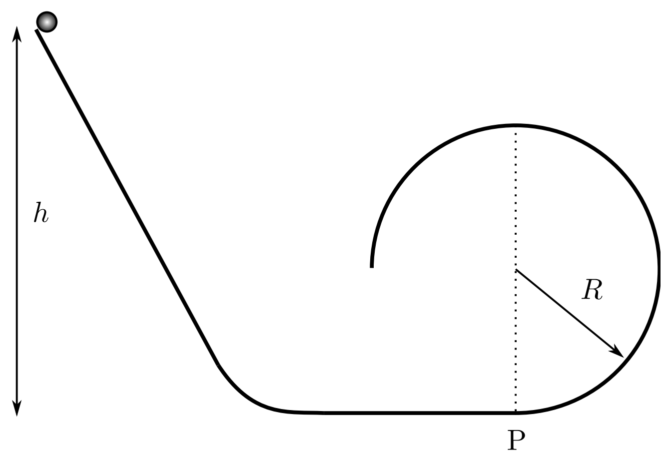
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

A uniform solid spherical ball starts from rest on a loop-the-loop track. It rolls without slipping along the track. However, it does not have enough speed to make it to the top of the loop. From what height h would the ball need to start in order to land at point P directly underneath the top of the loop? Express your answer in terms of R, the radius of the loop. Assume that the radius of the ball is very small compared to the radius of the loop, and that there are no energy losses due to friction.



Solution

We fix the origin at P. Assume the ball leaves at an angle θ away from the vertical. At this point, the x and y coordinates are

$$x = R\sin\theta, \quad y = R(1+\cos\theta).$$

By energy conservation, we have

$$mg(h-y) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m(1+\beta)v^2$$

where $\beta = 2/5$, and we used the fact that the ball rolls without slipping.

Let v be the speed of the ball when it leaves the loop. Then its velocity components at that moment are

$$v_x = -v\cos\theta, \quad v_y = v\sin\theta.$$

Assuming the ball impacts P at time t,

$$y = \frac{1}{2}gt^2 - v_y t, \quad x = -v_x t.$$

The second equation yields

$$t = \frac{R}{v}\frac{\sin\theta}{\cos\theta}$$

and plugging this into the first equation gives

$$R + R\cos\theta = \frac{1}{2}g\left(\frac{R}{v}\frac{\sin\theta}{\cos\theta}\right)^2 - v\sin\theta\frac{R}{v}\frac{\sin\theta}{\cos\theta}$$

which simplifies to

$$1 + \cos\theta = \frac{gR}{2v^2}\frac{\sin^2\theta}{\cos\theta}.$$

Now, the ball leaves the surface when the normal component of the force of the loop on the ball just drops to zero. This happens when

$$mg\cos\theta = m\frac{v^2}{R} \quad \Rightarrow \quad \frac{v^2}{gR} = \cos\theta$$

and plugging this into the previous equation gives

$$1 + \cos\theta = \frac{1}{2}\frac{1 - \cos^2\theta}{\cos^2\theta} \quad \Rightarrow \quad 2\cos^2\theta = 1 - \cos\theta.$$

This is a quadratic equation with solutions

$$\cos\theta = \frac{-1 \pm \sqrt{1+8}}{4} = -\frac{1}{4} \pm \frac{3}{4}$$

Only the positive answer of $\cos\theta = 1/2$ is relevant here, though the negative answer is still physical!

Now that we know θ, getting the final answer is straightforward. We combine the energy conservation equation and the condition

$$mg\cos\theta = m\frac{v^2}{R}$$

to find

$$h = 1 + \left(\frac{1}{2}(1+\beta) + 1\right)R\cos\theta = \frac{37}{20}R.$$

Question B2

a. A spherical region of space of radius R has a uniform charge density and total charge $+Q$. An electron of charge $-e$ is free to move inside or outside the sphere, under the influence of the charge density alone. For this first part ignore radiation effects.

i. Consider a circular orbit for the electron where $r < R$. Determine the period of the orbit T in terms of any or all of r, R, Q, e, and any necessary fundamental constants.

Solution

We apply Gauss's law,

$$\frac{Q_{\text{in}}}{\epsilon_0} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}.$$

This yields

$$\frac{Q}{\epsilon_0}\frac{r^3}{R^3} = 4\pi r^2 E \quad \Rightarrow \quad E = \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}$$

Since the motion is circular,

$$m\frac{4\pi^2 r}{T^2} = eE = \frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}$$

and solving for T gives

$$T = 2\pi\sqrt{\frac{4\pi\epsilon_0 m R^3}{eQ}}.$$

It is independent of r since the motion is simple harmonic.

ii. Consider a circular orbit for the electron where $r > R$. Determine the period of the orbit T in terms of any or all of r, R, Q, e, and any necessary fundamental constants.

Solution

Applying Gauss's law as in the previous part gives

$$E = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}$$

as expected by the shell theorem; one could also just write this down directly. Using the same circular motion equation,

$$m\frac{4\pi^2 r}{T^2} = e\frac{eQ}{4\pi\epsilon_0}\frac{1}{r^2}$$

and solving for T gives

$$T = 2\pi\sqrt{\frac{4\pi\epsilon_0 m r^3}{eQ}}.$$

It is proportional to $r^{3/2}$ in accordance with Kepler's third law.

iii. Assume the electron starts at rest at $r = 2R$. Determine the speed of the electron when it passes through the center in terms of any or all of R, Q, e, and any necessary fundamental constants.

Solution

We use the above results to compute the potential difference,

$$\begin{aligned}\Delta V &= -\int_{2R}^{0} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}, \\ &= \int_{2R}^{R} \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} + \int_{R}^{0} \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}, \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{-1}{2R} - \frac{-1}{R} + \frac{R^2}{2R^3}\right), \\ &= \frac{Q}{4\pi\epsilon_0 R}.\end{aligned}$$

By energy conservation,

$$v = \sqrt{\frac{2}{m}e\Delta V} = \sqrt{\frac{2eQ}{4\pi\epsilon_0 mR}}.$$

b. Accelerating charges radiate. The total power P radiated by charge q with acceleration a is given by

$$P = C\xi a^n$$

where C is a dimensionless numerical constant (which is equal to $1/6\pi$), ξ is a physical constant that is a function only of the charge q, the speed of light c, and the permittivity of free space ϵ_0, and n is a dimensionless constant. Determine ξ and n.

Solution

This is a dimensional analysis problem. The most straightforward method is to write out all the dimensions explicitly. Note that a has dimensions of $[\mathrm{L}]/[\mathrm{T}]^2$, P has dimensions of $[\mathrm{M}][\mathrm{L}]^2/[\mathrm{T}]^3$, c has dimensions of $[\mathrm{L}]/[\mathrm{T}]$, q has dimensions of $[\mathrm{C}]$, and ϵ_0 has dimensions of $[\mathrm{C}]^2[\mathrm{T}]^2/[\mathrm{M}][\mathrm{L}]^3$. The equation

$$P = a^\alpha c^\beta {\epsilon_0}^\gamma q^\delta$$

has dimensions

$$[\mathrm{M}][\mathrm{L}]^2/[\mathrm{T}]^3 = \left([\mathrm{L}]/[\mathrm{T}]^2\right)^\alpha ([\mathrm{L}]/[\mathrm{T}])^\beta \left([\mathrm{C}]^2[\mathrm{T}]^2/[\mathrm{M}][\mathrm{L}]^3\right)^\gamma ([\mathrm{C}])^\delta$$

Mass is only balanced if $\gamma = -1$. As a result, charge is balanced if $\delta = 2$. Proceeding similarly for length and time,

$$P = \frac{1}{6\pi}a^2 c^{-3} {\epsilon_0}^{-1} q^2$$

giving answers of $\xi = q^2/c^3\epsilon_0$ and $n = 2$.

c. Consider the electron in the first part, except now take into account radiation. Assume that the orbit remains circular and the orbital radius r changes by an amount $|\Delta r| \ll r$.

i. Consider a circular orbit for the electron where $r < R$. Determine the change in the orbital radius Δr during one orbit in terms of any or all of r, R, Q, e, and any necessary fundamental constants. Be very specific about the sign of Δr.

Solution

The energy radiated away is given by

$$\Delta E = -PT$$

where T is determined in the previous sections.

It is possible to compute the actual energy of each orbit, and it is fairly trivial to do for regions $r > R$, but perhaps there is an easier, more entertaining way. Consider

$$\Delta E = \Delta K + \Delta U$$

and for small changes in r,

$$\frac{\Delta U}{\Delta r} \approx -F = \frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}.$$

This implies the potential energy increases with increasing r, as expected. Now

$$\frac{\Delta K}{\Delta r} \approx \frac{d}{dr}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}\frac{d}{dr}\left|r\frac{mv^2}{r}\right|$$

but $mv^2/r = F$, so

$$\frac{\Delta K}{\Delta r} \approx \frac{1}{2}\frac{d}{dr}|rF| = \frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}.$$

This implies the kinetic energy increases with increasing r, also as expected, as this region acts like a multidimensional simple harmonic oscillator. Combining,

$$\frac{\Delta E}{\Delta r} \approx 2\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3} = 2ma$$

Finally,

$$\Delta r = -\left(\frac{1}{6\pi}\frac{a^2}{c^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mR^3}{eQ}}\right)\left(\frac{1}{2ma}\right).$$

Plugging in the value of a, this can be simplified to

$$\Delta r = -\frac{1}{6}\sqrt{\frac{e^5Q}{4\pi\epsilon_0{}^3R(mc^2)^3}}\frac{r}{R}.$$

Alternatively, we can write the result in terms of dimensionless groups,

$$\Delta r = -\frac{2\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 Rmc^2}\right)\sqrt{\frac{eQ}{4\pi\epsilon_0 Rmc^2}}r.$$

ii. Consider a circular orbit for the electron where $r > R$. Determine the change in the orbital radius Δr during one orbit in terms of any or all of r, R, Q, e, and any necessary fundamental constants. Be very specific about the sign of Δr.

Solution

Picking up where we left off,

$$\frac{\Delta U}{\Delta r} \approx -F = \frac{eQ}{4\pi\epsilon_0}\frac{1}{r^2}.$$

This implies the potential energy increases with increasing r.

$$\frac{\Delta K}{\Delta r} \approx \frac{1}{2}\frac{d}{dr}|rF| = \frac{\Delta K}{\Delta r} \approx -\frac{eQ}{8\pi\epsilon_0}\frac{1}{r^2}.$$

This implies the kinetic energy *decreases* with increasing r, a somewhat nonintuitive but true statement for circular orbits. Combining,

$$\frac{\Delta E}{\Delta r} \approx \frac{1}{2}\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3} = \frac{ma}{2}.$$

Using the same manipulations as before,

$$\Delta r = -\left(\frac{1}{6\pi}\frac{a^2}{c^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mr^3}{eQ}}\right)\left(\frac{2}{ma}\right).$$

Plugging in the value of a, this can be simplified to

$$\Delta r = -\frac{2}{3}\sqrt{\frac{e^5 Q}{4\pi\epsilon_0^3 r(mc^2)^3}}.$$

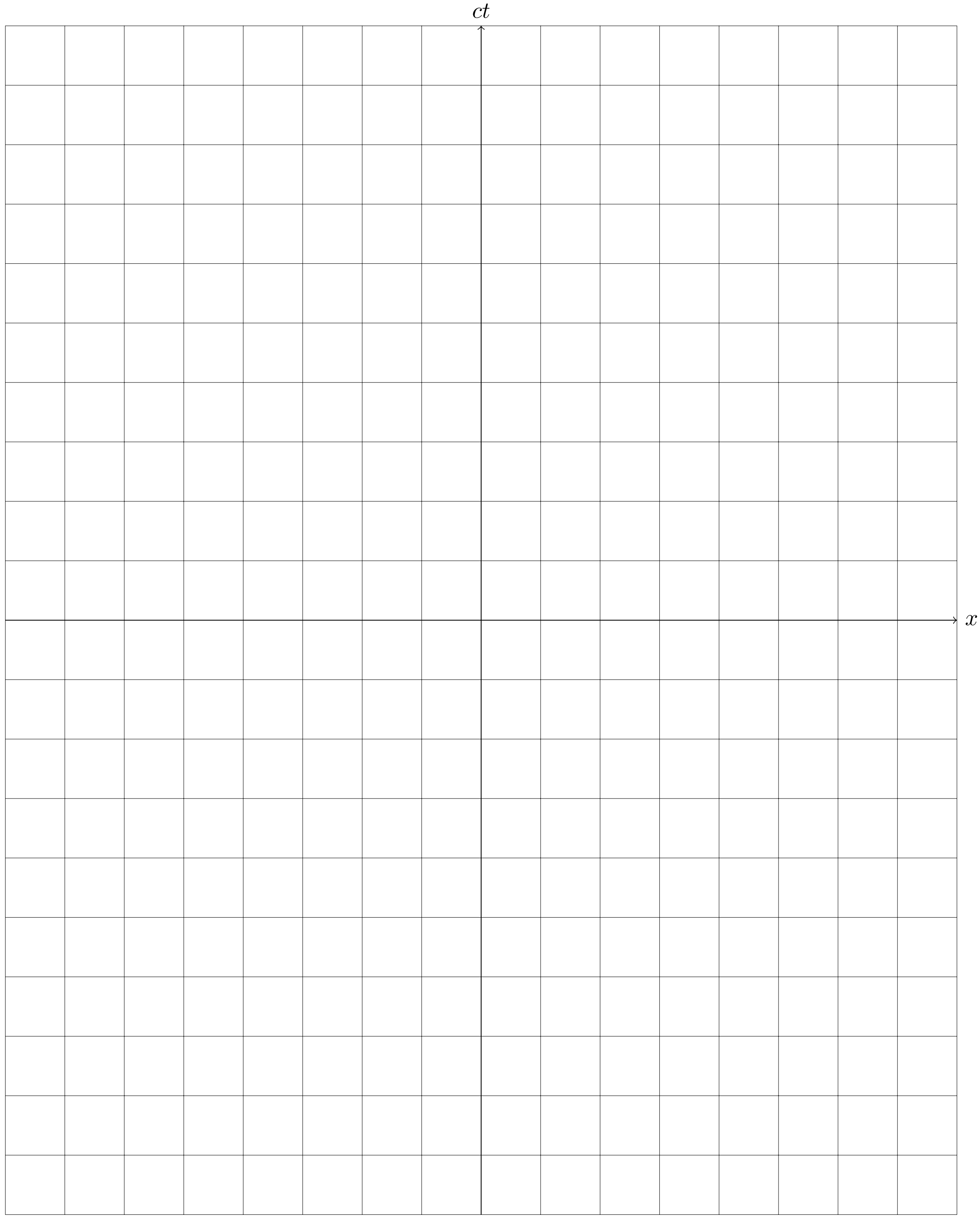
Alternatively, we can write the result in terms of dimensionless groups,

$$\Delta r = -\frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 Rmc^2}\right)\sqrt{\frac{eQ}{4\pi\epsilon_0 Rmc^2}}\frac{R^2}{r}.$$

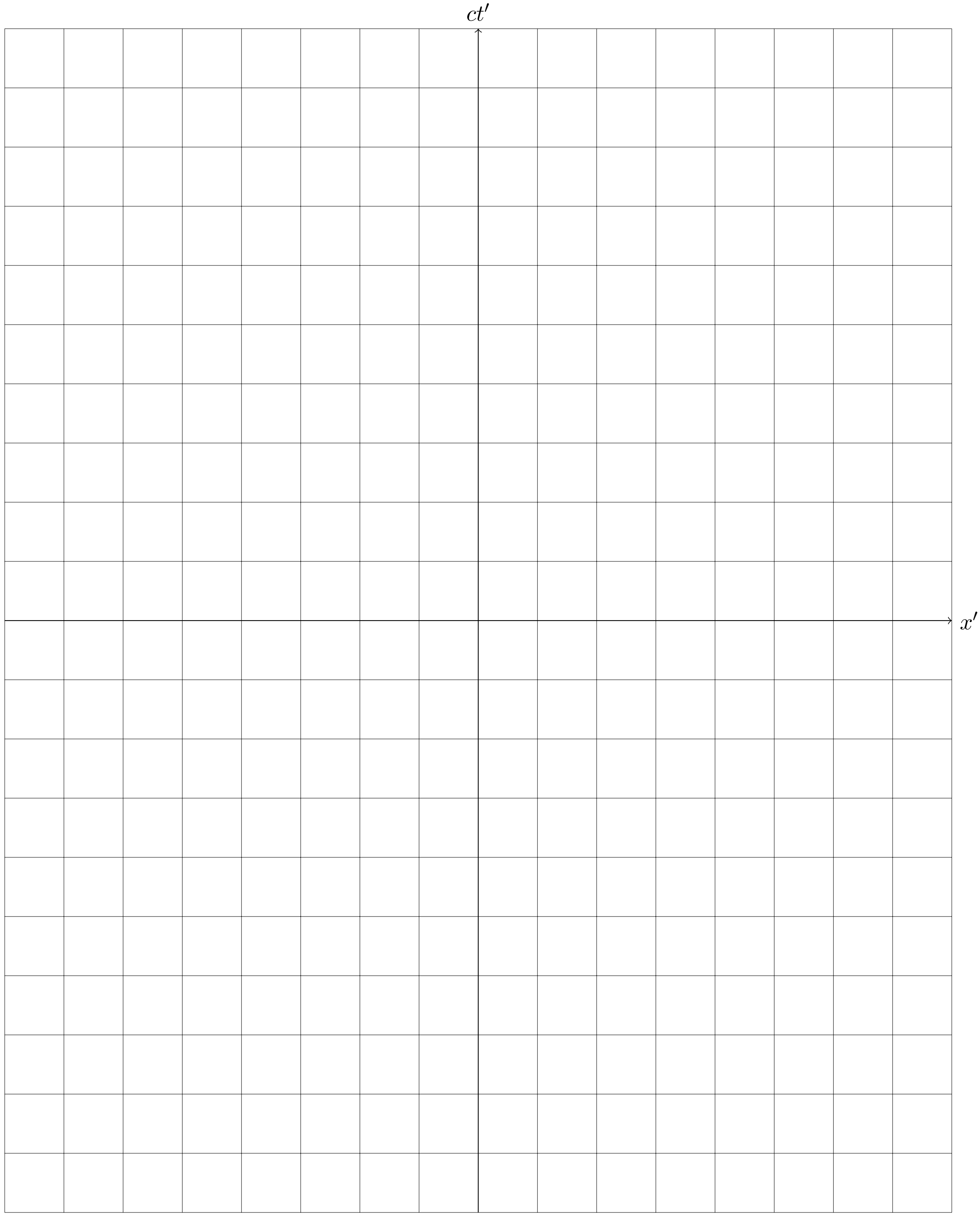
Answer Sheets

Following are answer sheets for some of the graphical portions of the test.

The Cylinder's Frame



The Rod's Frame



AAPT UNITED STATES PHYSICS TEAM
AIP 2017

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has four questions and is allowed 90 minutes; Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 3 parts: the cover sheet (page 2), Part A (pages 3-6), Part B (pages 8-10). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after April 15, 2017.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

AAPT UNITED STATES PHYSICS TEAM
AIP 2017

USA Physics Olympiad Exam

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all four problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your proctor's AAPT ID, your AAPT ID, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  Doe, Jamie
  student AAPT ID #
  proctor AAPT ID #
  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 8, 2017.**

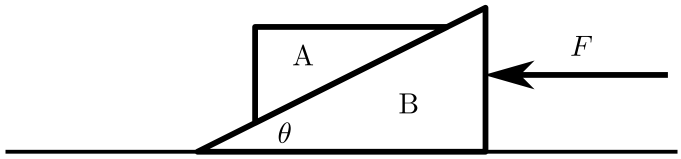
Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}$ (mol)$^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31$ J/(mol $\cdot$ K)
$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)	$e = 1.602 \times 10^{-19}$ C
1 eV $= 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s
$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2	$(1+x)^n \approx 1 + nx$ for $\lvert x\rvert \ll 1$
$m_p = 1.673 \times 10^{-27}$ kg $= 938$ MeV/c^2	$\ln(1+x) \approx x$ for $\lvert x\rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

A pair of wedges are located on a horizontal surface. The coefficient of friction (both sliding and static) between the wedges is μ, the coefficient of friction between the bottom wedge B and the horizontal surface is μ, and the angle of the wedge is θ. The mass of the top wedge A is m, and the mass of the bottom wedge B is $M = 2m$. A horizontal force F directed to the left is applied to the bottom wedge as shown in the figure.



Determine the range of values for F so that the the top wedge does not slip on the bottom wedge. Express your answer(s) in terms of any or all of m, g, θ, and μ.

Question A2

Consider two objects with equal heat capacities C and initial temperatures T_1 and T_2. A Carnot engine is run using these objects as its hot and cold reservoirs until they are at equal temperatures. Assume that the temperature changes of both the hot and cold reservoirs is very small compared to the temperature during any one cycle of the Carnot engine.

a. Find the final temperature T_f of the two objects, and the total work W done by the engine.

Now consider three objects with equal and constant heat capacity at initial temperatures $T_1 = 100$ K, $T_2 = 300$ K, and $T_3 = 300$ K. Suppose we wish to raise the temperature of the third object.

To do this, we could run a Carnot engine between the first and second objects, extracting work W. This work can then be dissipated as heat to raise the temperature of the third object. Even better, it can be stored and used to run a Carnot engine between the first and third object in reverse, which pumps heat into the third object.

Assume that all work produced by running engines can be stored and used without dissipation.

b. Find the minimum temperature T_L to which the first object can be lowered.

c. Find the maximum temperature T_H to which the third object can be raised.

Question A3

A ship can be thought of as a symmetric arrangement of soft iron. In the presence of an external magnetic field, the soft iron will become magnetized, creating a second, weaker magnetic field. We want to examine the effect of the ship's field on the ship's compass, which will be located in the middle of the ship.

Let the strength of the Earth's magnetic field near the ship be B_e, and the orientation of the field be horizontal, pointing directly toward true north.

The Earth's magnetic field B_e will magnetize the ship, which will then create a second magnetic field B_s in the vicinity of the ship's compass given by

$$\vec{\mathbf{B}}_s = B_e\left(-K_b \cos\theta\,\hat{\mathbf{b}} + K_s \sin\theta\,\hat{\mathbf{s}}\right)$$

where K_b and K_s are positive constants, θ is the angle between the heading of the ship and magnetic north, measured clockwise, $\hat{\mathbf{b}}$ and $\hat{\mathbf{s}}$ are unit vectors pointing in the forward direction of the ship (bow) and directly right of the forward direction (starboard), respectively.

Because of the ship's magnetic field, the ship's compass will no longer necessarily point North.

a. Derive an expression for the deviation of the compass, $\delta\theta$, from north as a function of K_b, K_s, and θ.

b. Assuming that K_b and K_s are both much smaller than one, at what heading(s) θ will the deviation $\delta\theta$ be largest?

A pair of iron balls placed in the same horizontal plane as the compass but a distance d away can be used to help correct for the error caused by the induced magnetism of the ship.

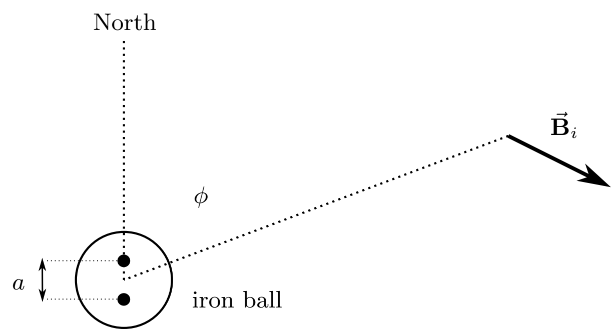
A binnacle, protecting the ship's compass in the center, with two soft iron spheres to help correct for errors in the compass heading. The use of the spheres was suggested by Lord Kelvin.

Just like the ship, the iron balls will become magnetic because of the Earth's field B_e. As spheres, the balls will individually act like dipoles. A dipole can be thought of as the field produced by two magnetic monopoles of strength $\pm m$ at two different points.

The magnetic field of a single pole is

$$\vec{\mathbf{B}} = \pm m \frac{\hat{\mathbf{r}}}{r^2}$$

where the positive sign is for a north pole and the negative for a south pole. The dipole magnetic field is the sum of the two fields: a north pole at $y = +a/2$ and a south pole at $y = -a/2$, where the y axis is horizontal and pointing north. a is a small distance much smaller than the radius of the iron balls; in general $a = K_i B_e$ where K_i is a constant that depends on the size of the iron sphere.



c. Derive an expression for the magnetic field $\vec{\mathbf{B}}_i$ from the iron a distance $d \gg a$ from the center of the ball. Note that there will be a component directed radially away from the ball and a component directed tangent to a circle of radius d around the ball, so using polar coordinates is recommended.

d. If placed directly to the right and left of the ship compass, the iron balls can be located at a distance d to cancel out the error in the magnetic heading for any angle(s) where $\delta\theta$ is largest. Assuming that this is done, find the resulting expression for the combined deviation $\delta\theta$ due to the ship and the balls for the magnetic heading for all angles θ.

Question A4

Relativistic particles obey the mass energy relation

$$E^2 = (pc)^2 + (mc^2)^2$$

where E is the relativistic energy of the particle, p is the relativistic momentum, m is the mass, and c is the speed of light.

A proton with mass m_p and energy E_p collides head on with a photon which is massless and has energy E_b. The two combine and form a new particle with mass m_Δ called Δ, or "delta". It is a one dimensional collision that conserves both relativistic energy and relativistic momentum.

a. Determine E_p in terms of m_p, m_Δ, and E_b. You may assume that E_b is small.

b. In this case, the photon energy E_b is that of the cosmic background radiation, which is an EM wave with wavelength 1.06 mm. Determine the energy of the photons, writing your answer in electron volts.

c. Assuming this value for E_b, what is the energy of the proton, in electron volts, that will allow the above reaction? This sets an upper limit on the energy of cosmic rays. The mass of the proton is given by $m_pc^2 = 938$ MeV and the mass of the Δ is given by $m_\Delta c^2 = 1232$ MeV.

The following relationships may be useful in solving this problem:

velocity parameter	$\beta = \frac{v}{c}$
Lorentz factor	$\gamma = \frac{1}{\sqrt{1-\beta^2}}$
relativistic momentum	$p = \gamma\beta mc$
relativistic energy	$E = \gamma mc^2$
relativistic doppler shift	$\frac{f}{f_0} = \sqrt{\frac{1-\beta}{1+\beta}}$

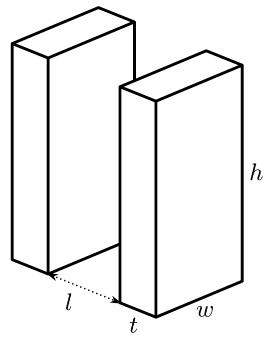
STOP: Do Not Continue to Part B

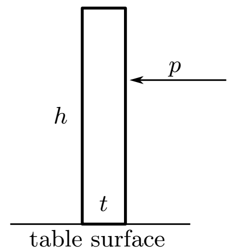
If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

Suppose a domino stands upright on a table. It has height h, thickness t, width w (as shown below), and mass m. The domino is free to rotate about its edges, but will not slide across the table.





a. Suppose we give the domino a sharp, horizontal impulsive push with total momentum p.

   i. At what height H above the table is the impulse p required to topple the domino smallest?

   ii. What is the minimum value of p to topple the domino?

b. Next, imagine a long row of dominoes with equal spacing l between the nearest sides of any pair of adjacent dominos, as shown above. When a domino topples, it collides with the next domino in the row. Imagine this collision to be completely inelastic. What fraction of the total kinetic energy is lost in the collision of the first domino with the second domino?

c. After the collision, the dominoes rotate in such a way so that they always remain in contact. Assume that there is no friction between the dominoes and the first domino was given the smallest possible push such that it toppled. What is the minimum l such that the second domino will topple?

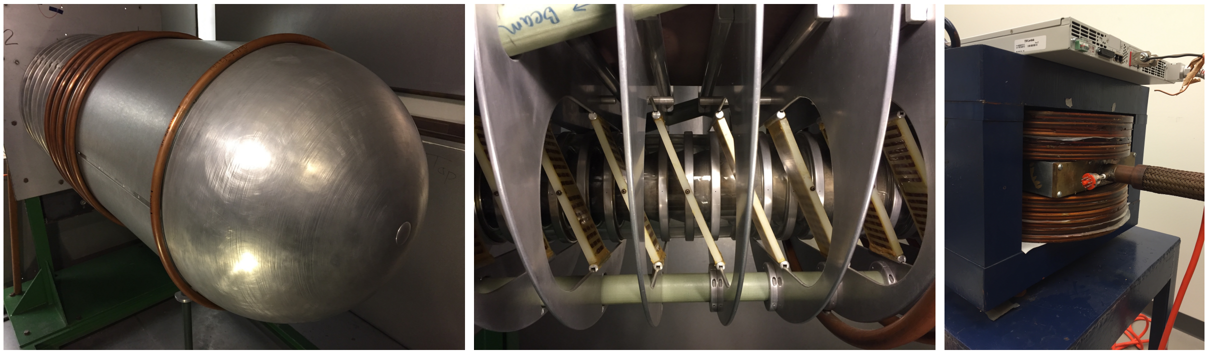
   You may work to lowest nontrivial order in the angles through which the dominoes have rotated. Equivalently, you may approximate $t,\ l \ll h$.

d. A row of toppling dominoes can be considered to have a propagation speed of the length $l + t$ divided by the time between successive collisions. When the first domino is given a minimal push just large enough to topple and start a chain reaction of toppling dominoes, the speed increases with each domino, but approaches an asymptotic speed v.

   Suppose there is a row of dominoes on another planet. These dominoes have the same density as the dominoes previously considered, but are twice as tall, wide, and thick, and placed with a spacing of $2l$ between them. If this row of dominoes topples with the same asymptotic speed v previously found, what is the local gravitational acceleration on this planet?

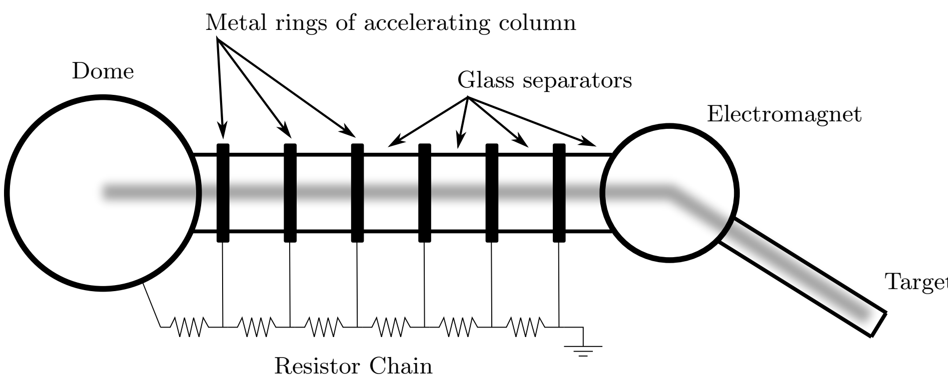
Question B2

Beloit College has a "homemade" 500 kV VanDeGraff proton accelerator, designed and constructed by the students and faculty.



Accelerator dome (assume it is a sphere); accelerating column; bending electromagnet

The accelerator dome, an aluminum sphere of radius $a = 0.50$ meters, is charged by a rubber belt with width $w = 10$ cm that moves with speed $v_b = 20$ m/s. The accelerating column consists of 20 metal rings separated by glass rings; the rings are connected in series with 500 MΩ resistors. The proton beam has a current of 25 μA and is accelerated through 500 kV and then passes through a tuning electromagnet. The electromagnet consists of wound copper pipe as a conductor. The electromagnet effectively creates a uniform field B inside a circular region of radius $b = 10$ cm and zero outside that region.



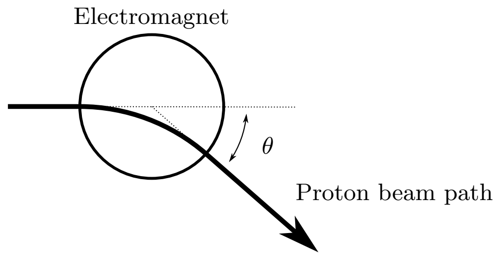
Only six of the 20 metals rings and resistors are shown in the figure. The fuzzy grey path is the path taken by the protons as they are accelerated from the dome, through the electromagnet, into the target.

a. Assuming the dome is charged to 500 kV, determine the strength of the electric field at the surface of the dome.

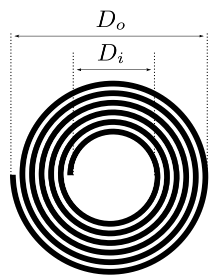
b. Assuming the proton beam is off, determine the time constant for the accelerating dome (the time it takes for the charge on the dome to decrease to $1/e \approx 1/3$ of the initial value.

c. Assuming the 25 μA proton beam is on, determine the surface charge density that must be sprayed onto the charging belt in order to maintain a steady charge of 500 kV on the dome.

d. The proton beam enters the electromagnet and is deflected by an angle $\theta = 10°$. Determine the magnetic field strength.



e. The electromagnet is composed of layers of spiral wound copper pipe; the pipe has inner diameter $d_i = 0.40$ cm and outer diameter $d_o = 0.50$ cm. The copper pipe is wound into this flat spiral that has an inner diameter $D_i = 20$ cm and outer diameter $D_o = 50$ cm. Assuming the pipe almost touches in the spiral winding, determine the length L in one spiral.



f. Hollow pipe is used instead of solid conductors in order to allow for cooling of the magnet. If the resistivity of copper is $\rho = 1.7 \times 10^{-8} \Omega \cdot \text{m}$, determine the electrical resistance of one spiral.

g. There are $N = 24$ coils stacked on top of each other. Tap water with an initial temperature of $T_c = 18°$ C enters the spiral through the copper pipe to keep it from over heating; the water exits at a temperature of $T_h = 31°$ C. The copper pipe carries a direct 45 Amp current in order to generate the necessary magnetic field. At what rate must the cooling water flow be provided to the electromagnet? Express your answer in liters per second with only one significant digit. The specific heat capacity of water is 4200 J/°C · kg; the density of water is 1000 kg/m^3.

h. The protons are fired at a target consisting of Fluorine atoms ($Z = 9$). What is the distance of closest approach to the center of the Fluorine nuclei for the protons? You can assume that the Fluorine does not move.

AAPT UNITED STATES PHYSICS TEAM
AIP 2017

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has four questions and is allowed 90 minutes; Part B has two questions and is allowed 90 minutes.

- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.

- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.

- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.

- Allow 90 minutes to complete Part B. Do not let students go back to Part A.

- Ideally the test supervisor will divide the question paper into 3 parts: the cover sheet (page 2), Part A (pages 3-13), Part B (pages 15-23). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!

- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after April 15, 2017.

- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

AAPT UNITED STATES PHYSICS TEAM
AIP 2017

USA Physics Olympiad Exam

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all four problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your proctor's AAPT ID, your AAPT ID, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  Doe, Jamie
  student AAPT ID #
  proctor AAPT ID #
  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 8, 2017.**

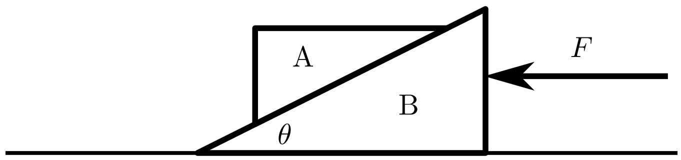
Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T\cdot m/A}$
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol\cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s\cdot m^2\cdot K^4)}$	$e = 1.602 \times 10^{-19}$ C
$1\ \mathrm{eV} = 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}\ \mathrm{J\cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV\cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x\rvert \ll 1$
$m_p = 1.673 \times 10^{-27}\ \mathrm{kg} = 938\ \mathrm{MeV/c^2}$	$\ln(1+x) \approx x$ for $\lvert x\rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

A pair of wedges are located on a horizontal surface. The coefficient of friction (both sliding and static) between the wedges is μ, the coefficient of friction between the bottom wedge B and the horizontal surface is μ, and the angle of the wedge is θ. The mass of the top wedge A is m, and the mass of the bottom wedge B is $M = 2m$. A horizontal force F directed to the left is applied to the bottom wedge as shown in the figure.



Determine the range of values for F so that the the top wedge does not slip on the bottom wedge. Express your answer(s) in terms of any or all of m, g, θ, and μ.

Solution

Solution 1. Assume the block does not slip. Considering the horizontal forces on the entire system gives

$$F - 3\mu mg = 3ma \quad \Rightarrow \quad a = \frac{F}{3m} - \mu g.$$

When F is small, the top wedge wants to slide downward, so static friction points up the ramp. Considering the horizontal and vertical forces on the block gives

$$N\cos\theta + f\sin\theta = mg, \quad N\sin\theta - f\cos\theta = ma.$$

When the minimal force is applied, the friction is maximal, $f = \mu N$. Eliminating N gives

$$ma = mg\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}$$

and plugging in our first equation gives

$$F_{\text{min}} = 3mg\left(\mu + \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right) = 3mg\frac{(1+\mu^2)\tan\theta}{1+\mu\tan\theta}.$$

When F is large, the top wedge wants to slide upward, so static friction points down the ramp, and

$$N\cos\theta - f\sin\theta = mg, \quad N\sin\theta + f\cos\theta = ma.$$

Now setting $f = \mu N$ gives

$$F_{\text{max}} = 3mg\left(\mu + \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right) = 3mg\frac{2\mu + (1-\mu^2)\tan\theta}{1-\mu\tan\theta}.$$

Therefore, naively the range of forces so that the block will not slip is

$$F \in [F_{\text{min}}, F_{\text{max}}].$$

However, to get full credit, students must account for two edge cases. First, when $\mu > \tan\theta$, no force is required at all to keep the block in place, so the minimum force is zero. Second, when $\mu > \cot\theta$, the block will not slip up under any circumstances, so there is no maximal force.

Solution 2. The problem can also be solved geometrically. In general, the no slip condition is

$$\mu > \tan\phi$$

where ϕ is the angle between the vertical and the normal to the plane. Working in the noninertial reference frame of the plane, the fictitious force due to the acceleration is equivalent to a tilting of the gravity vector by an angle

$$\tan\beta = \frac{a}{g}$$

where, as in solution 1,

$$a = \frac{F}{3m} - \mu g.$$

Then the top block will not slip as long as $|\theta - \beta| \leq \phi$. At the minimum acceleration a_{min}, $\beta = \theta - \phi$, and taking the tangent of both sides gives

$$\frac{a_{\text{min}}}{g} = \frac{\tan\theta - \mu}{1 + \mu\tan\theta}.$$

This is only meaningful for $\tan\theta > \mu$, otherwise the answer is simply $a_{\text{min}} = 0$. At the maximum acceleration a_{max}, $\beta = \theta + \phi$, which gives

$$\frac{a_{\text{max}}}{g} = \frac{\tan\theta + \mu}{1 - \mu\tan\theta}.$$

This is only meaningful for $\cot\theta > \mu$, otherwise the answer is simply $a_{\text{max}} = \infty$.

Question A2

Consider two objects with equal heat capacities C and initial temperatures T_1 and T_2. A Carnot engine is run using these objects as its hot and cold reservoirs until they are at equal temperatures. Assume that the temperature changes of both the hot and cold reservoirs is very small compared to the temperature during any one cycle of the Carnot engine.

a. Find the final temperature T_f of the two objects, and the total work W done by the engine.

Solution

Since a Carnot engine is reversible, it produces no entropy,

$$dS_1 + dS_2 = \frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} = 0.$$

By the definition of heat capacity, $dQ_i = C dT_i$, so

$$\frac{dT_1}{T_1} = -\frac{dT_2}{T_2}.$$

Integrating this equation shows that $T_1 T_2$ is constant, so the final temperature is

$$T_f = \sqrt{T_1 T_2}.$$

The change in thermal energy of the objects is

$$C(T_f - T_1) + C(T_f - T_2) = C\left[2\sqrt{T_1 T_2} - T_1 - T_2\right].$$

By the First Law of Thermodynamics, the missing energy has been used to do work, so

$$W = C\left[T_1 + T_2 - 2\sqrt{T_1 T_2}\right].$$

Now consider three objects with equal and constant heat capacity at initial temperatures $T_1 = 100$ K, $T_2 = 300$ K, and $T_3 = 300$ K. Suppose we wish to raise the temperature of the third object.

To do this, we could run a Carnot engine between the first and second objects, extracting work W. This work can then be dissipated as heat to raise the temperature of the third object. Even better, it can be stored and used to run a Carnot engine between the first and third object in reverse, which pumps heat into the third object.

Assume that all work produced by running engines can be stored and used without dissipation.

b. Find the minimum temperature T_L to which the first object can be lowered.

Solution

By the Second Law of Thermodynamics, we must have $T_L = 100$K. Otherwise, we would have a process whose sole effect was a net transfer of heat from a cold body to a warm one.

c. Find the maximum temperature T_H to which the third object can be raised.

Solution

The entropy of an object with constant heat capacity is

$$S = \int \frac{dQ}{T} = C \int \frac{dT}{T} = C \ln T.$$

Since the total entropy remains constant, $T_1 T_2 T_3$ is constant; this is a direct generalization of the result for T_f found in part (a). Energy is also conserved, as it makes no sense to leave stored energy unused, so $T_1 + T_2 + T_3$ is constant.

When one object is at temperature T_H, the other two must be at the same lower temperature T_0, or else further work could be extracted from their temperature difference, so

$$T_1 + T_2 + T_3 = T_H + 2T_0, \quad T_1 T_2 T_3 = T_H T_0^2.$$

Plugging in temperatures with values divided by 100 for convenience, and eliminating T_0 gives

$$T_H (7 - T_H)^2 = 36.$$

We know that $T_H = 1$ is one (spurious) solution, since this is the minimum possible final temperature as found in part (b). The other roots are $T_H = 4$ and $T_H = 9$ by the quadratic formula. The solution $T_H = 9$ is impossible by energy conservation, so

$$T_H = 400\text{K}.$$

It is also possible to solve the problem more explicitly. For example, one can run a Carnot cycle between the first two objects until they are at the same temperature, then run a Carnot cycle in reverse between the last two objects using the stored work. At this point, the first two objects will no longer be at the same temperature, so we can repeat the procedure; this yields an infinite series for T_H. Some students did this, and took only the first term of the series. This yields a fairly good approximation of $T_H \approx 395\text{K}$.

Another explicit method is to continuously switch between running one Carnot engine forward and another Carnot engine in reverse; this yields three differential equations for T_1, T_2, and T_3. Solving the equations and setting $T_1 = T_2$ yields $T_3 = T_H$.

Question A3

A ship can be thought of as a symmetric arrangement of soft iron. In the presence of an external magnetic field, the soft iron will become magnetized, creating a second, weaker magnetic field. We want to examine the effect of the ship's field on the ship's compass, which will be located in the middle of the ship.

Let the strength of the Earth's magnetic field near the ship be B_e, and the orientation of the field be horizontal, pointing directly toward true north.

The Earth's magnetic field B_e will magnetize the ship, which will then create a second magnetic field B_s in the vicinity of the ship's compass given by

$$\vec{\mathbf{B}}_s = B_e\left(-K_b\cos\theta\,\hat{\mathbf{b}} + K_s\sin\theta\,\hat{\mathbf{s}}\right)$$

where K_b and K_s are positive constants, θ is the angle between the heading of the ship and magnetic north, measured clockwise, $\hat{\mathbf{b}}$ and $\hat{\mathbf{s}}$ are unit vectors pointing in the forward direction of the ship (bow) and directly right of the forward direction (starboard), respectively.

Because of the ship's magnetic field, the ship's compass will no longer necessarily point North.

a. Derive an expression for the deviation of the compass, $\delta\theta$, from north as a function of K_b, K_s, and θ.

Solution

We add the fields to get the local field. The northward component is

$$B_{\text{north}} = B_e - B_eK_b\cos\theta\cos\theta - B_eK_s\sin\theta\sin\theta$$

while the eastward component is

$$B_{\text{east}} = -B_eK_b\sin\theta\cos\theta + B_eK_s\cos\theta\sin\theta$$

The deviation is given by

$$\tan\delta\theta = (K_s - K_b)\frac{\sin\theta\cos\theta}{1 - K_b\cos^2\theta - K_s\sin^2\theta}.$$

This form is particularly nice, because as we'll see below, K_b and K_s are small enough to ignore in the denominator.

b. Assuming that K_b and K_s are both much smaller than one, at what heading(s) θ will the deviation $\delta\theta$ be largest?

Solution

By inspection, $\theta = 45°$ will yield the largest deviation. It's also acceptable to list 45°, 135°, 225°, and 315°.

A pair of iron balls placed in the same horizontal plane as the compass but a distance d away can be used to help correct for the error caused by the induced magnetism of the ship.

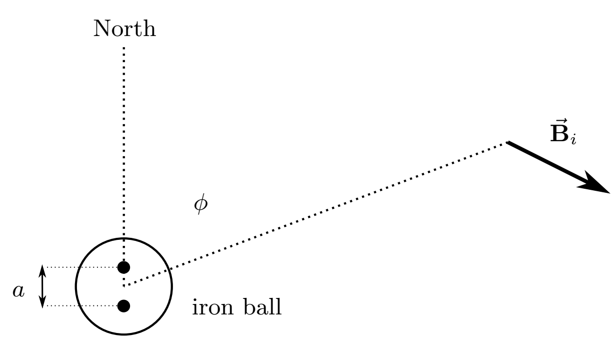
A binnacle, protecting the ship's compass in the center, with two soft iron spheres to help correct for errors in the compass heading. The use of the spheres was suggested by Lord Kelvin.

Just like the ship, the iron balls will become magnetic because of the Earth's field B_e. As spheres, the balls will individually act like dipoles. A dipole can be thought of as the field produced by two magnetic monopoles of strength $\pm m$ at two different points.

The magnetic field of a single pole is

$$\vec{\mathbf{B}} = \pm m \frac{\hat{\mathbf{r}}}{r^2}$$

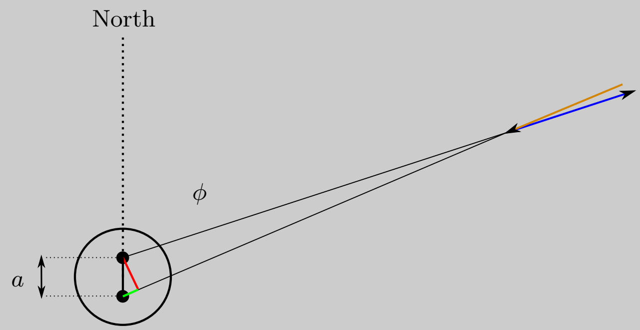
where the positive sign is for a north pole and the negative for a south pole. The dipole magnetic field is the sum of the two fields: a north pole at $y = +a/2$ and a south pole at $y = -a/2$, where the y axis is horizontal and pointing north. a is a small distance much smaller than the radius of the iron balls; in general $a = K_i B_e$ where K_i is a constant that depends on the size of the iron sphere.



c. Derive an expression for the magnetic field $\vec{\mathbf{B}}_i$ from the iron a distance $d \gg a$ from the center of the ball. Note that there will be a component directed radially away from the ball and a component directed tangent to a circle of radius d around the ball, so using polar coordinates is recommended.

Solution

This problem is not nearly as difficult as it looks.



Consider the colored triangle above. The black side has length a. The angle between the green and black sides is ϕ, so the length of the red side is $a\sin\phi$ and the length of the green side is $a\cos\phi$.

The magnetic field strength from one magnetic pole a distance d away is given by

$$B = \pm m\frac{1}{d^2}$$

The sum of the two fields has two components. The angular component is a measure of the "opening" of the triangle formed by the two vectors, and since the two vectors basically have the same length, we can use similar triangles to conclude

$$\frac{a\sin\phi}{d} \approx \frac{B_\phi}{B} \qquad \Rightarrow \qquad B_\phi = m\frac{a}{d^3}\sin\phi = B_e\frac{mK_i}{d^3}\sin\phi.$$

As expected, this component vanishes for $\phi = 0$.

The radial component is given by the difference in the lengths of the two field vectors, or

$$B_r = m\left(\frac{1}{d^2} - \frac{1}{(d+x)^2}\right) = \frac{m}{d^2}\left(1 - \frac{1}{(1+x/d)^2}\right) \approx \frac{m}{d^2}\frac{2x}{d},$$

where $x = a\cos\phi$ is the length of the green side, so

$$B_r = 2B_e\frac{mK_i}{d^3}\cos\phi.$$

That wasn't so bad, was it?

d. If placed directly to the right and left of the ship compass, the iron balls can be located at a distance d to cancel out the error in the magnetic heading for any angle(s) where $\delta\theta$ is largest. Assuming that this is done, find the resulting expression for the combined deviation $\delta\theta$ due to the ship and the balls for the magnetic heading for all angles θ.

Solution

Note that the two iron balls create a magnetic field near the compass that behaves like that of the ship as a whole. There is a component directed toward the bow given by

$$B_b = -2B_\theta \propto \sin\phi \propto \cos\theta$$

and a component directed toward the starboard given by

$$B_s = 2B_r \propto \cos\phi \propto \sin\theta$$

where the factors of 2 are because there are two balls. Note that θ is the ship heading while ϕ is the angle between North and the location of the compass relative to one of the balls. Thus, if the field is corrected for the maximum angles it will necessarily cancel out the induced ship field for all of the angles, so that

$$\delta\theta = 0$$

for all θ. Effectively, this means placing the balls to make $K_b = K_s$.

Question A4

Relativistic particles obey the mass energy relation

$$E^2 = (pc)^2 + (mc^2)^2$$

where E is the relativistic energy of the particle, p is the relativistic momentum, m is the mass, and c is the speed of light.

A proton with mass m_p and energy E_p collides head on with a photon which is massless and has energy E_b. The two combine and form a new particle with mass m_Δ called Δ, or "delta". It is a one dimensional collision that conserves both relativistic energy and relativistic momentum.

a. Determine E_p in terms of m_p, m_Δ, and E_b. You may assume that E_b is small.

Solution

Solution 1. We can solve the problem exactly, approximating only in the last step. This certainly isn't necessary; we do this to illustrate a useful technique. We set $c = 1$ throughout, and transform to an inertial frame where the proton is initially at rest. Before the collision,

$$E_p = m_p, \quad E_\gamma = |p_\gamma|.$$

For the Δ particle, we have the usual relativistic relation

$$E_\Delta^2 = p_\Delta^2 + m_\Delta^2.$$

By energy-momentum conservation,

$$E_p + E_\gamma = E_\Delta, \quad p_\gamma = p_\Delta.$$

Combining these results gives

$$(m_p + E_\gamma)^2 = E_\gamma^2 + m_\Delta^2 \quad \Rightarrow \quad E_\gamma = \frac{m_\Delta^2 - m_p^2}{2m_p}.$$

Now we need to transform back to the original frame, where the energy of the photon is E_b. We can use the Lorentz transformation for this, but it's a little easier to realize that $E = hf$ for photons, and apply the Doppler shift. Then

$$\alpha \equiv \frac{E_b}{E_\gamma} = \sqrt{\frac{1-\beta}{1+\beta}}$$

where β is the velocity parameter of the proton in the inertial frame where the photon has energy E_b. Solving for β in terms of the energy ratio α,

$$\beta = \frac{1-\alpha^2}{1+\alpha^2}.$$

To calculate the proton energy, we need the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1+\alpha^2}{2\alpha}.$$

Then the proton energy in the original frame is

$$E_p = \gamma m_p = \frac{m_p}{2}\left(\alpha + \frac{1}{\alpha}\right) = \frac{m_p}{2}\left(\frac{2m_pE_b}{m_\Delta^2 - m_p^2} + \frac{m_\Delta^2 - m_p^2}{2m_pE_b}\right)$$

which is the exact answer.

At this point, we can approximate. The second term is much larger than the first, so

$$E_p \approx \frac{m_\Delta^2 - m_p^2}{4E_b} = \frac{m_\Delta^2 c^4 - m_p^2 c^4}{4E_b}$$

where we put back the factors of c in the last step.

Solution 2. We now show another method that approximates throughout. We'll do everything in the lab frame, so the symbols in this solution don't mean the same things they did in solution 1. In the lab frame, energy-momentum conservation gives

$$p_p - p_b = p_\Delta, \quad E_p + E_b = E_\Delta.$$

Squaring both expressions and dropping E_b^2 terms, since E_b is small,

$$p_p^2 - 2p_pp_b \approx p_\Delta^2, \quad E_p^2 + 2E_pE_b \approx E_\Delta^2.$$

Combining these equations gives

$$m_p^2c^4 + 2E_pE_b + 2cp_pE_b = m_\Delta^2 c^4 \quad \Rightarrow \quad E_p + cp_p = \frac{m_\Delta^2c^4 - m_p^2c^4}{2E_b}.$$

Since E_b is small, this quantity must be large. But this means the protons are ultrarelativistic, so $E_p \approx cp_p$, giving

$$E_p \approx \frac{m_\Delta^2c^4 - m_p^2c^4}{4E_b}$$

as desired.

b. In this case, the photon energy E_b is that of the cosmic background radiation, which is an EM wave with wavelength 1.06 mm. Determine the energy of the photons, writing your answer in electron volts.

Solution

Plugging in the numbers gives

$$E = \frac{hc}{\lambda} = 0.00112 \text{ eV}.$$

c. Assuming this value for E_b, what is the energy of the proton, in electron volts, that will allow the above reaction? This sets an upper limit on the energy of cosmic rays. The mass of the proton is given by $m_p c^2 = 938$ MeV and the mass of the Δ is given by $m_\Delta c^2 = 1232$ MeV.

Solution

Plugging in the numbers gives

$$E_p = 1.4 \times 10^{20} \text{ eV}.$$

This is known as the GZK bound for cosmic rays.

The following relationships may be useful in solving this problem:

velocity parameter	$\beta = \frac{v}{c}$
Lorentz factor	$\gamma = \frac{1}{\sqrt{1-\beta^2}}$
relativistic momentum	$p = \gamma\beta mc$
relativistic energy	$E = \gamma mc^2$
relativistic doppler shift	$\frac{f}{f_0} = \sqrt{\frac{1-\beta}{1+\beta}}$

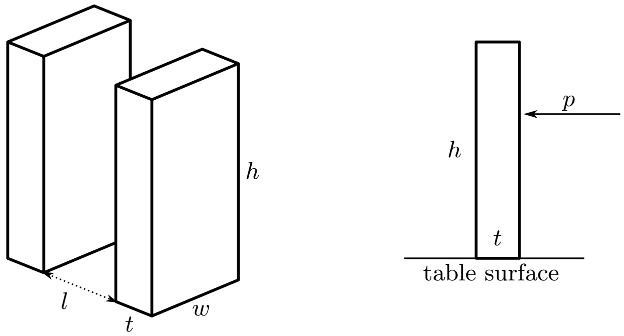
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

Suppose a domino stands upright on a table. It has height h, thickness t, width w (as shown below), and mass m. The domino is free to rotate about its edges, but will not slide across the table.



a. Suppose we give the domino a sharp, horizontal impulsive push with total momentum p.

i. At what height H above the table is the impulse p required to topple the domino smallest?

ii. What is the minimum value of p to topple the domino?

Solution

First we'll look for the height H at which we should push to topple the domino with the least momentum. A convenient method is to look at the angular momentum in the domino because this is easy to calculate and describes rotational motion. Because the push is horizontal, the moment arm of the push (about the domino's rotational axis) is purely vertical. That means the angular momentum of the push is pH, where p is the momentum imparted and H is the height of the push. There is some minimum angular momentum L_{min} to topple the domino, so we set $L_{\text{min}} = pH$. The bigger H, the smaller p, so we should choose the largest possible H. In other words, we should push at the very top of the domino, $H = h$. While we're on this part, note that if the push weren't constrained to be horizontal, the push could be a little bit smaller since the moment arm could be the entire diagonal of the thin edge of the domino.

Next we calculate the minimum p using energy. As the domino rotates, it converts kinetic energy to potential energy, so we'll calculate both. The domino's potential energy is greatest when its center of mass is directly over the contact point of the domino and the table. That height is half the diagonal of the domino, so the distance from the contact point to the center of the domino is $\frac{1}{2}\sqrt{t^2+h^2}$. From the push until it reaches this point, the domino's potential energy increases by

$$\Delta U = \frac{1}{2}mg\left(\sqrt{t^2+h^2} - h\right).$$

Then the domino topples. By conservation of energy, ΔU is how much rotational kinetic energy the domino must have begun with.

Next we find the kinetic energy using the rotational kinetic energy formula, $T = L^2/2I$. The moment of inertia of the domino about its contact point with the table is $I = \frac{1}{3}m(h^2+t^2)$.

You can find this with an integral, or if you know the moment of inertia about the center ($\frac{1}{12}m(h^2+t^2)$), you can use the parallel axis theorem. We know the momentum $L = ph$, so the kinetic energy is

$$T = \frac{1}{2}\frac{L^2}{I} = \frac{1}{2}\frac{(ph)^2}{(1/3)m(h^2+t^2)} = \frac{3}{2}\frac{p^2h^2}{m(h^2+t^2)}.$$

Setting the initial kinetic energy equal to the gain in potential energy and solving for p,

$$p_{\text{min}} = \frac{1}{\sqrt{3}}\frac{m}{h}\sqrt{g\left(\sqrt{t^2+h^2}-h\right)(h^2+t^2)}.$$

b. Next, imagine a long row of dominoes with equal spacing l between the nearest sides of any pair of adjacent dominos, as shown above. When a domino topples, it collides with the next domino in the row. Imagine this collision to be completely inelastic. What fraction of the total kinetic energy is lost in the collision of the first domino with the second domino?

Solution

Right after the collision, the dominoes touch at a height $\sqrt{h^2-l^2}$ above the table. The second domino is vertical, while the first is rotated so that the angle between its leading edge and the table is $\theta = \arccos(l/h)$. Let the dominoes' angular velocities be ω_1 and ω_2 respectively. After the dominoes collide, they stick together. If we take the two parts of the dominoes that are in contact, they must have the same horizontal velocity component in order for the dominoes to stay in contact. For the first domino, this velocity is $\omega_1 h \sin\theta = \omega_1\sqrt{h^2-l^2}$. For the second domino it is ω_2 times the height of the impact point, so $\omega_2\sqrt{h^2-l^2}$. Because these velocity components must be equal, $\omega_1 = \omega_2$.

This means the dominoes have the same angular momentum as each other, measured relative to their respective rotation axes. If we look at the collision, the forces between the dominoes are purely horizontal because the dominoes' faces are frictionless, so they only exert normal forces. The second domino is vertical, so all its normal forces are purely horizontal. These horizontal normal forces exchange angular momentum between the two dominoes. They have the same moment arm (i.e. the height of the collision), so the amount of angular momentum transferred out of the first domino is equal to the amount gained by the second domino, both measured relative to the dominoes' respective rotation axes. Because the dominoes are identical and have the same angular velocity, they have the same angular momentum. This means each domino has half as much angular momentum about its rotation axis as the first domino had just before the collision. (Note that forces from the table cannot change the angular momentum of the dominoes about their respective rotational axes because such forces have zero moment arm, so only the inter-domino forces need to be examined here.)

Kinetic energy scales with the square of angular momentum, so each domino has a quarter as much kinetic energy after the collision as the first domino had before it. That means the total kinetic energy after the collision is half what it was before the collision. The fraction of kinetic energy lost is one half.

c. After the collision, the dominoes rotate in such a way so that they always remain in contact. Assume that there is no friction between the dominoes and the first domino was given the smallest possible push such that it toppled. What is the minimum l such that the second domino will topple?

You may work to lowest nontrivial order in the angles through which the dominoes have rotated. Equivalently, you may approximate $t,\ l \ll h$.

Solution

As in part (a), we will find the highest potential energy of the system and make sure the initial kinetic energy is high enough to get the system to that point.

At any time after the collision, let us call the angle that the first domino has rotated past its point of highest potential energy α, and the angle that the second domino has rotated past its point of highest potential energy β. Also, let's call the angle a domino rotates from its standing position up to its point of highest potential energy ϕ, so the dominoes have rotated $\phi + \alpha$ and $\phi + \beta$ respectively.

The dominoes need to be in contact. The top right corner of the first domino has moved horizontally a distance $h \sin(\phi + \alpha)$ which we will approximate as $h(\phi + \alpha)$ using the small angle approximation. The top left corner of the second domino moves horizontally forward by $h \sin(\phi + \beta) + t(1 - \cos(\phi + \beta))$. We ignore the cosine term as second order in the rotation angle and approximate this as $h(\phi + \beta)$.

The y-coordinate of the upper right corner of the first domino is $h(1 - \cos(\phi + \alpha)) \approx h$. The y-coordinate of the upper left corner of the second domino is $h(1 - \cos(\phi + \beta)) + t \sin(\phi + \beta) \approx h + t(\phi + \beta)$ There is a first-order difference in y-coordinates of the two corners, but this means the difference in x coordinate between the top left corner of the second domino and the top right corner of the first domino is second-order in the rotation angles. We conclude that to first order

$$h(\phi + \alpha) = l + h(\phi + \beta) \quad \Rightarrow \quad \alpha = \frac{l}{h} + \beta$$

because this condition puts the top right corner of the first domino at the same position as the top left of the second domino.

The potential energy of the first domino, setting zero potential energy to be when the domino is upright, is $U_1 = \Delta U(1 - (\alpha/\phi)^2)$ to second order in α, and for the second domino, $U_2 = \Delta U(1 - (\beta/\phi)^2)$. These figures come from fitting a quadratic whose peak is when the center of mass is above the rotation point and which is zero when the domino is upright. The total potential energy is $U_1 + U_2$, and using the relation between α and β, it is minimized for

$$\beta_{\max} = -\frac{l}{2h}.$$

In other words, the second domino is as far away from rotating to the top of arc as the first domino has rotated past the top of its arc. This gives a maximum potential energy

$$U_{\max} = 2\Delta U \left(1 - \frac{l^2}{4t^2}\right)$$

where we have used the approximation $\phi \approx t/h$.

At impact, $U_{\text{impact}} \approx \Delta U(-l^2/t^2 + 2l/t)$. Before impact, the kinetic energy is $\Delta U - U_{\text{impact}}$ because the maximum potential energy before impact was ΔU, and the kinetic energy was zero there. The kinetic energy just after the collision is then $(\Delta U - U_{\text{impact}})/2$. Setting this equal to the potential energy gain as the two dominoes rotate to their highest potential energy U_{max},

$$\frac{1}{2}(\Delta U - U_{\text{impact}}) = U_{\text{max}} - U_{\text{impact}}$$

or

$$\frac{1}{2}\Delta U = U_{\text{max}} - \frac{1}{2}U_{\text{impact}}.$$

Plugging in the earlier expressions for all these gives

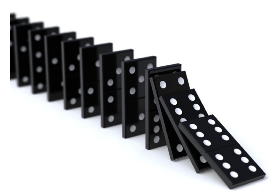
$$\frac{1}{2}\Delta U = 2\Delta U\left(1 - \frac{l^2}{4t^2}\right) - \Delta U\left(\frac{l}{t} - \frac{l^2}{2t^2}\right)$$

and solving this yields

$$l = \frac{3}{2}t$$

to first order in t.

d. A row of toppling dominoes can be considered to have a propagation speed of the length $l+t$ divided by the time between successive collisions. When the first domino is given a minimal push just large enough to topple and start a chain reaction of toppling dominoes, the speed increases with each domino, but approaches an asymptotic speed v.



Suppose there is a row of dominoes on another planet. These dominoes have the same density as the dominoes previously considered, but are twice as tall, wide, and thick, and placed with a spacing of $2l$ between them. If this row of dominoes topples with the same asymptotic speed v previously found, what is the local gravitational acceleration on this planet?

Solution

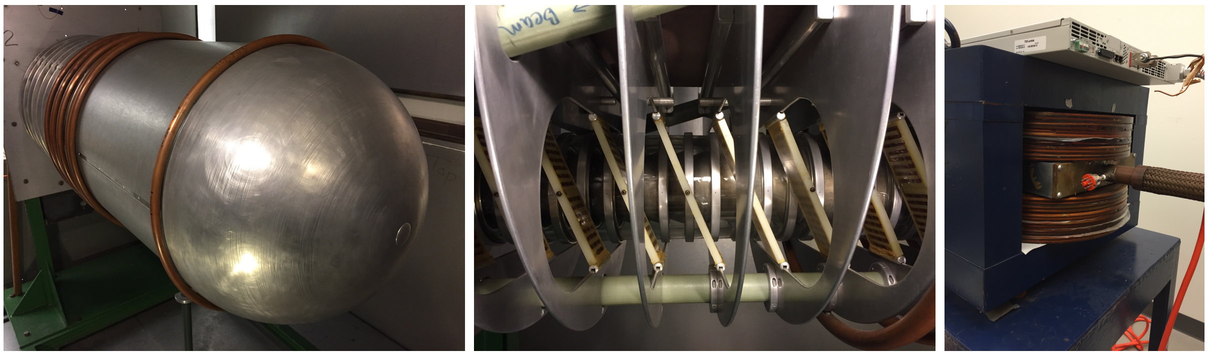
This part is independent of the others and requires only dimensional analysis. The speed v can depend on g, h, w, l. To get a quantity with dimensions $[LT^{-1}]$, we must take

$$v = c\sqrt{gL}$$

where L is some length made from h, w, and l. On the new planet, v is the same and L is twice as much, so g must be half as great on the new planet.

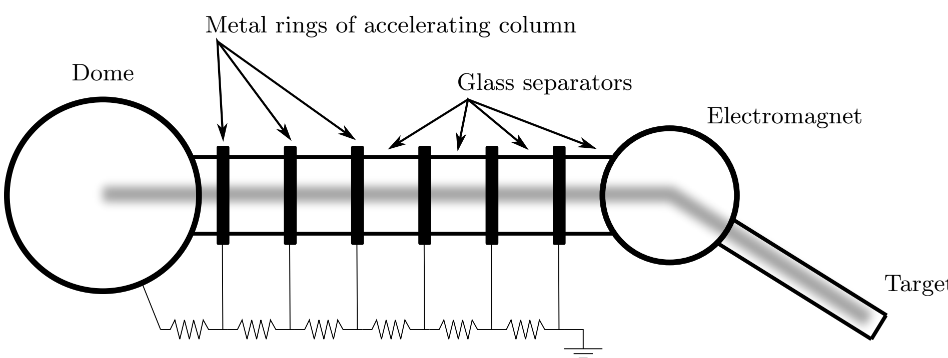
Question B2

Beloit College has a "homemade" 500 kV VanDeGraff proton accelerator, designed and constructed by the students and faculty.



Accelerator dome (assume it is a sphere); accelerating column; bending electromagnet

The accelerator dome, an aluminum sphere of radius $a = 0.50$ meters, is charged by a rubber belt with width $w = 10$ cm that moves with speed $v_b = 20$ m/s. The accelerating column consists of 20 metal rings separated by glass rings; the rings are connected in series with 500 MΩ resistors. The proton beam has a current of 25 μA and is accelerated through 500 kV and then passes through a tuning electromagnet. The electromagnet consists of wound copper pipe as a conductor. The electromagnet effectively creates a uniform field B inside a circular region of radius $b = 10$ cm and zero outside that region.



Only six of the 20 metals rings and resistors are shown in the figure. The fuzzy grey path is the path taken by the protons as they are accelerated from the dome, through the electromagnet, into the target.

a. Assuming the dome is charged to 500 kV, determine the strength of the electric field at the surface of the dome.

Solution

The electric potential is given by

$$V = \frac{q}{4\pi\epsilon_0 a}$$

and the electric field is given by

$$E = \frac{q}{4\pi\epsilon_0 a^2}$$

so

$$E = \frac{V}{a} = 10^6 \text{ V/m}.$$

b. Assuming the proton beam is off, determine the time constant for the accelerating dome (the time it takes for the charge on the dome to decrease to $1/e \approx 1/3$ of the initial value.

Solution

The time constant is given by

$$\tau = RC$$

where

$$C = Q/V = 4\pi\epsilon_0 a = 5.56 \times 10^{-11} \text{ F}$$

and

$$R = 20r_0 = 10^{10}\ \Omega$$

so

$$\tau = RC = 0.556 \text{ s}.$$

c. Assuming the 25 μA proton beam is on, determine the surface charge density that must be sprayed onto the charging belt in order to maintain a steady charge of 500 kV on the dome.

Solution

There are several ways the dome can discharge, two of which are along the resistors and the proton beam. We will ignore any other path.

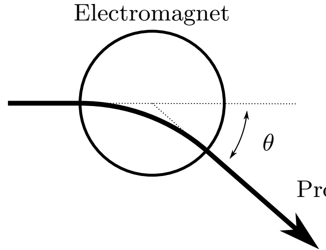
At 500kV, the current through the resistor chain is 50 μA, from $V = IR$. So the total current I needed to be supplied to the dome is 75 μA. This is sprayed onto the belt, which moves at a rate of

$$\frac{\delta A}{\Delta t} = v_b w$$

so the necessary surface charge density is

$$\sigma = \frac{I}{v_b w} = \frac{(75\ \mu\text{C/s})}{(20 \text{ m/s})(0.10 \text{ m})} = 37.5\ \mu\text{C/m}^2.$$

d. The proton beam enters the electromagnet and is deflected by an angle $\theta = 10°$. Determine the magnetic field strength.



Solution

Start with $F = qvB$ where F is the force on the protons, and v the velocity. The protons are non-relativistic, so

$$\frac{1}{2}mv^2 = qV.$$

They move in a circle of radius r inside the field, given by

$$\frac{mv^2}{r} = qvB.$$

Combining these equations gives

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{q}}.$$

Solving for the magnetic field strength,

$$B = \frac{1}{r}\sqrt{\frac{2mV}{q}}.$$

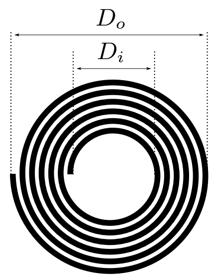
To relate this to the angle, we need to do some geometry. Sketch two circles, one of radius b, the other of radius r, that intersect perpendicular to each other, as in the diagram above. Then drawing a triangle gives

$$\tan\frac{\theta}{2} = \frac{b}{r}.$$

Combining our results, the answer is

$$B = \frac{\tan\theta/2}{b}\sqrt{\frac{2mV}{q}} = 0.0894 \text{ T}.$$

e. The electromagnet is composed of layers of spiral wound copper pipe; the pipe has inner diameter $d_i = 0.40$ cm and outer diameter $d_o = 0.50$ cm. The copper pipe is wound into this flat spiral that has an inner diameter $D_i = 20$ cm and outer diameter $D_o = 50$ cm. Assuming the pipe almost touches in the spiral winding, determine the length L in one spiral.



Solution

Treat the problem as two dimensional. The area of the spiral is

$$A = \frac{\pi}{4}(D_o^2 - D_i^2).$$

The area of the pipe is

$$A = Ld_o.$$

Equating and solving,

$$L = \frac{\pi(D_o^2 - D_i^2)}{4d_o} = 33 \text{ m}.$$

f. Hollow pipe is used instead of solid conductors in order to allow for cooling of the magnet. If the resistivity of copper is $\rho = 1.7 \times 10^{-8}\Omega \cdot \text{m}$, determine the electrical resistance of one spiral.

Solution

We have

$$r_s = \frac{\rho L}{A}$$

where A is the cross sectional area of the pipe, or

$$A = \frac{\pi}{4}(d_o^2 - d_i^2) = 7.1 \times 10^{-6} \text{ m}^2$$

Combining, we have

$$r_s = \frac{\rho}{d}\frac{D_o^2 - D_i^2}{d_o^2 - d_i^2} = 0.079\ \Omega.$$

g. There are $N = 24$ coils stacked on top of each other. Tap water with an initial temperature of $T_c = 18^\circ$ C enters the spiral through the copper pipe to keep it from over heating; the water exits at a temperature of $T_h = 31^\circ$ C. The copper pipe carries a direct 45 Amp current in order to generate the necessary magnetic field. At what rate must the cooling water flow be provided to the electromagnet? Express your answer in liters per second with only one significant digit. The specific heat capacity of water is 4200 J/°C · kg; the density of water is 1000 kg/m^3.

Solution

The rate of heat generation in the coils is given by

$$P = I^2 R = I^2 N r_s = 3850 \text{ W}.$$

This must be dissipated via the increase in water temperature,

$$P = C\Delta T Q$$

where C is the specific heat capacity in liters, and Q is the flow rate in liters per second. But since one liter of water is one kilogram, we can use either C. Combining, we have

$$Q = \frac{I^2 N r}{C\Delta T} = 0.07 \text{ l/s}.$$

h. The protons are fired at a target consisting of Fluorine atoms ($Z = 9$). What is the distance of closest approach to the center of the Fluorine nuclei for the protons? You can assume that the Fluorine does not move.

Solution

Conservation of energy gives

$$qV = \frac{1}{4\pi\epsilon_0}\frac{Zq^2}{r}$$

where r is the radius of closest approach. Then

$$r = \frac{1}{4\pi\epsilon_0}\frac{Zq}{V} = 2.59 \times 10^{-14} \text{ m}$$

Since this is about the size of a Fluorine nucleus, we can potentially get a nuclear reaction. Actually, the important reaction occurs at about 380 kV.

Exam Statistics

Question	A1	A2	A3	A4	B1	B2	Total
Mean	11	3	5	6	12	15	53
Standard Deviation	7	4	5	7	10	14	29
Maximum	25	21	25	25	42	50	163
Upper Quartile	18	4	9	8	18	26	73
Median	10	2	5	5	10	12	49
Lower Quartile	4	0	0	0	3	1	30
Minimum	0	0	0	0	0	0	5

Some Trivia:

- California had 121 test takers
- New Jersey had 42
- Texas had 38
- Florida, Illinois, Massachusetts, Maryland, New York, and Virginia each had between one dozen and two dozen test takers
- Alabama, Alaska, Colorado, Idaho, Iowa, Louisiana, Montana, North Dakota, South Dakota, and Wyoming did not have a test taker this year.

AAPT UNITED STATES PHYSICS TEAM
AIP 2018

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has three questions and is allowed 90 minutes; Part B also has three questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 3 parts: the cover sheet (page 2), Part A (pages 3-5), Part B (pages 7-9), and several answer sheets for one of the questions in Part A (pages 11-13). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after April 21, 2018.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

AAPT UNITED STATES PHYSICS TEAM
AIP 2018

USA Physics Olympiad Exam

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all three problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete three problems. Each question is worth 25 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your proctor's AAPT ID number, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  Student AAPT ID #

  Proctor AAPT ID #

  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 13, 2018.**

Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}$ C
$1\ \mathrm{eV} = 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x\rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

a. Suppose you drop a block of mass m vertically onto a fixed ramp with angle θ with coefficient of static and kinetic friction μ. The block is dropped in such a way that it does not rotate after colliding with the ramp. Throughout this problem, assume the time of the collision is negligible.

   i. Suppose the block's speed just before it hits the ramp is v and the block slides down the ramp immediately after impact. What is the speed of the block right after the collision?

   ii. What is the minimum μ such that the speed of the block right after the collision is 0?

b. Now suppose you drop a sphere with mass m, radius R and moment of inertia $\beta m R^2$ vertically onto the same fixed ramp such that it reaches the ramp with speed v.

   i. Suppose the sphere immediately begins to roll without slipping. What is the new speed of the sphere in this case?

   ii. What is the minimum coefficient of friction such that the sphere rolls without slipping immediately after the collision?

Question A2

For this problem, graphical answers should be drawn on the answer sheets graphs provided. Supporting work is to be written on blank answer sheets. Incorrect graphs without supporting work will receive no partial credit.

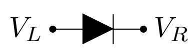
The current I as a function of voltage V for a certain electrical device is

$$I = I_0 e^{-qV_0/k_B T}\left(e^{qV/k_B T} - 1\right)$$

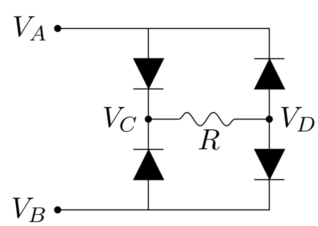
where q is the magnitude of the charge on an electron, k_B is Boltzmann's constant, and T is the absolute temperature. I_0 and V_0 are non-zero positive constants. Throughout this problem assume low temperature values $k_B T \ll qV_0$.

a. On the answer sheets, sketch a graph of the current versus voltage for low temperature values $k_B T \ll qV_0$, clearly indicating any asymptotic behavior.

Shown is a schematic for the device. Positive voltage means that the electric potential of the left hand side of the device is higher than the right hand side. For this device, $I_0 = 25\,\mu\text{A}$ and $V_0 = 1.0\,\text{V}$.

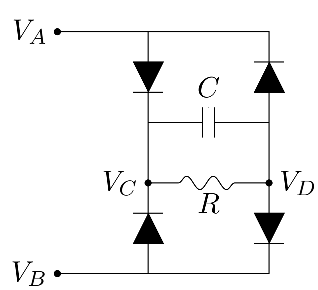


Below is a circuit made up of these elements. The voltage supplied the circuit is sinusoidal, $V_{AB} = V_A - V_B = V_s \sin\omega t$, and is also shown on answer sheets. The resistance is $R = 5.0\,\Omega$ and $V_s = 5.0\,\text{V}$.



b. Sketch the potential difference $V_{CD} = V_C - V_D$ as a function of time on the answer sheet. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time.

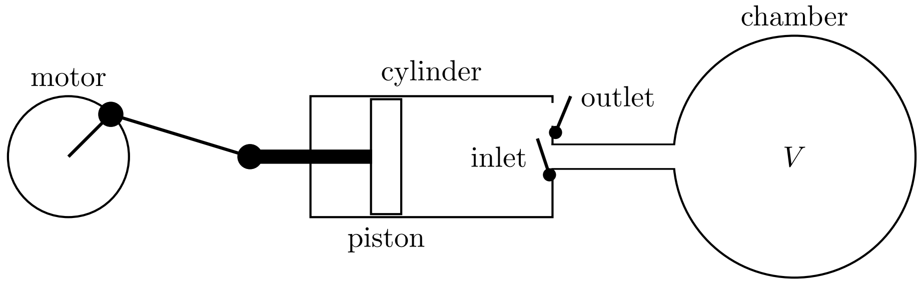
A capacitor is connected to the circuit as shown below. The capacitance is $C = 50\,\text{mF}$.



c. Sketch the new potential difference $V_{CD} = V_C - V_D$ as a function of time on the answer sheet. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time.

Question A3

A vacuum system consists of a chamber of volume V connected to a vacuum pump that is a cylinder with a piston that moves left and right. The minimum volume in the pump cylinder is V_0, and the maximum volume in the cylinder is $V_0 + \Delta V$. You should assume that $\Delta V \ll V$.



The cylinder has two valves. The inlet valve opens when the pressure inside the cylinder is lower than the pressure in the chamber, but closes when the piston moves to the right. The outlet valve opens when the pressure inside the cylinder is greater than atmospheric pressure P_a, and closes when the piston moves to the left. A motor drives the piston to move back and forth. The piston moves at such a rate that heat is not conducted in or out of the gas contained in the cylinder during the pumping cycle. One complete cycle takes a time Δt. You should assume that Δt is a very small quantity, but $\Delta V/\Delta t = R$ is finite. The gas in the chamber is ideal monatomic and remains at a fixed temperature of T_a.

Start with assumption that $V_0 = 0$ and there are no leaks in the system.

a. At $t = 0$ the pressure inside the chamber is P_a. Find an equation for the pressure at a later time t.

b. Find an expression for the temperature of the gas as it is emitted from the pump cylinder into the atmosphere. Your answer may depend on time.

For the remainder of this problem $0 < V_0 < \Delta V \ll V$.

c. Find an expression for the minimum possible pressure in the chamber, P_{min}.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

The electric potential at the center of a cube with uniform charge density ρ and side length a is

$$\Phi \approx \frac{0.1894 \rho a^2}{\epsilon_0}.$$

You do not need to derive this.[1]

For the entirety of this problem, any computed numerical constants should be written to three significant figures.

a. What is the electric potential at a corner of the same cube? Write your answer in terms of ρ, a, ϵ_0, and any necessary numerical constants.

b. What is the electric potential at the tip of a pyramid with a square base of side length a, height $a/2$, and uniform charge density ρ? Write your answer in terms of ρ, a, ϵ_0, and any necessary numerical constants.

c. What is the electric potential due to a square plate with side length a of uniform charge density σ at a height $a/2$ above its center? Write your answer in terms of σ, a, ϵ_0, and any necessary numerical constants.

d. Let $E(z)$ be the electric field at a height z above the center of a square with charge density σ and side length a. If the electric potential at the center of the square is approximately $\dfrac{0.281 a\sigma}{\epsilon_0}$, estimate $E(a/2)$ by assuming that $E(z)$ is linear in z for $0 < z < a/2$. Write your answer in terms of σ, a, ϵ_0, and any necessary numerical constants.

[1] See https://arxiv.org/pdf/chem-ph/9508002.pdf for more details if you are interested.

Question B2

In this problem, use a particle-like model of photons: they propagate in straight lines and obey the law of reflection, but are subject to the quantum uncertainty principle. You may use small-angle approximations throughout the problem.

A photon with wavelength λ has traveled from a distant star to a telescope mirror, which has a circular cross-section with radius R and a focal length $f \gg R$. The path of the photon is nearly aligned to the axis of the mirror, but has some slight uncertainty $\Delta\theta$. The photon reflects off the mirror and travels to a detector, where it is absorbed by a particular pixel on a charge-coupled device (CCD).

Suppose the telescope mirror is manufactured so that photons coming in parallel to each other are focused to the same pixel on the CCD, regardless of where they hit the mirror. Then all small cross-sectional areas of the mirror are equally likely to include the point of reflection for a photon.

a. Find the standard deviation Δr of the distribution for r, the distance from the center of the telescope mirror to the point of reflection of the photon.

b. Use the uncertainty principle, $\Delta r \Delta p_r \geq \hbar/2$, to place a bound on how accurately we can know the angle of the photon from the axis of the telescope. Give your answer in terms of R and λ. If you were unable to solve part a, you may also give your answer in terms of Δr.

c. Suppose we want to build a telescope that can tell with high probability whether a photon it detected from Alpha Centauri A came the left half or right half of the star. Approximately how large would a telescope have to be to achieve this? Alpha Centauri A is approximately 4×10^{16} m from Earth and has a radius approximately 7×10^{8} m. Assume visible light with $\lambda = 500$ nm.

Question B3

Radiation pressure from the sun is responsible for cleaning out the inner solar system of small particles.

a. The force of radiation on a spherical particle of radius r is given by

$$F = PQ\pi r^2$$

where P is the radiation pressure and Q is a dimensionless quality factor that depends on the relative size of the particle r and the wavelength of light λ. Throughout this problem assume that the sun emits a single wavelength λ_{max}; unless told otherwise, leave your answers in terms of symbolic variables.

i. Given that the total power radiated from the sun is given by $L_\odot$, find an expression for the radiation pressure a distance R from the sun.

ii. Assuming that the particle has a density ρ, derive an expression for the ratio $\dfrac{F_{radiation}}{F_{gravity}}$ in terms of $L_\odot$, mass of sun $M_\odot$, ρ, particle radius r, and quality factor Q.

iii. The quality factor is given by one of the following

- If $r \ll \lambda$, $Q \sim (r/\lambda)^2$
- If $r \sim \lambda$, $Q \sim 1$.
- If $r \gg \lambda$, $Q = 1$

Considering the three possible particle sizes, which is most likely to be blown away by the solar radiation pressure?

b. The **Poynting-Robertson** effect acts as another mechanism for cleaning out the solar system.

i. Assume that a particle is in a circular orbit around the sun. Find the speed of the particle v in terms of $M_\odot$, distance from sun R, and any other fundamental constants.

ii. Because the particle is moving, the radiation force is *not* directed directly away from the sun. Find the torque τ on the particle because of radiation pressure. You may assume that $v \ll c$.

iii. Since $\tau = dL/dt$, the angular momentum L of the particle changes with time. As such, develop a differential equation to find dR/dt, the rate of change of the radial location of the particle. You may assume the orbit is always quasi circular.

iv. Develop an expression for the time required to remove particles of size $r \approx 1\,\text{cm}$ and density $\rho \approx 1000\,\text{kg/m}^3$ originally in circular orbits at a distance $R = R_{earth}$, and use the numbers below to simplify your expression.

Some useful constants include

$$\begin{aligned} M_\odot &= 1.989 \times 10^{30}\,\text{kg} \\ L_\odot &= 3.828 \times 10^{26}\,\text{W} \\ R_{earth} &= 1.5 \times 10^{11}\,\text{m} \\ \lambda_{\text{max}} &= 500\,\text{nm} \end{aligned}$$

Answer Sheets

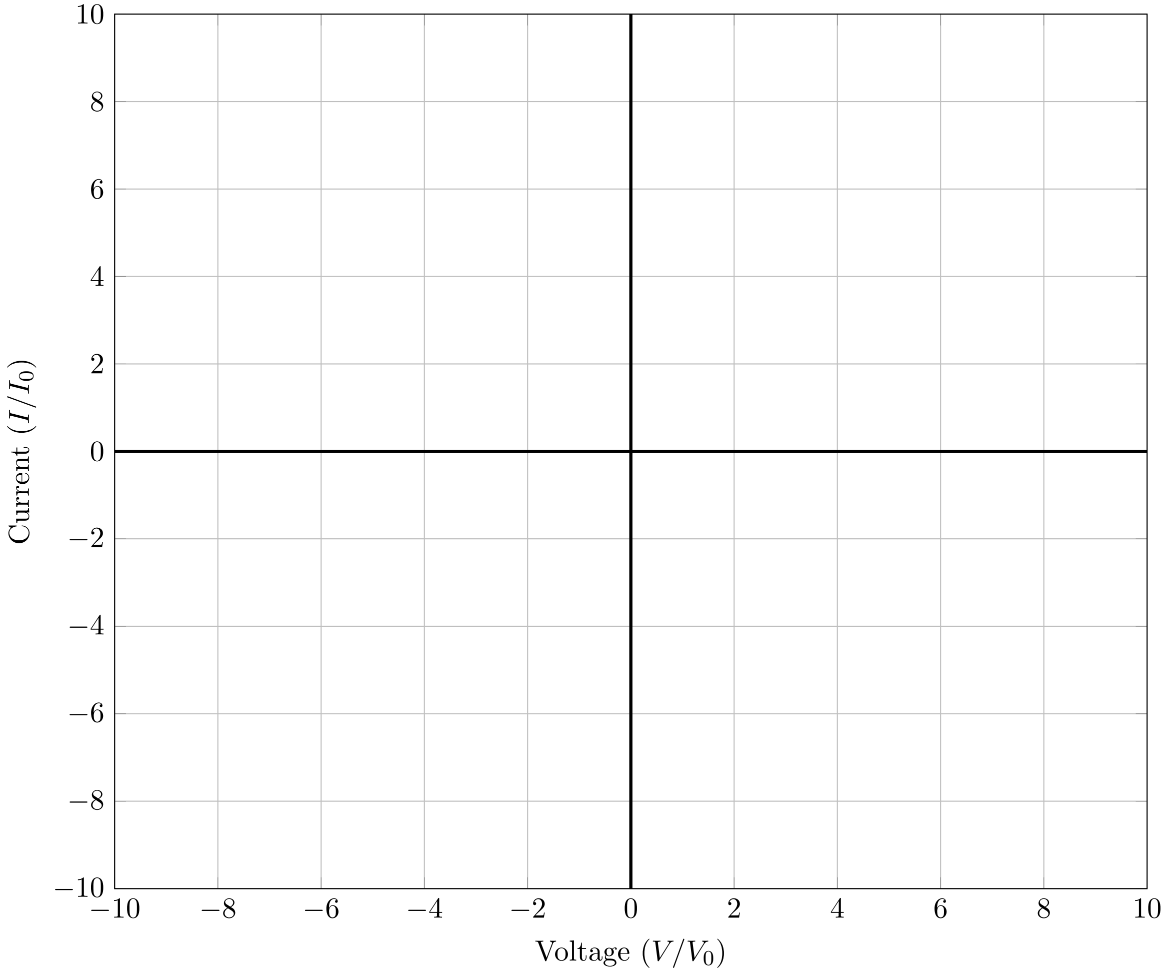
Following are answer sheets for some of the graphical portions of the test.

Student AAPT ID #:

Proctor AAPT ID #:

Question A2

a. Sketch a graph of the current versus voltage for low temperature values $k_B T \ll qV_0$, clearly indicating any asymptotic behavior.

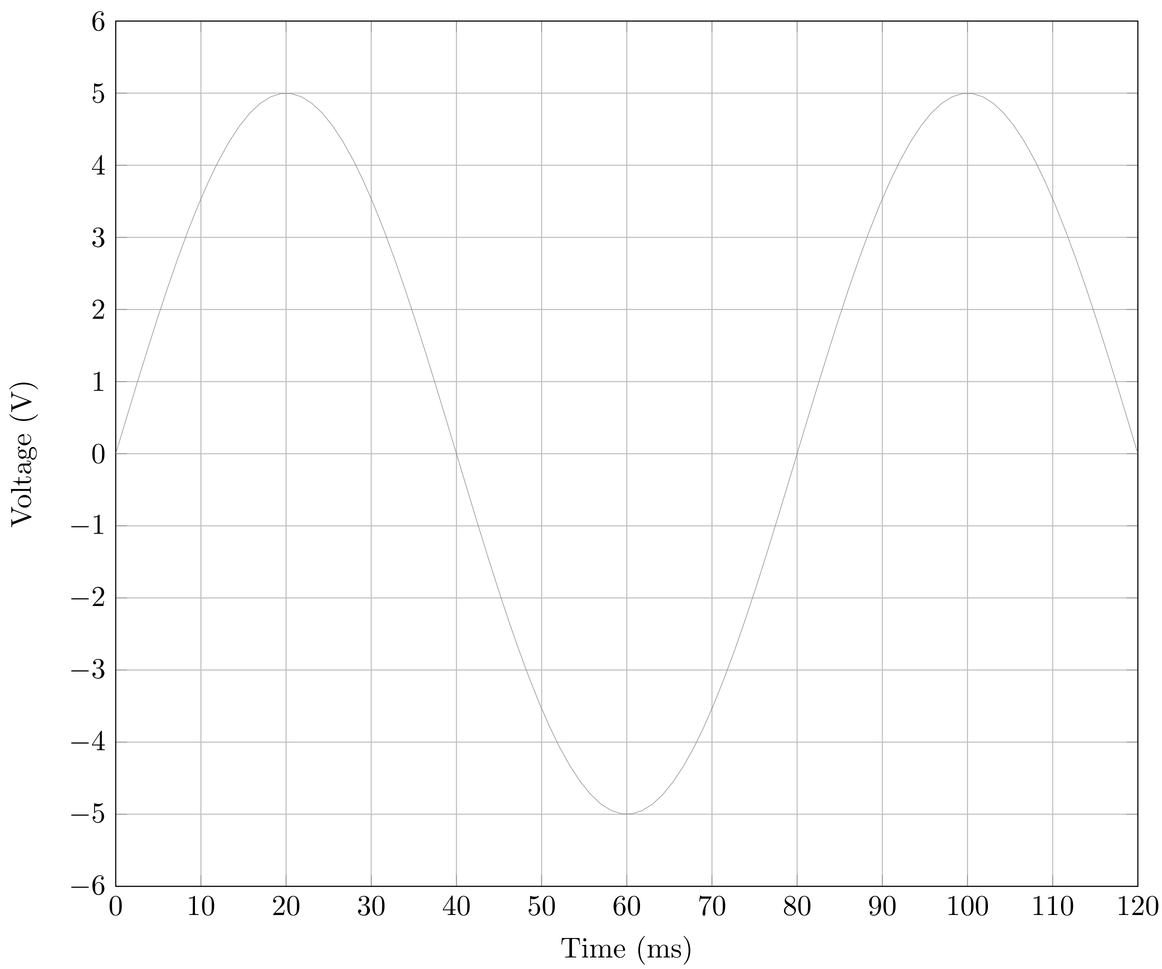


Student AAPT ID #:

Proctor AAPT ID #:

Question A2

b. Sketch the potential difference $V_{CD} = V_C - V_D$ as a function of time. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time. There is **no** capacitor in this circuit!



Student AAPT ID #:

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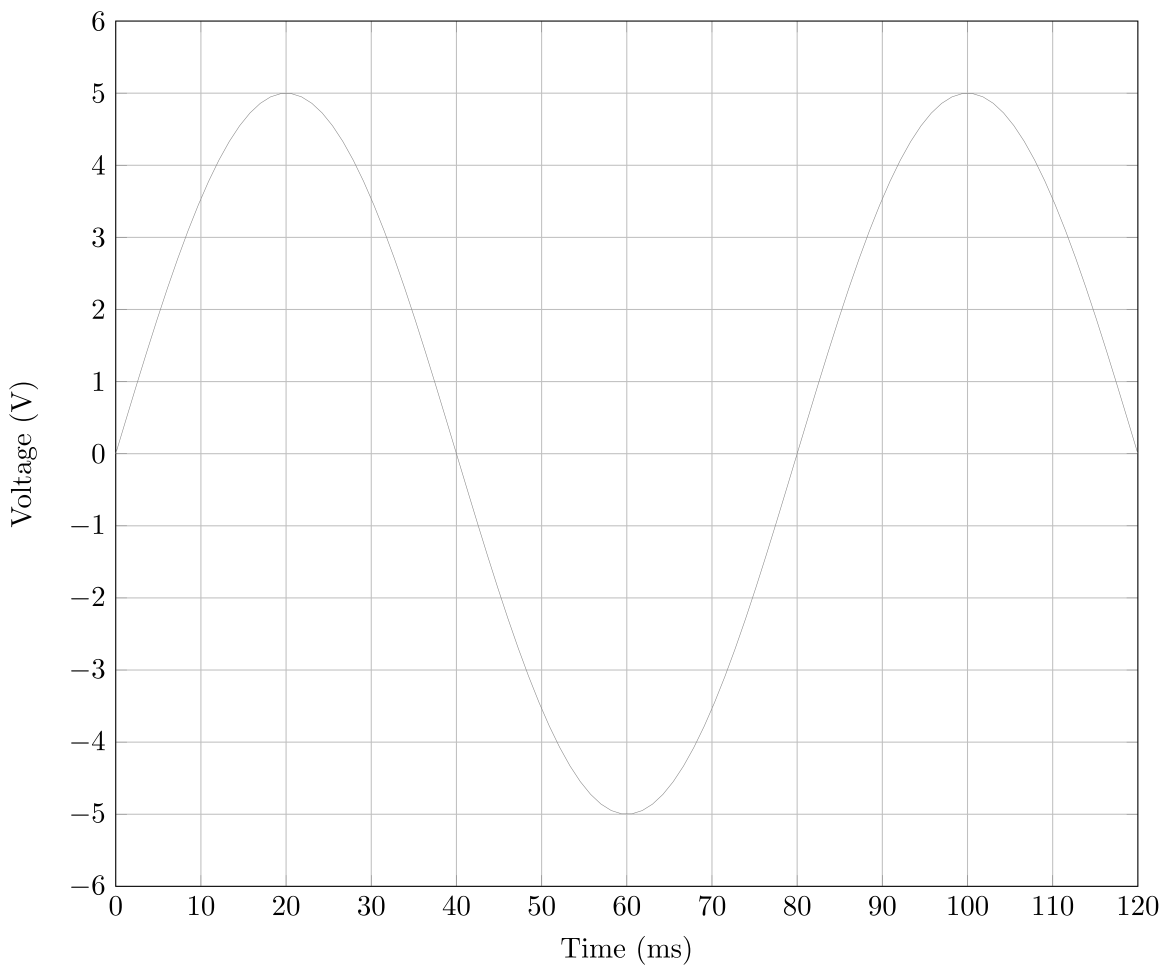
Question A2

c. Sketch the potential difference $V_{CD} = V_C - V_D$ as a function of time. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time. There is a capacitor in this circuit!



AAPT UNITED STATES PHYSICS TEAM
AIP 2018

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has three questions and is allowed 90 minutes; Part B also has three questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
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AAPT AIP UNITED STATES PHYSICS TEAM 2018

USA Physics Olympiad Exam

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  Student AAPT ID #

  Proctor AAPT ID #

  A1 - 1/3

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$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}$ C
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$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

a. Suppose you drop a block of mass m vertically onto a fixed ramp with angle θ with coefficient of static and kinetic friction μ. The block is dropped in such a way that it does not rotate after colliding with the ramp. Throughout this problem, assume the time of the collision is negligible.

i. Suppose the block's speed just before it hits the ramp is v and the block slides down the ramp immediately after impact. What is the speed of the block right after the collision?

Solution

During the collision, the block receives impulses from the normal force, friction, and gravity. Since the collision is very short, the impulse due to gravity is negligible. Let p_N and p_F be the magnitudes of the impulses from the normal force and friction force.

Since the block stays on the ramp after the collision, its final momentum is parallel to the ramp. Then the normal force must completely eliminate the block's initial momentum perpendicular to the ramp, so

$$p_N = mv\cos\theta.$$

Since the block still moves after the collision,

$$p_F = \mu p_N.$$

The block's initial momentum parallel to the ramp is $mv\sin\theta$, so

$$mv\sin\theta - p_F = mu,$$

where u is the final speed of the block. Solving for u gives

$$u = v(\sin\theta - \mu\cos\theta).$$

ii. What is the minimum μ such that the speed of the block right after the collision is 0?

Solution

We set $u = 0$ to obtain

$$\mu = \tan\theta.$$

Note that this is simply the no-slip condition for a block resting on an inclined plane! This is because in both cases, equality is achieved when the normal force and maximal friction force sum to a purely vertical force.

b. Now suppose you drop a sphere with mass m, radius R and moment of inertia βmR^2 vertically onto the same fixed ramp such that it reaches the ramp with speed v.

i. Suppose the sphere immediately begins to roll without slipping. What is the new speed of the sphere in this case?

Solution

If the sphere immediately begins to roll without slipping, we can calculate the frictional impulse independently of the normal impulse. We have

$$mv\sin\theta - p_F = mu.$$

The frictional impulse is responsible for the sphere's rotation, so its angular momentum about its center of mass is $L = p_F R$. But we also know that

$$L = \beta m R^2 \omega = \beta m R u.$$

Then

$$p_F = \beta m u.$$

Substituting into the previous expression gives

$$mv\sin\theta = (1+\beta)mu \quad \Rightarrow \quad u = \frac{v\sin\theta}{1+\beta}.$$

ii. What is the minimum coefficient of friction such that the sphere rolls without slipping immediately after the collision?

Solution

As in part (a), the normal impulse is $p_N = mv\cos\theta$ and the maximal frictional impulse is $p_F = \mu p_N$. From the previous part, we need

$$p_F = \frac{\beta m v \sin\theta}{1+\beta}$$

and equating these expressions gives

$$\mu = \frac{\beta\tan\theta}{1+\beta}.$$

Question A2

For this problem, graphical answers should be drawn on the answer sheets graphs provided. Supporting work is to be written on blank answer sheets. Incorrect graphs without supporting work will receive no partial credit.

The current I as a function of voltage V for a certain electrical device is

$$I = I_0 e^{-qV_0/k_BT}\left(e^{qV/k_BT} - 1\right)$$

where q is the magnitude of the charge on an electron, k_B is Boltzmann's constant, and T is the absolute temperature. I_0 and V_0 are non-zero positive constants. Throughout this problem assume low temperature values $k_BT \ll qV_0$.

a. On the answer sheets, sketch a graph of the current versus voltage for low temperature values $k_BT \ll qV_0$, clearly indicating any asymptotic behavior.

Solution

The current is simply proportional to $e^{qV/k_BT} - 1$, which is a shifted exponential. Then I always has the same sign as V, and vanishes when V vanishes. The current grows quickly for high V and approaches a constant for low V.

This answer is acceptable, but we can use the condition $k_BT \ll qV_0$ to simplify the graph. For negative V, we have

$$I/I_0 \approx e^{-qV_0/k_BT}$$

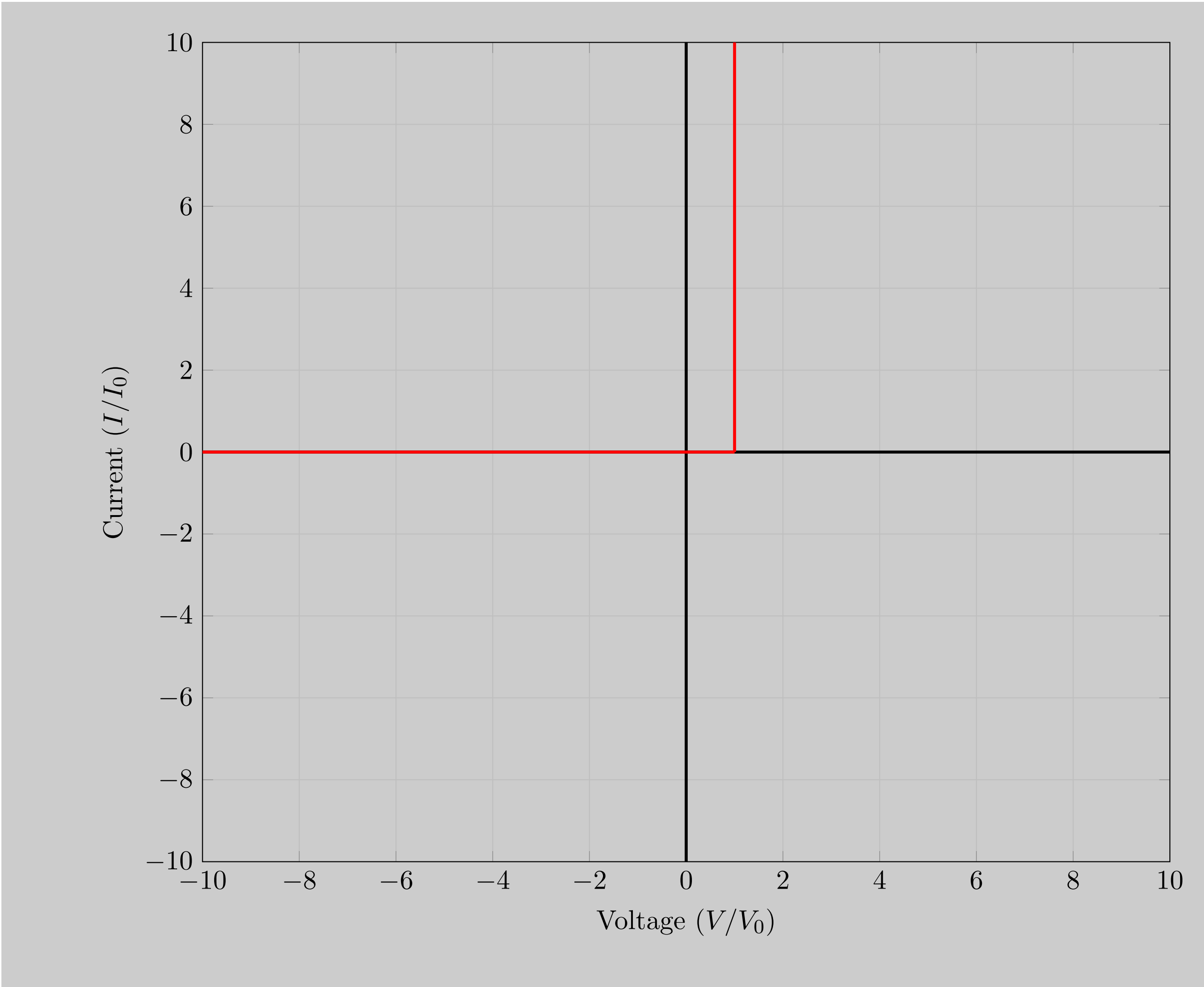
which is extremely small. For positive V, we have

$$I/I_0 \approx e^{q(V-V_0)/k_BT}$$

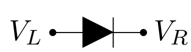
which is extremely small when $V < V_0$ and extremely large when $V > V_0$. Then

$$\frac{I}{I_0} \approx \begin{cases} 0 & V < V_0, \\ \infty & V > V_0 \end{cases}$$

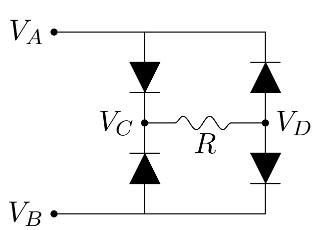
as shown below. Accounting for finite temperature, which is not necessary for full credit, simply rounds the corners in all of the graphs.



Shown is a schematic for the device. Positive voltage means that the electric potential of the left hand side of the device is higher than the right hand side. For this device, $I_0 = 25\,\mu\text{A}$ and $V_0 = 1.0\,\text{V}$.



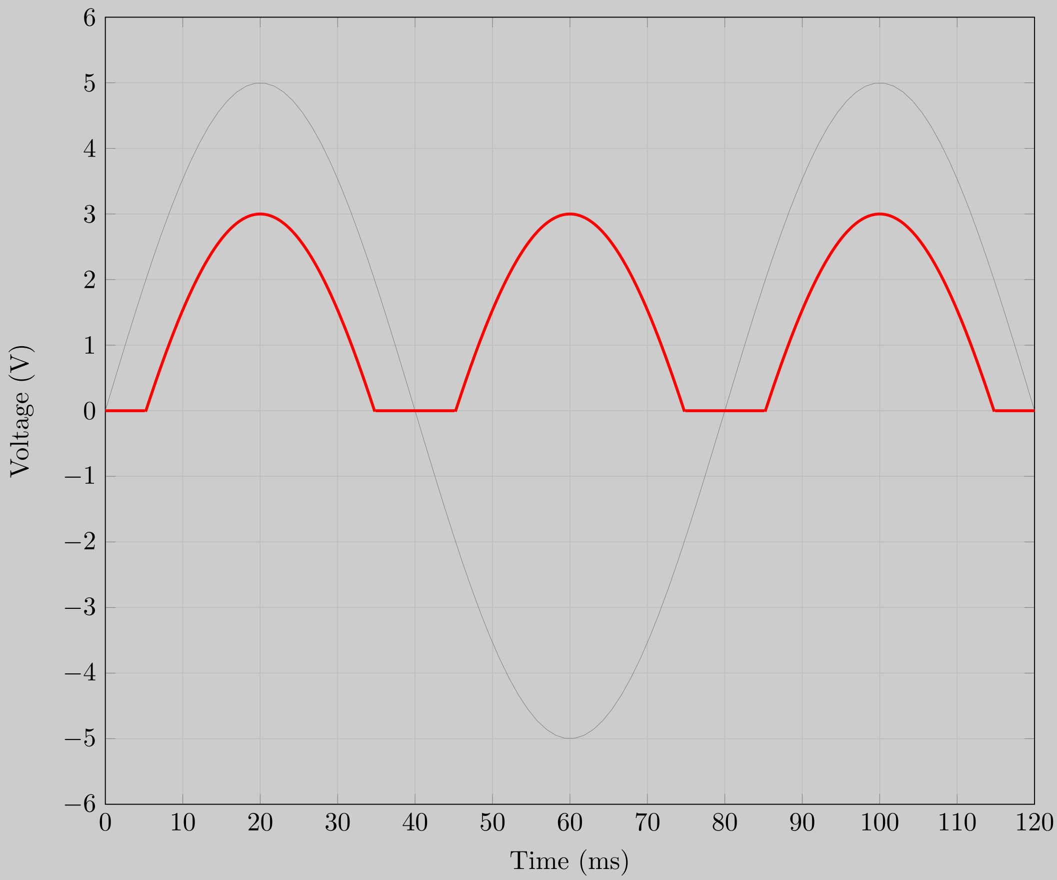
Below is a circuit made up of these elements. The voltage supplied the circuit is sinusoidal, $V_{AB} = V_A - V_B = V_s \sin \omega t$, and is also shown on answer sheets. The resistance is $R = 5.0\,\Omega$ and $V_s = 5.0\,\text{V}$.



b. Sketch the potential difference $V_{CD} = V_C - V_D$ as a function of time on the answer sheet. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time.

Solution

When $|V_{AB}| < 2V_0$, no current flows. When $|V_{AB}| > 2V_0$, current begins to flow, with each diode subtracting a potential difference of V_0. Note that the current flows in the same direction for both positive and negative V_{AB}. This device is a rectifier.



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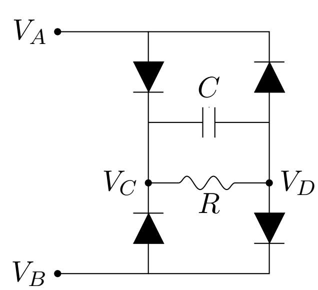


c. Sketch the new potential difference $V_{CD} = V_C - V_D$ as a function of time on the answer sheet. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time.

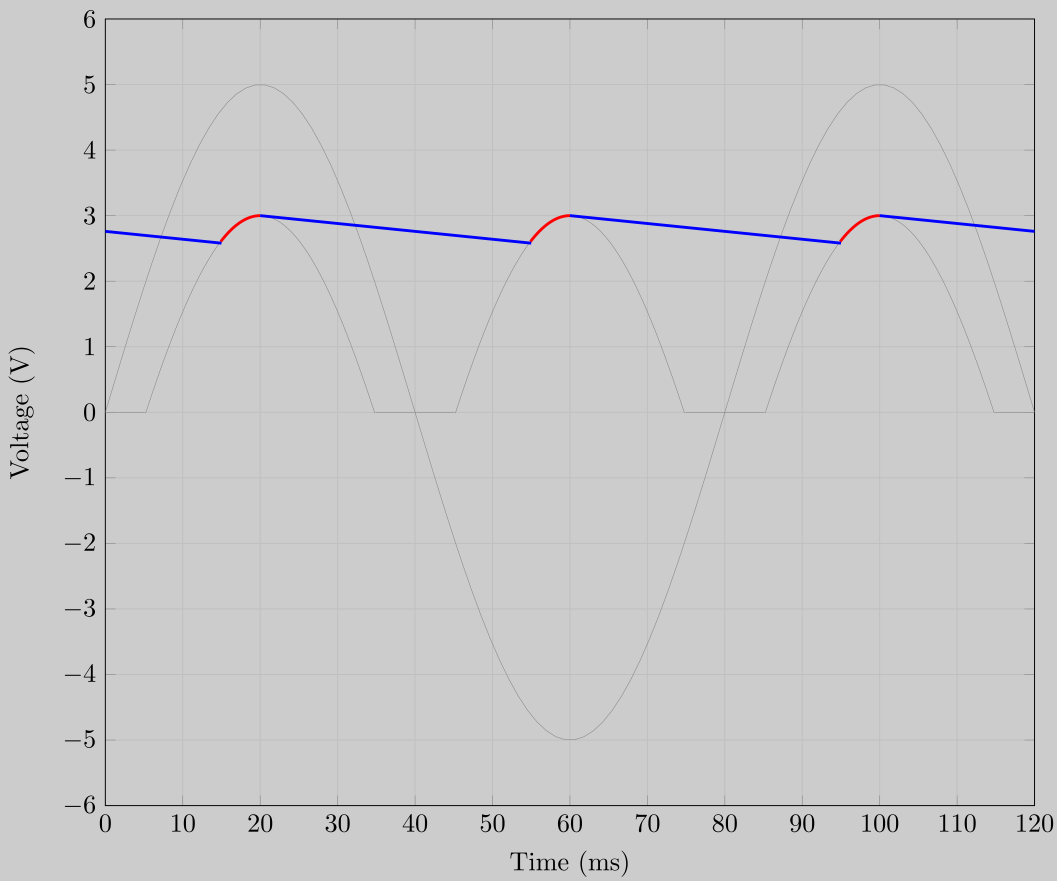
Solution

Let the voltage on the capacitor be V_s. Whenever $|V_{AB}| \geq V_s + 2V_0$, current flows through the diodes, charging the capacitor up to voltage $|V_{AB}| - 2V_0$. Whenever $|V_{AB}| < V_s + 2V_0$, no current flows through the diodes, and the capacitor and resistor simply discharge as an RC circuit with time constant $RC = 250\,\text{ms}$.

Since RC is much longer than the timescale on the answer sheets, the discharge is approximately linear, with

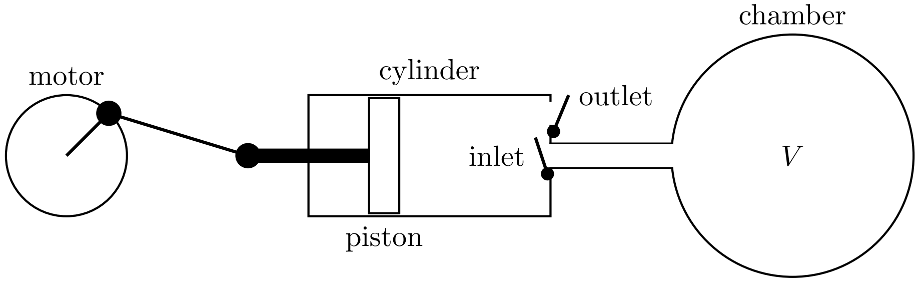
$$\frac{dV_s}{dt} = \frac{1.0\,\text{V}}{83.3\,\text{ms}}.$$

Thus the capacitor charges completely to $3.0\,\text{V}$ every cycle, then discharges approximately linearly by about a half volt during the reminder of the cycle. On the graph we show charging as red and discharging as blue.



Question A3

A vacuum system consists of a chamber of volume V connected to a vacuum pump that is a cylinder with a piston that moves left and right. The minimum volume in the pump cylinder is V_0, and the maximum volume in the cylinder is $V_0 + \Delta V$. You should assume that $\Delta V \ll V$.



The cylinder has two valves. The inlet valve opens when the pressure inside the cylinder is lower than the pressure in the chamber, but closes when the piston moves to the right. The outlet valve opens when the pressure inside the cylinder is greater than atmospheric pressure P_a, and closes when the piston moves to the left. A motor drives the piston to move back and forth. The piston moves at such a rate that heat is not conducted in or out of the gas contained in the cylinder during the pumping cycle. One complete cycle takes a time Δt. You should assume that Δt is a very small quantity, but $\Delta V/\Delta t = R$ is finite. The gas in the chamber is ideal monatomic and remains at a fixed temperature of T_a.

Start with assumption that $V_0 = 0$ and there are no leaks in the system.

a. At $t = 0$ the pressure inside the chamber is P_a. Find an equation for the pressure at a later time t.

Solution

During each cycle, the system sucks gas out of the chamber and pushes it into the atmosphere. Since $V_0 = 0$, the inlet valve opens the moment the piston starts moving to the left. When the piston is all the way to the left, a fraction $\Delta V/(V + \Delta V)$ of the gas is in the cylinder. As the piston moves to the right, all of this gas is pushed out, so after a single cycle,

$$P_f = P_i \left(\frac{V}{V + \Delta V} \right)$$

and in general,

$$P(t) = P_a \left(\frac{V}{V + \Delta V} \right)^{t/\Delta t}.$$

While this is technically correct, it can be simplified significantly. Write

$$P(t) = P_a \left(1 + \frac{\Delta V}{V} \right)^{-t/\Delta t} = P_a \left((1+x)^{1/x} \right)^{-Rt/V}$$

where $x = \Delta V/V \ll 1$. Then using the definition of e,

$$e = \lim_{x \to 0} (1+x)^{1/x}$$

we have

$$P(t) = P_a e^{-Rt/V}.$$

b. Find an expression for the temperature of the gas as it is emitted from the pump cylinder into the atmosphere. Your answer may depend on time.

Solution

When the piston is all the way to the left, the pressure is $P(t)$ and the temperature is T_a. As the piston moves to the right, the gas is adiabatically compressed until its pressure reaches P_a and the outlet valve opens. Since PV^γ is constant during adiabatic compression and PV/T is constant by the ideal gas law,

$$T_{\text{out}}(t) = T_a \left(\frac{P_a}{P(t)}\right)^{1-1/\gamma} = T_a \left(\frac{P_a}{P(t)}\right)^{2/5} = T_a e^{2Rt/5V}$$

where we used $\gamma = 5/3$ for a monatomic ideal gas.

For the remainder of this problem $0 < V_0 < \Delta V \ll V$.

c. Find an expression for the minimum possible pressure in the chamber, P_{min}.

Solution

Since $V_0 > 0$, the inlet valve will not open immediately when the piston begins moving to the left; instead it will open once the pressure in the cylinder equals the pressure in the chamber. Since the expansion of the cylinder is adiabatic, PV^γ is constant, so

$$P_{\text{min}} = P_a \left(\frac{V_0}{V_0 + \Delta V}\right)^\gamma = P_a \left(1 + \frac{\Delta V}{V_0}\right)^{-\gamma}.$$

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

The electric potential at the center of a cube with uniform charge density ρ and side length a is

$$\Phi \approx \frac{0.1894\rho a^2}{\epsilon_0}.$$

You do not need to derive this.[1]

For the entirety of this problem, any computed numerical constants should be written to three significant figures.

a. What is the electric potential at a corner of the same cube? Write your answer in terms of ρ, a, ϵ_0, and any necessary numerical constants.

Solution

By dimensional analysis, the answer takes the form

$$\Phi_c(a, \rho) \approx \frac{C\rho a^2}{\epsilon_0}$$

for a dimensionless constant C. Note that a cube of side length a consists of 8 cubes of side length $a/2$, each with a corner at the center of the larger cube. Then

$$\frac{0.1894\rho a^2}{\epsilon_0} = 8\frac{C\rho(a/2)^2}{\epsilon_0}$$

so $C = 0.1894/2 = 0.0947$.

b. What is the electric potential at the tip of a pyramid with a square base of side length a, height $a/2$, and uniform charge density ρ? Write your answer in terms of ρ, a, ϵ_0, and any necessary numerical constants.

Solution

A cube of side length a consists of 6 such pyramids. Then we simply compute 0.1894/6 for

$$\Phi_p(a, \rho) \approx \frac{0.0316\,\rho a^2}{\epsilon_0}$$

.

c. What is the electric potential due to a square plate with side length a of uniform charge density σ at a height $a/2$ above its center? Write your answer in terms of σ, a, ϵ_0, and any necessary numerical constants.

[1] See https://arxiv.org/pdf/chem-ph/9508002.pdf for more details if you are interested.

Solution

Let the potential due to such a square be $\Phi_s(a,\sigma)$. Note that adding a square plate of infinitesimal thickness dz and side length a to a square pyramid with base side length a and height $a/2$ yields a square pyramid with base side length $a+2dz$ and height $a/2+dz$.

The surface charge density of a square plate with thickness dz and volume charge density ρ is $\sigma = \rho\,dz$. Then by the principle of superposition,

$$\Phi_s(a,\rho\,dz) = \Phi_p(a+2dz,\rho) - \Phi_p(a,\rho) \approx \frac{0.0316\rho\,((a+2dz)^2 - a^2)}{\epsilon_0} = \frac{0.126\,a\rho\,dz}{\epsilon_0}.$$

so we have

$$\Phi_s(a,\sigma) \approx \frac{0.126\,a\sigma}{\epsilon_0}.$$

d. Let $E(z)$ be the electric field at a height z above the center of a square with charge density σ and side length a. If the electric potential at the center of the square is approximately $\frac{0.281a\sigma}{\epsilon_0}$, estimate $E(a/2)$ by assuming that $E(z)$ is linear in z for $0 < z < a/2$. Write your answer in terms of σ, a, ϵ_0, and any necessary numerical constants.

Solution

The potential difference between height 0 and $a/2$ is

$$\Delta\Phi = (0.281 - 0.126)\frac{a\sigma}{\epsilon_0} = \frac{0.155\,a\sigma}{\epsilon_0}.$$

On the other hand, we have

$$\Delta\Phi = \int_0^{a/2} E(z)\,dz \approx \frac{a}{2}\,\frac{E(0)+E(a/2)}{2}$$

where we approximated $E(z)$ as linear, and $E(0) = \sigma/2\epsilon_0$ by Gauss's law. Solving for $E(a/2)$,

$$E(a/2) \approx \frac{0.119\sigma}{\epsilon_0}$$

where the last significant digit is not important. Incidentally, the actual value is exactly $\sigma/6\epsilon_0$, and this fact has a slick calculation-free proof.

Question B2

In this problem, use a particle-like model of photons: they propagate in straight lines and obey the law of reflection, but are subject to the quantum uncertainty principle. You may use small-angle approximations throughout the problem.

A photon with wavelength λ has traveled from a distant star to a telescope mirror, which has a circular cross-section with radius R and a focal length $f \gg R$. The path of the photon is nearly aligned to the axis of the mirror, but has some slight uncertainty $\Delta\theta$. The photon reflects off the mirror and travels to a detector, where it is absorbed by a particular pixel on a charge-coupled device (CCD).

Suppose the telescope mirror is manufactured so that photons coming in parallel to each other are focused to the same pixel on the CCD, regardless of where they hit the mirror. Then all small cross-sectional areas of the mirror are equally likely to include the point of reflection for a photon.

a. Find the standard deviation Δr of the distribution for r, the distance from the center of the telescope mirror to the point of reflection of the photon.

Solution

The square of the standard deviation is the variance, so

$$(\Delta r)^2 = \langle r^2 \rangle - \langle r \rangle^2.$$

Computing the average value of r^2 is mathematically the exact same thing as computing the moment of inertia of a uniform disk; we have

$$\langle r^2 \rangle = \frac{1}{\pi R^2} \int_0^R r^2 \, (2\pi r dr) = \frac{1}{2} R^2.$$

Similarly, the average value of r is

$$\langle r \rangle = \frac{1}{\pi R^2} \int_0^R r \, (2\pi r dr) = \frac{2}{3} R.$$

Then we have

$$\Delta r = \frac{R}{\sqrt{18}}.$$

b. Use the uncertainty principle, $\Delta r \Delta p_r \geq \hbar/2$, to place a bound on how accurately we can know the angle of the photon from the axis of the telescope. Give your answer in terms of R and λ. If you were unable to solve part a, you may also give your answer in terms of Δr.

Solution

We have

$$\Delta p_r \approx p \Delta\theta = \frac{h}{\lambda} \Delta\theta.$$

Applying the uncertainty principle, we have

$$\Delta\theta \geq \frac{\sqrt{18}\lambda}{4\pi R}.$$

Since the factors of $\hbar$ canceled out, this is really a classical calculation; one could get a similar result by considering classical diffraction from a circular aperture.

c. Suppose we want to build a telescope that can tell with high probability whether a photon it detected from Alpha Centauri A came the left half or right half of the star. Approximately how large would a telescope have to be to achieve this? Alpha Centauri A is approximately $4 \times 10^{16}\,\mathrm{m}$ from Earth and has a radius approximately $7 \times 10^{8}\,\mathrm{m}$. Assume visible light with $\lambda = 500\,\mathrm{nm}$.

Solution

We need $\Delta\theta$ to be much smaller than the actual angular separation, or

$$\Delta\theta \ll \frac{7 \times 10^{8}\,\mathrm{m}}{4 \times 10^{16}\,\mathrm{m}} \approx 2 \times 10^{-8}.$$

This means that

$$R \approx \frac{\sqrt{18}\lambda}{4\pi\Delta\theta} \gg \frac{\sqrt{18}(5 \times 10^{-7}\,\mathrm{m})}{4\pi(2 \times 10^{-8})} = 8.4\,\mathrm{m}$$

Question B3

Radiation pressure from the sun is responsible for cleaning out the inner solar system of small particles.

a. The force of radiation on a spherical particle of radius r is given by

$$F = PQ\pi r^2$$

where P is the radiation pressure and Q is a dimensionless quality factor that depends on the relative size of the particle r and the wavelength of light λ. Throughout this problem assume that the sun emits a single wavelength λ_{max}; unless told otherwise, leave your answers in terms of symbolic variables.

i. Given that the total power radiated from the sun is given by $L_\odot$, find an expression for the radiation pressure a distance R from the sun.

Solution

We will assume that light from the sun is completely absorbed, and then re-radiated as blackbody isotropically. In that case,

$$P = \frac{I}{c} = \frac{L_\odot}{4\pi R^2 c}.$$

The relationship between the pressure P and the energy density I can be derived from the equation $p = E/c$ for photons, or simply postulated by dimensional analysis, since there is no other relevant speed.

Alternatively, this relation can be derived using classical electromagnetism, though this was not required. For a plane wave, the momentum density (i.e. pressure) of the electromagnetic field is $P = \epsilon_0|\mathbf{E}\times\mathbf{B}|$, while the energy density is $I = |\mathbf{S}|/c$ where $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$. Then $P = I/c$ since $c^2 = 1/\epsilon_0\mu_0$.

ii. Assuming that the particle has a density ρ, derive an expression for the ratio $\dfrac{F_{radiation}}{F_{gravity}}$ in terms of $L_\odot$, mass of sun $M_\odot$, ρ, particle radius r, and quality factor Q.

Solution

We have

$$F_{\text{gravity}} = \frac{GM_\odot}{R^2}\frac{4}{3}\pi\rho r^3, \quad F_{\text{radiation}} = \frac{L_\odot}{4\pi R^2 c}Q\pi r^2 = \frac{L_\odot}{4R^2 c}Qr^2$$

which gives

$$\frac{F_{\text{radiation}}}{F_{\text{gravity}}} = \frac{3L_\odot}{16\pi GcM_\odot\rho}\frac{Q}{r}.$$

iii. The quality factor is given by one of the following

- If $r \ll \lambda$, $Q \sim (r/\lambda)^2$

- If $r \sim \lambda$, $Q \sim 1$.
- If $r \gg \lambda$, $Q = 1$

Considering the three possible particle sizes, which is most likely to be blown away by the solar radiation pressure?

Solution

In order to be blown away, the ratio should be greater than one. Since it is independent of distance from the sun, if it is blown away, it will be blown away at any distance. For $r \gg \lambda$, the ratio is proportional to $1/r$, so smaller particles are more likely to be blown away. For $r \ll \lambda$, the ratio is proportional to r, so larger particles are more likely to be blown away. Thus particles of size near λ are most likely to be blown away, and even then, only if the density is small enough.

b. The **Poynting-Robertson** effect acts as another mechanism for cleaning out the solar system.

i. Assume that a particle is in a circular orbit around the sun. Find the speed of the particle v in terms of $M_\odot$, distance from sun R, and any other fundamental constants.

Solution

Using the circular motion equation

$$\frac{GM_\odot}{R^2} = \frac{v^2}{R}$$

we have

$$v = \sqrt{\frac{GM_\odot}{R}}.$$

ii. Because the particle is moving, the radiation force is *not* directed directly away from the sun. Find the torque τ on the particle because of radiation pressure. You may assume that $v \ll c$.

Solution

Work in the reference frame of the particle. In this frame, the radiation hits the particle at an angle $\theta = v/c$ from the radial direction. The particle then re-emits the radiation isotropically, contributing no additional radiation pressure. (We ignore relativistic effects because they occur at second order in v/c, while the effect we care about is first order.) The tangential component of the force is

$$F = \frac{v}{c}\frac{L_\odot}{4\pi R^2 c} Q\pi r^2 = \frac{v}{c}\frac{L_\odot}{4R^2 c} Q r^2$$

so

$$\tau = -\frac{v}{c}\frac{L_\odot}{4Rc} Q r^2$$

where the negative sign is because this tends to decrease the angular momentum. In other words, the Poynting-Robertson effect cleans out the solar system by making particles spiral into the Sun.

The problem can also be solved in the reference frame of the Sun. In this frame, the radiation hits the particle radially, but the particle does *not* re-emit the radiation isotropically since it is moving; instead the radiation is Doppler shifted. This eventually leads to the same result, after a much more complicated calculation.

iii. Since $\tau = dL/dt$, the angular momentum L of the particle changes with time. As such, develop a differential equation to find dR/dt, the rate of change of the radial location of the particle. You may assume the orbit is always quasi circular.

Solution

The angular momentum is

$$L = mvR = \frac{4}{3}\pi\rho r^3\sqrt{\frac{GM_\odot}{R}}R = \frac{4}{3}\pi\rho r^3\sqrt{GM_\odot R}.$$

Differentiating both sides with respect to time,

$$-\frac{v}{c}\frac{L_\odot}{4Rc}Qr^2 = \frac{4}{3}\pi\rho r^3\sqrt{\frac{GM_\odot}{R}}\frac{1}{2}\frac{dR}{dt}$$

which simplifies to

$$-\frac{1}{c^2}\frac{L_\odot}{R}Q = \frac{8}{3}\pi\rho r\frac{dR}{dt}.$$

iv. Develop an expression for the time required to remove particles of size $r \approx 1\,\text{cm}$ and density $\rho \approx 1000\,\text{kg/m}^3$ originally in circular orbits at a distance $R = R_{earth}$, and use the numbers below to simplify your expression.

Solution

Integrating both sides,

$$T\,\frac{L_\odot Q}{c^2} = \frac{4}{3}\pi\rho r R^2.$$

Since $r \gg \lambda$, we have $Q = 1$ and

$$T = \frac{4}{3}\frac{\pi c^2}{L_\odot}\rho r R^2 \approx 2\times10^{14}\,\text{s} \approx 7\times10^6\,\text{y}.$$

Some useful constants include

$$
\begin{aligned}
M_\odot &= 1.989 \times 10^{30}\,\mathrm{kg} \\
L_\odot &= 3.828 \times 10^{26}\,\mathrm{W} \\
R_{earth} &= 1.5 \times 10^{11}\,\mathrm{m} \\
\lambda_{\max} &= 500\,\mathrm{nm}
\end{aligned}
$$

Answer Sheets

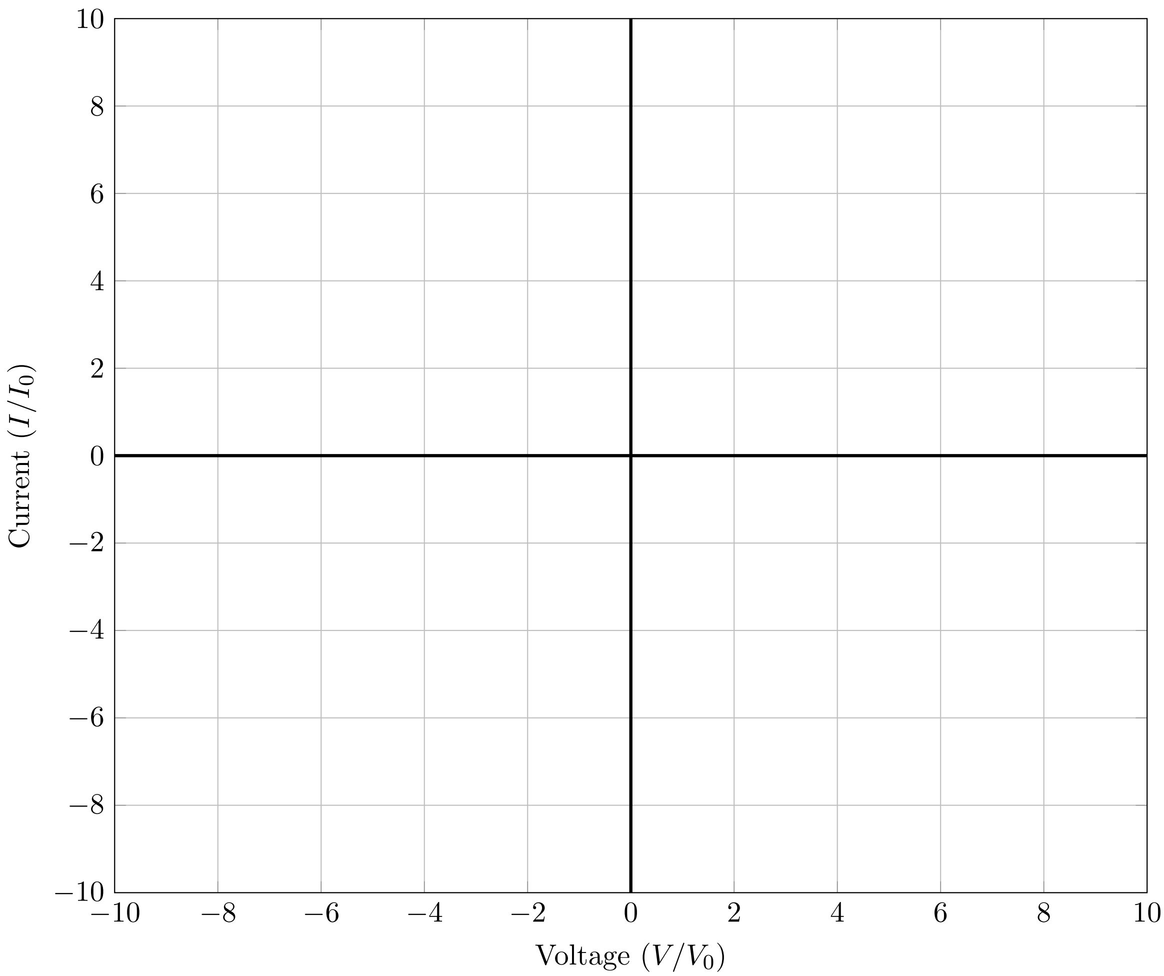
Following are answer sheets for some of the graphical portions of the test.

Student AAPT ID #:

Proctor AAPT ID #:

Question A2

a. Sketch a graph of the current versus voltage for low temperature values $k_B T \ll qV_0$, clearly indicating any asymptotic behavior.

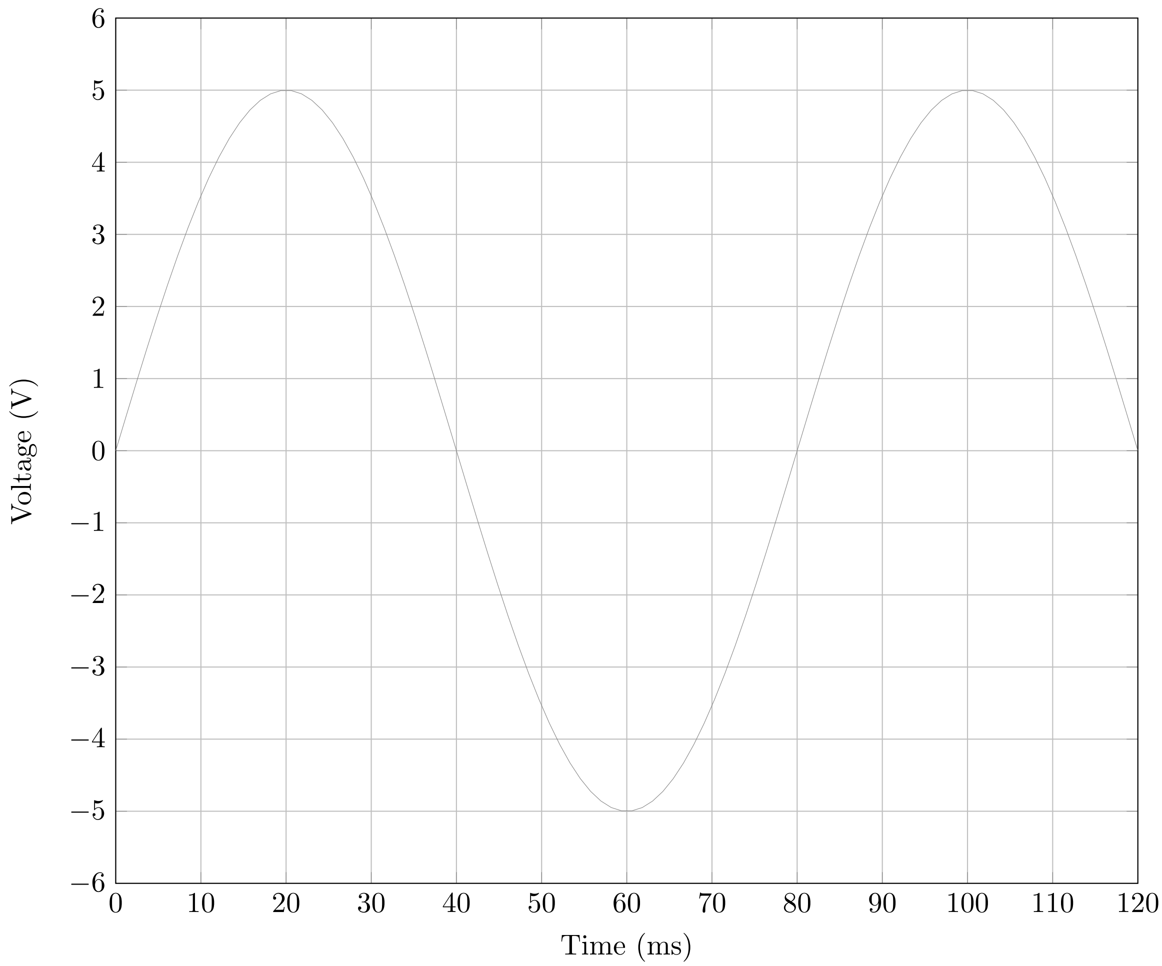


Student AAPT ID #:

Proctor AAPT ID #:

Question A2

b. Sketch the potential difference $V_{CD} = V_C - V_D$ as a function of time. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time. There is **no** capacitor in this circuit!

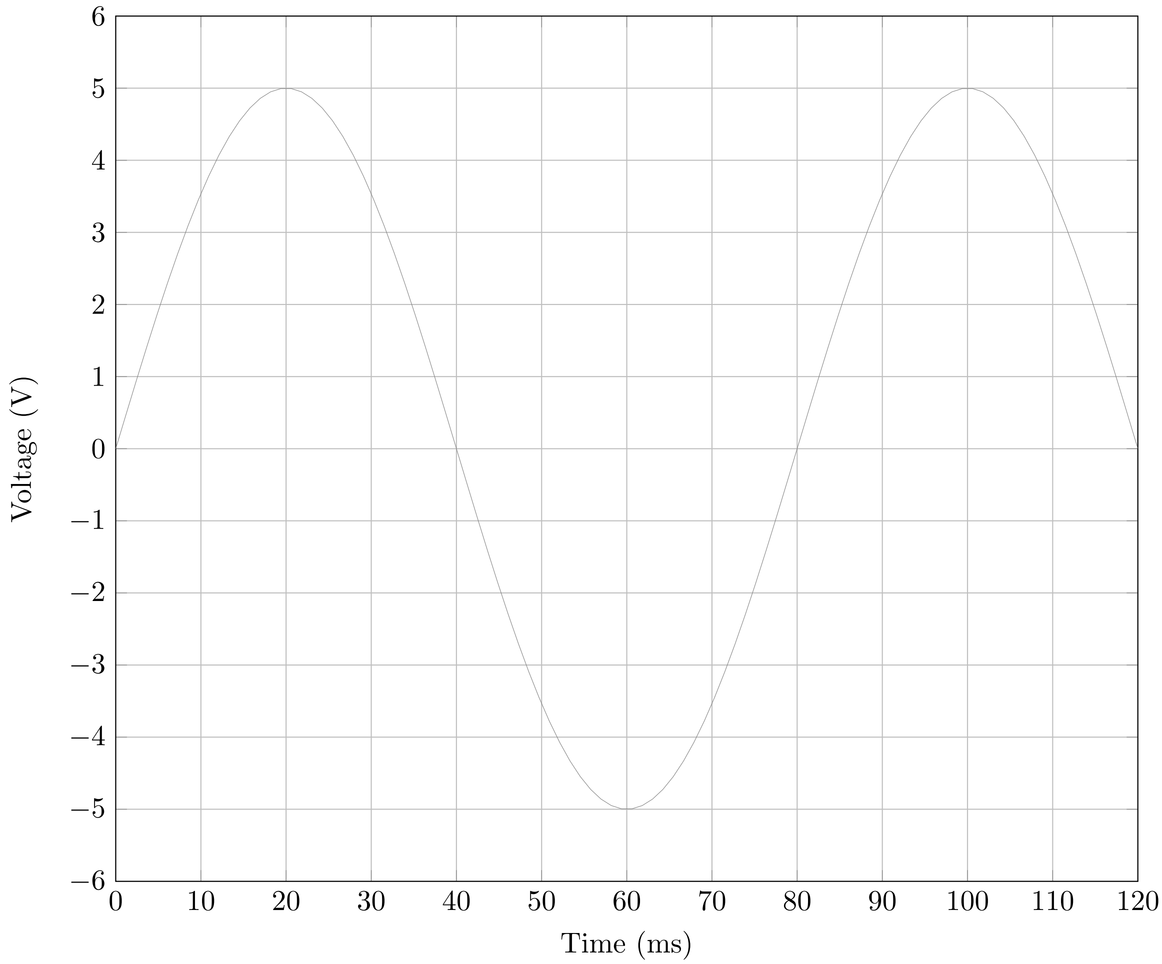


Student AAPT ID #:

Proctor AAPT ID #:

Question A2

c. Sketch the potential difference $V_{CD} = V_C - V_D$ as a function of time. For your convenience, V_{AB} is shown in light gray. Assume that V_{AB} has been running for a long time. There is a capacitor in this circuit!



AAPT UNITED STATES PHYSICS TEAM
AIP 2019

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts. Part A has three questions and is allowed 90 minutes. Part B also has three questions and is allowed 90 minutes.
- Divide the exam paper into 4 parts: the instructions (pages 2–3), Part A (pages 4–6), Part B (pages 8–10), and answer sheets for one of the questions in Part A (pages 12–13). The exam should be printed single-sided to facilitate dividing the test and scanning the answer sheets.
- Provide students with the instructions for the competition (pages 2–3). Students can keep the pages for both parts of the exam, as they contain a reference list of physical constants.
- Provide students with blank sheets of paper as scratch paper. Students are not allowed to bring their own papers.
- Then provide students with Part A and the associated answer sheets, and allow 90 minutes to complete Part A. Do not give students Part B during this time, even if they finish with time remaining. At the end of the 90 minutes, collect the solutions to Part A along with the answer sheets and questions.
- Students are allowed a 10 to 15 minute break between Parts A and B. Then allow 90 minutes to complete Part B. Do not let students go back to Part A.
- At the end of the exam, the supervisor *must* collect all papers, including the questions, the instructions, and any scratch paper used by the students. Students may *not* take the exam questions. The examination questions may be returned to the students after April 8th, 2019.
- Students are allowed calculators, but they may not use symbolic math, programming, or graphical features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDAs, or cameras may not be used during the exam or while the exam papers are present. Students may not use any tables, books, or collections of formulas.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

JiaJia Dong, Abijith Krishnan, Brian Skinner, and Kevin Zhou.

AAPT UNITED STATES PHYSICS TEAM
AIP 2019

USA Physics Olympiad Exam

Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- At the beginning of the exam, you shall be provided with the instruction sheets, blank papers (both for your answers and scratch work), and the exam packet.
- Work on Part A first. You have 90 minutes to complete three problems. Each question is worth 25 points, but they are not necessarily of the same difficulty. Do not look at Part B during this time.
- After you have completed Part A you may take a break. You may consider checking your answers to Part A with the remaining time as you will not be allowed to return to Part A once you start Part B.
- Then work on Part B. You have 90 minutes to complete three problems. Each question is worth 25 points. Do not look at Part A during this time.
- Show your work and reasoning. Partial credit will be given if you make your reasoning clear. Do not write on the back of any page. Further guidance is given on the next page.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDAs or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- **In order to maintain exam security, do not communicate any information about the questions of this exam, or their solutions until after April 8th, 2019.**

Possibly useful information. You may use this sheet for both parts of the exam.

$g = 9.8\ \mathrm{N/kg}$	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8\ \mathrm{m/s}$	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J/K}$
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \theta^3/6$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \theta^2/2$ for $\lvert\theta\rvert \ll 1$

Following is some further guidance on formatting your solutions. Start each question on a new sheet of paper. Put your AAPT ID number, your proctor's AAPT ID number, the problem number and the page number and total number of pages for this problem, in the upper right hand corner of each page. As an example, the second page of your solution to B3 might look as follows.

student AAPT #29485
proctor AAPT #1038
B3: 2/2

Remember to also write the AAPT ID numbers on the provided answer sheets. Write single-sided to facilitate scanning. You may use either pencil or pen, but in either case, make sure to write sufficiently clearly so your work will be legible after scanning. To preserve anonymity of grading, do **not** write your name on any sheet.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

Question A1

Collision Course

Two blocks, A and B, of the same mass are on a fixed inclined plane, which makes a 30° angle with the horizontal. At time $t = 0$, A is a distance $\ell = 5\,\mathrm{cm}$ along the incline above B, and both blocks are at rest. Suppose the coefficients of static and kinetic friction between the blocks and the incline are

$$\mu_A = \frac{\sqrt{3}}{6}, \qquad \mu_B = \frac{\sqrt{3}}{3},$$

and that the blocks collide perfectly elastically. Let $v_A(t)$ and $v_B(t)$ be the speeds of the blocks down the incline. For this problem, use $g = 10\,\mathrm{m/s^2}$, assume both blocks stay on the incline for the entire time, and neglect the sizes of the blocks.

a. Graph the functions $v_A(t)$ and $v_B(t)$ for t from 0 to 1 second on the provided answer sheet, with a solid and dashed line respectively. Mark the times at which collisions occur.

b. Derive an expression for the total distance block A has moved from its original position right after its n^{th} collision, in terms of ℓ and n.

Now suppose that the coefficient of block B is instead $\mu_B = \sqrt{3}/2$, while $\mu_A = \sqrt{3}/6$ remains the same.

c. Again, graph the functions $v_A(t)$ and $v_B(t)$ for t from 0 to 1 second on the provided answer sheet, with a solid and dashed line respectively. Mark the times at which collisions occur.

d. At time $t = 1\,\mathrm{s}$, how far has block A moved from its original position?

Question A2

Green Revolution[1]

In this problem, we will investigate a simple thermodynamic model for the conversion of solar energy into wind. Consider a planet of radius R, and assume that it rotates so that the same side always faces the Sun. The bright side facing the Sun has a constant uniform temperature T_1, while the dark side has a constant uniform temperature T_2. The orbit radius of the planet is R_0, the Sun has temperature T_s, and the radius of the Sun is R_s. Assume that outer space has zero temperature, and treat all objects as ideal blackbodies.

a. Find the solar power P received by the bright side of the planet. (Hint: the Stefan-Boltzmann law states that the power emitted by a blackbody with area A is $\sigma A T^4$.)

In order to keep both T_1 and T_2 constant, heat must be continually transferred from the bright side to the dark side. By viewing the two hemispheres as the two reservoirs of a reversible heat engine, work can be performed from this temperature difference, which appears in the form of wind power. For simplicity, we assume all of this power is immediately captured and stored by windmills.

b. The equilibrium temperature ratio T_2/T_1 depends on the heat transfer rate between the hemispheres. Find the minimum and maximum possible values of T_2/T_1. In each case, what is the wind power P_w produced?

c. Find the wind power P_w in terms of P and the temperature ratio T_2/T_1.

d. Estimate the maximum possible value of P_w as a fraction of P, to one significant figure. Briefly explain how you obtained this estimate.

[1] This question inspired by De Vos, Alexis, and Guust Flater, American Journal of Physics 59.8 (1991): 751-754.

Question A3

Electric Slide

Two large parallel plates of area A are placed at $x = 0$ and $x = d \ll \sqrt{A}$ in a semiconductor medium. The plate at $x = 0$ is grounded, and the plate at $x = d$ is at a fixed potential $-V_0$, where $V_0 > 0$. Particles of positive charge q flow between the two plates. You may neglect any dielectric effects of the medium.

a. For large V_0, the velocity of the positive charges is determined by a strong drag force, so that

$$v = \mu E$$

where E is the local electric field and μ is the charge mobility.

i. In the steady state, there is a nonzero but time-independent density of charges between the two plates. Let the charge density at position x be $\rho(x)$. Use charge conservation to find a relationship between $\rho(x)$, $v(x)$, and their derivatives.

ii. Let $V(x)$ be the electric potential at x. Derive an expression relating $\rho(x)$, $V(x)$, and their derivatives. (Hint: start by using Gauss's law to relate the charge density $\rho(x)$ to the derivative of the electric field $E(x)$.)

iii. Suppose that in the steady state, conditions have been established so that $V(x)$ is proportional to x^b, where b is an exponent you must find, and the current is nonzero. Derive an expression for the current in terms of V_0 and the other given parameters.

b. For small V_0, the positive charges move by diffusion. The current due to diffusion is given by Fick's Law,

$$I = -AD\frac{\mathrm{d}\rho}{\mathrm{d}x}.$$

Here, D is the diffusion constant, which you can assume to be described by the Einstein relation

$$D = \frac{\mu k_B T}{q},$$

where T is the temperature of the system.

i. Assume that in the steady state, conditions have been established so that a nonzero, steady current flows, and the electric potential again satisfies $V(x) \propto x^{b'}$, where b' is another exponent you must find. Derive an expression for the current in terms of V_0 and the other given parameters.

ii. At roughly what voltage V_0 does the system transition from this regime to the high voltage regime of the previous part?

STOP: Do Not Continue to Part B

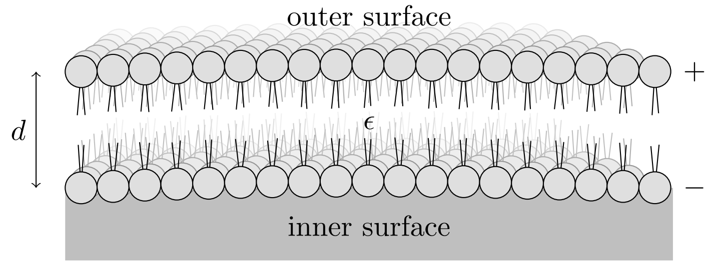
If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

Strain in the Membrane[2]



The wall of a neuron is made from an elastic membrane, which resists compression in the same way as a spring. It has an effective spring constant k and an equilibrium thickness d_0. Assume that the membrane has a very large area A and negligible curvature.

The neuron has "ion pumps" that can move ions across the membrane. In the resulting charged state, positive and negative ionic charge is arranged uniformly along the outer and inner surfaces of the membrane, respectively. The permittivity of the membrane is ϵ.

a. Suppose that, after some amount of work is done by the ion pumps, the charges on the outer and inner surfaces are Q and $-Q$, respectively. What is the thickness d of the membrane?

b. Derive an expression for the voltage difference V between the outer and inner surfaces of the membrane in terms of Q and the other parameters given.

c. Suppose that the ion pumps are first turned on in the uncharged state, and the membrane is charged very slowly (quasistatically). The pumps will only turn off when the voltage difference across the membrane becomes larger than a particular value V_{th}. How large must the spring constant k be so that the ion pumps turn off before the membrane collapses?

d. How much work is done by the ion pumps in each of the following situations? Express your answers in terms of k and d_0.

   i. k is infinitesimally larger than the value derived in part (c).

   ii. k is infinitesimally smaller than the value derived in part (c).

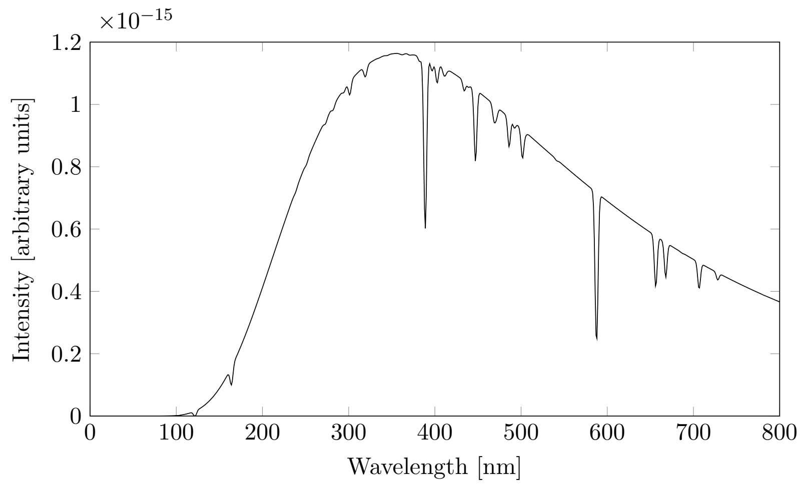
   Assume in each case that the membrane thickness d cannot become negative.

[2] This question inspired by Partenskii and Jordan, Physical Review E 80, 011112 (2009).

Question B2

Stellar Black Box

Scientists have recently detected a new star, the MAR-Kappa. The star is almost a perfect blackbody, and its measured light spectrum is shown below.



The total measured light intensity from MAR-Kappa is $I = 1.12 \times 10^{-8}\,\mathrm{W/m^2}$. The mass of MAR-Kappa is estimated to be $3.5 \times 10^{30}\,\mathrm{kg}$. It is stationary relative to the sun. You may find the Stefan-Boltzmann law useful, which states the power emitted by a blackbody with area A is $\sigma A T^4$.

a. The spectrum of wavelengths λ emitted from a blackbody only depends on h, c, k_B, λ, and T. Given that the sun has a surface temperature of 5778 K and peak emission at 500 nm, what is the approximate surface temperature of MAR-Kappa?

b. The "lines" in the spectrum result from atoms in the star absorbing specific wavelengths of the emitted light. One contribution to the width of the spectral lines is the Doppler shift associated with the thermal motion of the atoms in the star. The spectral line at $\lambda = 389\,\mathrm{nm}$ is due to helium. Estimate to within an order of magnitude the thermal broadening $\Delta\lambda$ of this line. The mass of a helium atom is $6.65 \times 10^{-27}\,\mathrm{kg}$.

c. Over the course of a year, MAR-Kappa appears to oscillate between two positions in the background night sky, which are an angular distance of $1.6 \times 10^{-6}\,\mathrm{rad}$ apart. How far away is MAR-Kappa? Assume that MAR-Kappa lies in the same plane as the Earth's orbit, which is circular with radius $1.5 \times 10^{11}\,\mathrm{m}$.

d. What is the radius of MAR-Kappa?

Over the course of some time, you observe that the star's intensity periodically dips from I to $(1 - 10^{-5})I$ and then rises back to I, with period t. One possible explanation for this observation is that an exoplanet is orbiting the star and blocking the starlight for some time.

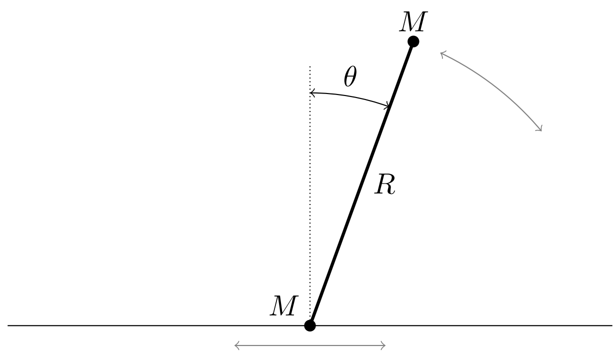
e. Estimate the exoplanet's radius, assuming that it is much closer to the star than to the Earth.

f. Assume the exoplanet is a blackbody with uniform temperature in a circular orbit around the star. What must t be so that the planet has a temperature of 250 K? (If this were true, and the planet had an appropriate atmosphere, the temperature would increase enough to support life.)

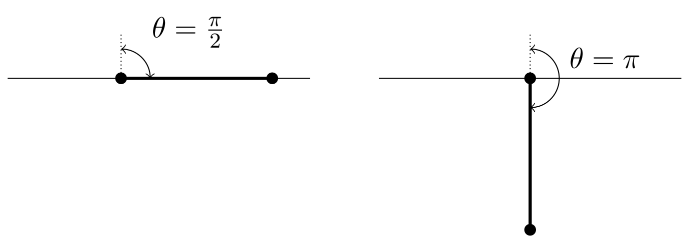
Question B3

Pitfall

A bead is placed on a horizontal rail, along which it can slide frictionlessly. It is attached to the end of a rigid, massless rod of length R. A ball is attached at the other end. Both the bead and the ball have mass M. The system is initially stationary, with the ball directly above the bead. The ball is then given an infinitesimal push, *parallel* to the rail.



Assume that the rod and ball are designed in such a way (not shown explicitly in the diagram) so that they can pass through the rail without hitting it. In other words, the rail only constrains the motion of the bead. Two subsequent states of the system are shown below.



a. Derive an expression for the force in the rod when it is horizontal, as shown at left above, and indicate whether it is tension or compression.

b. Derive an expression for the force in the rod when the ball is directly below the bead, as shown at right above, and indicate whether it is tension or compression.

c. Let θ be the angle the rod makes with the vertical, so that the rod begins at $\theta = 0$. Find the angular velocity $\omega = d\theta/dt$ as a function of θ.

Answer Sheets

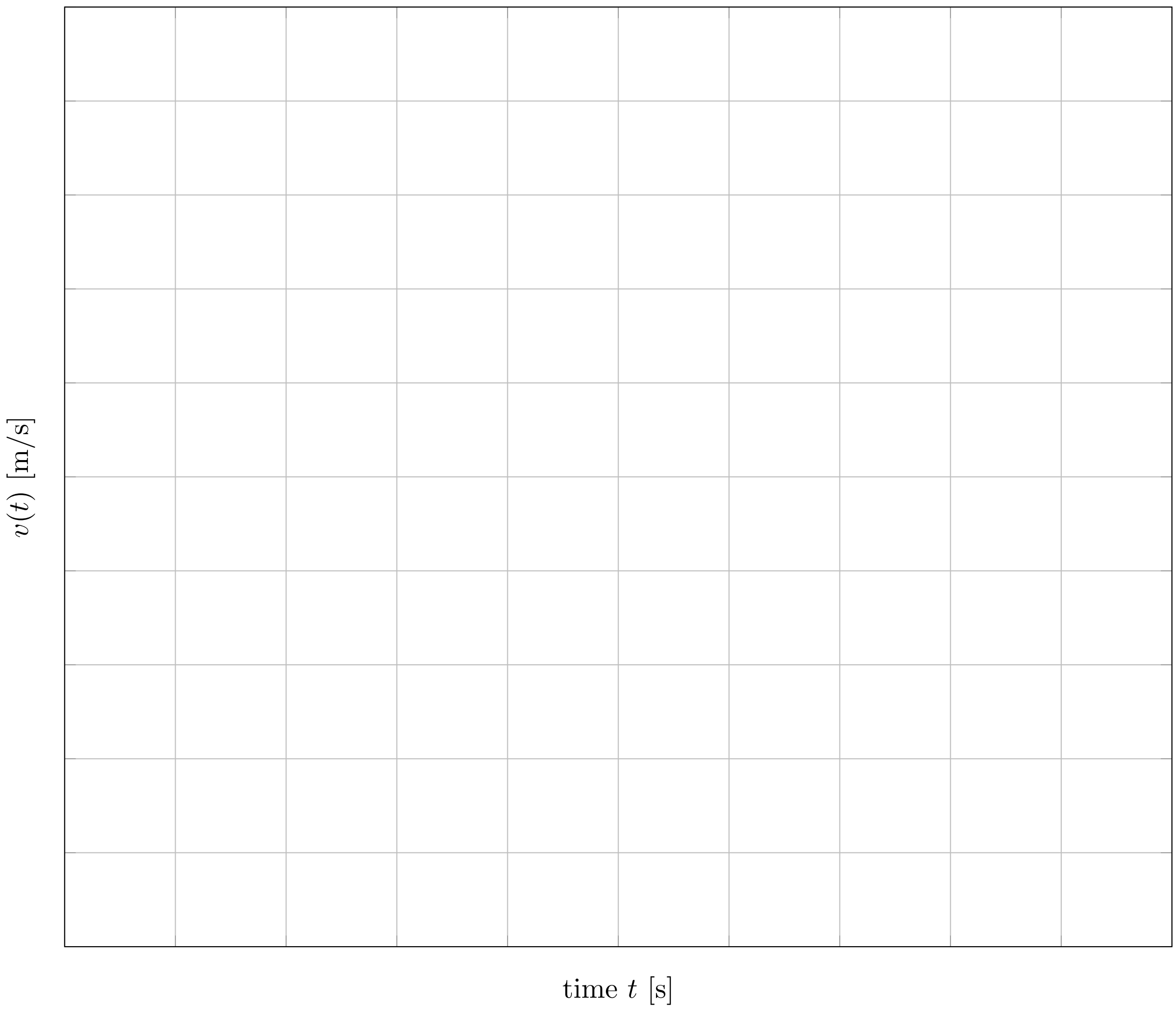
Following are answer sheets for the graphing portion of Problem A1.

Student AAPT ID #:

Proctor AAPT ID #:

A1: Collision Course

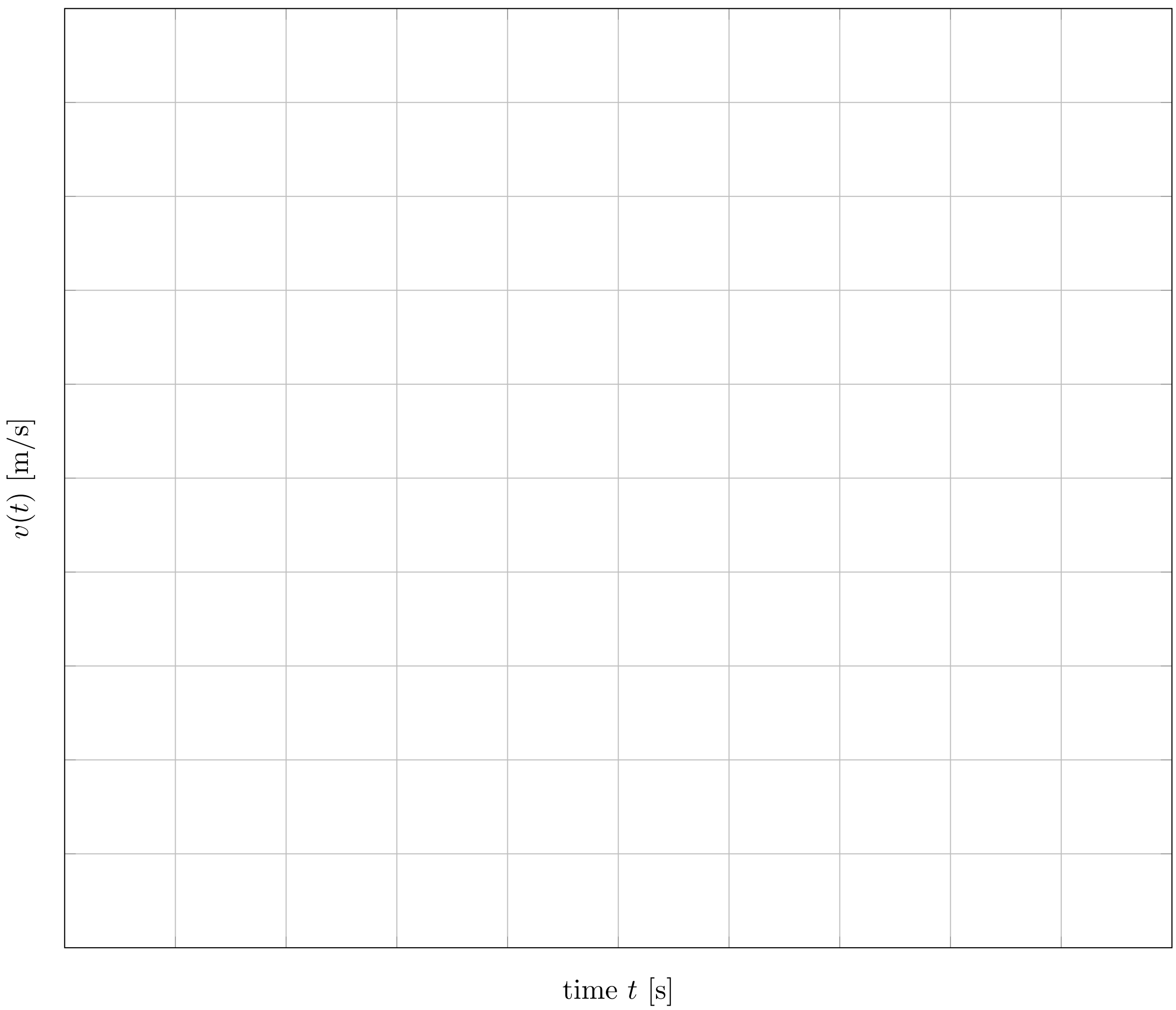
(a)



Student AAPT ID #:

Proctor AAPT ID #:

(c)



AAPT UNITED STATES PHYSICS TEAM
AIP 2019

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts. Part A has three questions and is allowed 90 minutes. Part B also has three questions and is allowed 90 minutes.
- Divide the exam paper into 4 parts: the instructions (pages 2–3), Part A (pages 4–12), Part B (pages 14–23), and answer sheets for one of the questions in Part A (pages 25–26). The exam should be printed single-sided to facilitate dividing the test and scanning the answer sheets.
- Provide students with the instructions for the competition (pages 2–3). Students can keep the pages for both parts of the exam, as they contain a reference list of physical constants.
- Provide students with blank sheets of paper as scratch paper. Students are not allowed to bring their own papers.
- Then provide students with Part A and the associated answer sheets, and allow 90 minutes to complete Part A. Do not give students Part B during this time, even if they finish with time remaining. At the end of the 90 minutes, collect the solutions to Part A along with the answer sheets and questions.
- Students are allowed a 10 to 15 minute break between Parts A and B. Then allow 90 minutes to complete Part B. Do not let students go back to Part A.
- At the end of the exam, the supervisor *must* collect all papers, including the questions, the instructions, and any scratch paper used by the students. Students may *not* take the exam questions. The examination questions may be returned to the students after April 8th, 2019.
- Students are allowed calculators, but they may not use symbolic math, programming, or graphical features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDAs, or cameras may not be used during the exam or while the exam papers are present. Students may not use any tables, books, or collections of formulas.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

JiaJia Dong, Abijith Krishnan, Brian Skinner, and Kevin Zhou.

AAPT UNITED STATES PHYSICS TEAM
AIP 2019

USA Physics Olympiad Exam

Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- At the beginning of the exam, you shall be provided with the instruction sheets, blank papers (both for your answers and scratch work), and the exam packet.
- Work on Part A first. You have 90 minutes to complete three problems. Each question is worth 25 points, but they are not necessarily of the same difficulty. Do not look at Part B during this time.
- After you have completed Part A you may take a break. You may consider checking your answers to Part A with the remaining time as you will not be allowed to return to Part A once you start Part B.
- Then work on Part B. You have 90 minutes to complete three problems. Each question is worth 25 points. Do not look at Part A during this time.
- Show your work and reasoning. Partial credit will be given if you make your reasoning clear. Do not write on the back of any page. Further guidance is given on the next page.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDAs or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- **In order to maintain exam security, do not communicate any information about the questions of this exam, or their solutions until after April 8th, 2019.**

Possibly useful information. You may use this sheet for both parts of the exam.

$g = 9.8\ \mathrm{N/kg}$	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8\ \mathrm{m/s}$	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J/K}$
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \theta^3/6$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \theta^2/2$ for $\lvert\theta\rvert \ll 1$

Following is some further guidance on formatting your solutions. Start each question on a new sheet of paper. Put your AAPT ID number, your proctor's AAPT ID number, the problem number and the page number and total number of pages for this problem, in the upper right hand corner of each page. As an example, the second page of your solution to B3 might look as follows.

student AAPT #29485
proctor AAPT #1038
B3: 2/2

Remember to also write the AAPT ID numbers on the provided answer sheets. Write single-sided to facilitate scanning. You may use either pencil or pen, but in either case, make sure to write sufficiently clearly so your work will be legible after scanning. To preserve anonymity of grading, do **not** write your name on any sheet.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

Question A1

Collision Course

Two blocks, A and B, of the same mass are on a fixed inclined plane, which makes a $30°$ angle with the horizontal. At time $t = 0$, A is a distance $\ell = 5\,\text{cm}$ along the incline above B, and both blocks are at rest. Suppose the coefficients of static and kinetic friction between the blocks and the incline are

$$\mu_A = \frac{\sqrt{3}}{6}, \qquad \mu_B = \frac{\sqrt{3}}{3},$$

and that the blocks collide perfectly elastically. Let $v_A(t)$ and $v_B(t)$ be the speeds of the blocks down the incline. For this problem, use $g = 10\,\text{m/s}^2$, assume both blocks stay on the incline for the entire time, and neglect the sizes of the blocks.

a. Graph the functions $v_A(t)$ and $v_B(t)$ for t from 0 to 1 second on the provided answer sheet, with a solid and dashed line respectively. Mark the times at which collisions occur.

Solution

Draw a free-body diagram for both blocks and we can find that the acceleration of A points down along the incline: $a_A = g\sin\theta - \mu_A g\cos\theta = \frac{1}{2}g - \frac{1}{4}g = 2.5\,\text{m/s}^2$. Similarly, $a_B = g\sin\theta - \mu_B g\cos\theta = 0$.

Let's first look at a **qualitative** picture of the collisions: When A slides down the incline before colliding with B, it moves with acceleration. When the blocks collide, the total momentum of the system is conserved. Because $m_A = m_B$, the blocks exchange velocity, and thus B slides down with constant velocity. After a momentary stop, A will again accelerate down the incline, and catches up with B, and another collision occurs.

Quantitatively, the first collision happens when A travels $\ell = 5\,\text{cm} = 0.05\,\text{m}$.

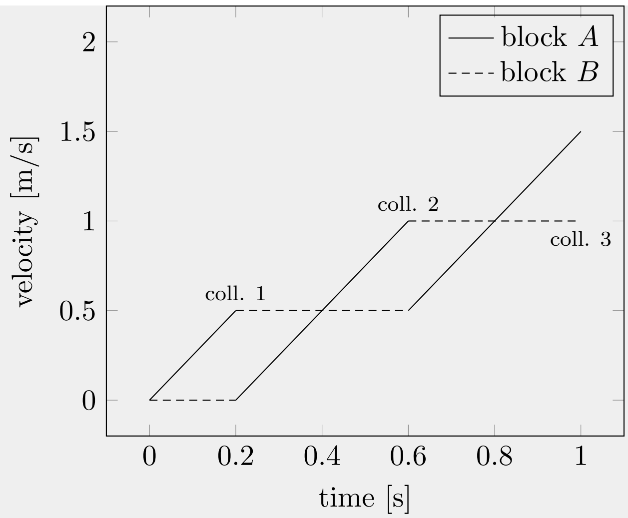
$$t_1 = \sqrt{2\ell/a_A} = 0.2\,\text{s}; \qquad v_{A1} = a_A t_1 = 0.5\,\text{m/s}$$

This is when A and B first collide. Then B moves down the incline at constant velocity $v_B = v_{A_1} = 0.5\text{m/s}$, while A starts from rest and accelerates down the incline with $a_A = 2.5\text{m/s}^2$, until catches up with B at $t_2 = 0.6\text{s}$. At that point,

$$v_{A2} = a_A(t_2 - t_1) = 1\,\text{m/s}$$

Using a similar approach, we can find that at $t_3 = 1\,\text{s}$.

Graphically, the $v_{A/B}(t)$ graphs during the first second are included below.



b. Derive an expression for the total distance block A has moved from its original position right after its n^{th} collision, in terms of ℓ and n.

Solution

The easiest way to calculate the total distance after n collisions is through the graph, namely the area enclosed by the blue line: When $n = 1$, the distance traveled is $d_A(1) = \ell$, for $n = 2, d_A(2) = d_A(1) + 4\ell$, and so on, so

$$d_A(n) = \ell + 4\ell + 8\ell + ... + (n-1) \times 4\ell = \ell(2n^2 - 2n + 1).$$

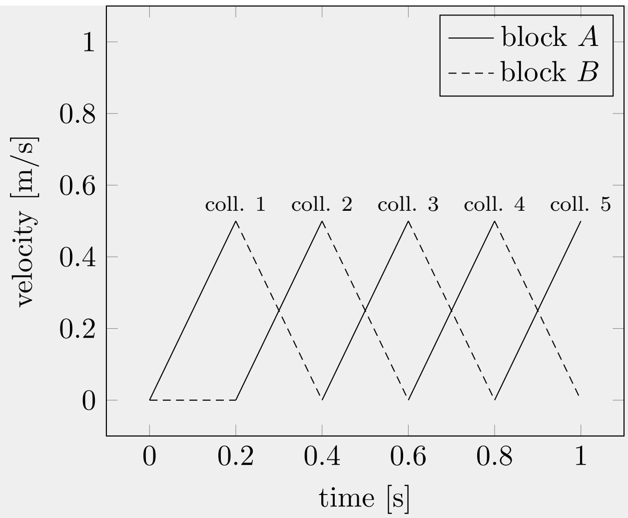
Now suppose that the coefficient of block B is instead $\mu_B = \sqrt{3}/2$, while $\mu_A = \sqrt{3}/6$ remains the same.

c. Again, graph the functions $v_A(t)$ and $v_B(t)$ for t from 0 to 1 second on the provided answer sheet, with a solid and dashed line respectively. Mark the times at which collisions occur.

Solution

In this case, $a_B = -g/4$. In other words, friction is larger than the component of gravity. Once B moves, friction acts to stop it.

In this case, A will again move down the incline and hit B with $v_{A1} = 0.5\,\text{m/s}$ at $t_1 = 0.2\,\text{s}$. As C moves with a deceleration of $g/4$, A accelerates with the same magnitude, colliding again at $t_2 = 2t_1$, and the process repeats. The relevant graphs are shown below.



d. At time $t = 1\,\text{s}$, how far has block A moved from its original position?

Solution

Again, using the graph, it is easy to see that at $t = 1\,\text{s}$, A just finished the 5th collision, and the total distance it moved is: $5\ell = 25\,\text{cm}$.

Question A2

Green Revolution[1]

In this problem, we will investigate a simple thermodynamic model for the conversion of solar energy into wind. Consider a planet of radius R, and assume that it rotates so that the same side always faces the Sun. The bright side facing the Sun has a constant uniform temperature T_1, while the dark side has a constant uniform temperature T_2. The orbit radius of the planet is R_0, the Sun has temperature T_s, and the radius of the Sun is R_s. Assume that outer space has zero temperature, and treat all objects as ideal blackbodies.

a. Find the solar power P received by the bright side of the planet. (Hint: the Stefan-Boltzmann law states that the power emitted by a blackbody with area A is $\sigma A T^4$.)

Solution

The intensity of solar radiation at the surface of the sun is σT_s^4, so the intensity at the planet's orbit radius is

$$I = \sigma T_s^4 \frac{R_s^2}{R_0^2}.$$

The area subtended by the planet is πR^2, so

$$P = \pi \sigma T_s^4 \frac{R^2 R_s^2}{R_0^2}.$$

In order to keep both T_1 and T_2 constant, heat must be continually transferred from the bright side to the dark side. By viewing the two hemispheres as the two reservoirs of a reversible heat engine, work can be performed from this temperature difference, which appears in the form of wind power. For simplicity, we assume all of this power is immediately captured and stored by windmills.

b. The equilibrium temperature ratio T_2/T_1 depends on the heat transfer rate between the hemispheres. Find the minimum and maximum possible values of T_2/T_1. In each case, what is the wind power P_w produced?

Solution

The minimum value is simply zero; in this case zero heat is transferred to the dark side of the planet. Since no heat is transferred, the heat engine can't run, so $P_w = 0$. (To show this a bit more carefully, note that the entropy exhausted by the heat engine is $Q_2/T_2 \propto T_2^3$ by the Stefan-Boltzmann law. In the limit $T_2 \to 0$, the entropy out goes to zero, so the entropy in and hence the heat intake also goes to zero.)

The maximum value is $T_2/T_1 = 1$. It cannot be any higher by the second law of thermodynamics. In this case, there is no temperature difference, so the heat engine has zero efficiency and $P_w = 0$. Power $P/2$ is simply transferred from the bright side to the dark side as heat.

c. Find the wind power P_w in terms of P and the temperature ratio T_2/T_1.

[1] This question inspired by De Vos, Alexis, and Guust Flater, American Journal of Physics 59.8 (1991): 751-754.

Solution

Let heat be transferred from the bright side at a rate Q_1 and transferred to the dark side at a rate Q_2. Then by conservation of energy,

$$Q_1 = P_w + Q_2.$$

Since the two hemispheres have constant temperatures, energy balance for each gives

$$P = Q_1 + (2\pi R^2\sigma)T_1^4, \quad Q_2 = AT_2^4.$$

Finally, since the engine is reversible,

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}.$$

By combining the first three equations, and defining $x = T_2/T_1$, we have

$$P_w = Q_1 - Q_2 = P - (2\pi R^2\sigma)(T_1^4 + T_2^4) = P - (2\pi R^2\sigma)T_1^4(1 + x^4).$$

This is not yet in terms of P and x, so now we use the reversibility condition,

$$\frac{P - AT_1^4}{T_1} = \frac{AT_2^4}{T_2}$$

which simplifies to

$$P = (2\pi R^2\sigma)(T_1^4 + T_1T_2^3) = (2\pi R^2\sigma)T_1^4(1 + x^3).$$

Plugging this in above, we find

$$P_w = P - \frac{P}{1 + x^3}(1 + x^4) = \frac{x^3(1 - x)}{1 + x^3}P.$$

d. Estimate the maximum possible value of P_w as a fraction of P, to one significant figure. Briefly explain how you obtained this estimate.

Solution

There are many ways to get the required answer. For example, by sketching the function, one can see that there is a unique maximum at an intermediate value of x, and furthermore that this maximum is at $x > 0.5$, because of the rapid rise of the x^3 factor. One could then compute P_w/P with a calculator at a few trial values such as $x = 0.5, 0.7, 0.9$, which are already enough to get the desired accuracy.

The optimum value is $x = 0.69$, at which point

$$P_w^{\max} = 0.077\,P.$$

Hence in this model, at most 7.7% of solar energy can be converted into wind energy. Any answer within 15% of this value was accepted.

Question A3

Electric Slide

Two large parallel plates of area A are placed at $x = 0$ and $x = d \ll \sqrt{A}$ in a semiconductor medium. The plate at $x = 0$ is grounded, and the plate at $x = d$ is at a fixed potential $-V_0$, where $V_0 > 0$. Particles of positive charge q flow between the two plates. You may neglect any dielectric effects of the medium.

a. For large V_0, the velocity of the positive charges is determined by a strong drag force, so that

$$v = \mu E$$

where E is the local electric field and μ is the charge mobility.

i. In the steady state, there is a nonzero but time-independent density of charges between the two plates. Let the charge density at position x be $\rho(x)$. Use charge conservation to find a relationship between $\rho(x)$, $v(x)$, and their derivatives.

Solution

In the steady state, the current is the same everywhere. Consider the region $(x, x+dx)$. The time it takes for the charge in the second region to leave is $\frac{\mathrm{d}x}{v(x)}$. The amount of charge that leaves is $\rho A\,\mathrm{d}x$. The current is thus given by $\rho A v$, so ρv is constant. Alternatively, one can write this as

$$v\frac{\mathrm{d}\rho}{\mathrm{d}x} + \rho\frac{\mathrm{d}v}{\mathrm{d}x} = 0.$$

Both forms were accepted.

ii. Let $V(x)$ be the electric potential at x. Derive an expression relating $\rho(x)$, $V(x)$, and their derivatives. (Hint: start by using Gauss's law to relate the charge density $\rho(x)$ to the derivative of the electric field $E(x)$.)

Solution

Let us find an expression for the electric field at position x. The position x is effectively in between two uniform sheets of charge density. The sheet on the left has charge density $\int_0^x \rho\,\mathrm{d}x + \sigma_0$, where σ_0 is the charge density on the left plate, and the sheet on the right has charge density $\int_x^d \rho\,\mathrm{d}x + \sigma_d$, where σ_d is the charge density on the left plate. Then, the electric field is given by

$$E = \sigma_0/(2\epsilon_0) + \int_0^x \rho/(2\epsilon_0)\ \mathrm{d}x - \int_x^d \rho/(2\epsilon_0)\ \mathrm{d}x - \sigma_d/(2\epsilon_0).$$

Then, by the Fundamental Theorem of Calculus

$$\frac{dE}{dx} = \frac{\rho}{\epsilon_0},$$

so

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0}.$$

This can also be derived from the differential form of Gauss's Law more easily and is known as Poisson's equation.

iii. Suppose that in the steady state, conditions have been established so that $V(x)$ is proportional to x^b, where b is an exponent you must find, and the current is nonzero. Derive an expression for the current in terms of V_0 and the other given parameters.

Solution

We have that

$$\rho\frac{dv}{dx} + v\frac{d\rho}{dx} = 0,$$

and now that $v = -\mu\frac{dV}{dx}$, so substituting in Poisson's equation gives us that

$$\left(\frac{d^2V}{dx^2}\right)^2 + \frac{dV}{dx}\left(\frac{d^3V}{dx^3}\right) = 0.$$

Using $V(x) = -V_0(x/d)^b$ gives

$$b(b-1)b(b-1) = -bb(b-1)(b-2).$$

The solution with $b = 0$ cannot satisfy the boundary conditions, while $b = 1$ has zero current. Assuming b is neither of these values, we have $b - 1 = -(b-2)$, so $b = 3/2$. Substituting gives

$$v = -\frac{3V_0\mu x^{1/2}}{2d^{3/2}}$$

and

$$\rho = -\frac{3V_0\epsilon_0}{4d^{3/2}x^{1/2}},$$

so

$$I = \rho A v = \frac{9\epsilon_0\mu A V_0^2}{8d^3},$$

with the current flowing from left to right.

b. For small V_0, the positive charges move by diffusion. The current due to diffusion is given by Fick's Law,

$$I = -AD\frac{\mathrm{d}\rho}{\mathrm{d}x}.$$

Here, D is the diffusion constant, which you can assume to be described by the Einstein relation

$$D = \frac{\mu k_B T}{q},$$

where T is the temperature of the system.

i. Assume that in the steady state, conditions have been established so that a nonzero, steady current flows, and the electric potential again satisfies $V(x) \propto x^{b'}$, where b' is another exponent you must find. Derive an expression for the current in terms of V_0 and the other given parameters.

Solution

We again have $V(x) = -V_0(x/d)^b$. Note that from Poisson's equation, $\frac{d\rho}{dx} = -\epsilon_0 \frac{d^3V}{dx^3}$, so we need $b = 3$ for this expression to be constant. Therefore,

$$I = -\frac{6\mu k_B T A \epsilon_0 V_0}{qd^3}.$$

The negative sign indicates that the current is flowing from right to left, i.e. from low to high potential; this is possible because it is governed by diffusion. (Similarly, molecules of perfume can diffuse upward in air, even though gravity always pulls them downward.) Since the question didn't ask for the direction of the current, we accepted either sign.

ii. At roughly what voltage V_0 does the system transition from this regime to the high voltage regime of the previous part?

Solution

The two contributions to the current are equally important when

$$\frac{6\mu k_B T A \epsilon_0 V_0}{qd^3} = \frac{9\epsilon_0 \mu A V_0^2}{8d^3},$$

at which point

$$V_0 = \frac{16 k_B T}{3q}.$$

This rough estimate tells us that the transition occurs at $V_0 \sim k_B T/q$, where $\sim$ means "within an order of magnitude". For credit, any answer of the form $a\, k_B T/q$ for an order-one number a is acceptable.

STOP: Do Not Continue to Part B

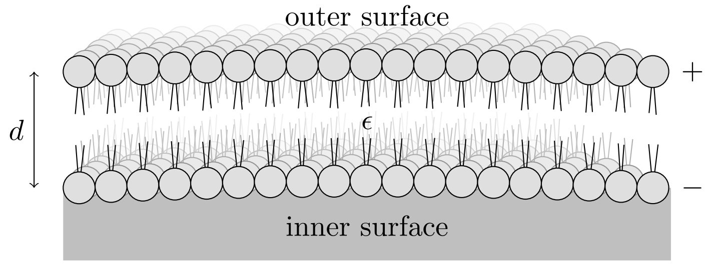
If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

Strain in the Membrane[2]



The wall of a neuron is made from an elastic membrane, which resists compression in the same way as a spring. It has an effective spring constant k and an equilibrium thickness d_0. Assume that the membrane has a very large area A and negligible curvature.

The neuron has "ion pumps" that can move ions across the membrane. In the resulting charged state, positive and negative ionic charge is arranged uniformly along the outer and inner surfaces of the membrane, respectively. The permittivity of the membrane is ϵ.

a. Suppose that, after some amount of work is done by the ion pumps, the charges on the outer and inner surfaces are Q and $-Q$, respectively. What is the thickness d of the membrane?

Solution

One charge layer by itself creates an electric field $E_1 = Q/(2\epsilon A)$ in each direction. So the force between the two sides of the membrane $F_E = QE_1 = Q^2/(2\epsilon A)$.

This electric force is balanced by the spring force $F_s = kx$, where $x = d_0 - d$. Equating these two forces and solving for d gives gives

$$d = d_0 - \frac{Q^2}{2\epsilon Ak}.$$

b. Derive an expression for the voltage difference V between the outer and inner surfaces of the membrane in terms of Q and the other parameters given.

Solution

The electric field inside the membrane (as produced by both the left and right plates) is $E = Q/(\epsilon_0 \kappa A)$. So the voltage between them is

$$V = Ed = \frac{Q}{\epsilon A}d.$$

[2]This question inspired by Partenskii and Jordan, Physical Review E 80, 011112 (2009).

Inserting the expression for Q from part (a) gives

$$V = \frac{Q}{\epsilon A}\left(d_0 - \frac{Q^2}{2\epsilon A k}\right).$$

This equation implies that as the charge Q is increased, the voltage first increases and then decreases again.

c. Suppose that the ion pumps are first turned on in the uncharged state, and the membrane is charged very slowly (quasistatically). The pumps will only turn off when the voltage difference across the membrane becomes larger than a particular value V_{th}. How large must the spring constant k be so that the ion pumps turn off before the membrane collapses?

Solution

The voltage V first increases and then decreases as a function of Q, which implies that there is a maximum voltage $V_{\max}$ to which the membrane can be charged. This voltage can be found by taking the derivative dV/dQ and setting it equal to zero. This procedure gives

$$V_{\max} = \sqrt{\frac{kd_0^3}{\epsilon A}}\left(\frac{2}{3}\right)^{3/2}.$$

The corresponding charge at the maximum voltage is given by

$$Q^2_{\text{Vmax}} = \frac{2}{3}\epsilon A k d_0.$$

For the ion pumps to turn off, we must have $V_{\max} > V_{\text{th}}$. Otherwise the pumps will continue to move charge across the membrane until it collapses. Setting $V_{\max} > V_{\text{th}}$ and solving for k gives

$$k > \left(\frac{3}{2}\right)^3 \frac{V_{\text{th}}^2 \epsilon A}{d_0^3}.$$

d. How much work is done by the ion pumps in each of the following situations? Express your answers in terms of k and d_0.

i. k is infinitesimally larger than the value derived in part (c).

Solution

If k is larger than the value in part (c), then the threshold voltage $V_{\text{th}} > V_{\max}$, and the ion pumps turn off before the membrane thickness d reaches zero. The work W done by the ion pumps is equal to the potential energy of the system relative to the uncharged state (with $Q = 0$ and $d = 0$). That is, $W = \frac{1}{2}kx^2 + Q^2/(2C)$, where $C = \epsilon A/d$ is the capacitance of the membrane. Writing this equation in terms of Q gives $W = Q^2 d_0/(2\epsilon A) - Q^4/(8\epsilon^2 A^2 k)$.

k being infinitesimally larger than the critical value means that the ion pumps turn off just as the voltage maximum V_{th} is reached. At this point the charge Q approaches

Q_{Vmax}, derived in the previous answer. Inserting the value into the expression for W gives

$$W = \frac{5}{18} k d_0^2.$$

ii. k is infinitesimally smaller than the value derived in part (c).

Solution

If k is smaller than the value in part (c), then the threshold voltage is larger than the stopping voltage V_{th}, and the ion pumps continue to work until the membrane collapses to $d = 0$. At this point there is no electrostatic energy in the membrane (the membrane capacitance is infinite), and

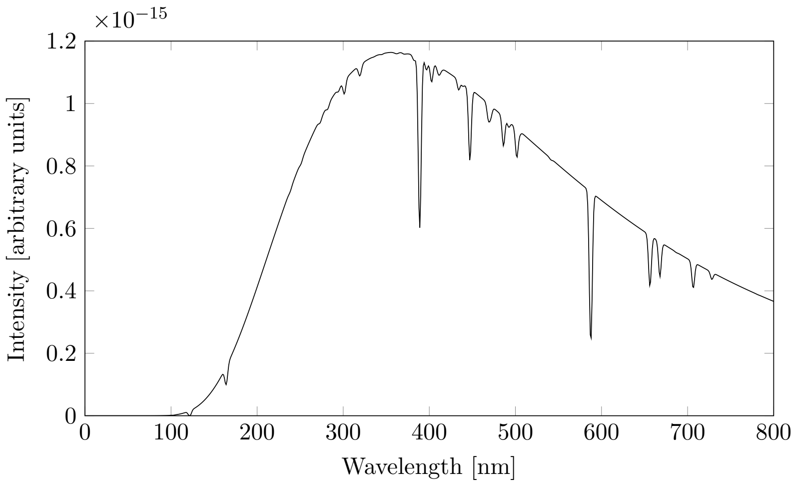
$$W = \frac{1}{2} k d_0^2.$$

Assume in each case that the membrane thickness d cannot become negative.

Question B2

Stellar Black Box

Scientists have recently detected a new star, the MAR-Kappa. The star is almost a perfect blackbody, and its measured light spectrum is shown below.



The total measured light intensity from MAR-Kappa is $I = 1.12 \times 10^{-8}\,\mathrm{W/m^2}$. The mass of MAR-Kappa is estimated to be $3.5 \times 10^{30}\,\mathrm{kg}$. It is stationary relative to the sun. You may find the Stefan-Boltzmann law useful, which states the power emitted by a blackbody with area A is $\sigma A T^4$.

Solution

Note: because a range of answers were accepted for part (a), credit was given for later subparts if they were consistent with the value given in part (a).

a. The spectrum of wavelengths λ emitted from a blackbody only depends on h, c, k_B, λ, and T. Given that the sun has a surface temperature of 5778 K and peak emission at 500 nm, what is the approximate surface temperature of MAR-Kappa?

Solution

By dimensional analysis, we have two energy scales – hc/λ and k_BT. Then, we have that $hc/\lambda_{\text{max}} \propto k_BT$. The diagram has a peak around 360 nm, so, because $\lambda \propto T^{-1}$, we see a peak around 8030 K. We accepted any answer within ± 500 K of 8000 K.

b. The "lines" in the spectrum result from atoms in the star absorbing specific wavelengths of the emitted light. One contribution to the width of the spectral lines is the Doppler shift associated with the thermal motion of the atoms in the star. The spectral line at $\lambda = 389$ nm is due to helium. Estimate to within an order of magnitude the thermal broadening $\Delta\lambda$ of this line. The mass of a helium atom is 6.65×10^{-27} kg.

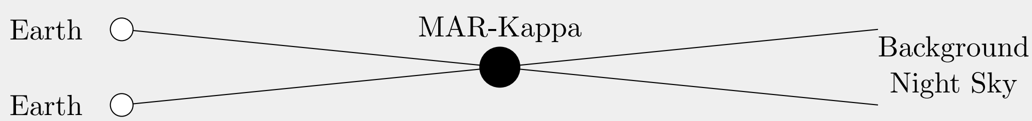
Solution

The rms velocity of the atoms in the star, in the direction towards Earth, is about $\sqrt{\frac{k_B T}{m}} = 4080\,\mathrm{m/s}$. The red shift is thus $\frac{v}{c} \approx 1.4 \times 10^{-5}$, and so $\Delta\lambda \approx 5.3 \times 10^{-3}\,\mathrm{nm}$. This thermal broadening is very insignificant.

c. Over the course of a year, MAR-Kappa appears to oscillate between two positions in the background night sky, which are an angular distance of $1.6 \times 10^{-6}\,\mathrm{rad}$ apart. How far away is MAR-Kappa? Assume that MAR-Kappa lies in the same plane as the Earth's orbit, which is circular with radius $1.5 \times 10^{11}\,\mathrm{m}$.

Solution

This is a parallax effect, as shown in the below diagram:



The angle Earth makes with MAR-Kappa is the angular shift it makes in the night sky. Then we have that $D = R_{ES}/(\theta/2) = 1.9 \times 10^{17}\,\mathrm{m}$.

d. What is the radius of MAR-Kappa?

Solution

The luminosity (total power given off by the star) is given by

$$4\pi D^2 I = L = 4\pi(1.9 \times 10^{17}\,\mathrm{m})^2 \cdot (1.12 \times 10^{-8}\,\mathrm{W/m^2}) = 5.1 \times 10^{27}\,\mathrm{W},$$

with D the distance from Earth to MAR-Kappa. The radius of the star is given by $\sqrt{L/(4\pi\sigma T^4)} = 1.3 \times 10^9\,\mathrm{m}$.

Over the course of some time, you observe that the star's intensity periodically dips from I to $(1 - 10^{-5})I$ and then rises back to I, with period t. One possible explanation for this observation is that an exoplanet is orbiting the star and blocking the starlight for some time.

e. Estimate the exoplanet's radius, assuming that it is much closer to the star than to the Earth.

Solution

The planet is sufficiently close to the star that we can just take the ratio of areas. Then, $R_p^2/R_s^2 = 10^{-5}$, so $R_p = 10^{-2.5} R_s = 4.1 \times 10^6\,\mathrm{m}$.

f. Assume the exoplanet is a blackbody with uniform temperature in a circular orbit around the star. What must t be so that the planet has a temperature of $250\,\mathrm{K}$? (If this were true, and the planet had an appropriate atmosphere, the temperature would increase enough to support life.)

Solution

Suppose the planet is r away from the star. Then, the power from the star absorbed by the planet is

$$\frac{L(\pi R_p^2)}{4\pi r^2} = \frac{LR_p^2}{4r^2}.$$

The planet radiates heat with power $4\pi R_p^2 \sigma T_p^4$. Setting the two equal gives $\frac{L}{16\pi r^2} = \sigma T_p^4$. Then,

$$r = \frac{1}{4T_p^2}\sqrt{\frac{L}{\pi\sigma}} = 6.75 \times 10^{11}\,\mathrm{m}.$$

Recall that for a circular orbit,

$$v = \sqrt{\frac{M_S G}{r}},$$

so

$$\frac{2\pi r}{t} = \sqrt{\frac{M_S G}{r}}.$$

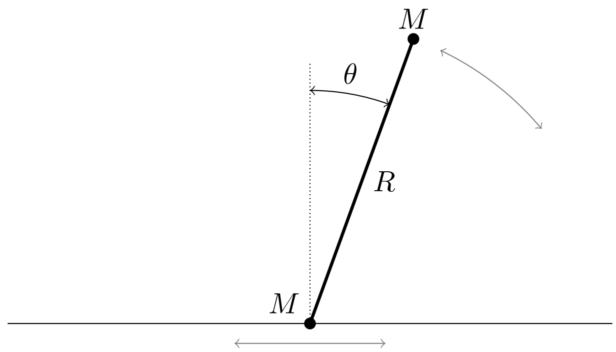
Therefore,

$$t = 2\pi\sqrt{\frac{r^3}{M_S G}} = 7.3\,\mathrm{yr}.$$

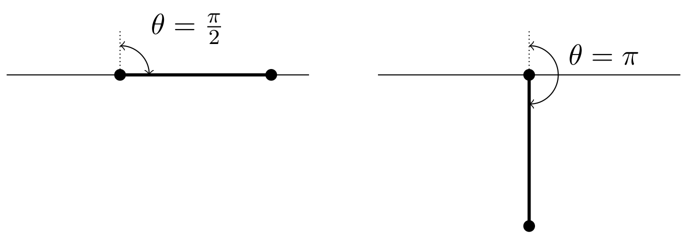
Question B3

Pitfall

A bead is placed on a horizontal rail, along which it can slide frictionlessly. It is attached to the end of a rigid, massless rod of length R. A ball is attached at the other end. Both the bead and the ball have mass M. The system is initially stationary, with the ball directly above the bead. The ball is then given an infinitesimal push, *parallel* to the rail.



Assume that the rod and ball are designed in such a way (not shown explicitly in the diagram) so that they can pass through the rail without hitting it. In other words, the rail only constrains the motion of the bead. Two subsequent states of the system are shown below.



a. Derive an expression for the force in the rod when it is horizontal, as shown at left above, and indicate whether it is tension or compression.

Solution

Since this problem is subtle, we will present multiple solutions.

First solution: The motion of the rod can be decomposed as a superposition of rotation about the center of mass, and translation of the center of mass. We claim that the bead is stationary at this moment. To see this, note that the bead cannot have a vertical velocity component since it is fixed to the rod. Moreover, it cannot have a horizontal velocity component, because since the system experiences no external forces in the horizontal direction, the motion of the center of mass has no horizontal component.

Therefore, the bead is stationary, and since the released gravitational potential energy is MgR, the speed v of the ball obeys

$$\frac{1}{2}Mv^2 = MgR \quad \Rightarrow \quad v = \sqrt{2gR}.$$

The motion of the system consists of a downward speed $v/2$ of the center of mass, superposed on a rotation about the center of mass which gives the masses speed $v/2$.

Now consider the forces on the ball. The only horizontal force on the ball is the force in the rod, so we need to find the horizontal acceleration of the ball. Part of this is due to the centripetal force associated with the rotational motion,

$$F = \frac{M(v/2)^2}{R/2} = Mg.$$

Another part of the ball's acceleration is due to the acceleration of the center of mass, but this is purely vertical and hence irrelevant here. A third part is due to the angular acceleration about the center of mass, but this also is associated with a vertical acceleration of the ball and hence irrelevant. Thus the force in the rod is just a tension

$$T = Mg.$$

A common pitfall to think of the motion of the system as pure rotation about the bead, giving the incorrect answer $F = Mv^2/R = 2Mg$. The reason this is incorrect is that, while it correctly describes the instantaneous velocity, to compute the acceleration one must also account for the rate of change of the pivot point. It can be easy to miss this, because the point at which the pivot is currently located (the bead, in this case) always has zero instantaneous velocity. Yet the pivot point itself *does* have an instantaneous velocity, which must be accounted for. It is possible to solve the problem this way, but it's subtle.

Second solution: While the first solution required only two lines of algebra, it also required careful reasoning. Here we present a more complicated, but more straightforward alternative. Let x be the horizontal position of the bead. The position of the ball is

$$\mathbf{r} = (x + R\sin\theta, R\cos\theta).$$

By conservation of momentum, the center of mass must be at the same horizontal position as it started, so

$$2x + R\sin\theta = 0.$$

Therefore, the position of the ball is

$$\mathbf{r} = ((R/2)\sin\theta, R\cos\theta).$$

That is, it follows an ellipse whose major axis is twice the minor axis.

Differentiating this twice, the acceleration of the ball is

$$a_x = \frac{R}{2}(\alpha\cos\theta - \omega^2\sin\theta), \quad a_y = -R(\alpha\sin\theta + \omega^2\cos\theta).$$

In this case, $\theta = 90^\circ$ and we are only interested in a_x, giving

$$a_x = -\frac{R\omega^2}{2}.$$

At this point, one needs to find $\omega(\theta)$ using conservation of energy. Plugging in the result from either part (c) below or the first solution above gives $a_x = -g$, which indicates a force of Mg to the left, and hence a tension in the rod. Incidentally, yet another solution is to use $F = Mv^2/R$ where R is the instantaneous radius of curvature of the ellipse, which may be computed directly.

Third solution: We can work in the *noninertial* reference frame of the bead. In this frame, the ball simply rotates around the bead, with the naive centripetal force

$$F = \frac{Mv^2}{R} = 2Mg.$$

However, since the tension accelerates the bead to the right, there is a fictitious force of magnitude T pulling the ball to the left in this frame. The real tension force and the fictitious force add up to the centripetal force, so $T + T = F$, giving $T = Mg$.

b. Derive an expression for the force in the rod when the ball is directly below the bead, as shown at right above, and indicate whether it is tension or compression.

Solution

First solution: At this point the released gravitational potential energy is $2MgR$, and both masses are moving horizontally with speed v, where

$$2\,\frac{1}{2}Mv^2 = 2MgR \quad \Rightarrow \quad v = \sqrt{2gR}.$$

Work in the frame moving to the right with speed v. In this frame the bead is stationary and the ball has velocity $2v$ and is instantaneously rotating about the bead, so it must be experiencing a centripetal force

$$\frac{M(2v)^2}{R} = 8Mg.$$

Unlike in part (a), there are no additional contribution from the acceleration of the rotation center, because the bead can only ever accelerate horizontally, and the force in the rod at this moment is vertical. Since the ball also experiences a downward force of Mg due to gravity, the force in the rod is a tension

$$T = 9Mg.$$

Second solution: We can also solve the problem using the algebraic approach. Following what we found in the second solution to part (a),

$$a_y = -R(\alpha \sin\theta + \omega^2 \cos\theta) = R\omega^2.$$

Plugging in the result for ω^2, one finds

$$a_y = 8g$$

which, as before, indicates a tension of $T = 9Mg$.

Third solution: Work in the original frame, where the masses are instantaneously rotating about the midpoint of the rod. Because of this motion, the ball must be experiencing a centripetal force

$$\frac{Mv^2}{R/2} = 4Mg.$$

This makes answering $5Mg$ a tempting pitfall; to get the right answer, we must also account for the vertical acceleration of the center of mass. The height of the center of mass is $y_{\text{CM}} = (R/2)\cos\theta$, so

$$a_{\text{CM}} = \frac{d}{dt}\left(-\frac{R}{2}\omega\sin\theta\right) = -\frac{R}{2}\left(\omega^2\cos\theta + \alpha\sin\theta\right).$$

At this point, $\sin\theta = 0$ and $\cos\theta = -1$, so

$$a_{\text{CM}} = \frac{R}{2}\omega^2 = \frac{v^2}{R/2} = 4g$$

where we used $v = \omega(R/2)$. Therefore, the total upward acceleration of the ball is $8g$, as we found earlier, so the force in the rod is a tension $T = 9Mg$.

In summary, there are many valid ways to arrive at the correct answers of Mg and $9Mg$, but all of them require careful thought. When this exam was given, only about 2% of students successfully found these answers.

c. Let θ be the angle the rod makes with the vertical, so that the rod begins at $\theta = 0$. Find the angular velocity $\omega = d\theta/dt$ as a function of θ.

Solution

The height of the ball is $R\cos\theta$, so the released gravitational potential energy is

$$MgR(1-\cos\theta).$$

The kinetic energy may be decomposed into rotation about the center of mass and translation of the center of mass; this translation is purely vertical by conservation of momentum. The vertical speed of the center of mass is $(R/2)\omega\sin\theta$, so the kinetic energy is

$$\frac{1}{2}I\omega^2 + \frac{1}{2}(2M)v_{\text{CM}}^2 = \frac{1}{4}MR^2\omega^2 + \frac{1}{4}MR^2\omega^2\sin^2\theta.$$

Equating these two expressions and simplifying gives

$$\omega^2 = \frac{4g}{R}\frac{1-\cos\theta}{1+\sin^2\theta}.$$

Some students instead treated the motion of the bead and ball separately; this can lead to the correct answer, but it is easy to make a mistake. Another common route was to apply Lagrangian mechanics, solving the Euler-Lagrange equations, or equivalently to solve the $F = ma$ equations. These are quite complicated, and nobody managed to integrate them to get the correct answer.

Answer Sheets

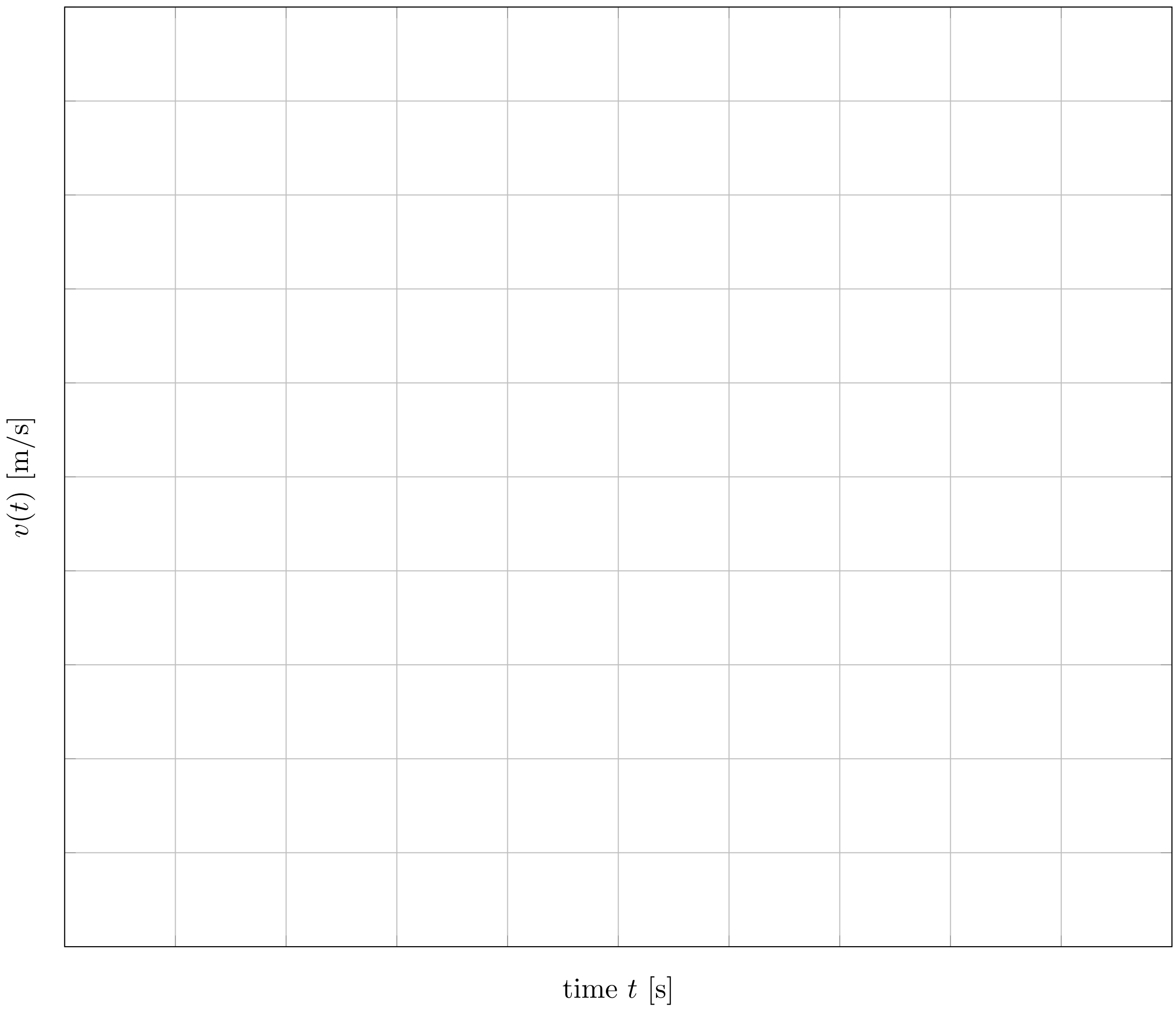
Following are answer sheets for the graphing portion of Problem A1.

Student AAPT ID #:

Proctor AAPT ID #:

A1: Collision Course

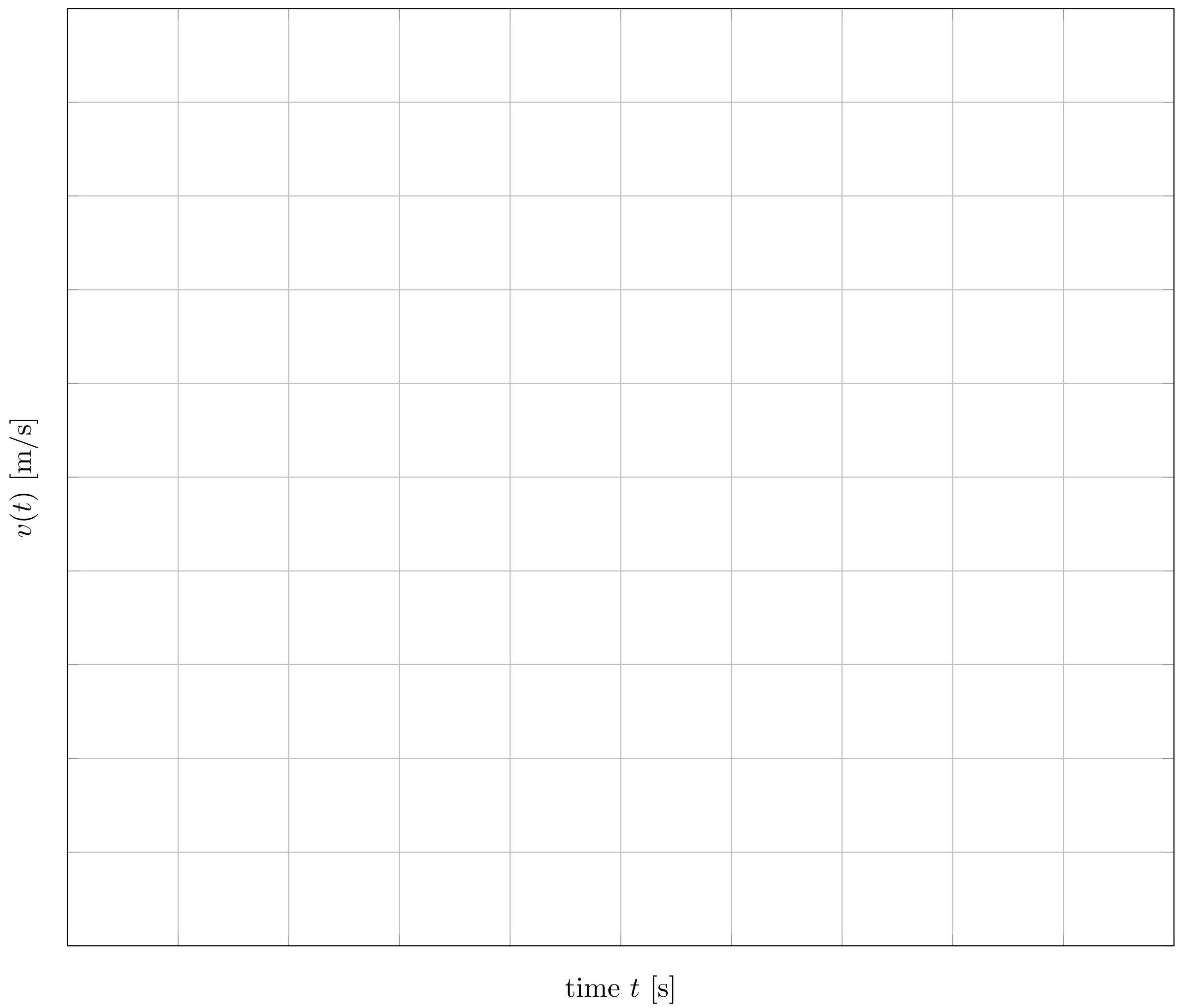
(a)



Student AAPT ID #:

Proctor AAPT ID #:

(c)



AAPT UNITED STATES PHYSICS TEAM
AIP 2020

USA Physics Olympiad Exam

Information about USAPhO Test

- This examination consists of two parts and six problems. Ordinarily, you will be given 90 minutes to complete Part A, take a 10-15 minute break, and return to complete Part B in 90 minutes.

  You may choose to observe the above guideline if you'd like to practice for future USAPhO tests.

- Ordinarily, you are allowed calculators, but they are not allowed to use symbolic math, programming, or graphical features of these calculators. Calculators must not be shared and their memory must be cleared of data and programs. Cell phones, smart watches, PDAs, or cameras cannot be used during the exam or while the exam papers are present. Students are not allowed to bring any tables, books, or collections of formulas.

  You may choose to observe the above guideline if you'd like to practice for future USAPhO tests.

Thank you for participating USAPhO this year under such extraordinary circumstances. We hope that you and your family stay safe, and that you continue to encourage more students like you to study physics and try out $F = ma$ test hosted by AAPT.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Ariel Amir, JiaJia Dong, Mark Eichenlaub, Abijith Krishnan, Kye W. Shi, and Mike Winer.

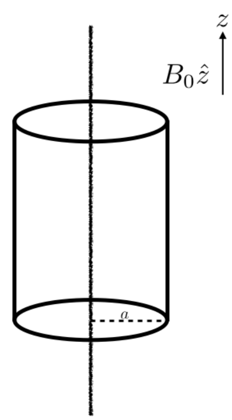
Part A

Question A1

Braking up

An infinitely long wire with linear charge density $-\lambda$ lies along the z-axis. An infinitely long insulating cylindrical shell of radius a is concentric with the wire and can rotate freely about the z-axis. The shell has moment of inertia per unit length I. Charge is uniformly distributed on the shell, with surface charge density $\dfrac{\lambda}{2\pi a}$.

The system is immersed in an external magnetic field $B_0\hat{\mathbf{z}}$, and is initially at rest. Starting at $t = 0$, the external magnetic field is slowly reduced to zero over a time $T \gg a/c$, where c is the speed of light.



a. Find an expression of the final angular velocity ω of the cylinder in terms of the symbols given and other constants.

b. You may be surprised that the expression you find above is not zero! However, the electric and magnetic fields can have angular momentum. Analogous to the "regular" angular momentum definition, the EM field angular momentum per unit volume at a displacement $\mathbf{r}$ from the axis of rotation is:

$$\mathcal{L}(\mathbf{r}) = \mathbf{r} \times \mathcal{P}(\mathbf{r}).$$

$\mathcal{P}(\mathbf{r})$ is a vector analogous to momentum, given by

$$\mathcal{P}(\mathbf{r}) = \alpha \cdot \big(\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\big)\,.$$

where α is some proportionality constant. Find an expression for α in terms of given variables and fundamental constants.

Question A2

Swoosh!

In 1851, Léon Foucault built a pendulum 67 metres tall with a 28-kg weight. He connected it to the top of the Panthéon in Paris with a bearing that enabled it to freely change its plane of oscillation. Because of the Earth's rotation, the plane of oscillation slowly moved over time: if we imagine a large horizontal clock under the pendulum, if initially the oscillations went from "12" to "6", later on they would move to the "3-9" plane, for example, as shown in the figure below. Perhaps surprisingly, the time it took the oscillations to go back to their original plane is longer than 12 hours. In this problem we will investigate why this is the case, and what the shape the pendulum traces out.

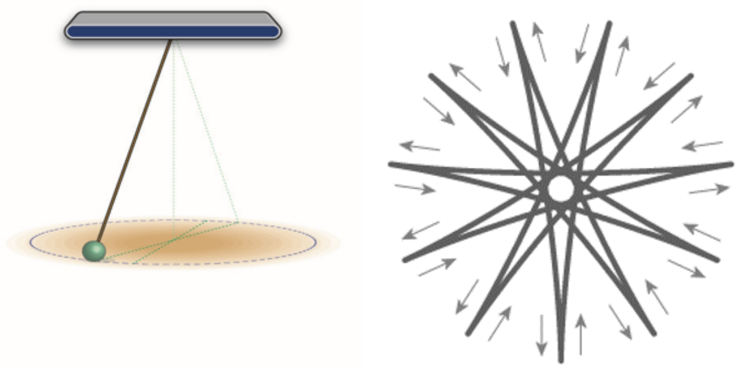


Figure 1: Left: A schematic of Foucault's pendulum. Right: The pendulum motion projected on a horizontal plane in the *rotating* lab frame.

First, consider the case of a Foucault pendulum installed precisely at *the North Pole*, with length l. We denote $\sqrt{g/l} = \omega$. The angular velocity of the Earth is Ω.

John is an observer looking at the pendulum from a *fixed point in space*. At $t = 0$, he sees the pendulum at position $(A, 0)$ and with velocity $(0, V)$ in the x-y (horizontal) plane.

a. For John, what are the approximate equations describing the motion of the pendulum in the x-y plane? You may assume that the amplitude of the oscillations is small. We define the coordinates of the pendulum at rest as $(0, 0)$.

b. What will the coordinates x, y in Jonh's frame be at a later time t?

c. Ella, an observer resides at the North Pole, is also looking at the pendulum. What are the coordinates, $\tilde{x}(t)$ and $\tilde{y}(t)$, as observed by Ella? Assume that at time $t = 0$, the coordinate systems of John's and Ella's overlap.

d. What is the speed of the pendulum bob observed by Ella at $t = 0$?

e. Find the initial conditions for A, V, such that as *measured in Ella's frame*:

  i. the pendulum passes precisely through its resting position.

  ii. it has a "spike" at the points of maximal amplitude (see figure below) instead of a "rounded" trajectory.

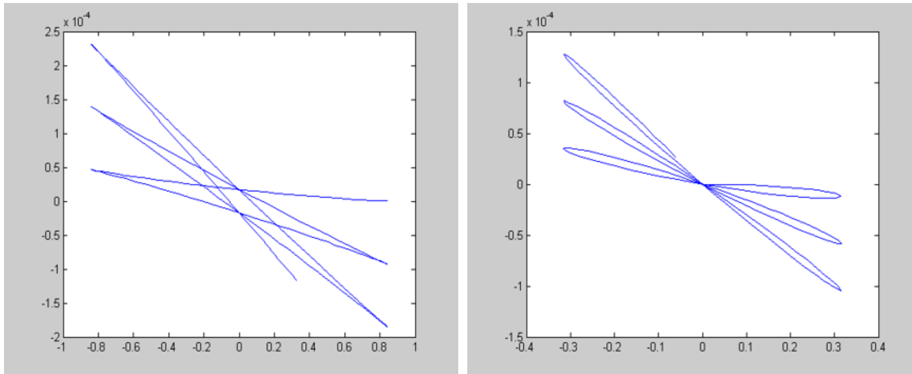


Figure 2: Two possible trajectories with "spike"(left) and more "rounded" (right).

In a rotating frame, a fictitious force known as the Coriolis force acts on the particles. For Foucault's pendulum, the Coriolis force acts primarily in the horizontal plane, in a direction *perpendicular to the velocity of the mass in the Earth's frame* with magnitude:

$$F = 2m\Omega v \cdot \sin\theta, \tag{A2-1}$$

where m and v are the pendulum's mass and its velocity, and θ the latitude (90° for the North Pole). *Note that when the velocity changes sign, so does the Coriolis force.*

f. How long would it take for the plane of oscillation of Foucault's pendulum to return to its initial value in Paris, which has a latitude of about 49°.

Question A3

Spin Cycle

Cosmonaut Carla is preparing for the Intergalactic 5000 race. She practices for her race on her handy race track of radius R, carrying a stopwatch with her. Her racecar maintains a constant speed v during her practices. For this problem, you can assume that $v > 0.1c$, where c is the speed of light.

a. How much time elapses on Carla's stopwatch with each revolution?

Carla decides to do a fun experiment during her training. She places two stationary clocks down: Clock A at the center of the race track, i.e. the origin; and Clock B at a point on the race track denoted as $(R, 0)$. She then begins her training.

For parts (b) through (d), we define Carla's inertial reference frame (CIRF) as an inertial reference frame in which Carla is momentarily at rest, and which has the same origin of coordinates as the lab frame. Thus, CIRF is a new inertial frame each moment. The times on the clocks and stopwatch are all calibrated such that they all read 0 in CIRF when she passes by Clock B for the first time.

b. In the lab frame (the reference frame of the clocks, which are at rest), what is the offset between Clock A and Clock B?

c. If Carla's stopwatch measures an elapsed time τ, what does Clock A measure in CIRF?

d. If Carla's stopwatch measures an elapsed time τ, what does Clock B measure in CIRF?

Part B

Question B1

String Cheese

a. When a faucet is turned on, a stream of water flows down with initial speed v_0 at the spout. For this problem, we define y to be the vertical coordinate with its positive direction pointing up.

Assuming the water speed is only affected by gravity as the water falls, find the speed of water $v(y)$ at height y. Define the zero of y such that the equation for v^2 has only one term and find y_0, the height of the spout.

b. Assume that the stream of water falling from the faucet is cylindrically symmetric about a vertical axis through the center of the stream. Also assume that the volume of water per unit time exiting the spout is constant, and that the shape of the stream of water is constant over time.

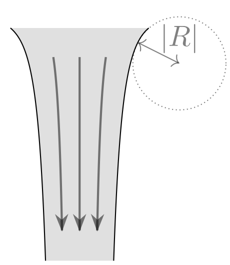
In this case, the radius r of the stream of water is a function of vertical position y. Let the radius at the faucet be r_0. Using your result from part (a), find $r(y)$.

If $r(y)$ is not constant, it implies that the water has some radial velocity during its fall, in contradiction to our assumptions in part (a) that the motion is purely vertical. You may assume throughout the problem that any such radial velocity is negligibly small.

c. The water-air interface has some surface tension, σ. The effect of surface tension is to change the pressure in the stream according to the *Young-Laplace equation*,

$$\Delta P = \sigma \left(\frac{1}{r} + \frac{1}{R} \right),$$

where ΔP is the difference in pressure between the stream and the atmosphere and R is the radius of curvature of the vertical profile of the stream, visualized below. ($R < 0$ for the stream of water; the radius of curvature would be positive only if the stream profile curved inwards.)



For this part of the problem, we assume that $|R| \gg |r|$, so that the curvature of the vertical profile of the stream can be ignored. Also assume that water is incompressible.

Accounting for the pressure in the stream, find a new equation relating for $r(y)$ in terms of σ, r_0, v_0, and ρ, the density of water. You do not need to solve the equation for r.

d. After falling for some distance, the water stream usually breaks into smaller droplets. This occurs because small random perturbations to the shape of the stream grow over time, eventually breaking the stream into apart.

For the rest of this problem we ignore the change in the radius of the stream due to changing speed of the water, as considered earlier. Instead, we examine small random variations in the radius of the stream.

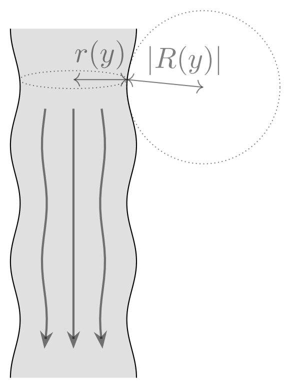
Random variations can be broken down into a sum of sinusoidal variations in stream radius, each with a different wavenumber k. We can analyze these different sinusoidal variations independently.

Consider a stream of water whose radius obeys

$$r(y) = r_0 + A\cos(ky),$$

where $A \ll r_0$ is the perturbation amplitude. To analyze such a stream, it is sufficient to consider only the thickest and thinnest parts of the stream.

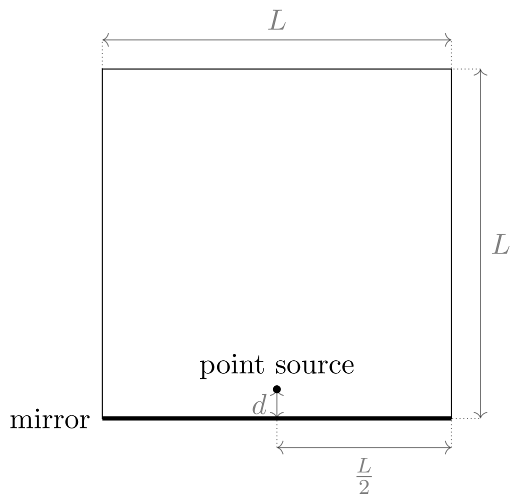
Accounting for both sources of curvature, find a condition on r_0 and k such that the size of perturbations increases with time.



Question B2

Mirror Mirror on the Wall

Consider a square room with side length L. The bottom wall of the room is a perfect mirror.* A perfect monochromatic point source with wavelength λ is placed a distance d above the center of the mirror, where $\lambda \ll d \ll L$.

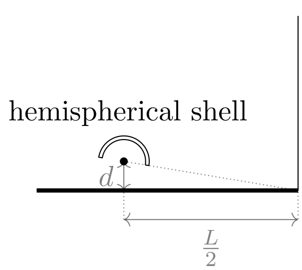


*Remember that the phase of light reflected by a mirror changes by 180°.

a. On the right wall, an interference pattern emerges. What is the distance y between the bottom corner and the closest bright fringe above it? Hint: you may assume $\lambda \ll y \ll L$ as well.

b. You plan on running an experiment to determine λ in a room with $L = 40$ m, and you know that λ is between 550 and 750 nm. You will measure d and y_{10} (the distance of the tenth fringe from the corner) with the same ruler (with markings of 1 mm). At what d should you place the point source to minimize your error in your λ measurement? Roughly what is that minimum error?

c. Now suppose we place a transparent hemispherical shell of thickness s and index of refraction n over the source such that all light from the source that directly strikes the right wall passes through the shell, and all light from the source that strikes the mirror first does not pass through the shell.



At what y is the fringe closest to the bottom-most corner now? (You may find it convenient to use $\lfloor x \rfloor$, the largest integer below x.) What is the spacing between the fringes now? Ignore any reflections or diffraction from the hemispherical shell.

d. Now, suppose the hemispherical shell is removed, and we instead observe the interference pattern on the top wall. To the nearest integer, what is the total number of fringes that appear on the top wall? You may assume that $d \ll L$.

Question B3

Real Expansion

Consider a “real” monatomic gas consisting of N atoms of negligible volume and mass m in equilibrium inside a closed cubical container of volume V. In this “real” gas, the attractive forces between atoms is small but not negligible. Because these atoms have negligible volume, you can assume that the atoms do not collide with each other for the entirety of the problem.

a. Consider an atom in the interior of this container of volume V. Suppose the potential energy of the interaction is given by

$$u(r) = \begin{cases} 0 & r < d \\ -\epsilon \left(\frac{d}{r}\right)^6 & r \geq d \end{cases}$$

where $d \ll V^{1/3}$ is the minimum allowed distance between two atoms. Assume the gas is uniformly distributed within the container, what is the average potential energy of this atom? Write your answer in terms of $a' = \dfrac{2\pi d^3 \epsilon}{3}$, N, and V.

b. What is the average potential energy of an atom near the boundary of the box? Assume that there is no interaction between atoms near the boundary and the box itself.

c. Using Bernoulli's law $P + U + \rho v^2/2 =$ constant, with pressure P, potential energy density U, mass density ρ and fluid velocity v, what is the pressure at the boundary of the box? Assume the interior pressure is given by the ideal gas law.

d. Assuming most atoms are in the interior of the box, what is the total energy of the atoms in the box?

Now consider an insulated partitioned container with two sections, each of volume V. We fill one side of the container with N atoms of this “real” gas at temperature T, which the other side being a vacuum. We then quickly remove the partition and let the gas expand to fill the entirety of the partitioned container. During this expansion, the energy of the gas remains unchanged.

e. What is the final temperature of the gas after the expansion?

f. What is the increase in the entropy of the universe as a result of the free expansion? Give your answer to first order in $\frac{a'N}{Vk_BT}$.

AAPT UNITED STATES PHYSICS TEAM
AIP 2020

USA Physics Olympiad Exam

Information about USAPhO Test

- This examination consists of two parts and six problems. Ordinarily, you will be given 90 minutes to complete Part A, take a 10-15 minute break, and return to complete Part B in 90 minutes.

  You may choose to observe the above guideline if you'd like to practice for future USAPhO tests.

- Ordinarily, you are allowed calculators, but they are not allowed to use symbolic math, programming, or graphical features of these calculators. Calculators must not be shared and their memory must be cleared of data and programs. Cell phones, smart watches, PDAs, or cameras cannot be used during the exam or while the exam papers are present. Students are not allowed to bring any tables, books, or collections of formulas.

  You may choose to observe the above guideline if you'd like to practice for future USAPhO tests.

Thank you for participating USAPhO this year under such extraordinary circumstances. We hope that you and your family stay safe, and that you continue to encourage more students like you to study physics and try out $F = ma$ test hosted by AAPT.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Ariel Amir, JiaJia Dong, Mark Eichenlaub, Abijith Krishnan, Kye W. Shi, and Mike Winer.

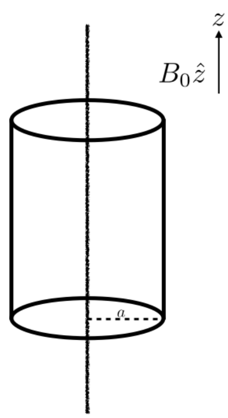
Part A

Question A1

Braking up

An infinitely long wire with linear charge density $-\lambda$ lies along the z-axis. An infinitely long insulating cylindrical shell of radius a is concentric with the wire and can rotate freely about the z-axis. The shell has moment of inertia per unit length I. Charge is uniformly distributed on the shell, with surface charge density $\frac{\lambda}{2\pi a}$.

The system is immersed in an external magnetic field $B_0\hat{\mathbf{z}}$, and is initially at rest. Starting at $t = 0$, the external magnetic field is slowly reduced to zero over a time $T \gg a/c$, where c is the speed of light.



a. Find an expression of the final angular velocity ω of the cylinder in terms of the symbols given and other constants.

Solution

From Faraday's law, you can find the induced electric field inside the cylinder at a distance r from the wire:

$$\oint \vec{E}_{\text{ind}} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \tag{A1-1}$$

$$E_{\text{ind}}(r) = -\frac{r}{2}\frac{dB}{dt} \tag{A1-2}$$

This induced field exerts a torque on the cylinder, causing it to rotate:

$$\tau = 2\pi a \cdot \frac{\lambda}{2\pi a} \cdot E_{\text{ind}}(a) \cdot a = I \cdot \frac{d\omega}{dt} \tag{A1-3}$$

$$\Rightarrow \frac{d\omega}{dt} = -\frac{\lambda a^2}{2I} \cdot \frac{dB}{dt} \tag{A1-4}$$

Integrate on both sides, and noting that $\omega(t = 0) = 0$, we have:

$$\omega(T) = -\frac{\lambda a^2}{2I}\big(B(T) - B_0\big) \tag{A1-5}$$

It is important to note that $B(T) \neq 0$. Even though the external field decreases to zero, the now-rotating charged cylinder generates a magnetic field. Using Ampere's Law, you can find at $t = T$, the magnetic field is:

$$\oint \vec{B}_{\text{ind}} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}, \tag{A1-6}$$

$$\text{where} \quad I_{\text{enc}} = \frac{\lambda}{2\pi a} \cdot \omega(T) \cdot a \tag{A1-7}$$

$$\Rightarrow B(T) = \mu_0 \frac{\lambda}{4\pi^2 a} \omega(T) \tag{A1-8}$$

Combining equations (A1-5) and (A1-8), we have:

$$\omega(T) = \frac{\frac{\lambda a^2}{2I} B_0}{1 + \mu_0 \frac{\lambda^2 a}{8\pi I}}$$

b. You may be surprised that the expression you find above is not zero! However, the electric and magnetic fields can have angular momentum. Analogous to the "regular" angular momentum definition, the EM field angular momentum per unit volume at a displacement $\mathbf{r}$ from the axis of rotation is:

$$\mathcal{L}(\mathbf{r}) = \mathbf{r} \times \mathcal{P}(\mathbf{r}).$$

$\mathcal{P}(\mathbf{r})$ is a vector analogous to momentum, given by

$$\mathcal{P}(\mathbf{r}) = \alpha \cdot \big(\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\big) .$$

where α is some proportionality constant. Find an expression for α in terms of given variables and fundamental constants.

Solution

The electric field inside the cylindrical shell is given by $\mathbf{E}(r) = -\frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}}$ inward. The magnetic field is given by $B(t)\hat{\mathbf{z}}$. Then:

$$\mathcal{P}(\mathbf{r}) = \alpha \frac{\lambda B(t)}{2\pi\epsilon_0 r}\hat{\theta}.$$

The angular momentum per unit volume is then:

$$\mathcal{L}(\mathbf{r}) = -\alpha \frac{\lambda}{2\pi\epsilon_0}\hat{\mathbf{z}}$$

The angular momentum *per unit length is then*:

$$\mathbf{L} = -\frac{\alpha\lambda B(t) a^2}{2}\hat{\mathbf{z}}.$$

Comparing this to Equation (A1-5) shows that $\alpha = \epsilon_0$.

Question A2

Swoosh!

In 1851, Léon Foucault built a pendulum 67 metres tall with a 28-kg weight. He connected it to the top of the Panthéon in Paris with a bearing that enabled it to freely change its plane of oscillation. Because of the Earth's rotation, the plane of oscillation slowly moved over time: if we imagine a large horizontal clock under the pendulum, if initially the oscillations went from "12" to "6", later on they would move to the "3-9" plane, for example, as shown in the figure below. Perhaps surprisingly, the time it took the oscillations to go back to their original plane is longer than 12 hours. In this problem we will investigate why this is the case, and what the shape the pendulum traces out.

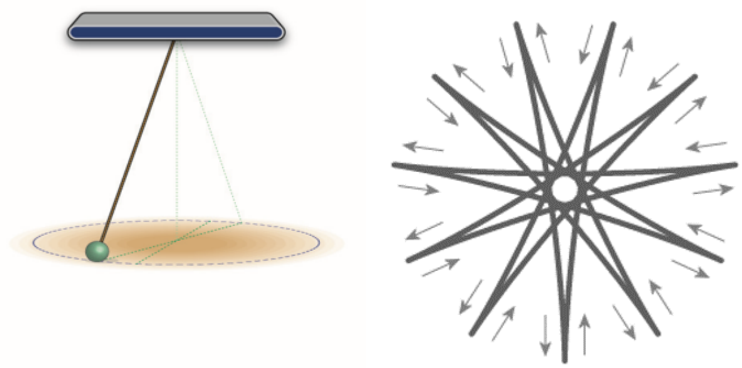


Figure 1: Left: A schematic of Foucault's pendulum. Right: The pendulum motion projected on a horizontal plane in the *rotating* lab frame.

First, consider the case of a Foucault pendulum installed precisely at *the North Pole*, with length l. We denote $\sqrt{g/l} = \omega$. The angular velocity of the Earth is Ω.

John is an observer looking at the pendulum from a *fixed point in space*. At $t = 0$, he sees the pendulum at position $(A, 0)$ and with velocity $(0, V)$ in the x-y (horizontal) plane.

a. For John, what are the approximate equations describing the motion of the pendulum in the x-y plane? You may assume that the amplitude of the oscillations is small. We define the coordinates of the pendulum at rest as $(0, 0)$.

Solution

John's reference frame is inertial and the point of attachment stationary, so this is a free-moving pendulum obeying simple harmonic motion in each axis:

$$a_x + \omega^2 x = 0; a_y + \omega^2 y = 0, \tag{A2-1}$$

b. What will the coordinates x, y in Jonh's frame be at a later time t?

Solution

The solution to Eq.(A2-1) is the familiar simple harmonic motion. In general, if the displacement is $r = A\cos\omega t$, then the velocity is $v = -A\omega \sin\omega t$. Using the inital conditions provided, we have:

$$x(t) = A\cos(\omega t); y(t) = \frac{V}{\omega}\sin(\omega t). \tag{A2-2}$$

Note that this corresponds to an ellipse.

c. Ella, an observer resides at the North Pole, is also looking at the pendulum. What are the coordinates, $\tilde{x}(t)$ and $\tilde{y}(t)$, as observed by Ella? Assume that at time $t = 0$, the coordinate systems of John's and Ella's overlap.

Solution

In the rotating frame, we have $\tilde{x} = x\cos(\Omega t) + y\sin(\Omega t)$, $\tilde{y} = -x\sin(\Omega t) + y\cos(\Omega t)$ (with $2\pi/\Omega = 24$ hrs). Plugging in the form of $x(t)$ and $y(t)$ we find:

$$\tilde{x} = A\cos(\omega t)\cos(\Omega t) + \frac{V}{\omega}\sin(\omega t)\sin(\Omega t), \tag{A2-3}$$

and:

$$\tilde{y} = -A\cos(\omega t)\sin(\Omega t) + \frac{V}{\omega}\sin(\omega t)\cos(\Omega t). \tag{A2-4}$$

d. What is the speed of the pendulum bob observed by Ella at $t = 0$?

Solution

In John's frame, the velocity at this time is $(0, V)$. To get the velocity in Ella's frame, we can either take the derivative of the result of part (c) directly, or transform the velocity obtained in John's frame to Ella's frame, not forgetting to add the term $-\Omega A$ to the initial velocity in the y axis. This gives

$$\tilde{v}_x \approx (V - \Omega A)\sin(\Omega t); \tilde{v}_y = (V - \Omega A)\cos(\Omega t). \tag{A2-5}$$

At $t = 0$, $\tilde{v}_x = 0$ and $\tilde{v}_y = V - \Omega A$.

e. Find the initial conditions for A, V, such that as *measured in Ella's frame*:

i. the pendulum passes precisely through its resting position.

Solution

Considering the motion in John's frame, clearly the pendulum will pass through the resting position if and only if $V = 0$.

ii. it has a "spike" at the points of maximal amplitude (see figure below) instead of a "rounded" trajectory.

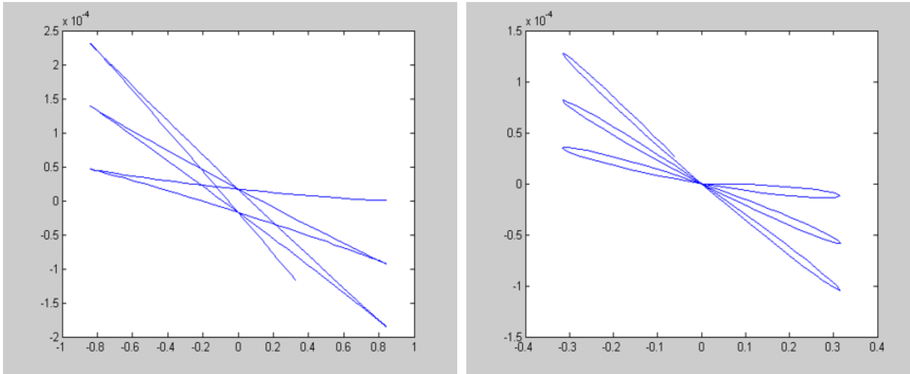


Figure 2: Two possible trajectories with “spike”(left) and more “rounded” (right).

Solution

To have a spike, we need the velocity to *vanish* at the extremal points in *Ella's* frame. This gives the condition:

$$V = \Omega A. \tag{A2-6}$$

Note that in Ella's frame, this implies releasing the pendulum from rest at some amplitude.

In a rotating frame, a fictitious force known as the Coriolis force acts on the particles. For Foucault's pendulum, the Coriolis force acts primarily in the horizontal plane, in a direction *perpendicular to the velocity of the mass in the Earth's frame* with magnitude:

$$F = 2m\Omega v \cdot \sin\theta, \tag{A2-7}$$

where m and v are the pendulum's mass and its velocity, and θ the latitude (90° for the North Pole). *Note that when the velocity changes sign, so does the Coriolis force.*

f. How long would it take for the plane of oscillation of Foucault's pendulum to return to its initial value in Paris, which has a latitude of about 49°.

Solution

Since the expression for the Coriolis force only depends on the combination $\Omega\sin(\theta)$, and since the solution at the North Pole must be $\pi/\Omega = 12$hours, the time at a general latitude must be:

$$T = \frac{\pi}{\Omega\sin(\theta)}. \tag{A2-8}$$

For Paris, the time is about 16 hours.

Question A3

Spin Cycle

Cosmonaut Carla is preparing for the Intergalactic 5000 race. She practices for her race on her handy race track of radius R, carrying a stopwatch with her. Her racecar maintains a constant speed v during her practices. For this problem, you can assume that $v > 0.1c$, where c is the speed of light.

a. How much time elapses on Carla's stopwatch with each revolution?

Solution

From time dilation, her clock ticks slower by a factor γ. Therefore, each revolution takes

$$\frac{2\pi R}{\gamma v} = \frac{2\pi R\sqrt{1 - v^2/c^2}}{v}$$

when measured by Carla's stopwatch.

Carla decides to do a fun experiment during her training. She places two stationary clocks down: Clock A at the center of the race track, i.e. the origin; and Clock B at a point on the race track denoted as $(R, 0)$. She then begins her training.

For parts (b) through (d), we define Carla's inertial reference frame (CIRF) as an inertial reference frame in which Carla is momentarily at rest, and which has the same origin of coordinates as the lab frame. Thus, CIRF is a new inertial frame each moment. The times on the clocks and stopwatch are all calibrated such that they all read 0 in CIRF when she passes by Clock B for the first time.

b. In the lab frame (the reference frame of the clocks, which are at rest), what is the offset between Clock A and Clock B?

Solution

Carla's motion is perpendicular to the displacement between Clock A and Clock B when they are synchronized in CIRF. Therefore, the simultaneous synchronization in CIRF is also simultaneous in the lab frame. Thus, the offset is 0.

To understand why this offset is 0, you can also imagine placing an lightbulb halfway between the two clocks and having it send a light pulse at some known time. In both Carla's frame and the lab frame, the light pulse reaches the two clocks simultaneously.

c. If Carla's stopwatch measures an elapsed time τ, what does Clock A measure in CIRF?

Solution

By symmetry, the speed at which the center clock ticks according to CIRF cannot change. In one revolution, Carla's stopwatch measures $\frac{2\pi R\sqrt{1-v^2/c^2}}{v}$, while the center clock measures $\frac{2\pi R}{v}$. Then,

$$t_A(\tau) = \frac{\tau}{\sqrt{1 - v^2/c^2}}.$$

d. If Carla's stopwatch measures an elapsed time τ, what does Clock B measure in CIRF?

Solution

The readings on Clock B and on Clock A are not necessarily identical once Carla moves through the circle (because her motion becomes more parallel with the displacement between the two clocks, and thus simultaneity is lost).

Suppose Carla is at $(R\cos\theta, R\sin\theta)$, so her velocity is given by $(-v\sin\theta, v\cos\theta)$. Suppose we place a light bulb between the two clocks and having it propagate a light pulse. In the lab frame, the light pulse reaches the two clocks simultaneously. In CIRF, the math is a little more complicated.

We first rotate our lab coordinates so that $\hat{\mathbf{a}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}$, and $\hat{\mathbf{b}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$. We now give the coordinates of the clocks and bulb in the rotated lab frame: Clock A, $(a, b) = (0, 0)$; Clock B, $(a, b) = (-R\sin\theta, R\cos\theta)$; bulb, $(a, b) = (-R\sin\theta, R\cos\theta)/2$. In the lab frame, a light pulse is emitted at

$$t = 0, a = -(R/2)\sin\theta, b = (R/2)\cos\theta.$$

The light pulse reaches Clock A at

$$t = R/2, a = 0, b = 0,$$

and Clock B at

$$t = R/2, a = -R\sin\theta, b = R\cos\theta.$$

Under a Lorentz tranformation from the lab frame to CIRF, we have that the light pulse reaches Clock A at $t' = \gamma R/2$ and Clock B at $t' = \gamma R/2 + \gamma v R\sin\theta$. Thus, Clock B reads the same time as Clock A with offset $\gamma v R\sin\theta$ in the reference frame moving at $v_a = v$, $v_b = 0$. Note that Clock A ticks slower by a factor of γ in this frame. Therefore, the time on clock B is $vR\sin\theta$ behind the time on clock A.

Then,

$$t_B(\tau) = t_A(\tau) - vR\sin\theta = \frac{\tau}{\sqrt{1 - v^2/c^2}} - vR\sin\theta.$$

(This is the answer we expect from the rear clock ahead effect!) Finally, we use that $\theta = \omega\tau$ and $\omega = \frac{2\pi}{T}$, where T is the period in Carla's frame. Then,

$$t_B(\tau) = \frac{\tau}{\sqrt{1 - v^2/c^2}} - \frac{vR}{c^2}\sin\left(\frac{v\tau}{R\sqrt{1 - v^2/c^2}}\right).$$

Part B

Question B1

String Cheese

a. When a faucet is turned on, a stream of water flows down with initial speed v_0 at the spout. For this problem, we define y to be the vertical coordinate with its positive direction pointing up.

Assuming the water speed is only affected by gravity as the water falls, find the speed of water $v(y)$ at height y. Define the zero of y such that the equation for v^2 has only one term and find y_0, the height of the spout.

Solution

We can use energy conservation to answer this question. For a bit of water with mass m, the total energy E is the sum of the kinetic and gravitational potential energies,

$$E = \frac{1}{2}mv^2 + mgy. \tag{B1-1}$$

(With this sign convention, $g \approx 10\ \text{m/s}^2$ is positive. As y decreases, so does the potential energy.)

As the bit of water falls, its energy remains constant, and is equal to the initial value of

$$E = \frac{1}{2}mv_0^2 + mgy_0. \tag{B1-2}$$

Equating eliminating E from equations B1-1 and B1-2, we have

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 + mgy_0,$$

and solving for v, we get

$$v = \sqrt{v_0^2 + 2g(y_0 - y)}$$

The equation for v^2 has three terms, but we were asked to choose the zero of y such that there is only one. Evidently, two of the terms must cancel, and these must be the two constant terms, since the final term varies with y.

That means we need

$$v_0^2 + 2gy_0 = 0.$$

Solving for y_0, the vertical position of the spout is

$$\boxed{y_0 = \frac{-v_0^2}{2g}}.$$

With this choice of the zero of y, the equation for v simplifies to

$$\boxed{v = \sqrt{-2gy}}. \tag{B1-3}$$

We note that the result of this equation is real because $y < 0$ at the spout, and decreases as the water falls, so this equation shows that v is real and increases as the water falls.

b. Assume that the stream of water falling from the faucet is cylindrically symmetric about a vertical axis through the center of the stream. Also assume that the volume of water per unit time exiting the spout is constant, and that the shape of the stream of water is constant over time.

In this case, the radius r of the stream of water is a function of vertical position y. Let the radius at the faucet be r_0. Using your result from part (a), find $r(y)$.

If $r(y)$ is not constant, it implies that the water has some radial velocity during its fall, in contradiction to our assumptions in part (a) that the motion is purely vertical. You may assume throughout the problem that any such radial velocity is negligibly small.

Solution

The same volume of water must fall through any horizontal cross-section of the stream each second because water doesn't disappear during its fall, and its density if constant. That volume per unit time Q is the cross-sectional area of the stream multiplied by the speed of the water in the vertical direction. As an equation,

$$Q = v\pi r^2. \tag{B1-4}$$

Q is the same at all y, and is equal to its initial value of

$$Q = v_0 \pi r_0^2. \tag{B1-5}$$

Eliminating Q from B1-4 and B1-5 and solving for r gives

$$r = r_0\sqrt{\frac{v_0}{v}}.$$

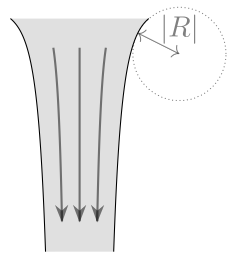
Plugging in our equation B1-3 for v,

$$\boxed{r = r_0 \sqrt[4]{\frac{v_0^2}{-2gy}}}.$$

c. The water-air interface has some surface tension, σ. The effect of surface tension is to change the pressure in the stream according to the *Young-Laplace equation*,

$$\Delta P = \sigma\left(\frac{1}{r} + \frac{1}{R}\right),$$

where ΔP is the difference in pressure between the stream and the atmosphere and R is the radius of curvature of the vertical profile of the stream, visualized below. ($R < 0$ for the stream of water; the radius of curvature would be positive only if the stream profile curved inwards.)



For this part of the problem, we assume that $|R| \gg |r|$, so that the curvature of the vertical profile of the stream can be ignored. Also assume that water is incompressible.

Accounting for the pressure in the stream, find a new equation relating for $r(y)$ in terms of σ, r_0, v_0, and ρ, the density of water. You do not need to solve the equation for r.

Solution

Our conservation of energy approach from part (b) needs to be modified to account for the work done against pressure. As we look further down in the stream, the radius is smaller. This means the pressure is higher there, and the water is slowed compared to when we assumed only gravity acted on the water.

The result of accounting for changes in pressure in a flow where no energy is dissipated is the *Bernoulli equation*,

$$\frac{1}{2}\rho v^2 + \rho g y + P = \frac{1}{2}\rho v_0^2 + \rho g y_0 + P_0$$

where P_0 is the pressure in the stream at the spout.

Using the Young-Laplace equation to replace P and P_0, we have

$$\frac{1}{2}\rho v^2 + \rho g y + \frac{\sigma}{r} = \frac{1}{2}\rho v_0^2 + \rho g y_0 + \frac{\sigma}{r_0}.$$

If we substitute in $y_0 = -\frac{v_0^2}{2g}$ and $v = v_0 \frac{r_0^2}{r^2}$, this becomes

$$\frac{1}{2}\rho v_0^2 \frac{r_0^4}{r^4} + \rho g y + \frac{\sigma}{r} = \frac{1}{2}\rho v_0^2 - \rho g \frac{v_0^2}{2g} + \frac{\sigma}{r_0}.$$

This may be simplified to

$$\boxed{\frac{1}{2}\rho v_0^2 \frac{r_0^4}{r^4} + \rho g y = \sigma\left(\frac{1}{r_0} - \frac{1}{r}\right)}.$$

d. After falling for some distance, the water stream usually breaks into smaller droplets. This occurs because small random perturbations to the shape of the stream grow over time, eventually breaking the stream into apart.

For the rest of this problem we ignore the change in the radius of the stream due to changing speed of the water, as considered earlier. Instead, we examine small random variations in the radius of the stream.

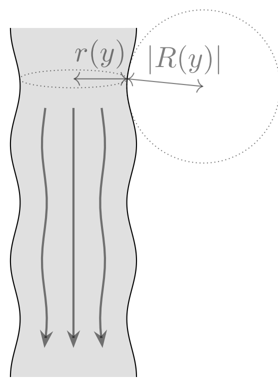
Random variations can be broken down into a sum of sinusoidal variations in stream radius, each with a different wavenumber k. We can analyze these different sinusoidal variations independently.

Consider a stream of water whose radius obeys

$$r(y) = r_0 + A\cos(ky),$$

where $A \ll r_0$ is the perturbation amplitude. To analyze such a stream, it is sufficient to consider only the thickest and thinnest parts of the stream.

Accounting for both sources of curvature, find a condition on r_0 and k such that the size of perturbations increases with time.



Solution

If the size of the perturbation increases with time, water must be flowing from the thin parts of the stream to the thick parts. For that to happen, the pressure needs to be higher in the thin parts of the stream than in the thick parts of the stream so that the pressure gradient will force water towards the thick parts, eventually breaking the stream into droplets.

We consider a small patch with side lengths h on the surface of the stream at the thinnest part of the stream. The pressure is

$$\Delta P_{\text{thin}} = \sigma\left(\frac{1}{r_{\text{thin}}} + \frac{1}{R_{\text{thin}}}\right).$$

And at the thickest part of the stream,

$$\Delta P_{\text{thick}} = \sigma\left(\frac{1}{r_{\text{thick}}} + \frac{1}{R_{\text{thick}}}\right).$$

We are looking for the wavenumbers such that

$$\Delta P_{\text{thin}} > \Delta P_{\text{thick}}.$$

Using the Young-Laplace equation, this becomes

$$\sigma\left(\frac{1}{r_{\text{thin}}}+\frac{1}{R_{\text{thin}}}\right)>\sigma\left(\frac{1}{r_{\text{thick}}}+\frac{1}{R_{\text{thick}}}\right).$$

Dropping the common factor σ,

$$\left(\frac{1}{r_{\text{thin}}}+\frac{1}{R_{\text{thin}}}\right)>\left(\frac{1}{r_{\text{thick}}}+\frac{1}{R_{\text{thick}}}\right).$$

To simplify this further, we will need to find r and R in terms of A and k, the variables given in the problem statement.

r is the thickness of the stream, which from the equation given, varies sinusoidally. So

$$r_{\text{thin}} = r_0 - A.$$
$$r_{\text{thick}} = r_0 + A.$$

We are going to need $\frac{1}{r}$ to use in the Young Laplace equation, so we make the approximations

$$\frac{1}{r_{\text{thin}}} \approx \frac{1}{r_0}+\frac{A}{r_0^2}.$$
$$\frac{1}{r_{\text{thick}}} \approx \frac{1}{r_0}-\frac{A}{r_0^2}.$$

(To find these, recall $\frac{1}{1-\epsilon}\approx 1+\epsilon$ for small ϵ.)

The inequality now becomes

$$\frac{1}{r_0}+\frac{A}{r_0^2}+\frac{1}{R_{\text{thin}}}>\frac{1}{r_0}-\frac{A}{r_0^2}+\frac{1}{R_{\text{thick}}}.$$

This simplifies to

$$\frac{2A}{r_0^2}>\frac{1}{R_{\text{thick}}}-\frac{1}{R_{\text{thin}}}.$$

Next we need to determine the radius of curvature R of the sinusoidal as a function of k and A.

To do this, we compare the sinusoidal function and a circle at small deviations from the thickest part of the stream.

Recall that, for small θ,

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2,$$

which means that for small x,

$$y_{\text{sinusoidal}} = A\cos(kx) \approx A\left(1 - \frac{1}{2}k^2x^2\right).$$

Next we consider a circle of radius R. If a particle moves along such a circle at speed v, its acceleration is v^2/R. This means that if the particle moves forward for a short time t, it moves forward a distance vt and falls a distance $\frac{1}{2}\frac{v^2}{R}t^2$. If we set $vt = x$, then the y position of the particle is given by

$$y_{\text{circle}} \approx y_0 - \frac{1}{2}\frac{x^2}{R}.$$

Comparing y_{circle} and $y_{\text{sinusoidal}}$, they give the same motion if $Ak^2 = \frac{1}{R}$.

Then

$$\frac{1}{R_{\text{thin}}} = -Ak^2.$$
$$\frac{1}{R_{\text{thick}}} = Ak^2.$$

Putting these into the inequality,

$$\frac{2A}{r_0^2} > 2Ak^2.$$

This simplifies to

$$k < \frac{1}{r_0}.$$

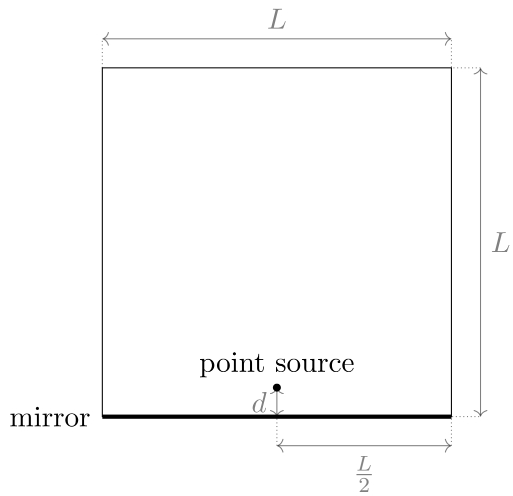
So the perturbations will grow as long as they have a wavenumber greater than one over the radius, or equivalently when the wavelength of the perturbation is longer than the circumference of the stream.

This result was discovered experimentally by Plateau and derived theoretically by Rayleigh. The breaking up of a stream into droplets is called the *Plateau-Rayleigh instability.*

Question B2

Mirror Mirror on the Wall

Consider a square room with side length L. The bottom wall of the room is a perfect mirror.* A perfect monochromatic point source with wavelength λ is placed a distance d above the center of the mirror, where $\lambda \ll d \ll L$.



*Remember that the phase of light reflected by a mirror changes by 180°.

a. On the right wall, an interference pattern emerges. What is the distance y between the bottom corner and the closest bright fringe above it? Hint: you may assume $\lambda \ll y \ll L$ as well.

Solution

This setup is essentially a double-slit experiment with the second slit being the image of the point source on the other side of the mirror, with the additional phase shift from the mirror. The distance between the source and a spot y on the wall is given by $\sqrt{(d-y)^2+(L/2)^2}$ and the distance between the image and a spot y is given by $\sqrt{(d+y)^2+(L/2)^2}$. Subtracting the two distances and adding in the phase shift gives us approximately

$$L/2\left(\frac{2(d+y)^2}{L^2}-\frac{2(d-y)^2}{L^2}\right)+\lambda/2.$$

This distance must be a multiple of λ for interference to occur. Then,

$$\frac{4dy}{L}+\lambda/2=m\lambda.$$

Substituting $m=1$ gives us $y=\frac{\lambda L}{8d}$.

b. You plan on running an experiment to determine λ in a room with $L = 40$ m, and you know that λ is between 550 and 750 nm. You will measure d and y_{10} (the distance of the tenth fringe from the corner) with the same ruler (with markings of 1 mm). At what d should you place the point source to minimize your error in your λ measurement? Roughly what is that minimum error?

Solution

Our error is given by

$$\frac{\Delta\lambda}{\lambda} = \sqrt{\left(\frac{\Delta d}{d}\right)^2 + \left(\frac{\Delta y_{10}}{y_{10}}\right)^2}.$$

Note that $\Delta d = \Delta y_{10} \sim 0.5$ mm. From earlier, note that after substituting $m = 10$, $y_{10} = \frac{19\lambda L}{8d}$.

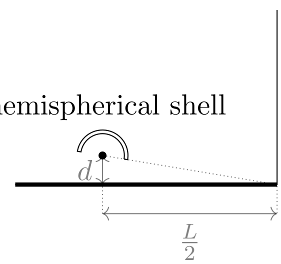
If we assume that $\lambda \sim 650$ nm, note that

$$y_{10}d = 6.2 \times 10^{-5}\ \text{m}^2.$$

Choosing $d = y_{10}$ minimizes our error, so we get that $d = y_{10} = 8$ mm. Then, $\Delta\lambda \approx 60$ nm.

Note: Accept any reasonable uncertainty in tick spacing ~ 0.5 mm or ~ 1 mm.

c. Now suppose we place a transparent hemispherical shell of thickness s and index of refraction n over the source such that all light from the source that directly strikes the right wall passes through the shell, and all light from the source that strikes the mirror first does not pass through the shell.



At what y is the fringe closest to the bottom-most corner now? (You may find it convenient to use $\lfloor x \rfloor$, the largest integer below x.) What is the spacing between the fringes now? Ignore any reflections or diffraction from the hemispherical shell.

Solution

Now the optical distance between the source and a spot y on the wall is increased by $(n-1)s$. Then, we need

$$\frac{4dy}{L} - (n-1)s + \lambda/2 = m\lambda.$$

To minimize y, we take m to be $-\left\lfloor \frac{(n-1)s}{\lambda} - \frac{1}{2} \right\rfloor$. Then,

$$y = \frac{L}{4d}\left((n-1)s - \lambda\left\lfloor \frac{(n-1)s}{\lambda} - \frac{1}{2}\right\rfloor - \frac{\lambda}{2}\right).$$

Because $(n-1)s$ is just an offset, the spacing between the fringes does not change, i.e., the spacing is still $\lambda L/(4d)$.

d. Now, suppose the hemispherical shell is removed, and we instead observe the interference pattern on the top wall. To the nearest integer, what is the total number of fringes that appear on the top wall? You may assume that $d \ll L$.

Solution

Now, the distance between the source and a spot x on the wall is given by $\sqrt{(L-d)^2+x^2}$ and the distance between the image and a spot on the wall is $\sqrt{(L+d)^2+x^2}+\lambda/2$. We do not assume $x \ll L$ this time. Subtracting the two distances gives us roughly

$$\sqrt{L^2+x^2}\sqrt{1+\frac{2dL}{L^2+x^2}}-\sqrt{L^2+x^2}\sqrt{1-\frac{2dL}{L^2+x^2}}+\lambda/2=m\lambda.$$

Taylor expanding gives us

$$\frac{2dL}{\sqrt{L^2+x^2}}=(m-1/2)\lambda.$$

Then,

$$x=\pm L\sqrt{\frac{4d^2}{(m-1/2)^2\lambda^2}-1}.$$

For x to be physical, we require that $m-1/2 \leq 2d/\lambda$.

The maximum allowed x is $L/2$. Then,

$$\sqrt{\frac{4d^2}{(m-1/2)^2\lambda^2}-1} \leq \frac{1}{2},$$

so

$$\frac{4d^2}{(m-1/2)^2\lambda^2} \leq \frac{5}{4}.$$

Thus, we have that

$$m-1/2 \geq \frac{4d}{\sqrt{5}\lambda}.$$

Then, the number of fringes is

$$2\cdot\frac{2d}{\lambda}\left(1-\frac{2}{\sqrt{5}}\right),$$

where the extra factor of 2 comes from there being two sides to the interference pattern.

Question B3

Real Expansion

Consider a "real" monatomic gas consisting of N atoms of negligible volume and mass m in equilibrium inside a closed cubical container of volume V. In this "real" gas, the attractive forces between atoms is small but not negligible. Because these atoms have negligible volume, you can assume that the atoms do not collide with each other for the entirety of the problem.

a. Consider an atom in the interior of this container of volume V. Suppose the potential energy of the interaction is given by

$$u(r) = \begin{cases} 0 & r < d \\ -\epsilon \left(\frac{d}{r}\right)^6 & r \geq d \end{cases}$$

where $d \ll V^{1/3}$ is the minimum allowed distance between two atoms. Assume the gas is uniformly distributed within the container, what is the average potential energy of this atom? Write your answer in terms of $a' = \dfrac{2\pi d^3 \epsilon}{3}$, N, and V.

Solution

The density of the gas is given by N/V. In a spherical shell of radius r and thickness Δr, there are $(4\pi r^2 \Delta r)N/V$ atoms. The potential energy is given by

$$\Delta U = -(4\pi r^2 \Delta r)N/V \epsilon d^6/r^6.$$

Then, the total potential energy is given by

$$U = \int_d^\infty -(4\pi r^2 \,\mathrm{d}r)N/V \epsilon d^6/r^6 = -2a'N/V.$$

b. What is the average potential energy of an atom near the boundary of the box? Assume that there is no interaction between atoms near the boundary and the box itself.

Solution

Now only half of the shell of radius r is full of gas, and the other half is outside of the box. This mean that the potential energy is lessened by a factor of two, to $-a'N/V$.

c. Using Bernoulli's law $P + U + \rho v^2/2 =$ constant, with pressure P, potential energy density U, mass density ρ and fluid velocity v, what is the pressure at the boundary of the box? Assume the interior pressure is given by the ideal gas law.

Solution

The potential energy density difference is $-a' \frac{N^2}{V^2}$. Since there is no velocity difference, this is also the pressure difference. If the pressure on the interior is $\frac{NkT}{V}$, then the pressure on the box is $\frac{NkT}{V} - a'\frac{N^2}{V^2}$

d. Assuming most atoms are in the interior of the box, what is the total energy of the atoms in the box?

Solution

The total kinetic energy is $\frac{3}{2}NkT$. The total potential energy is $-a'N^2/V$ (we drop a factor of two to avoid double-counting). So the total energy is $\frac{3}{2}NkT - a'N^2/V$.

Now consider an insulated partitioned container with two sections, each of volume V. We fill one side of the container with N atoms of this "real" gas at temperature T, which the other side being a vacuum. We then quickly remove the partition and let the gas expand to fill the entirety of the partitioned container. During this expansion, the energy of the gas remains unchanged.

e. What is the final temperature of the gas after the expansion?

Solution

Naively, we might say that the total potential energy of the gas is $-2a'N^2/V$, but to avoid double-counting, we divide by 2 and instead arrive at $-a'N^2/V$. Then, the quantity

$$E = \frac{3}{2}Nk_BT - \frac{a'N^2}{V}$$

is conserved. Therefore,

$$T' = T - \frac{a'N}{3k_BV}.$$

f. What is the increase in the entropy of the universe as a result of the free expansion? Give your answer to first order in $\frac{a'N}{Vk_BT}$.

Solution

The entropy of the surroundings do not increase as a result of the free expansion (no heat is dumped to the surroundings, and the surroundings remain in thermal equilibrium). However, the entropy of the gas does increase because the gas is momentarily not in equilibrium. Therefore, we just have to compute the increase in entropy of the gas.

Because entropy is a state function, we compute this change in entropy by constructing a reversible process between the initial and final states of the expansion, and computing the change in entropy for this process. Consider constant energy reversible expansion of this gas. For this process, the work done by the gas is equal to the heat the gas takes in. Therefore,

$$\mathrm{d}S = \frac{p\mathrm{d}v}{t},$$

where we use lowercase letters to denote the quantities during the reversible expansion.

Recall that

$$pv + \frac{a'N^2}{v} = Nk_Bt.$$

If the energy of the system is E, then,

$$\frac{3}{2}pv + \frac{3a'N^2}{2v} - \frac{a'N^2}{v} = E.$$

Then,

$$p = \frac{2E}{3v} - \frac{a'N^2}{3v^2}.$$

From our expression of energy,

$$t = \frac{2}{3}\frac{E + a'N^2/v}{Nk_B}.$$

Then,

$$\Delta S = \int_V^{2V} \frac{ENk_B}{Ev + a'N^2} - \frac{a'N^3k_B}{2(Ev^2 + a'N^2v)}\,\mathrm{d}v\,.$$

Taylor expanding gives us

$$\Delta S = \int_V^{2V} \frac{Nk_B}{v} - \frac{3a'N^3k_B}{2Ev^2}\,\mathrm{d}v\,.$$

Integrating gives us

$$\Delta S = Nk_B \log 2 - \frac{3a'N^3k_B}{4EV}.$$

Using that $E \approx 3/2Nk_BT$, we arrive at

$$\Delta S = Nk_B \log 2 - \frac{a'N^2}{2VT}.$$

AAPT UNITED STATES PHYSICS TEAM
AIP 2021

Traveling Team Selection Exam

Information About The 2021 USAPhO+

- The 2021 USAPhO+ is a 5-hour exam taking place on Saturday, May 8 from noon to 5 PM, Eastern time.
- The exam is hosted by AAPT on the platform provided by Art of Problem Solving. It will be proctored by the US Physics Team coaches via Zoom.
- Before you start the exam, make sure you are provided with blank paper, both for your answers and scratch work, writing utensils, **graph paper and a ruler**, a hand-held scientific calculator with memory and programs erased, and a computer for you to log into the USAPhO+ testing page.
- At the end of the exam, you have 20 minutes to upload solutions to all of the problems for that part. For each problem, scan or photograph each page of your solution, combine them into a single PDF file, and upload them on the testing platform.
- USAPhO+ graders are not responsible for missing pages or illegible handwriting. No late submissions will be accepted.

Congratulations again on your qualification for the USAPhO+. We wish you the best of luck on the challenging problems to follow.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

JiaJia Dong, Mark Eichenlaub, Abijith Krishnan, Kye W. Shi, Brian Skinner, Mike Winer, and Kevin Zhou.

Question 1

The Jet Stream

The jet stream is an eastward wind current that moves over the continental United States at an altitude of 23,000 to 35,000 feet (the range of typical cruising altitudes of commercial airlines). This strong current affects flight times significantly: flights traveling eastward fly significantly faster than flights traveling westward.

This problem consists of two independent parts. In the first part, you will consider a simple model for airplane flight. In the second part, you will determine the jet stream speed on a fictitious planet called Orb.

1. The power that a plane expends is used both to combat drag and to generate lift. Throughout this part of the problem, you may assume that the plane travels with horizontal velocity $\mathbf{v}_{\text{rel}}$ relative to the air, the density of air is ρ_{air}, the mass of the plane is m, and the cross-sectional area of the plane is A_{cs}.

   (a) The drag force on an airplane is given by

   $$\mathbf{F}_{\text{drag}} = -\frac{1}{2} c_d \rho_{\text{air}} A_{\text{cs}} |\mathbf{v}_{\text{rel}}| \mathbf{v}_{\text{rel}},$$

   where c_d is the drag coefficient (which depends on the shape of the plane). Write an expression for the power expended by the airplane to combat the drag force from the air.

   (b) Airplanes generate lift by deflecting air downward.

   i. Estimate the air mass per unit time which is deflected by the wings of the plain.

   ii. Estimate the power expended by the plane for lift.

   (c) Estimate the speed at which an airplane flies relative to the air by minimizing the power expended by the plane. To get a numeric answer, you may use the following parameters:

   $$m_{\text{plane}} \sim 8 \times 10^4 \text{ kg}, \quad \rho_{\text{air}} \sim 1 \text{ kg/m}^3, \quad c_d \sim 10^{-2}, \quad A_{\text{cs}} \sim 100 \text{ m}^2.$$

Next, we estimate the jet stream speed using flight times. Because the jet stream speed on Earth varies greatly with location, time of year, and climate effects (such as El Niño and La Niña), you will instead consider the fictitious planet Orb, where the jet stream is **eastward** and uniform in the region of interest. At the end of the problem is a map of the region, whose area is much smaller than the surface area of Orb (i.e., you can neglect the curvature of Orb).

2. At cruising altitude, we asume all airplanes travel at a fixed speed v_{rel} relative to the air. (This is not necessarily the same as your answer to 1(c), which was just a rough estimate.) Additionally, we assume that flights occur in three stages – (1) taxi and takeoff, (2) flight at cruising altitude, (3) landing and taxi – and that stages (1) and (3) take a fixed total time t_0 for every flight.

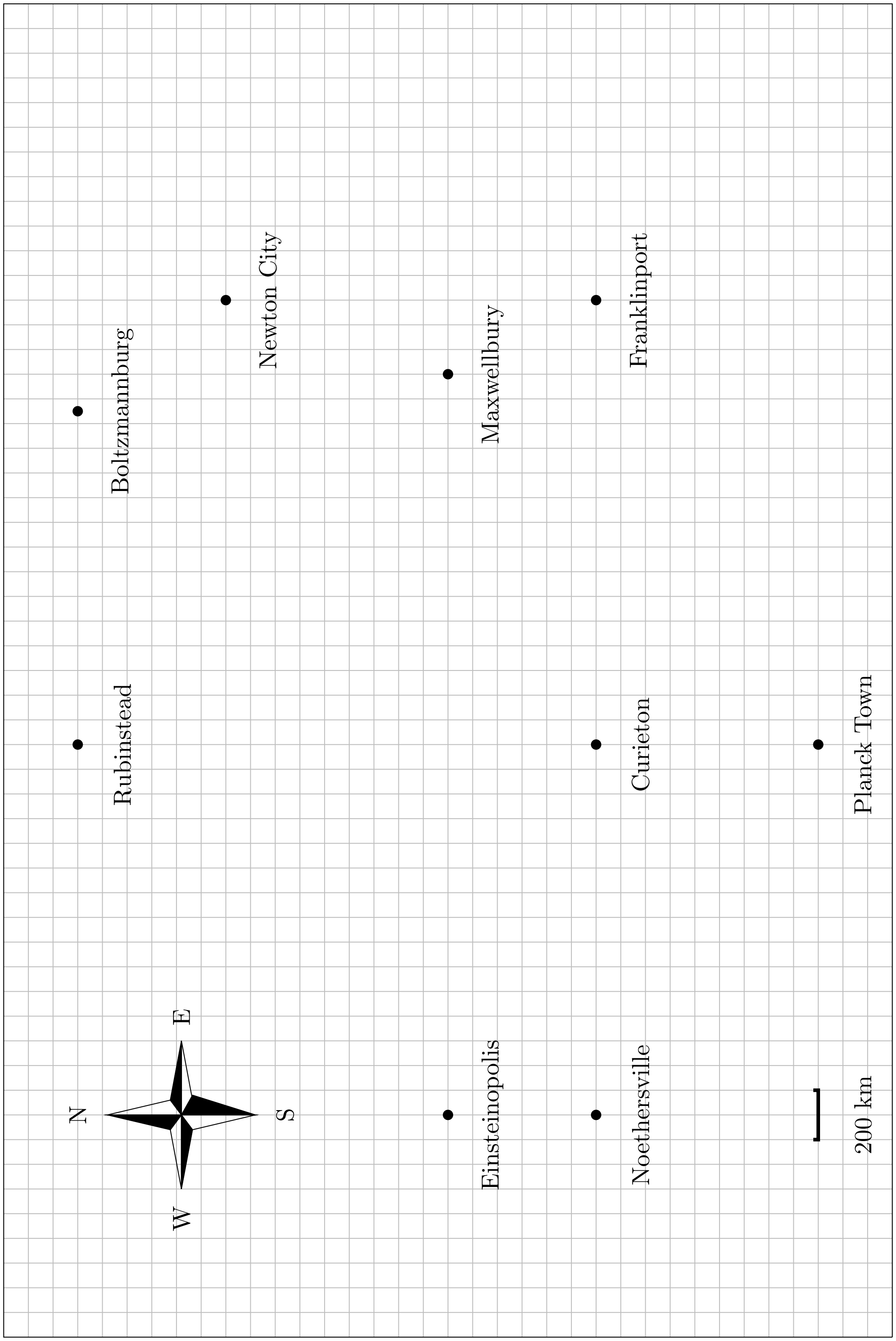
   (a) Suppose a plane, at cruising altitude, is traveling at an angle θ away from due east relative to the ground. What is the speed of the plane relative to the ground? Give your answer in terms of v_{rel}, θ, and v_w, the speed of the jet stream relative to the Earth's surface.

   (b) If the plane travels a distance D, what is the total travel time t, including taxi, takeoff, and landing?

(c) Below, we present some data on airplane flights on Planet Orb. Each of the flight times shown below has an independent uncertainty of $\Delta t = 5\,\text{min}$. From the data and the map, determine v_w and v_{rel}, giving your answers in km per hour with uncertainties. Indicate clearly what two quantities you are plotting against each other on each graph that you plot.

Departure City	**Arrival City**	t **(min)**
Noethersville	Rubinstead	185
Rubinstead	Noethersville	286
Curieton	Franklinport	107
Franklinport	Curieton	244
Planck Town	Maxwellbury	143
Maxwellbury	Planck Town	256
Rubinstead	Boltzmannburg	92
Boltzmannburg	Rubinstead	190
Einsteinopolis	Maxwellbury	160
Maxwellbury	Einsteinopolis	384
Planck Town	Franklinport	128
Franklinport	Planck Town	266
Einsteinopolis	Franklinport	188
Franklinport	Einsteinopolis	431
Boltzmannburg	Maxwellbury	135
Maxwellbury	Boltzmannburg	150

Departure City	**Arrival City**	t **(min)**
Noethersville	Einsteinopolis	68
Einsteinopolis	Noethersville	74
Franklinport	Newton City	144
Newton City	Franklinport	129
Curieton	Rubinstead	186
Rubinstead	Curieton	175
Planck Town	Curieton	95
Curieton	Planck Town	102
Planck Town	Rubinstead	249
Rubinstead	Planck Town	250


Boltzmannburg
Newton City
Maxwellbury
Franklinport
Rubinstead
Curieton
Planck Town
N
E
S
W
Einsteinopolis
Noethersville
200 km

Question 2

The Dark Forest

Dark matter could be made of hypothetical, extremely light particles called axions. Because individual axions are so light, experiments do not search for individual axions, but rather for the classical axion field formed by a large collection of axions, which oscillates as

$$a(t) = a_0 \sin(\omega t).$$

This is analogous to how a large collection of photons can form a classical electromagnetic field. In the presence of a magnetic field $\mathbf{B}$ and an axion field a, the axion field produces an effective current

$$\mathbf{J} = g\dot{a}\mathbf{B}$$

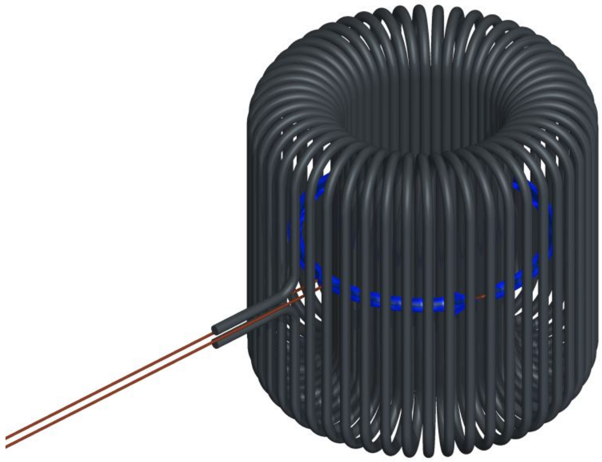
where we define $\dot{a} = da/dt$. The effective current produces electromagnetic fields in exactly the same way as ordinary current, though it does not come from the motion of actual charges. Experiments can search for axion dark matter using systems which are resonantly driven by this current.

You may use fundamental constants in your answers, such as

$$c = 3.00 \times 10^8\,\mathrm{m/s} \qquad \hbar = 1.055 \times 10^{-34}\,\mathrm{J\cdot s} \qquad e = 1.602 \times 10^{-19}\,\mathrm{C}$$
$$G = 6.67 \times 10^{-11}\,\mathrm{N\cdot m^2/kg^2} \qquad \mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}.$$

You do not have to provide numeric answers unless asked. When asked to "estimate", you may drop constants of order one. The numeric values provided below are from standard references where $\hbar$, c, μ_0, and ϵ_0 are set to one; to get correct numeric results, you must restore these factors yourself.

1. First, we will describe some physical properties of the axion field.
   (a) Consider a single axion at rest, with mass m. Find its associated angular frequency ω. This will be the angular frequency of the corresponding classical field, when there are many axions.
   (b) Suppose dark matter is distributed spherically symmetrically in the galaxy with uniform density ρ. The solar system is a distance r from the center of the galaxy and orbits around it with period T. Neglecting everything besides dark matter, find the dark matter density ρ.
   (c) The energy density of the axion field is $m^2 a_0^2/(2\hbar^3 c)$. Find the axion field amplitude a_0.
   (d) The radius and period of the Sun's orbit, as well as a typical axion mass, are
   $$r = 2.5 \times 10^{20}\ \mathrm{m}, \quad T = 7.1 \times 10^{15}\ \mathrm{s}, \quad m = 1.0 \times 10^{-9}\ \mathrm{eV}.$$
   Numerically compute the axion field amplitude a_0.
   (e) In this problem, we treat the axion field as spatially uniform within a terrestrial laboratory. To verify that this assumption is reasonable, numerically estimate the axion field's wavelength λ, assuming the axions have the same galactic speed as the Sun.
   (f) In part (a), you found ω by neglecting the axion's speed. In reality, the axion's finite speed changes the frequency to $\omega + \Delta\omega$, in a frame at rest with respect to the galactic center. Numerically estimate $\Delta\omega/\omega$ to show that it is reasonable to neglect this effect.



The ABRACADABRA[1] experiment, currently taking data at MIT, is a toroidal solenoid with inner and outer radius R_{in} and R_{out} and height h. You may assume $h \gg R_{\text{out}}$ for simplicity. A superconducting wire carrying current I wraps N times around the toroid, where N is high enough to neglect the discreteness of the wires. A circular pickup loop with radius slightly less than R_{in} is placed at the center of the toroid.

2. Now, we will find the axion signal generated in the ABRACADABRA apparatus.

   (a) Find the magnetic field $\mathbf{B}(\mathbf{r})$ inside the toroid due to the superconducting current.

   (b) The superconducting wires lose their superconductivity when exposed to a magnetic field greater than B_{max}. Find the maximum possible current I_{max} that can be used, and assume this current is used in later parts.

   (c) Assuming that ω is small, find the magnetic flux $\Phi_B(t)$ through the pickup loop due to the axion field in terms of a_0, g, ω, B_{max}, and the dimensions of the apparatus. (You may ignore any currents induced on the surfaces of the superconducting wires. Accounting for them makes the problem much harder, but does not substantially affect the final result.)

   (d) If ω is too large, the result above breaks down due to radiation effects. Estimate the frequency ω_c where this happens.

   (e) Using the design values

   $$R_{\text{in}} = 0.5 \text{ m}, \quad R_{\text{out}} = 1.0 \text{ m}, \quad h = 2.0 \text{ m}$$

   estimate the numerical value of ω/ω_c.

   (f) Let Φ_0 be the amplitude of the time-varying axion flux. Using the typical values

   $$B_{\text{max}} = 5.0 \text{ T}, \quad g = 1.0 \times 10^{-16} \text{ GeV}^{-1}$$

   and your previous results, compute the numerical value of Φ_0.

[1] aka, A Broadband/Resonant Approach to Cosmic Axion Detection with an Amplifying B-field Ring Apparatus.

The pickup loop has inductance L and is attached to a capacitor, forming a circuit with resonant frequency equal to the axion frequency ω. The circuit also has a small internal resistance R in series, and is at temperature T. The axion signal can be detected by monitoring the current in the circuit. The main source of noise is thermal noise, which causes fluctuations in the current.

3. We will now estimate the sensitivity of ABRACADABRA to axions.

   (a) The axion produces a current which oscillates sinusoidally. Find the signal current amplitude I_s in terms of ω, Φ_0, and the circuit parameters.

   (b) Find the average value of the current squared $\langle I^2 \rangle$ in the circuit due to thermal noise.

   (c) At any moment in time, the noise current is oscillating sinusoidally with typical amplitude $I_n = \sqrt{\langle I^2 \rangle}$, which is much larger than I_s. However, the phase of the noise current also fluctuates randomly, so that after a typical time t_c, its phase will be roughly independent of the phase it had before. Find an estimate for t_c in terms of ω and the circuit parameters. (Hint: at any given moment, the thermal noise current is simultaneously being produced by the random motion of electrons in the circuit, and damped by the resistor.)

   (d) Suppose the experiment runs for a total time $t_e \gg t_c$. Roughly estimate the average amplitude of the noise current over this period of time.

   (e) The axion is detectable if the signal current amplitude is larger than the averaged noise current amplitude, and the circuit parameters are

   $$L = 1 \text{ mH}, \quad R = 10 \text{ m}\Omega, \quad T = 0.1 \text{ K}.$$

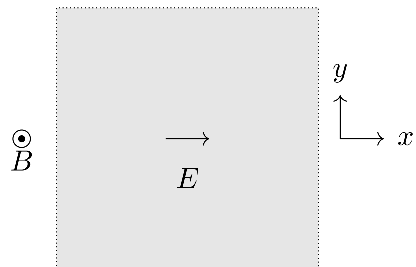
   Roughly numerically estimate the time needed to potentially detect the axion. (Hint: if your answer seems strange, note that in reality, the axion's phase also fluctuates over time, because of the effect of part 1(f). In addition, we don't know ω ahead of time, so the experiment needs to be run many times. We ignored these effects here to keep things simple.)

Question 3

Great Hall

The classical Hall effect was first measured by Edwin Hall in 1879, shortly after the publication of Maxwell's equations. In all parts of this problem, materials contain n_V electrons per unit volume, and each electron has charge $q_e < 0$ and mass m_e. You may use these quantities in all of your answers. We will begin by investigating the implications of the classical Hall effect.

1. An infinite plate in the xy plane, with thickness d in the z direction, is placed in a uniform magnetic field $\mathbf{B} = B\hat{z}$ as shown. An electric field $\mathbf{E} = E\hat{x}$ is applied in the plane of the plate and the system is allowed to reach a steady state.



(a) If the electrons have velocity $\mathbf{v}$ at steady state, what is the current density $\mathbf{J}$? Recall that $\mathbf{J}$ is defined as the total flow of charge through a unit cross-section area per unit time.

(b) In the Drude model, electrons are subject to both the Lorentz force and a damping force $-\gamma\mathbf{v}$, where γ is a constant that depends on the material. In the above system, what is the current density in the steady state? Give both the magnitude and direction of $\mathbf{J}$, e.g. in polar coordinates.

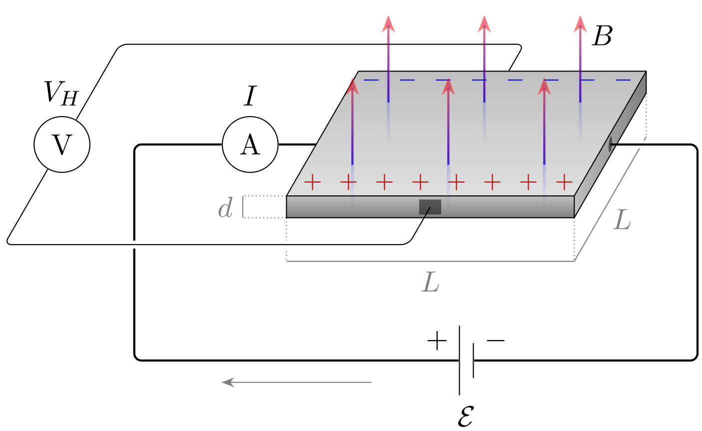
(c) Compute the electrical resistivity,

$$\rho_0 = \lim_{B \to 0} \frac{E}{|J_x|}$$

and the transverse Hall resistivity

$$\rho_H = \lim_{\gamma \to 0} \frac{E}{|J_y|}.$$

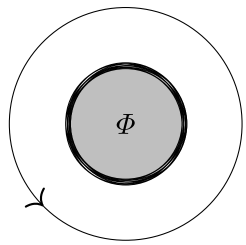
(d) A Hall effect sensor detects the strength of magnetic fields. Consider the following circuit consisting of a square plate of side length L and thickness d in a perpendicular uniform magnetic field B.



A longitudinal emf $\mathcal{E}$ is applied to the plate. At steady state, a Hall voltage V_H is measured across the plate due to the buildup of charge on either side of the plate. If the electrical resistivity of the plate at zero magnetic field is ρ_0, what is the Hall voltage V_H and the current I through the plate? Express your answer in terms of ρ_0, $\mathcal{E}$, B, and the dimensions of the plate.

Experiments in the 20th century revealed that in many materials, the Hall resistivity could only take certain discrete values. We will now show how this follows from Bohr quantization. (These next parts are independent of the first part of the problem.)

2. A zero-resistance loop of wire of radius R and cross-sectional area A_w carries a counterclockwise current I. A solenoid through the middle of the loops carries magnetic flux Φ out of the page, which we define to be the positive $\hat{z}$ direction.



(a) If the electrons all have the same speed, what is the angular momentum of each electron?

(b) If we allow the flux in the solenoid to change, the usual, "mechanical" angular momentum L of each electron is not conserved. Instead, a quantity called the canonical angular momentum, $L_{\text{can}} = L + Cq_e\Phi$, for some constant C, is conserved. Find C.

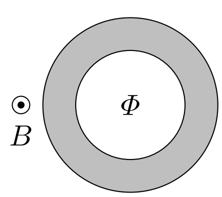
(c) The Bohr quantization condition says that for a closed circular orbit, an integer number of de Broglie wavelengths must fit in its circumference. The de Broglie wavelength is

$$\lambda = \frac{h}{p_{\text{can}}},$$

where h is Planck's constant, and $p_{\text{can}} = L_{\text{can}}/R$ is the canonical momentum. For a given solenoid flux Φ, what is the set of allowed mechanical angular momenta L?

(d) What is the minimum possible change in the magnetic flux for which the same set of mechanical angular momenta is allowed? This is known as the flux quantum.

3. Now, consider an annulus held perpendicular to a fixed, uniform external magnetic field B, and suppose an additional, tunable magnetic flux Φ threads the center of the annulus, with both pointing out of the page. The annulus has a transverse Hall resistance R_H (i.e., an EMF of $\mathcal{E}$ around the annulus generates a perpendicular current $\mathcal{E}/R_H$ via the Hall effect) and you may neglect its self-inductance.



(a) Suppose Φ begins to increase slowly and steadily in time. After a short time, the electrons will begin flowing steadily from one side of the annulus to the other. Do the electrons move inward or outward? Justify your answer.

(b) If the threaded flux increases by $\Delta\Phi$, how many electrons pass from one edge of the annulus to the other? You may use R_H, among other variables, in your answer.

(c) As we showed in 2(d), if the magnetic flux changes by the flux quantum Φ_q, the allowed orbits from Bohr quantization are unchanged. Quantum mechanics thus tells us that in conventional materials, if the magnetic flux changes by Φ_q, an integer number k of electrons must pass from one edge to another. What constraint does this place on the Hall resistance?

AAPT UNITED STATES PHYSICS TEAM
AIP 2021

Traveling Team Selection Exam

Information About The 2021 USAPhO+

- The 2021 USAPhO+ is a 5-hour exam taking place on Saturday, May 8 from noon to 5 PM, Eastern time.
- The exam is hosted by AAPT on the platform provided by Art of Problem Solving. It will be proctored by the US Physics Team coaches via Zoom.
- Before you start the exam, make sure you are provided with blank paper, both for your answers and scratch work, writing utensils, **graph paper and a ruler**, a hand-held scientific calculator with memory and programs erased, and a computer for you to log into the USAPhO+ testing page.
- At the end of the exam, you have 20 minutes to upload solutions to all of the problems for that part. For each problem, scan or photograph each page of your solution, combine them into a single PDF file, and upload them on the testing platform.
- USAPhO+ graders are not responsible for missing pages or illegible handwriting. No late submissions will be accepted.

Congratulations again on your qualification for the USAPhO+. We wish you the best of luck on the challenging problems to follow.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

JiaJia Dong, Mark Eichenlaub, Abijith Krishnan, Kye W. Shi, Brian Skinner, Mike Winer, and Kevin Zhou.

Question 1

The Jet Stream

The jet stream is an eastward wind current that moves over the continental United States at an altitude of $23,000$ to $35,000$ feet (the range of typical cruising altitudes of commercial airlines). This strong current affects flight times significantly: flights traveling eastward fly significantly faster than flights traveling westward.

This problem consists of two independent parts. In the first part, you will consider a simple model for airplane flight. In the second part, you will determine the jet stream speed on a fictitious planet called Orb.

1. The power that a plane expends is used both to combat drag and to generate lift. Throughout this part of the problem, you may assume that the plane travels with horizontal velocity $\mathbf{v}_{\text{rel}}$ relative to the air, the density of air is ρ_{air}, the mass of the plane is m, and the cross-sectional area of the plane is A_{cs}.

   (a) The drag force on an airplane is given by

   $$\mathbf{F}_{\text{drag}} = -\frac{1}{2}c_d \rho_{\text{air}} A_{\text{cs}} |\mathbf{v}_{\text{rel}}| \mathbf{v}_{\text{rel}},$$

   where c_d is the drag coefficient (which depends on the shape of the plane). Write an expression for the power expended by the airplane to combat the drag force from the air.

   Solution

   We have that $P = \mathbf{F} \cdot \mathbf{v}$, so

   $$P_{\text{drag}} = \frac{1}{2}c_d \rho_{\text{air}} A_{\text{cs}} v_{\text{rel}}^3.$$

   (b) Airplanes generate lift by deflecting air downward.

   i. Estimate the air mass per unit time which is deflected by the wings of the plain.

   Solution

   The mass flux is given by ρv_{rel}, so the rate is

   $$\rho_{\text{air}} v_{\text{rel}} A_{\text{cs}}.$$

   ii. Estimate the power expended by the plane for lift.

   Solution

   Suppose the deflected air has velocity u downward. Then the lift force is given by

   $$mg = \rho v_{\text{rel}} A u.$$

   The power is given by

   $$P = Fu = \frac{m^2 g^2}{\rho_{\text{air}} v_{\text{rel}} A_{\text{cs}}}.$$

This is the power that goes into accelerating the air downward, so by energy conservation it must have come from the plane's engine.

(c) Estimate the speed at which an airplane flies relative to the air by minimizing the power expended by the plane. To get a numeric answer, you may use the following parameters:

$$m_{\text{plane}} \sim 8 \times 10^4 \text{ kg}, \quad \rho_{\text{air}} \sim 1 \text{ kg/m}^3, \quad c_d \sim 10^{-2}, \quad A_{\text{cs}} \sim 100 \text{ m}^2.$$

Solution

The total power is given by

$$P \sim \frac{m^2 g^2}{\rho_{\text{air}} v_{\text{rel}} A_{\text{cs}}} + \frac{1}{2} c_d \rho A_{\text{cs}} v_{\text{rel}}^3.$$

The minimum power occurs when the derivative vanishes, which roughly gives

$$v_{\text{rel}} \sim \left(\frac{m^2 g^2}{c_d \rho^2 A^2} \right)^{1/4} \sim 300 \text{ m/s}.$$

Next, we estimate the jet stream speed using flight times. Because the jet stream speed on Earth varies greatly with location, time of year, and climate effects (such as El Niño and La Niña), you will instead consider the fictitious planet Orb, where the jet stream is **eastward** and uniform in the region of interest. At the end of the problem is a map of the region, whose area is much smaller than the surface area of Orb (i.e., you can neglect the curvature of Orb).

2. At cruising altitude, we asume all airplanes travel at a fixed speed v_{rel} relative to the air. (This is not necessarily the same as your answer to 1(c), which was just a rough estimate.) Additionally, we assume that flights occur in three stages – (1) taxi and takeoff, (2) flight at cruising altitude, (3) landing and taxi – and that stages (1) and (3) take a fixed total time t_0 for every flight.

(a) Suppose a plane, at cruising altitude, is traveling at an angle θ away from due east relative to the ground. What is the speed of the plane relative to the ground? Give your answer in terms of v_{rel}, θ, and v_w, the speed of the jet stream relative to the Earth's surface.

Solution

The velocity of the plane relative to the ground, $\mathbf{v}$, the velocity of the plane relative to the air, $\mathbf{v}_{\text{rel}}$, and the jet stream velocity, $\mathbf{v}_w$, all form a triangle under tip-tail addition. Then, from law of cosines,

$$v_{\text{rel}}^2 = v^2 + v_w^2 - 2 v v_w \cos\theta.$$

Solving gives us

$$v = v_w \cos\theta + \sqrt{v_{\text{rel}}^2 - v_w^2 \sin^2\theta}.$$

Of course, we assumed $v_w < v_{\text{rel}}$, as otherwise flying westward would not even be possible.

(b) If the plane travels a distance D, what is the total travel time t, including taxi, takeoff, and landing?

Solution

The answer is given by

$$t = \frac{D}{v} + t_0,$$

where v is our answer to the previous part.

(c) Below, we present some data on airplane flights on Planet Orb. Each of the flight times shown below has an independent uncertainty of $\Delta t = 5\,\text{min}$. From the data and the map, determine v_w and v_{rel}, giving your answers in km per hour with uncertainties. Indicate clearly what two quantities you are plotting against each other on each graph that you plot.

Departure City	**Arrival City**	t **(min)**
Noethersville	Rubinstead	185
Rubinstead	Noethersville	286
Curieton	Franklinport	107
Franklinport	Curieton	244
Planck Town	Maxwellbury	143
Maxwellbury	Planck Town	256
Rubinstead	Boltzmannburg	92
Boltzmannburg	Rubinstead	190
Einsteinopolis	Maxwellbury	160
Maxwellbury	Einsteinopolis	384
Planck Town	Franklinport	128
Franklinport	Planck Town	266
Einsteinopolis	Franklinport	188
Franklinport	Einsteinopolis	431
Boltzmannburg	Maxwellbury	135
Maxwellbury	Boltzmannburg	150

Departure City	**Arrival City**	t **(min)**
Noethersville	Einsteinopolis	68
Einsteinopolis	Noethersville	74
Franklinport	Newton City	144
Newton City	Franklinport	129
Curieton	Rubinstead	186
Rubinstead	Curieton	175
Planck Town	Curieton	95
Curieton	Planck Town	102
Planck Town	Rubinstead	249
Rubinstead	Planck Town	250

Solution

Note that the pairs of cities given in the second table are directly north/south of each other. For such cities, we expect the times in either direction to be equal, up to the timing uncertainty, and the expression for the time simplifies to

$$t_{NS} = \frac{D}{\sqrt{v_{\text{rel}}^2 - v_w^2}} + t_0.$$

We can read t_{NS} off the second table, and calculate D from the figure. The unknown quantity t_0 goes into the intercept, while the slope is

$$\frac{1}{\sqrt{v_{\text{rel}}^2 - v_w^2}} = 0.0768 \pm 0.0048\,\text{min/km}.$$

Next, consider the pairs of cities given in the second table. We can cancel out the effect of t_0 by considering the difference in flight times in the two directions,

$$t_{12} - t_{21} = -\frac{D}{v_w \cos\theta + \sqrt{v_{\text{rel}}^2 - v_w^2 \sin^2\theta}} + \frac{D}{-v_w \cos\theta + \sqrt{v_{\text{rel}}^2 - v_w^2 \sin^2\theta}}.$$

After some simplification, we find

$$t_{12} - t_{21} = \frac{2Dv_w \cos\theta}{v_{\text{rel}}^2 - v_w^2}.$$

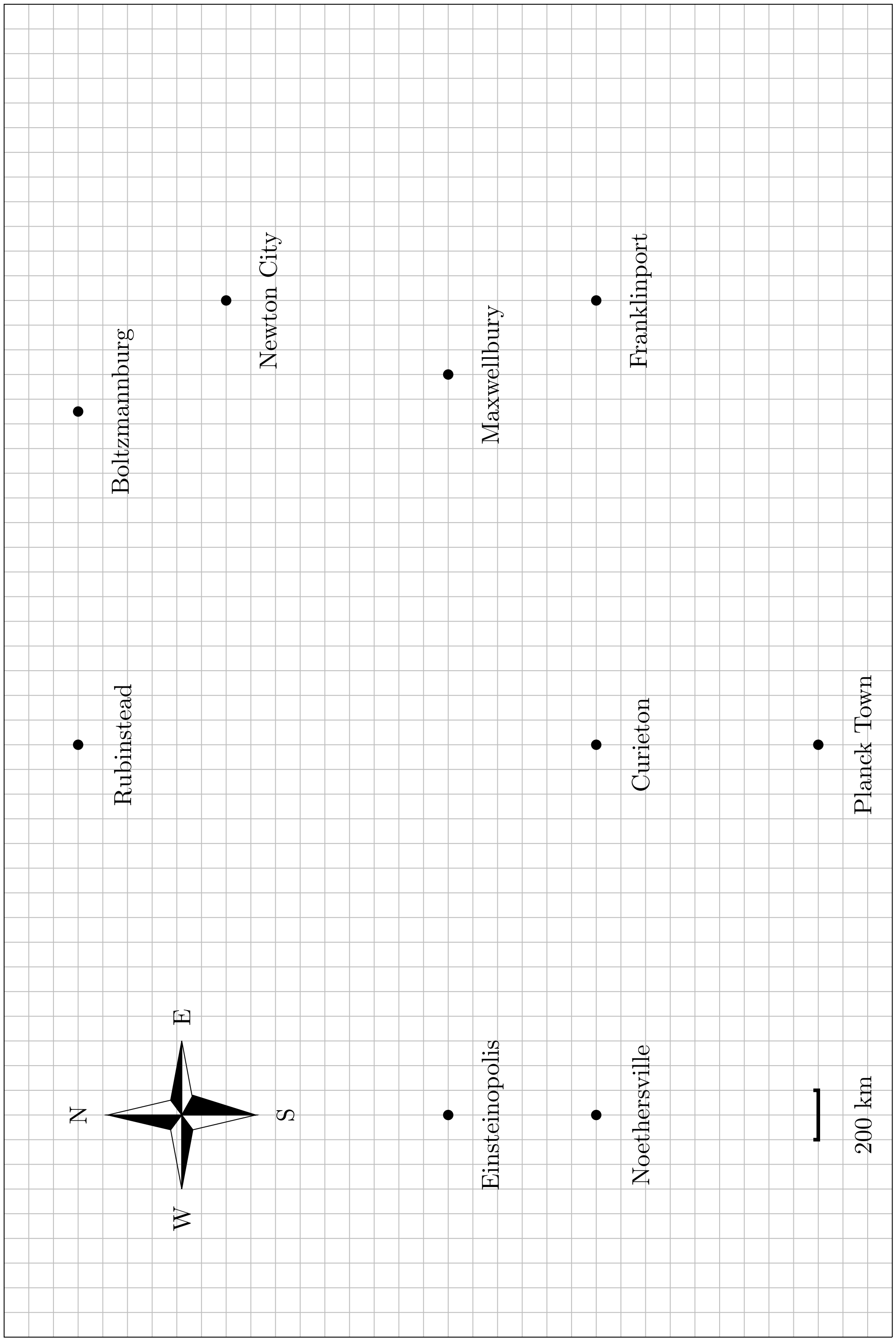
Therefore, plotting the time difference versus $D\cos\theta$, where θ can also be measured from the graph, gives

$$\frac{2v_w}{v_{\text{rel}}^2 - v_w^2} = 0.0722 \pm 0.0003\,\text{min/km}.$$

Combining these results gives the answers,

$$v_{\text{rel}} = 860 \pm 50\,\text{km/h}, \quad v_w = 370 \pm 30\,\text{km/h}.$$

Uncertainty on the times should be propagated through the calculation using the usual rules, while the uncertainties on the slopes are found by drawing the steepest and shallowest fit lines.


Boltzmannburg
Newton City
Maxwellbury
Franklinport
Rubinstead
Curieton
Planck Town
N
E
S
W
Einsteinopolis
Noethersville
200 km

Question 2

The Dark Forest

Dark matter could be made of hypothetical, extremely light particles called axions. Because individual axions are so light, experiments do not search for individual axions, but rather for the classical axion field formed by a large collection of axions, which oscillates as

$$a(t) = a_0 \sin(\omega t).$$

This is analogous to how a large collection of photons can form a classical electromagnetic field. In the presence of a magnetic field $\mathbf{B}$ and an axion field a, the axion field produces an effective current

$$\mathbf{J} = g\dot{a}\mathbf{B}$$

where we define $\dot{a} = da/dt$. The effective current produces electromagnetic fields in exactly the same way as ordinary current, though it does not come from the motion of actual charges. Experiments can search for axion dark matter using systems which are resonantly driven by this current.

You may use fundamental constants in your answers, such as

$$c = 3.00 \times 10^8 \,\mathrm{m/s} \qquad \hbar = 1.055 \times 10^{-34}\,\mathrm{J \cdot s} \qquad e = 1.602 \times 10^{-19}\,\mathrm{C}$$
$$G = 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2} \qquad \mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}.$$

You do not have to provide numeric answers unless asked. When asked to "estimate", you may drop constants of order one. The numeric values provided below are from standard references where $\hbar$, c, μ_0, and ϵ_0 are set to one; to get correct numeric results, you must restore these factors yourself.

1. First, we will describe some physical properties of the axion field.

   (a) Consider a single axion at rest, with mass m. Find its associated angular frequency ω. This will be the angular frequency of the corresponding classical field, when there are many axions.

   Solution

   The axion is a quantum particle and satisfies the de Broglie relation $E = \hbar\omega$, and its energy comes from its rest mass, $E = mc^2$. Thus,

   $$\omega = \frac{mc^2}{\hbar}.$$

   (b) Suppose dark matter is distributed spherically symmetrically in the galaxy with uniform density ρ. The solar system is a distance r from the center of the galaxy and orbits around it with period T. Neglecting everything besides dark matter, find the dark matter density ρ.

   Solution

   Setting the centripetal acceleration equal to the gravitational acceleration,

   $$r\omega^2 = \frac{GM}{r^2}, \quad M = \frac{4}{3}\pi r^3 \rho.$$

Solving for ρ, we find

$$\rho = \frac{3\pi}{GT^2}.$$

(c) The energy density of the axion field is $m^2 a_0^2/(2\hbar^3 c)$. Find the axion field amplitude a_0.

Solution

The dark matter density comes from the mass-energy of the axion field, so

$$\rho c^2 = \frac{m^2 a_0^2}{2\hbar^3 c}, \quad a_0 = \sqrt{\frac{2\rho\hbar^3 c^3}{m^2}}.$$

(d) The radius and period of the Sun's orbit, as well as a typical axion mass, are

$$r = 2.5 \times 10^{20}\ \text{m}, \quad T = 7.1 \times 10^{15}\ \text{s}, \quad m = 1.0 \times 10^{-9}\ \text{eV}.$$

Numerically compute the axion field amplitude a_0.

Solution

Plugging in numbers and noting that the provided value of the axion mass is really its energy (and hence dividing it by c^2) gives the value

$$a_0 = 2.36 \times 10^{-4}\,\text{kg}\,\text{m}^3/\text{s}^3.$$

(e) In this problem, we treat the axion field as spatially uniform within a terrestrial laboratory. To verify that this assumption is reasonable, numerically estimate the axion field's wavelength λ, assuming the axions have the same galactic speed as the Sun.

Solution

The axion speed is

$$v = \frac{2\pi r}{T} = 2 \times 10^5\,\text{m/s}.$$

Using the de Broglie relation again,

$$\lambda = \frac{h}{p} = \frac{h}{mv} \sim 10^6\,\text{m}.$$

As expected, this is much larger than the apparatus considered below. (Note: a common mistake was to write $v = f\lambda$, but this is only true for waves which travel at a constant speed, such as light. To see the problem, note that if you apply the de Broglie relations, this equation becomes $v = E/p$. That's true for photons, where $E = pc$, but certainly not true for massive particles. Some students also tried the Compton wavelength $\lambda = h/mc$, but this is a different quantity, namely the wavelength of a photon if it had the same energy as an axion at rest.)

(f) In part (a), you found ω by neglecting the axion's speed. In reality, the axion's finite speed changes the frequency to $\omega + \Delta\omega$, in a frame at rest with respect to the galactic center. Numerically estimate $\Delta\omega/\omega$ to show that it is reasonable to neglect this effect.

Solution

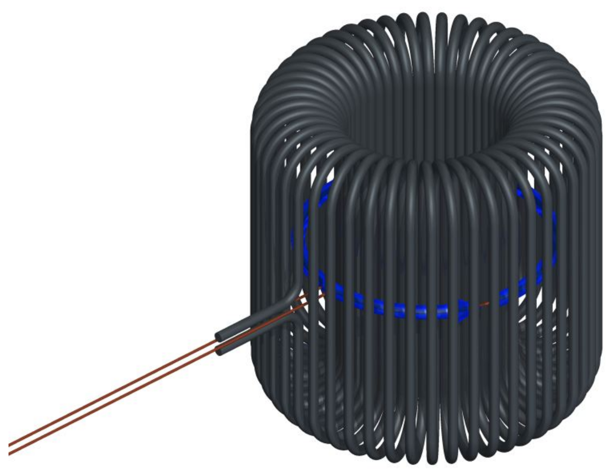
Because $E = \hbar\omega$, we need to see how the energy is changed. We have

$$\frac{\Delta E}{E} \approx \frac{mv^2/2}{mc^2} = \frac{v^2}{2c^2}$$

where we used the fact that v, computed above, is nonrelativistic. We thus find

$$\frac{\Delta\omega}{\omega} \sim 3 \times 10^{-7}$$

which is small as expected. (Note that the usual Doppler shift formula would not work here, and trying to use it would get the wrong answer. The Doppler shift formula applies to particles with constant speed, which satisfy $\omega = vk$. The axion is massive; when at rest, it has $\omega \neq 0$ and $k = 0$. However, you could also get the answer by Lorentz transforming the four-vector $(\omega, \mathbf{k})$.)



The ABRACADABRA[1] experiment, currently taking data at MIT, is a toroidal solenoid with inner and outer radius R_{in} and R_{out} and height h. You may assume $h \gg R_{\text{out}}$ for simplicity. A superconducting wire carrying current I wraps N times around the toroid, where N is high enough to neglect the discreteness of the wires. A circular pickup loop with radius slightly less than R_{in} is placed at the center of the toroid.

2. Now, we will find the axion signal generated in the ABRACADABRA apparatus.

   (a) Find the magnetic field $\mathbf{B}(\mathbf{r})$ inside the toroid due to the superconducting current.

[1] aka, A Broadband/Resonant Approach to Cosmic Axion Detection with an Amplifying B-field Ring Apparatus.

Solution

Using Ampere's law,

$$\mathbf{B} = \frac{\mu_0 N I}{2\pi r}\,\hat{\theta}.$$

(b) The superconducting wires lose their superconductivity when exposed to a magnetic field greater than B_{max}. Find the maximum possible current I_{max} that can be used, and assume this current is used in later parts.

Solution

The maximum field is at $r = R_{\text{in}}$, so

$$B_{\text{max}} = \frac{\mu_0 N I_{\text{max}}}{2\pi R_{\text{in}}}, \quad I_{\text{max}} = \frac{2\pi R_{\text{in}} B_{\text{max}}}{\mu_0 N}.$$

(c) Assuming that ω is small, find the magnetic flux $\Phi_B(t)$ through the pickup loop due to the axion field in terms of a_0, g, ω, B_{max}, and the dimensions of the apparatus. (You may ignore any currents induced on the surfaces of the superconducting wires. Accounting for them makes the problem much harder, but does not substantially affect the final result.)

Solution

The effective current is

$$\mathbf{J} = g\dot{a}\mathbf{B} = g a_0 B_{\text{max}} \frac{R_{\text{in}}}{r} \omega \cos(\omega t)\hat{\theta}.$$

The resulting field is like that of an array of concentric solenoids, where we may neglect fringe fields because $h \gg R_{\text{out}}$. Therefore, the axion-produced field $\mathbf{B}_{\text{ax}}$ inside the pickup loop is uniform. For simplicity, we define

$$J_0 = g a_0 B_{\text{max}} \omega.$$

Then using Ampere's law, we have

$$B_{\text{ax}} = \mu_0 J_0 \cos(\omega t) \int \frac{R_{\text{in}}}{r}\, dr = \mu_0 J_0 R_{\text{in}} \log\big(R_{\text{out}}/R_{\text{in}}\big) \cos(\omega t).$$

The magnetic flux is then

$$\Phi_B = \pi R_{\text{in}}^2 B_{\text{ax}} = \pi \mu_0 J_0 R_{\text{in}}^3 \log\big(R_{\text{out}}/R_{\text{in}}\big) \cos(\omega t).$$

(d) If ω is too large, the result above breaks down due to radiation effects. Estimate the frequency ω_c where this happens.

Solution

In the calculations above, we have assumed that the fields are quasistatic, neglecting the radiation propagation time. This approximation breaks down when

$$\omega_c \sim \frac{c}{R_{\text{in}}}$$

past which the flux will be diminished. Since this calculation is approximate, the answers c/R_{out} or c/h are also acceptable.

(e) Using the design values

$$R_{\text{in}} = 0.5 \text{ m}, \quad R_{\text{out}} = 1.0 \text{ m}, \quad h = 2.0 \text{ m}$$

estimate the numerical value of ω/ω_c.

Solution

Plugging numbers in, we find

$$\omega_c \sim 6.0 \times 10^8 \, \text{s}^{-1}, \quad \omega \sim 1.5 \times 10^6 \, \text{s}^{-1}.$$

Then we have

$$\omega/\omega_c \sim 0.0025$$

which is small as expected. Any answer within an order of magnitude is acceptable.

(f) Let Φ_0 be the amplitude of the time-varying axion flux. Using the typical values

$$B_{\text{max}} = 5.0 \text{ T}, \quad g = 1.0 \times 10^{-16} \text{ GeV}^{-1}$$

and your previous results, compute the numerical value of Φ_0.

Solution

First, we have to restore the dimensions of g. Notice that since current $g\dot{a}B$ has the same units as displacement current $\epsilon_0 \dot{E}$, the quantities ga and $(E/B)\epsilon_0$ must have the same dimensions, and furthermore E/B has dimensions of speed.

Now, in 1(c) we found that a has dimensions of energy times velocity, while the value of g given here has dimensions of inverse energy. Thus, the combination ga only has dimensions of speed, which means a factor of ϵ_0 was left out. In other words, if we denote the value given above as g', the true value of g is $\epsilon_0 g'$.

To avoid mistakes, it's best to group terms in the long expression for Φ_0, so that each piece has simple units and a reasonable magnitude. We have

$$\Phi_0 = (\pi R_{\text{in}}^2 B_0) \log\left(\frac{R_{\text{out}}}{R_{\text{in}}}\right) R_{\text{in}} \, \mu_0 \epsilon_0 g' a_0 \omega$$

and using $c^2 = 1/\mu_0\epsilon_0$ gives

$$\begin{aligned}\Phi_0 &= (\pi R_{\text{in}}^2 B_0)\,\log\left(\frac{R_{\text{out}}}{R_{\text{in}}}\right)\frac{\omega R_{\text{in}}}{c}\,\frac{g' a_0}{c} \\ &= (3.9\,\text{Wb})(0.693)(0.0025)(4.91\times10^{-19}) \\ &= 3.3\times10^{-21}\,\text{Wb}.\end{aligned}$$

The pickup loop has inductance L and is attached to a capacitor, forming a circuit with resonant frequency equal to the axion frequency ω. The circuit also has a small internal resistance R in series, and is at temperature T. The axion signal can be detected by monitoring the current in the circuit. The main source of noise is thermal noise, which causes fluctuations in the current.

3. We will now estimate the sensitivity of ABRACADABRA to axions.

(a) The axion produces a current which oscillates sinusoidally. Find the signal current amplitude I_s in terms of ω, Φ_0, and the circuit parameters.

Solution

Since the axion is at the resonant frequency, the impedance of the circuit is approximately R. Therefore the current is

$$I = \frac{\mathcal{E}}{R} = -\frac{1}{R}\frac{d\Phi_B}{dt}.$$

The amplitude is

$$I_s = \frac{\omega\Phi_0}{R}.$$

(b) Find the average value of the current squared $\langle I^2\rangle$ in the circuit due to thermal noise.

Solution

The energy stored in the inductor is $LI^2/2$. By the equipartition theorem, the average energy stored in any such quadratic degree of freedom is $k_BT/2$, so

$$\langle I^2\rangle = \frac{k_BT}{L}.$$

This effect is also called Johnson noise.

(c) At any moment in time, the noise current is oscillating sinusoidally with typical amplitude $I_n = \sqrt{\langle I^2\rangle}$, which is much larger than I_s. However, the phase of the noise current also fluctuates randomly, so that after a typical time t_c, its phase will be roughly independent of the phase it had before. Find an estimate for t_c in terms of ω and the circuit parameters. (Hint: at any given moment, the thermal noise current is simultaneously being produced by the random motion of electrons in the circuit, and damped by the resistor.)

Solution

The noise current that exists in the circuit at any given moment, by definition, came from the thermal motion of electrons in the circuit up to a time t_c ago. Therefore, t_c is roughly the characteristic time for decay of current in the circuit. There are many ways to calculate this, but a simple way is to note that

$$t_c \sim \frac{Q}{\omega}, \quad Q = \frac{\omega L}{R}$$

where Q is the quality factor. Thus,

$$t_c \sim \frac{L}{R}.$$

(d) Suppose the experiment runs for a total time $t_e \gg t_c$. Roughly estimate the average amplitude of the noise current over this period of time.

Solution

Uncertainty goes down by $\sqrt{n}$ when averaging n independent trials. Here we have $n = t_e/t_c$, so

$$\bar{I}_n = I_n \sqrt{\frac{t_c}{t_e}}.$$

(e) The axion is detectable if the signal current amplitude is larger than the averaged noise current amplitude, and the circuit parameters are

$$L = 1 \text{ mH}, \quad R = 10 \text{ m}\Omega, \quad T = 0.1 \text{ K}.$$

Roughly numerically estimate the time needed to potentially detect the axion. (Hint: if your answer seems strange, note that in reality, the axion's phase also fluctuates over time, because of the effect of part 1(f). In addition, we don't know ω ahead of time, so the experiment needs to be run many times. We ignored these effects here to keep things simple.)

Solution

By setting $I_s = \bar{I}_n$ and solving for t_e, we find

$$t_e \sim \frac{k_B T R}{\omega^2 \Phi_0^2} = 550\,\text{s}.$$

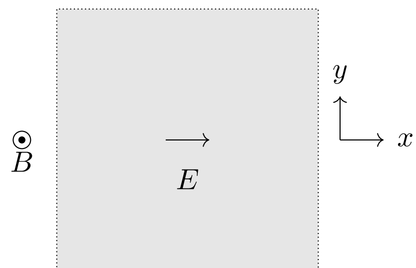
In reality, you would need to repeat this experiment millions of times to scan a sufficient range of ω, and the time in each step would be longer, because after some point the average axion signal also starts going down as $1/\sqrt{n}$.

Question 3

Great Hall

The classical Hall effect was first measured by Edwin Hall in 1879, shortly after the publication of Maxwell's equations. In all parts of this problem, materials contain n_V electrons per unit volume, and each electron has charge $q_e < 0$ and mass m_e. You may use these quantities in all of your answers. We will begin by investigating the implications of the classical Hall effect.

1. An infinite plate in the xy plane, with thickness d in the z direction, is placed in a uniform magnetic field $\mathbf{B} = B\hat{z}$ as shown. An electric field $\mathbf{E} = E\hat{x}$ is applied in the plane of the plate and the system is allowed to reach a steady state.



(a) If the electrons have velocity $\mathbf{v}$ at steady state, what is the current density $\mathbf{J}$? Recall that $\mathbf{J}$ is defined as the total flow of charge through a unit cross-section area per unit time.

Solution

The current density is given by $n_V q_e \mathbf{v}$.

(b) In the Drude model, electrons are subject to both the Lorentz force and a damping force $-\gamma\mathbf{v}$, where γ is a constant that depends on the material. In the above system, what is the current density in the steady state? Give both the magnitude and direction of $\mathbf{J}$, e.g. in polar coordinates.

Solution

The equation of motion for an electron is

$$q_e E\hat{\mathbf{x}} - q_e v_x B\hat{\mathbf{y}} + q_e v_y B\hat{\mathbf{x}} - \gamma v_x\hat{\mathbf{x}} - \gamma v_y\hat{\mathbf{y}} = 0.$$

Solving the system of equations gives us

$$v_x = \frac{\gamma q_e E}{\gamma^2 + q_e^2 B^2}, \quad v_y = \frac{-q_e^2 EB}{\gamma^2 + q_e^2 B^2}.$$

Then,

$$J_x = \frac{\gamma n_V q_e^2 E}{\gamma^2 + q_e^2 B^2}, \quad J_y = \frac{-n_V q_e^3 EB}{\gamma^2 + q_e^2 B^2}.$$

As requested, we compute the magnitude and direction:

$$|\mathbf{J}| = \frac{q_e^2 n_V E}{\sqrt{\gamma^2 + B^2 q_e^2}}, \quad \theta = -\arctan\left(\frac{Bq_e}{\gamma}\right).$$

(c) Compute the electrical resistivity,

$$\rho_0 = \lim_{B\to 0} \frac{E}{|J_x|}$$

and the transverse Hall resistivity

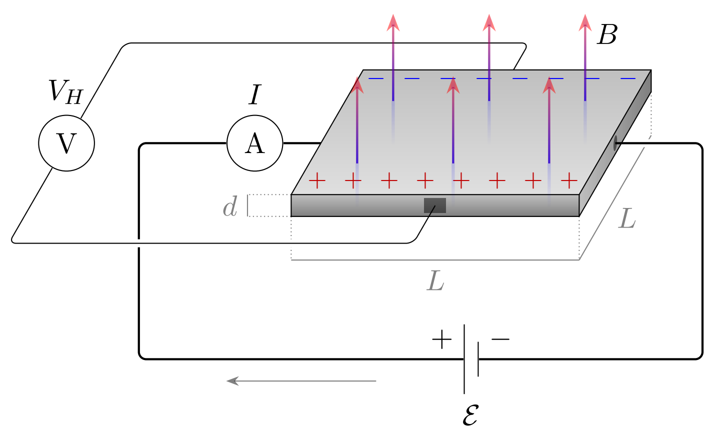
$$\rho_H = \lim_{\gamma\to 0} \frac{E}{|J_y|}.$$

Solution

The two resistivities are

$$\rho_0 = \frac{\gamma}{n_V q_e^2}, \quad \rho_H = \frac{B}{n_V q_e}.$$

(d) A Hall effect sensor detects the strength of magnetic fields. Consider the following circuit consisting of a square plate of side length L and thickness d in a perpendicular uniform magnetic field B.



A longitudinal emf $\mathcal{E}$ is applied to the plate. At steady state, a Hall voltage V_H is measured across the plate due to the buildup of charge on either side of the plate. If the electrical resistivity of the plate at zero magnetic field is ρ_0, what is the Hall voltage V_H and the current I through the plate? Express your answer in terms of ρ_0, $\mathcal{E}$, B, and the dimensions of the plate.

Solution

The Hall voltage is such that the electric field from the external EMF and from the charge buildup causes the current to flow in the x direction. Our equation of motion is thus

$$q_e\mathcal{E}/L\hat{\mathbf{x}} + q_e V_H/L\hat{\mathbf{y}} - q_e v_x B\hat{\mathbf{y}} - \gamma v_x\hat{\mathbf{x}} = 0.$$

Then, we have the two equations

$$V_H = Lv_x B. \quad Lv_x = \frac{q_e \mathcal{E}}{\gamma}.$$

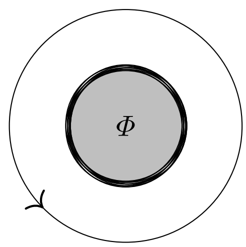
Substituting gives us

$$V_H = \frac{q_e \mathcal{E} B}{\gamma} = \frac{\mathcal{E} B}{n_V q_e \rho_0},$$

$$I = n_V q_e v_x L w = \frac{n_V q_e^2 \mathcal{E} d}{\gamma} = \frac{\mathcal{E} d}{\rho_0}.$$

Experiments in the 20th century revealed that in many materials, the Hall resistivity could only take certain discrete values. We will now show how this follows from Bohr quantization. (These next parts are independent of the first part of the problem.)

2. A zero-resistance loop of wire of radius R and cross-sectional area A_w carries a counterclockwise current I. A solenoid through the middle of the loops carries magnetic flux Φ out of the page, which we define to be the positive $\hat{z}$ direction.



(a) If the electrons all have the same speed, what is the angular momentum of each electron?

Solution

The current is $I = q_e n_V a A_w$, where v is the speed of each electron. Then the angular momentum is

$$m_e v R \hat{\mathbf{z}} = \frac{I m_e R}{q_e n_V A_w} \hat{\mathbf{z}}.$$

Note that the angular momentum points into the page, because the electrons have negative charge.

(b) If we allow the flux in the solenoid to change, the usual, "mechanical" angular momentum L of each electron is not conserved. Instead, a quantity called the canonical angular momentum, $L_{\text{can}} = L + C q_e \Phi$, for some constant C, is conserved. Find C.

Solution

We compute the change in angular momentum associated with a change in flux $\Delta\Phi$. By Faraday's Law, there is an induced electric field resulting from the change in magnetic flux.

$$2\pi R E = -\frac{d\Phi}{dt}.$$

The electric field results in a force $q_e E$ on each electron, and thus we find that

$$2\pi R \frac{m_e}{q_e}\frac{dv}{dt} = -\frac{d\Phi}{dt}.$$

Then, the rate of change in angular momentum is

$$R m_e \frac{dv}{dt} = \frac{q_e}{2\pi}\frac{d\Phi}{dt}.$$

Therefore,

$$\Delta L = -\frac{q_e}{2\pi}\Delta\Phi.$$

Then, for $L + C q_e \Phi$ to be conserved,

$$C = \frac{1}{2\pi}.$$

(c) The Bohr quantization condition says that for a closed circular orbit, an integer number of de Broglie wavelengths must fit in its circumference. The de Broglie wavelength is

$$\lambda = \frac{h}{p_{\text{can}}},$$

where h is Planck's constant, and $p_{\text{can}} = L_{\text{can}}/R$ is the canonical momentum. For a given solenoid flux Φ, what is the set of allowed mechanical angular momenta L?

Solution

The circumference of such an orbit is $2\pi R$. A de Broglie wavelength is given by

$$\lambda = \frac{hR}{L_{\text{can}}}.$$

Then,

$$n = \frac{2\pi L_{\text{can}}}{h} = \frac{L_{\text{can}}}{\hbar},$$

or $L_{\text{can}} = n\hbar$ where $\hbar = h/2\pi$. Then the allowed mechanical momenta satisfy

$$L = n\hbar - \frac{q_e \Phi}{2\pi},$$

where n is an integer.

(d) What is the minimum possible change in the magnetic flux for which the same set of mechanical angular momenta is allowed? This is known as the flux quantum.

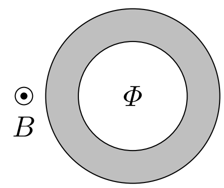
Solution

We require $\frac{|q_e|\Delta\Phi}{2\pi} = \hbar$ such that we get the same set of allowed mechanical angular momenta. Then,

$$\Delta\Phi = \frac{h}{|q_e|}.$$

Note that for a superconducting ring, the flux quantum is $h/2|q_e|$ because the electrons come in Cooper pairs, but this wasn't part of this problem.

3. Now, consider an annulus held perpendicular to a fixed, uniform external magnetic field B, and suppose an additional, tunable magnetic flux Φ threads the center of the annulus, with both pointing out of the page. The annulus has a transverse Hall resistance R_H (i.e., an EMF of $\mathcal{E}$ around the annulus generates a perpendicular current $\mathcal{E}/R_H$ via the Hall effect) and you may neglect its self-inductance.



(a) Suppose Φ begins to increase slowly and steadily in time. After a short time, the electrons will begin flowing steadily from one side of the annulus to the other. Do the electrons move inward or outward? Justify your answer.

Solution

The increasing flux induces a clockwise EMF. The Lorentz force on an electron is

$$q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Then, $\mathbf{v} \times \mathbf{B}$ must oppose $\mathbf{E}$. Therefore, $\mathbf{v}$ must point radially inward by the right hand rule. Therefore, electrons must be flowing radially inward.

(b) If the threaded flux increases by $\Delta\Phi$, how many electrons pass from one edge of the annulus to the other? You may use R_H, among other variables, in your answer.

Solution

We have that

$$|I| = \frac{1}{R_H}\left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right| \implies \Delta Q = \frac{1}{R_H}\Delta\Phi.$$

Therefore, the number of electrons that pass from one edge to another is

$$\frac{\Delta\Phi}{R_H|q_e|}.$$

(c) As we showed in 2(d), if the magnetic flux changes by the flux quantum Φ_q, the allowed orbits from Bohr quantization are unchanged. Quantum mechanics thus tells us that in conventional materials, if the magnetic flux changes by Φ_q, an integer number k of electrons must pass from one edge to another. What constraint does this place on the Hall resistance?

Solution

We set

$$|q_e|\,\nu = \frac{h}{R_H|q_e|} \implies R_H = \frac{h}{kq_e^2}.$$

This is known as the integer quantum Hall effect, and it applies to any material shape.

AAPT UNITED STATES PHYSICS TEAM
AIP 2022

USA Physics Olympiad Exam

DO NOT DISTRIBUTE THIS PAGE

Important Instructions for the Exam Supervisor

- This examination has two parts. Part A has three questions and lasts for 90 minutes. Part B also has three questions and lasts for 90 minutes.
- For each student, print out one copy of the exam and one copy of the answer sheets. Print everything single-sided, and do not staple anything. Divide the exam into the instructions (pages 2–3), Part A questions (pages 4–7), and Part B questions (pages 8–10).
- Begin by giving students the instructions and all of the answer sheets. Let the students read the instructions and fill out their information on the answer sheets. They can keep the instructions for both parts of the exam. Also give students blank sheets of paper to use as scratch paper throughout the exam.
- Students may bring calculators, but they may not use symbolic math, programming, or graphing features of these calculators. Calculators may not be shared, and their memory must be cleared of data and programs. Cell phones or other electronics may not be used during the exam or while the exam papers are present. Students may not use books or other references.
- To start the exam, give students the Part A questions, and allow 90 minutes to complete Part A. Do not give students Part B during this time, even if they finish with time remaining. At the end of the 90 minutes, collect the Part A questions and answer sheets.
- Then give students a 5 to 10 minute break. Then give them the Part B questions, and allow 90 minutes to complete Part B. Do not let students go back to Part A.
- At the end of the exam, collect everything, including the questions, the instructions, the answer sheets, and the scratch paper. Students may *not* keep the exam questions, but they may be returned to the students after April 19th, 2022.
- After the exam, sort each student's answer sheets by page number. Scan every answer sheet, including blank ones.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Tengiz Bibilashvilli, Abijith Krishnan, Brian Skinner, and Kevin Zhou.

AAPT UNITED STATES PHYSICS TEAM
AIP 2022

USA Physics Olympiad Exam

Instructions for the Student

- You should receive these instructions, the table on the next page, answer sheets, and blank paper for scratch work. Read this page carefully before the exam begins.
- You may use a calculator, but its memory must be cleared of data and programs, and you may not use symbolic math, programming, or graphing features. Calculators may not be shared. Cell phones or other electronics may not be used during the exam or while the exam papers are present. You may not use books or other outside references.
- When the exam begins, your proctor will give you the questions for Part A. You will have 90 minutes to complete three problems. Each question is worth 25 points, but they are not necessarily of the same difficulty. If you finish all of the questions, you may check your work, but you may not look at Part B during this time.
- After 90 minutes, your proctor will collect the questions and answer sheets for Part A. You may then take a short break.
- Then you will work on Part B. You have 90 minutes to complete three problems. Each question is worth 25 points. Do not look at Part A during this time. When the exam ends, you must return all papers to the proctor, including the exam questions.
- **Do not discuss the questions of this exam, or their solutions, until after April 19th, 2022. Violations of this rule may result in disqualification.**

Below are instructions for writing your solutions.

- This year, all of your solutions will be written on official answer sheets. Nothing outside these answer sheets will be graded. Before the exam begins, write your name, student AAPT number, and proctor AAPT number as directed on the answer sheets.
- There are several answer sheets per problem. If you run out of space for a problem, you may use the extra answer sheets, which are at the end of the answer sheet packet. To ensure this work is graded, you must indicate, at the bottom of your last answer sheet for that problem, that you are using these extra answer sheets.
- Only write within the frame of each answer sheet. To simplify grading, we recommend drawing a $\boxed{\text{box}}$ around your final answer for each subpart. You should organize your work linearly and briefly explain your reasoning, which will help you earn partial credit. You may use either pencil or pen, but sure to write clearly so your work will be legible after scanning.
- You may have to fit a line to data, which you can plot on the answer sheet with grid lines. You may use a ruler, pencil, pen, or piece of paper as a straightedge.

Reference table of possibly useful information

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}$ C
$1\ \mathrm{eV} = 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \theta^3/6$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \theta^2/2$ for $\lvert\theta\rvert \ll 1$

You may use this sheet for both parts of the exam.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

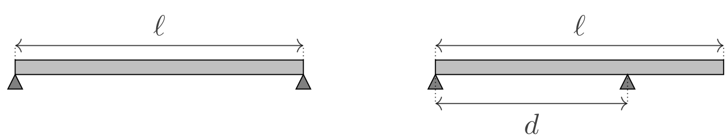
Question A1

Moment of Clarity

Hold your pencil horizontally by its tip. To keep it still, you will have to exert a combination of forces on its bottom and top. These forces can be viewed as a superposition of a net upward force, and a pair of opposite forces. The former ensures the forces on the pencil are balanced, while the latter provides a torque, called the bending moment, which ensures the torques on the pencil are balanced. Since the bending moment arises from a pair of opposite forces, it doesn't depend on the choice of origin.

a. Consider a rod of length ℓ and mass per length λ. What is the bending moment you must exert to hold the rod horizontally at its end?

Just as each piece of a string exerts a tension on neighboring pieces in equilibrium, each piece of a solid rod also exerts a bending moment on its neighbors. For thin rods under heavy loads, this bending moment can be the limiting factor that causes them to break.



Suppose a rod is supported at both ends, so that it forms a bridge, as shown at left above. Assume the supports are simple, so that they only provide an upward force, and no bending moment. In equilibrium, a bending moment will appear throughout the rod. The magnitude of the maximum bending moment the rod can exert at any point without breaking is M_0, and the length of the rod is ℓ. The bridge is loaded uniformly, with a mass per length of λ (including its own mass).

b. Find the maximum possible value of λ before the bridge collapses.

c. Now suppose that one support remains at the left end, while the other is a distance $d > \ell/2$ away from the left end, as shown at right above. In static equilibrium, find the bending moment $M(x_0)$ at a distance $x_0 < d$ from the left end.

d. Find the value of d that maximizes the load λ that the bridge can take before collapsing.

Question A2

Death Metal

A droplet of liquid metal has constant mass density and constant surface tension σ, which causes it to form into a sphere of radius R. (Throughout this problem, you may neglect gravity.) A thin wire is inserted into the droplet and connected to an electric current source which slowly charges the droplet. There is a critical value of the charge, Q_0, that causes the droplet to split in half. Each half takes half the total charge, $Q_0/2$, and half the mass of the original droplet. The "ejected half" is repelled far away from the other half, which remains in contact with the wire.

a. For simplicity, assume the droplet splits as soon as the final state (after the droplet has split in half and the two halves are well separated) has a lower total energy than that of the initial, single droplet. What is the value of Q_0? Give your answer in terms of a dimensionless constant A multiplied by a product of powers of σ, R, and the vacuum permittivity ϵ_0, and give the numeric value of A to three significant figures.

b. As more charge is added to the droplet by the current source, it continues to split in half repeatedly. What is the charge q_n on the n^{th} ejected droplet? Give your answer in terms of Q_0.

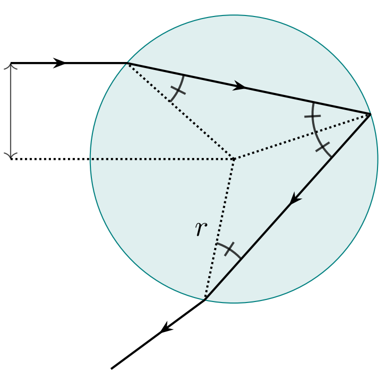
c. In the limit where all of the initial mass of the droplet has been ejected, what is the total work done by the current source? Give your answer in terms of a dimensionless constant B multiplied by a product of powers of σ, R, and ϵ_0, and give the numeric value of B to three significant figures.

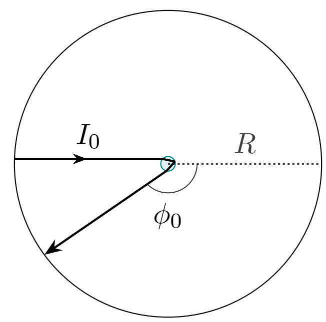
Question A3

Rainbow Road

In geometric optics, a caustic is a bright curve of light that appears when many incoming light rays are focused in the same outgoing direction. The most famous example of a caustic is a rainbow, which occurs when light interacts with spherical water droplets. Consider a spherical liquid droplet of radius r with index of refraction $1 < n < 2$, suspended in air with index of refraction $n = 1$.

a. Consider a light ray that enters the droplet with impact parameter b, reflects once off the inside surface of the droplet, then exits, as shown at left below. Give your answers in terms of the dimensionless impact parameter $x = b/r$. (Hint: the four marked angles are congruent.)





i. Find the angle by which the light ray is deflected at the first refraction.
ii. Find the angle by which the light ray is deflected at the reflection.
iii. Find the angle by which the light ray is deflected at the second refraction.

The sum of these three quantities is the net deflection angle $\phi(x)$. (The light can also reflect inside more than once, or never enter at all, but for simplicity we will ignore these other paths.)

b. Next, we consider when caustics form in general. Suppose the droplet is uniformly illuminated by parallel light rays of intensity I_0, and sits at the center of a spherical screen of radius $R \gg r$, as shown at right above. Consider the light that enters near dimensionless impact parameter x_0, and exits near angle $\phi_0 = \phi(x_0)$.

i. What is the power incident on the droplet at $x_0 \leq x \leq x_0 + dx$?
ii. What is the area on the screen illuminated by the outgoing rays, at $\phi_0 \leq \phi \leq \phi_0 + d\phi$?
iii. A caustic occurs when the intensity of light on the screen diverges. Assume that $\phi_0 \neq 0$ and $\phi_0 \neq \pi$. Under what conditions does light incident at x_0 lead to a caustic at ϕ_0? Express your answer as a condition on the function $\phi(x)$.

c. Find the angle ϕ_0 of the rainbow in terms of n. (Hint: the derivative of $\arcsin(x)$ is $1/\sqrt{1-x^2}$.)

d. For water, the index of refraction of red light is 1.331, and the index of refraction of blue light is 1.340. Find the angular width of the rainbow on the screen and give your answer in degrees.

e. A glory is an optical phenomenon which involves light scattered directly backward, at $\phi = \pi$, leading to an apparent halo around the shadow of an observer's head. For what values of n is there a caustic at $\phi = \pi$? Can glories from water droplets be explained in terms of caustics?

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

Virial Reality

The ideal gas law states that $PV_m = RT$, where $V_m = V/n$ is the volume per mole of gas. However, any real gas will exhibit deviations from the ideal gas law, described by the virial expansion,

$$PV_m = RT\left(1 + \frac{B(T)}{V_m} + \frac{C(T)}{V_m^2} + \dots\right).$$

For gases with low density, the higher-order terms are negligible, so in this problem we will neglect all of the temperature-dependent terms in parentheses except for $B(T)/V_m$. The table below shows measurements of B for nitrogen gas (N_2) at atmospheric pressure, $P = 1.01 \times 10^5\,\mathrm{Pa}$.

T (K)	B (cm^3/mol)
100	-160
200	-35
300	-4.2
400	9.0
500	16.9
600	21.3

a. According to the ideal gas law, what is the value of V_m at temperatures 100 K, 300 K, and 600 K? Give your answers in SI units.

b. What is the percentage change in V_m at these temperatures if one accounts for $B(T)$?

c. In 1910, van der Waals was awarded the Nobel Prize for formulating the equation

$$\left(P + \frac{a}{V_m^2}\right)(V_m - b) = RT$$

which accurately describes many real gases. According to this equation, what is the form of $B(T)$? You may assume that $b \ll V_m$.

d. Using the data above, extract the values of a and b. Give your answers in SI units.

e. In this problem, we have neglected terms in the virial expansion beyond $B(T)$, which is a good approximation as long as the volume correction due to $B(T)$ itself is small. Assuming the van der Waals equation holds, numerically estimate the temperature range within which the volume correction due to $B(T)$ is at most 10%, for nitrogen gas at atmospheric pressure.

Question B2

Broken Vase

A uniformly charged ring of radius d has total charge Q and is fixed in place. A point charge $-q$ of mass m is placed at its center. Both Q and q are positive. As a result, if the point charge is given a small velocity along the ring's axis of symmetry, it will oscillate about the ring's center.

a. Find the period T of the oscillations.

For the rest of the problem, we will consider this situation in a reference frame where the ring is moving along its axis of symmetry with a constant speed v, which may be comparable to the speed of light c. In this frame, the ring and point charge have charge Q' and q', and the point charge still oscillates about the ring's center.

b. What is the period T' of oscillation of the point charge in this frame?

c. When the charge is a small distance Δx from the center of the ring, find the restoring force in terms of q', Q', v, d, Δx, and fundamental constants.

d. Suppose the restoring force has the form $F = -k\,\Delta x$. Find the period of the resulting oscillations in terms of k, m, v, and fundamental constants.

e. Suppose the electric charge transforms between reference frames as $Q' = \gamma^n Q$ and $q' = \gamma^n q$. By combining your answers to parts (c) and (d), and comparing to part (b), find the value of n.

To solve this problem, you will need the following results from relativity:

- The Lorentz factor is defined as $\gamma = 1/\sqrt{1 - v^2/c^2}$.
- The momentum of a particle is $\mathbf{p} = \gamma m \mathbf{v}$, and m is the same in all frames.
- The electromagnetic force on a charge q is $\mathbf{F} = d\mathbf{p}/dt = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.
- The electric field of a charge q at the origin with constant velocity $\mathbf{v}$ is radial, with magnitude

$$E = \frac{q}{4\pi\epsilon_0 r^2} \frac{1 - v^2/c^2}{(1 - (v^2/c^2)\sin^2\theta)^{3/2}}$$

where θ is the angle of $\mathbf{r}$ to $\mathbf{v}$.

Question B3

Time Crystal

The kinetic energy E, momentum p, and velocity v of a particle moving in one dimension satisfy

$$F = \frac{dp}{dt} = \frac{dE}{dx}, \quad v = \frac{dE}{dp}$$

where F is the external force. For a free particle, the momentum and energy are related by $E = p^2/2m$. However, when an electron moves inside a metal, its interactions with the crystal lattice of positively charged ions lead to a different relationship between momentum and energy. All of the above identities still apply, but now suppose that

$$E(p) = V(1 - \cos(pb))$$

where V and b are constants that depend on the metal. This result is inherently quantum mechanical in origin, and as we will see, it leads to some rather strange behavior.

a. First, we investigate the motion of the electron in general.

   i. Find the velocity as a function of p.

   ii. The effective mass m_* of the electron is defined so that it satisfies $F = m_* a$. Find the effective mass as a function of p.

b. Now suppose a metal rod of infinite length, aligned with the x-axis, contains conducting electrons of charge $-e$, initially with zero momentum. At time $t = 0$, an electric field $\mathbf{E} = E_0\hat{\mathbf{x}}$ is turned on, and is experienced by every electron in the rod. Ignore the interactions of the electrons with each other.

   i. For an electron that starts at $x = 0$ at time $t = 0$, find its position $x(t)$.

   ii. If the number of conducting electrons per unit volume is n, and the cross-sectional area of the rod is A, find the average current in the rod over a long time.

   iii. Now suppose that every time τ, each electron suffers a collision with the crystal lattice, causing its momentum to reset to zero. In the limit of frequent collisions, $eE_0 b\tau \ll 1$, find the average current in the rod over a long time.

   iv. If τ can be freely adjusted, estimate the maximum possible average current in the rod, up to a dimensionless constant.

AAPT | UNITED STATES PHYSICS TEAM
AIP | 2022

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination has two parts. Part A has three questions and lasts for 90 minutes. Part B also has three questions and lasts for 90 minutes.
- For each student, print out one copy of the exam and one copy of the answer sheets. Print everything single-sided, and do not staple anything. Divide the exam into the instructions (pages 2–3), Part A questions (pages 4–13), and Part B questions (pages 14–22).
- Begin by giving students the instructions and all of the answer sheets. Let the students read the instructions and fill out their information on the answer sheets. They can keep the instructions for both parts of the exam. Also give students blank sheets of paper to use as scratch paper throughout the exam.
- Students may bring calculators, but they may not use symbolic math, programming, or graphing features of these calculators. Calculators may not be shared, and their memory must be cleared of data and programs. Cell phones or other electronics may not be used during the exam or while the exam papers are present. Students may not use books or other references.
- To start the exam, give students the Part A questions, and allow 90 minutes to complete Part A. Do not give students Part B during this time, even if they finish with time remaining. At the end of the 90 minutes, collect the Part A questions and answer sheets.
- Then give students a 5 to 10 minute break. Then give them the Part B questions, and allow 90 minutes to complete Part B. Do not let students go back to Part A.
- At the end of the exam, collect everything, including the questions, the instructions, the answer sheets, and the scratch paper. Students may *not* keep the exam questions, but they may be returned to the students after April 19th, 2022.
- After the exam, sort each student's answer sheets by page number. Scan every answer sheet, including blank ones.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Tengiz Bibilashvilli, Abijith Krishnan, Brian Skinner, and Kevin Zhou.

AAPT UNITED STATES PHYSICS TEAM
AIP 2022

USA Physics Olympiad Exam

Instructions for the Student

- You should receive these instructions, the table on the next page, answer sheets, and blank paper for scratch work. Read this page carefully before the exam begins.
- You may use a calculator, but its memory must be cleared of data and programs, and you may not use symbolic math, programming, or graphing features. Calculators may not be shared. Cell phones or other electronics may not be used during the exam or while the exam papers are present. You may not use books or other outside references.
- When the exam begins, your proctor will give you the questions for Part A. You will have 90 minutes to complete three problems. Each question is worth 25 points, but they are not necessarily of the same difficulty. If you finish all of the questions, you may check your work, but you may not look at Part B during this time.
- After 90 minutes, your proctor will collect the questions and answer sheets for Part A. You may then take a short break.
- Then you will work on Part B. You have 90 minutes to complete three problems. Each question is worth 25 points. Do not look at Part A during this time. When the exam ends, you must return all papers to the proctor, including the exam questions.
- **Do not discuss the questions of this exam, or their solutions, until after April 19th, 2022. Violations of this rule may result in disqualification.**

Below are instructions for writing your solutions.

- This year, all of your solutions will be written on official answer sheets. Nothing outside these answer sheets will be graded. Before the exam begins, write your name, student AAPT number, and proctor AAPT number as directed on the answer sheets.
- There are several answer sheets per problem. If you run out of space for a problem, you may use the extra answer sheets, which are at the end of the answer sheet packet. To ensure this work is graded, you must indicate, at the bottom of your last answer sheet for that problem, that you are using these extra answer sheets.
- Only write within the frame of each answer sheet. To simplify grading, we recommend drawing a $\boxed{\text{box}}$ around your final answer for each subpart. You should organize your work linearly and briefly explain your reasoning, which will help you earn partial credit. You may use either pencil or pen, but sure to write clearly so your work will be legible after scanning.
- You may have to fit a line to data, which you can plot on the answer sheet with grid lines. You may use a ruler, pencil, pen, or piece of paper as a straightedge.

Reference table of possibly useful information

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}$ (mol)$^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31$ J/(mol $\cdot$ K)
$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)	$e = 1.602 \times 10^{-19}$ C
1 eV $= 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s
$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2	$(1+x)^n \approx 1 + nx$ for $\lvert x\rvert \ll 1$
$\sin\theta \approx \theta - \theta^3/6$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \theta^2/2$ for $\lvert\theta\rvert \ll 1$

You may use this sheet for both parts of the exam.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

Question A1

Moment of Clarity

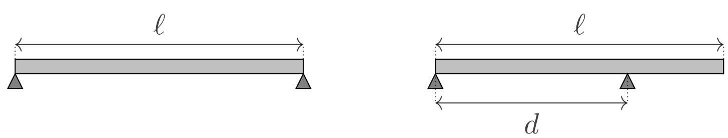
Hold your pencil horizontally by its tip. To keep it still, you will have to exert a combination of forces on its bottom and top. These forces can be viewed as a superposition of a net upward force, and a pair of opposite forces. The former ensures the forces on the pencil are balanced, while the latter provides a torque, called the bending moment, which ensures the torques on the pencil are balanced. Since the bending moment arises from a pair of opposite forces, it doesn't depend on the choice of origin.

a. Consider a rod of length ℓ and mass per length λ. What is the bending moment you must exert to hold the rod horizontally at its end?

Solution

Balance torques about the end of the rod. The weight is $\lambda g\ell$, applied uniformly along the rod, so the torque due to gravity is $\lambda g\ell^2/2$. This can only be balanced by the bending moment, so it is $\lambda g\ell^2/2$.

Just as each piece of a string exerts a tension on neighboring pieces in equilibrium, each piece of a solid rod also exerts a bending moment on its neighbors. For thin rods under heavy loads, this bending moment can be the limiting factor that causes them to break.



Suppose a rod is supported at both ends, so that it forms a bridge, as shown at left above. Assume the supports are simple, so that they only provide an upward force, and no bending moment. In equilibrium, a bending moment will appear throughout the rod. The magnitude of the maximum bending moment the rod can exert at any point without breaking is M_0, and the length of the rod is ℓ. The bridge is loaded uniformly, with a mass per length of λ (including its own mass).

b. Find the maximum possible value of λ before the bridge collapses.

Solution

Placing the origin at the left support, consider the system consisting of the points on the rod at $0 < x < x_0$. The external forces are:

- An upward force $\lambda g\ell/2$ at the left end, due to the left support.
- A weight of $\lambda g x_0$.
- An upward force $\lambda g(x_0 - \ell/2)$ at the right end, due to the rest of the rod.

Now consider balancing torques on this system about $x = x_0$. The external torques are:

- A clockwise torque $\lambda g\ell x_0/2$ due to the left support.

- A counterclockwise torque $\lambda g x_0^2/2$ due to the weight.
- A bending moment $M(x_0)$ due to the rest of the rod.

For torque balance, the bending moment must be

$$M(x_0) = \frac{\lambda g x_0^2}{2} - \frac{\lambda g \ell x_0}{2} = \frac{\lambda g x_0(x_0 - \ell)}{2}.$$

The rod is most likely to break where M is the largest, which occurs at the center, so

$$M_0 = \frac{\lambda g(\ell/2)^2}{2}.$$

Solving for λ gives

$$\lambda = \frac{8M_0}{g\ell^2}.$$

c. Now suppose that one support remains at the left end, while the other is a distance $d > \ell/2$ away from the left end, as shown at right above. In static equilibrium, find the bending moment $M(x_0)$ at a distance $x_0 < d$ from the left end.

Solution

To keep the expressions simple, let $W = \lambda g \ell$ be the weight of the rod. By balancing torques about the left end of the rod, the external forces on the entire rod are

- A weight force W.
- An upward force $W\ell/2d$ from the right support.
- An upward force $W - W\ell/2d$ from the left support.

Note that the reason we specified $d > \ell/2$ is because otherwise, the force from the left support would become negative, in which case the bridge would tip over to the right.

Now consider the system consisting of the points on the rod with $0 < x < x_0$, where $x_0 < d$. To balance the forces, the upward force from the rest of the rod must be

$$F_r = W\left(\frac{x_0}{\ell} + \frac{\ell}{2d} - 1\right).$$

The external torques on this system, about the left end, are

- A counterclockwise torque $Wx_0^2/2\ell$ due to the weight.
- A clockwise torque $F_r x_0$ due to the rest of the rod.
- A bending moment $M(x_0)$ due to the rest of the rod.

Balancing the torques thus gives

$$M(x_0) = -W\left(\frac{x_0^2}{2\ell} + \frac{\ell x_0}{2d} - x_0\right) = -\lambda g\left(\frac{x_0^2}{2} + \frac{\ell^2 x_0}{2d} - x_0\ell\right).$$

The sign of M is convention-dependent, so either sign is acceptable.

d. Find the value of d that maximizes the load λ that the bridge can take before collapsing.

Solution

If the bridge doesn't collapse, then the magnitude of the bending moment must be less than M_0 everywhere throughout the rod. Consider how the bending moment varies as one moves from the left end of the rod to the right. It begins at zero at $x_0 = 0$, then reaches a positive maximum somewhere between the two supports, when $M'(x_0)$ vanishes. This occurs when

$$x_0 = \ell - \frac{\ell^2}{2d}$$

and which point there is a bending moment of

$$M_1 = \frac{\lambda g}{2} \left(\ell - \frac{\ell^2}{2d} \right)^2 .$$

As a check, we recover the answer to part (b) when $d = \ell$.

Continuing rightward, the bending moment decreases, passes through zero at some point $x_0 < d$, and then becomes negative. (This change in sign is necessary to support the part of the rod hanging to the right of the right support.) At $x = d$, the bending moment does not change discontinuously, because the supports provide no bending moment. It then smoothly returns to zero as x_0 approaches ℓ. Therefore, the most negative value of the bending moment occurs at $x_0 = d$, giving a bending moment of magnitude

$$M_2 = -M(d) = \frac{\lambda g}{2} (\ell - d)^2 .$$

As another check, this matches the answer to part (a) for a rod of length $\ell - d$.

As we move the right support closer to the center, the middle of the bridge becomes more stable, since M_1 decreases. At the same time, M_2 increases, because more of the bridge is hanging off the edge of the right support. The bridge will not collapse as long as both of these quantities are less than M_0. Thus, when the right support is placed most efficiently, both of them will become equal to M_0 just before the bridge collapses, which means we should have $M_1 = M_2$. Equating them yields

$$1 - \frac{\ell}{2d} = 1 - \frac{d}{\ell}$$

which gives the answer,

$$d = \frac{\ell}{\sqrt{2}}.$$

Question A2

Death Metal

A droplet of liquid metal has constant mass density and constant surface tension σ, which causes it to form into a sphere of radius R. (Throughout this problem, you may neglect gravity.) A thin wire is inserted into the droplet and connected to an electric current source which slowly charges the droplet. There is a critical value of the charge, Q_0, that causes the droplet to split in half. Each half takes half the total charge, $Q_0/2$, and half the mass of the original droplet. The "ejected half" is repelled far away from the other half, which remains in contact with the wire.

a. For simplicity, assume the droplet splits as soon as the final state (after the droplet has split in half and the two halves are well separated) has a lower total energy than that of the initial, single droplet. What is the value of Q_0? Give your answer in terms of a dimensionless constant A multiplied by a product of powers of σ, R, and the vacuum permittivity ϵ_0, and give the numeric value of A to three significant figures.

Solution

Note that σ has units of energy/length^2, R has units of length, and ϵ_0 has units of charge^2/(energy $\times$ length). There is only one combination with dimensions of charge, so we must have

$$Q_0 = A\sqrt{\epsilon_0 \sigma R^3}.$$

To find the value of A, note that a droplet of radius R and charge Q_0 has a surface tension energy $4\pi R^2\sigma$ and an electrostatic energy $Q_0^2/2C$, where $C = 4\pi\epsilon_0 R$ is the capacitance of a conducting sphere relative to infinity. Thus, the total energy of the drop is

$$E_0 = 4\pi\sigma R^2 + \frac{Q_0^2}{8\pi\epsilon_0 R}.$$

When the droplet splits, the new halves both have charge $Q_0/2$ and a radius $R_1 = R/2^{1/3}$, since the total volume stays the same. The total energy of each of the halves is thus

$$E_1 = 4\pi\sigma R_1^2 + \frac{(Q_0/2)^2}{8\pi\epsilon_0 R_1}.$$

Setting $E_0 = 2E_1$ and solving for Q_0 gives

$$A = 8\pi\sqrt{\frac{2^{1/3}-1}{2-2^{1/3}}} \approx 14.9.$$

This model of the instability of a charged liquid drop is not exactly accurate, and was chosen to keep the problem simple. In reality, it isn't enough for the final state to have lower energy; the repulsive electrostatic force needs to exceed the surface tension force, or else the bubble can't get to this state of lower energy. Accounting for this requires a different calculation, first performed by Lord Rayleigh in 1882, which gives the "Rayleigh limit" of $A = 8\pi \approx 25.1$. Students who successfully derived this result also received full credit for part (a).

b. As more charge is added to the droplet by the current source, it continues to split in half repeatedly. What is the charge q_n on the n^{th} ejected droplet? Give your answer in terms of Q_0.

Solution

Consider the situation just before the n^{th} split. At this point, the initial droplet has split in half $n-1$ times, so it has a volume that is 2^{n-1} times smaller than the original volume, and thus a radius $R_{n-1} = R/2^{(n-1)/3}$. By part (a), the charge required to split this drop is

$$Q_n = A\sqrt{\epsilon_0 \sigma R_{n-1}^3} = \frac{Q_0}{2^{(n-1)/2}}.$$

Once the split happens, the n^{th} ejected droplet takes half of this charge, so

$$q_n = \frac{Q_n}{2} = \frac{Q_0}{2^{(n+1)/2}}.$$

Also note that the radius of this droplet is $R_n = R/2^{n/3}$.

c. In the limit where all of the initial mass of the droplet has been ejected, what is the total work done by the current source? Give your answer in terms of a dimensionless constant B multiplied by a product of powers of σ, R, and ϵ_0, and give the numeric value of B to three significant figures.

Solution

By dimensional analysis, the total work W done by the voltage source must be

$$W = B\sigma R^2.$$

The work W is equal to the difference between the energy E_f of the final state, where there are many small droplets that have been dispersed to far away, and the energy $E_i = 4\pi\sigma R^2$ of the initial, uncharged droplet. The final energy includes the surface tension and electrostatic energy of all the small droplets, which have radius R_n and charge q_n found in the previous problem. This means that the final energy is

$$\begin{aligned}
E_f &= \sum_{n=1}^{\infty} \left(4\pi\sigma R_n^2 + \frac{q_n^2}{8\pi\epsilon_0 R_n} \right) \\
&= 4\pi\sigma R^2 \sum_{n=1}^{\infty} \frac{1}{2^{2n/3}} + \frac{Q_0^2}{16\pi R_0 \epsilon_0} \sum_{n=1}^{\infty} \frac{1}{2^{2n/3}} \\
&= 4\pi\sigma R^2 \left(1 + \frac{A^2}{64\pi^2} \right) \sum_{n=1}^{\infty} \frac{1}{2^{2n/3}} \\
&= 4\pi\sigma R^2 \left(1 + \frac{A^2}{64\pi^2} \right) \frac{1}{2^{2/3}-1} = 4\pi\sigma R^2 \times 2.30.
\end{aligned}$$

Here we have inserted the results found in part (b), then used the result of part (a) and summed the geometric series. Thus,

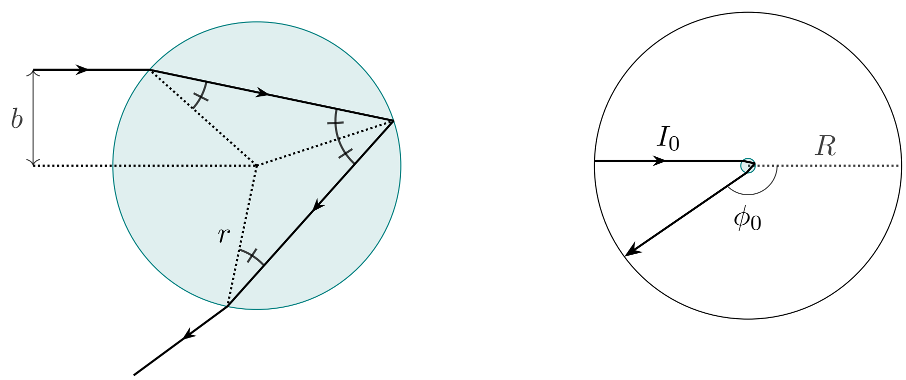
$$W = E_f - E_i \approx (4\pi\sigma R^2) \times 1.30 = 16.3\,\sigma R^2.$$

Question A3

Rainbow Road

In geometric optics, a caustic is a bright curve of light that appears when many incoming light rays are focused in the same outgoing direction. The most famous example of a caustic is a rainbow, which occurs when light interacts with spherical water droplets. Consider a spherical liquid droplet of radius r with index of refraction $1 < n < 2$, suspended in air with index of refraction $n = 1$.

a. Consider a light ray that enters the droplet with impact parameter b, reflects once off the inside surface of the droplet, then exits, as shown at left below. Give your answers in terms of the dimensionless impact parameter $x = b/r$. (Hint: the four marked angles are congruent.)



i. Find the angle by which the light ray is deflected at the first refraction.
ii. Find the angle by which the light ray is deflected at the reflection.
iii. Find the angle by which the light ray is deflected at the second refraction.

The sum of these three quantities is the net deflection angle $\phi(x)$. (The light can also reflect inside more than once, or never enter at all, but for simplicity we will ignore these other paths.)

Solution

The angle the incoming light ray makes with to the normal of the droplet is $\theta_1 = \arcsin(x)$. By Snell's law, the angle to the normal inside the droplet is $\theta_2 = \arcsin\big(x/n\big)$.

i. The first refraction causes a deflection of $\theta_1 - \theta_2$.
ii. The reflection keeps the angle to the normal the same, and causes a deflection of $\pi - 2\theta_2$.
iii. At this point, the angle to the normal is still θ_2, and by Snell's law, it reverts to θ_1 outside the droplet. This causes a deflection of $\theta_1 - \theta_2$, with the same sign as in part (i).

Thus, we conclude

$$\phi(x) = \pi - 4\theta_2 + 2\theta_1 = \pi - 4\arcsin\big(x/n\big) + 2\arcsin(x).$$

The definition of ϕ is convention-dependent, so answers that differed by a minus sign and/or a factor of 2π were also accepted.

b. Next, we consider when caustics form in general. Suppose the droplet is uniformly illuminated by parallel light rays of intensity I_0, and sits at the center of a spherical screen of radius $R \gg r$, as shown at right above. Consider the light that enters near dimensionless impact parameter x_0, and exits near angle $\phi_0 = \phi(x_0)$.

i. What is the power incident on the droplet at $x_0 \leq x \leq x_0 + dx$?

Solution

The region with impact parameters in this region is a thin annulus with radius rx and cross-sectional width $r\,dx$. Then, the power is

$$2\pi(I_0 r^2)\, x_0\, |dx|.$$

ii. What is the area on the screen illuminated by the outgoing rays, at $\phi_0 \leq \phi \leq \phi_0 + d\phi$?

Solution

By similar reasoning to the previous part, the area is

$$dA = 2\pi R^2 \sin\phi\, |d\phi|.$$

iii. A caustic occurs when the intensity of light on the screen diverges. Assume that $\phi_0 \neq 0$ and $\phi_0 \neq \pi$. Under what conditions does light incident at x_0 lead to a caustic at ϕ_0? Express your answer as a condition on the function $\phi(x)$.

Solution

By dividing our two answers, the intensity on the screen is

$$I(\phi) = \frac{I_0 r^2}{R^2} \frac{x}{|d\phi/dx|\, \sin\phi}.$$

This diverges when the denominator becomes zero, which in this case only occurs when $d\phi/dx$ vanishes. That is, a caustic will occur at $\phi_0 = \phi(x)$ if

$$\phi'(x_0) = 0.$$

Intuitively, this condition means that a broad range of incoming light rays end up focused on a narrow curve on the screen.

c. Find the angle ϕ_0 of the rainbow in terms of n. (Hint: the derivative of $\arcsin(x)$ is $1/\sqrt{1-x^2}$.)

Solution

The condition for a caustic is $\phi'(x_0) = 0$, which implies

$$\frac{2}{\sqrt{1-x_0^2}} - \frac{4}{\sqrt{n^2 - x_0^2}} = 0.$$

Solving for x_0 gives

$$x_0 = \sqrt{\frac{4-n^2}{3}}.$$

In particular, this implies that a caustic will appear for the entire range of n considered in this problem. Now, substituting this back into $\phi(x)$ gives

$$\phi_0 = \pi - 4\arcsin\left(\sqrt{\frac{4-n^2}{3n^2}}\right) + 2\arcsin\left(\sqrt{\frac{4-n^2}{3}}\right).$$

d. For water, the index of refraction of red light is 1.331, and the index of refraction of blue light is 1.340. Find the angular width of the rainbow on the screen and give your answer in degrees.

Solution

Substituting these two indices of refraction into the above equation and subtracting the results yields 1.30°.

e. A glory is an optical phenomenon which involves light scattered directly backward, at $\phi = \pi$, leading to an apparent halo around the shadow of an observer's head. For what values of n is there a caustic at $\phi = \pi$? Can glories from water droplets be explained in terms of caustics?

Solution

Referring to our answer above, we have a caustic whenever the denominator $|d\phi/dx|\sin\phi$ vanishes, and $\sin\phi$ vanishes for $\phi = \pi$. Therefore, we will have a caustic at $\phi = \pi$ as long as light can be reflected backwards at all, i.e. whenever there is a solution to $\phi(x) = \pi$. That is because at this angle, all the outgoing light is directed at a single point on the screen.

We thus need to solve

$$\arcsin(x) = 2\arcsin\big(x/n\big).$$

Taking the sine of both sides and using the double angle formula gives

$$x = 2(x/n)\cos\Big(\arcsin\big(x/n\big)\Big) = \frac{2x\sqrt{n^2-x^2}}{n^2}.$$

Solving for n gives

$$n = \frac{\sqrt{2}\,x}{\sqrt{1-\sqrt{1-x^2}}}.$$

As the impact parameter x yielding the backward caustic ranges from 0 to 1, the value of n ranges from a minimum of 2 to a minimum of $\sqrt{2}$. Thus,

$$\sqrt{2} < n < 2.$$

The index of refraction of water is outside this range, so glories from water droplets <u>cannot</u> be explained in terms of caustics. The reason glories are visible is still under debate, though all proposed mechanisms rely on the wave nature of light. For example, one proposed solution invoking “light tunneling” is explored in this paper. This problem was inspired by Berry, Contemporary Physics 56:1 (2015): 2-16.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

Virial Reality

The ideal gas law states that $PV_m = RT$, where $V_m = V/n$ is the volume per mole of gas. However, any real gas will exhibit deviations from the ideal gas law, described by the virial expansion,

$$PV_m = RT\left(1 + \frac{B(T)}{V_m} + \frac{C(T)}{V_m^2} + \dots\right).$$

For gases with low density, the higher-order terms are negligible, so in this problem we will neglect all of the temperature-dependent terms in parentheses except for $B(T)/V_m$. The table below shows measurements of B for nitrogen gas (N_2) at atmospheric pressure, $P = 1.01 \times 10^5$ Pa.

T (K)	B (cm^3/mol)
100	-160
200	-35
300	-4.2
400	9.0
500	16.9
600	21.3

a. According to the ideal gas law, what is the value of V_m at temperatures 100 K, 300 K, and 600 K? Give your answers in SI units.

Solution

Plugging in the values and converting to m^3/mol gives

$$V_m = \frac{RT}{P} = \begin{cases} 8.23 \times 10^{-3}\,\mathrm{m^3/mol} & T = 100\,\mathrm{K} \\ 2.47 \times 10^{-2}\,\mathrm{m^3/mol} & T = 300\,\mathrm{K} \\ 4.94 \times 10^{-2}\,\mathrm{m^3/mol} & T = 600\,\mathrm{K} \end{cases}$$

b. What is the percentage change in V_m at these temperatures if one accounts for $B(T)$?

Solution

The fractional change in volume is

$$\frac{\Delta V_m}{V_m} \approx \frac{B(T)}{V_m} \approx \begin{cases} -1.9\% & T = 100\,\mathrm{K} \\ -0.02\% & T = 300\,\mathrm{K} \\ 0.04\% & T = 600\,\mathrm{K} \end{cases}$$

c. In 1910, van der Waals was awarded the Nobel Prize for formulating the equation

$$\left(P + \frac{a}{V_m^2}\right)(V_m - b) = RT$$

which accurately describes many real gases. According to this equation, what is the form of $B(T)$? You may assume that $b \ll V_m$.

Solution

We have

$$P + \frac{a}{V_m^2} = \frac{RT}{V_m - b} \approx \frac{RT}{V_m}\left(1 + \frac{b}{V_m}\right).$$

Rearranging yields

$$PV_m \approx RT\left(1 + \frac{b}{V_m} - \frac{a}{V_m RT}\right)$$

from which we conclude

$$B(T) = b - \frac{a}{RT}.$$

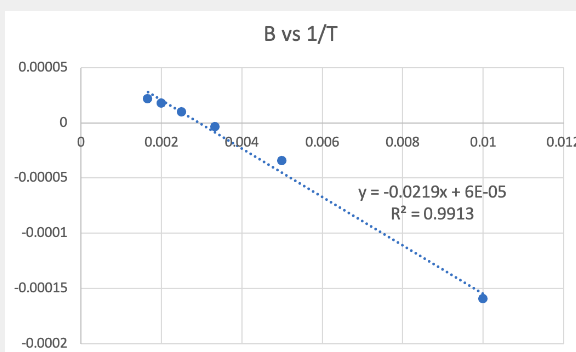
d. Using the data above, extract the values of a and b. Give your answers in SI units.

Solution

First, we convert the data points to SI units, giving

T(K)	B(m^3/mol)
100	-1.6×10^{-4}
200	-3.5×10^{-5}
300	-4.2×10^{-6}
400	9.0×10^{-6}
500	1.69×10^{-5}
600	2.13×10^{-5}

Now, plotting B versus $1/T$, the slope is $-a/R$ and the y-intercept is b.



The final result is

$$a = 0.18\,\mathrm{J\,m^3/mol^2}, \quad b = 6 \times 10^{-5}\,\mathrm{m^3/mol}.$$

Any answer within 20% of this is acceptable.

e. In this problem, we have neglected terms in the virial expansion beyond $B(T)$, which is a good approximation as long as the volume correction due to $B(T)$ itself is small. Assuming the van der Waals equation holds, numerically estimate the temperature range within which the volume correction due to $B(T)$ is at most 10%, for nitrogen gas at atmospheric pressure.

Solution

We are looking for the temperature range where

$$|B(T)| \lesssim \frac{V_m}{10} \approx \frac{RT}{10P}.$$

At high temperatures, $B(T)$ always yields a small correction for nitrogen at atmospheric pressure. At low temperatures, $B(T)$ is dominated by the $-a/RT$ term, so we require

$$T \gtrsim \frac{\sqrt{10Pa}}{R} = 50\,\mathrm{K}.$$

Any answer within 25% of this is acceptable.

Question B2

Broken Vase

A uniformly charged ring of radius d has total charge Q and is fixed in place. A point charge $-q$ of mass m is placed at its center. Both Q and q are positive. As a result, if the point charge is given a small velocity along the ring's axis of symmetry, it will oscillate about the ring's center.

a. Find the period T of the oscillations.

Solution

Suppose the charge is displaced by a small distance Δx. The horizontal component of the electric field of the ring, at the charge, is

$$E_x \approx E\,\frac{\Delta x}{d} = \frac{Q}{4\pi\epsilon_0 d^3}\,\Delta x.$$

The restoring force is linear in Δx, so it acts like a spring with spring constant

$$k = \frac{Qq}{4\pi\epsilon_0 d^3}.$$

Thus, the period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{k}} = \sqrt{\frac{16\pi^3\epsilon_0 m d^3}{Qq}}.$$

For the rest of the problem, we will consider this situation in a reference frame where the ring is moving along its axis of symmetry with a constant speed v, which may be comparable to the speed of light c. In this frame, the ring and point charge have charge Q' and q', and the point charge still oscillates about the ring's center.

b. What is the period T' of oscillation of the point charge in this frame?

Solution

By directly applying time dilation to the answer of part (a),

$$T' = \gamma T = \frac{1}{\sqrt{1 - v^2/c^2}}\sqrt{\frac{16\pi^3\epsilon_0 m d^3}{Qq}}.$$

c. When the charge is a small distance Δx from the center of the ring, find the restoring force in terms of q', Q', v, d, Δx, and fundamental constants.

Solution

The position of the charge, relative to each charge in the ring, is at $\theta \approx 90^\circ$. Thus, using the provided information, we have

$$E_x \approx \frac{Q'}{4\pi\epsilon_0 d^2} \frac{1 - v^2/c^2}{(1 - v^2/c^2)^{3/2}} \sin\theta.$$

Thus, the restoring force is

$$F = -\frac{Q'q'}{4\pi\epsilon_0 d^3} \frac{1}{\sqrt{1 - v^2/c^2}} \Delta x.$$

d. Suppose the restoring force has the form $F = -k\,\Delta x$. Find the period of the resulting oscillations in terms of k, m, v, and fundamental constants.

Solution

By the definition of force given in the information below, we have

$$F = \frac{dp}{dt} = m\frac{d}{dt}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right).$$

Carrying out the derivative and simplifying gives

$$F = \frac{m}{(1 - v^2/c^2)^{3/2}} \frac{dv}{dt} = -k\Delta x$$

For small oscillations, the velocity is always approximately equal to the original velocity v. Thus, the motion is still simple harmonic, except that the mass is effectively

$$m_{\text{eff}} = \frac{m}{(1 - v^2/c^2)^{3/2}}.$$

Thus, we conclude

$$T' = 2\pi\sqrt{\frac{m_{\text{eff}}}{k}} = 2\pi\sqrt{\frac{m}{k}\frac{1}{(1 - v^2/c^2)^{3/2}}}.$$

e. Suppose the electric charge transforms between reference frames as $Q' = \gamma^n Q$ and $q' = \gamma^n q$. By combining your answers to parts (c) and (d), and comparing to part (b), find the value of n.

Solution

The final result of part (c) tells us that

$$k = \frac{Q'q'}{4\pi\epsilon_0 d^3} \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Plugging this into the answer to part (d) gives

$$T' = \frac{1}{\sqrt{1 - v^2/c^2}} \sqrt{\frac{16\pi^3\epsilon_0 m d^3}{Q'q'}}.$$

This is precisely the same as the answer to part (b) if $Q'q' = Qq$, or in other words,

$$n = 0.$$

That is, the electric charge is Lorentz invariant.

To solve this problem, you will need the following results from relativity:

- The Lorentz factor is defined as $\gamma = 1/\sqrt{1 - v^2/c^2}$.
- The momentum of a particle is $\mathbf{p} = \gamma m \mathbf{v}$, and m is the same in all frames.
- The electromagnetic force on a charge q is $\mathbf{F} = d\mathbf{p}/dt = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.
- The electric field of a charge q at the origin with constant velocity $\mathbf{v}$ is radial, with magnitude

  $$E = \frac{q}{4\pi\epsilon_0 r^2} \frac{1 - v^2/c^2}{(1 - (v^2/c^2)\sin^2\theta)^{3/2}}$$

  where θ is the angle of $\mathbf{r}$ to $\mathbf{v}$.

Question B3

Time Crystal

The kinetic energy E, momentum p, and velocity v of a particle moving in one dimension satisfy

$$F = \frac{dp}{dt} = \frac{dE}{dx}, \quad v = \frac{dE}{dp}$$

where F is the external force. For a free particle, the momentum and energy are related by $E = p^2/2m$. However, when an electron moves inside a metal, its interactions with the crystal lattice of positively charged ions lead to a different relationship between momentum and energy. All of the above identities still apply, but now suppose that

$$E(p) = V(1 - \cos(pb))$$

where V and b are constants that depend on the metal. This result is inherently quantum mechanical in origin, and as we will see, it leads to some rather strange behavior.

a. First, we investigate the motion of the electron in general.

i. Find the velocity as a function of p.

Solution

Using the provided identity,

$$v = \frac{dE}{dp} = Vb\sin(pb).$$

ii. The effective mass m_* of the electron is defined so that it satisfies $F = m_* a$. Find the effective mass as a function of p.

Solution

By definition, we have

$$\frac{dp}{dt} = m_* \frac{dv}{dt}$$

so by the chain rule, we conclude

$$m_* = \left(\frac{dv}{dp}\right)^{-1} = \frac{1}{Vb^2}\frac{1}{\cos(pb)}.$$

Note that at some points, this mass is negative! Applying a forward force can cause an electron to accelerate backwards, which leads to the strange result we'll find in part b.ii. (This doesn't contradict conservation of momentum, because the forward momentum ends up imparted to the crystal lattice, causing the solid as a whole to recoil.)

b. Now suppose a metal rod of infinite length, aligned with the x-axis, contains conducting electrons of charge $-e$, initially with zero momentum. At time $t = 0$, an electric field $\mathbf{E} = E_0\hat{\mathbf{x}}$ is turned on, and is experienced by every electron in the rod. Ignore the interactions of the electrons with each other.

i. For an electron that starts at $x = 0$ at time $t = 0$, find its position $x(t)$.

Solution

The momentum is $p = Ft = -eE_0 t$, so using the result of part a.i,

$$v(t) = -Vb \sin(eE_0 tb).$$

Integrating both sides gives

$$x(t) = \frac{V}{eE_0}\left(\cos(eE_0 tb) - 1\right).$$

ii. If the number of conducting electrons per unit volume is n, and the cross-sectional area of the rod is A, find the average current in the rod over a long time.

Solution

The electrons just oscillate, with zero average velocity. Therefore, the average current they supply is zero! This strange phenomenon is called a Bloch oscillation. It doesn't occur in ordinary metals because of the frequent collisions between electrons and ions, considered in the parts below.

iii. Now suppose that every time τ, each electron suffers a collision with the crystal lattice, causing its momentum to reset to zero. In the limit of frequent collisions, $eE_0 b\tau \ll 1$, find the average current in the rod over a long time.

Solution

For small times, the velocity is approximately

$$v(t) \approx -(eE_0 Vb^2)\, t$$

where we applied the small angle approximation to the result of part b.i. Thus, if collisions happen every time τ, the average velocity is

$$\bar{v} = -\frac{1}{2}(eE_0 Vb^2)\, \tau.$$

Therefore, the average current is

$$I = -nAe\bar{v} = \frac{1}{2} nAe^2 E_0 Vb^2 \tau.$$

The plus sign makes sense, since it means current flows parallel to the applied field.

iv. If τ can be freely adjusted, estimate the maximum possible average current in the rod, up to a dimensionless constant.

Solution

If τ is small, then more frequent collisions slow down the current, as we showed in part b.iii. But if τ is very large, then the current cancels itself out due to Bloch oscillations, as we showed in part b.ii. Therefore, the highest possible current occurs when the electrons have time to accelerate to a substantial fraction of their maximum possible speed Vb, but collide before their effective mass goes negative. This occurs when

$$eE_0\tau b \sim 1$$

and gives an average current

$$I \sim nAeVb.$$

Some students tried to answer this part using dimensional analysis, but that won't work, since there are 5 variables and only 4 independent dimensions (mass, length, time, and charge). Incidentally, you can find the dimensionless coefficient by solving a simple equation numerically, though this wasn't required.

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You may use this sheet throughout the exam.

Take Five

The five parts of this question are unrelated.

1. Consider a partially polarized light beam, containing a mix of unpolarized and linearly polarized light. The intensity of the beam is analyzed using a linear polarizer. At a particular orientation of the polarizer, the outgoing beam has maximum intensity $I_{\max}$. Turning the polarizer by a 30° angle reduces the outgoing beam's intensity by 10%. Find the degree of polarization of the beam,

$$V \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

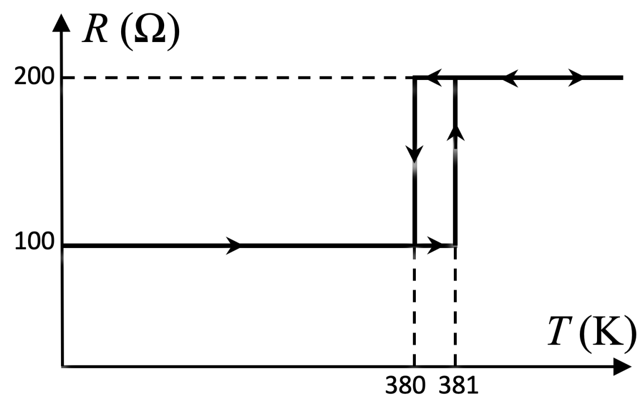
   where $I_{\min}$ is the minimum intensity for any orientation of the polarizer.

2. According to Newton's law of cooling, a hot object transfers heat to the environment at a rate

$$P = k(T_o - T_l),$$

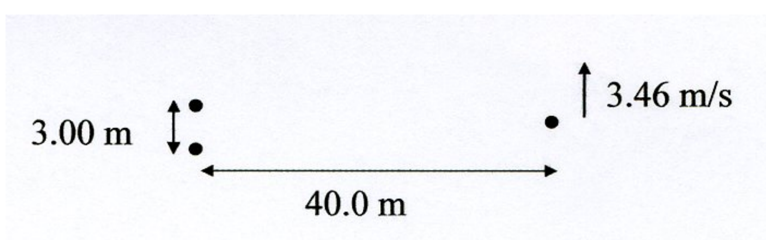
   where T_o is the temperature of the object, T_l is the temperature of the environment, and k is a constant.

   Consider a circuit element whose resistance R depends on its temperature T as shown below (not to scale), with heat capacity $C = 2$ J/K in a lab of temperature $T_l = 270$ K. Note that the curve is multivalued, which indicates hysteresis: the resistance takes the lower value when increasing from low temperatures, and the higher value when decreasing from high temperatures.

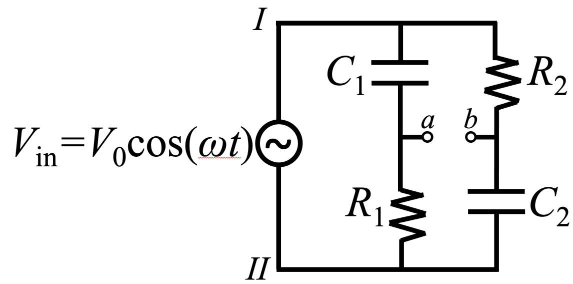


   When this component is placed in series with a voltage $V_1 = 50$ V, its temperature stabilizes at $T_1 = 350$ K. When it is instead placed in series with a voltage $V_2 = 70$ V, its temperature does not stabilize, and the current through it instead oscillates. Find the period of these oscillations.

3. Two speakers are 3.00 m apart. They both emit perfect sinusoids, whose frequencies differ by 0.250 Hz. Spaceman Fred, who is standing 40.0 m away in the direction shown in the diagram, must run at 3.46 m/s to avoid hearing beats. The speed of sound in air is 343 m/s. Approximately what frequency are the speakers emitting?

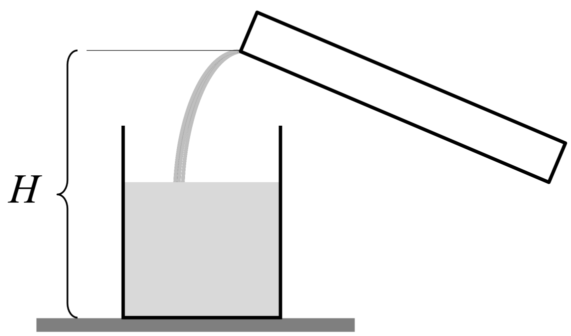


4. The input voltage is the voltage difference between points I and II. What is the voltage difference between the points a and b in the circuit below as a function of time?



You should assume $R_1 C_1 = R_2 C_2$, and simplify your answer as much as possible.

5. An empty cylindrical glass of cross-sectional area A is resting on a table. Water of density ρ is slowly poured into the glass from a beaker, at a constant volume per unit time Q.



The beaker nozzle is at height H, and the water exits the beaker with negligible speed. Let $t = 0$ at the moment the water first hits the bottom of the glass. Find the force of the water on the glass as a function of time, until the glass overflows. Assume the water does not splash and the atmospheric pressure is P_0. Furthermore, assume the glass is wide, so that the rate at which the water level rises is negligible compared to the speed at which water enters the glass.

Chain Reaction

The three parts of this question are unrelated.

1. A frictionless circular cylinder of radius R is placed with its axis horizontal, and a flexible, inextensible string of uniform linear mass density λ is wrapped around it. When the length of the string is slightly longer than $2\pi R$, part of the string will sag below the cylinder. Now suppose the string is slowly shortened, until the entire string just touches the cylinder. At this moment, find the tension at the top of the string.

2. Model a grappling hook as a point mass m attached to the end of a uniform chain of linear mass density λ. Initially, the chain is loosely coiled on the ground. Then the mass is launched directly upward from the ground, with an initial speed v_0. The chain is flexible, so that when the mass is at a height y, a length y of the chain dangles directly beneath it, while the rest of the chain remains at rest on the ground. Find the maximum height reached by the mass, assuming this is less than the length of the chain. (Hint: if you directly compute the acceleration, you will find an intractable differential equation, but it can be solved with a clever change of variable.)

3. A uniform rod of length $2R$ is placed inside a fixed, frictionless hemispherical bowl of radius R. In equilibrium, the rod makes an angle θ with the horizontal. Assume that the rod and bowl are ideally rigid, but that the lip of the bowl and the end of the rod are both slightly rounded, so that there is a well-defined normal direction at the points they touch. Find an analytic expression for θ and evaluate it to three significant figures, in degrees.

Lego Movie

Gravitational waves are predicted by general relativity, but can be modeled with Newtonian physics and a few small assumptions. Throughout this problem, assume classical Newtonian physics, and ignore special relativistic effects. The mass of the sun is $M_\odot = 2.0 \times 10^{30}$ kg, and the luminosity of the sun is $L_\odot = 3.8 \times 10^{26}$ W. In all parts of this problem, you may use the fundamental constants c and G in your answers.

1. Consider a spherically symmetric body with mass M. Determine the Schwarzschild radius R_s of such a body so that the escape velocity would be equal to the speed of light c.

2. Two such bodies, with masses M_1 and M_2, are in circular orbits about their common center of mass. The separation of the bodies is R, and the total mass is $M = M_1 + M_2$.

   (a) Find the frequency f of the orbital motion in terms of M and R.

   (b) Find the total energy E of the system in terms of M_1, M_2, and R.

   (c) The minimum possible orbital separation is $R_{\text{min}} = R_1 + R_2$, where R_1 and R_2 are the Schwarzschild radii for masses 1 and 2. Find the maximum possible orbital frequency f_{max} in terms of M.

3. We would like to estimate the rate at which the system loses energy due to the emission of gravitational waves. In classical electromagnetism, the simplest form of radiation is dipole radiation, which results from a second time derivative of the electric dipole moment. However, for gravity the analogue of the electric dipole moment is the center of mass, which always moves at constant velocity by momentum conservation. Thus, the leading source of gravitational radiation is quadrupole radiation, which depends on a time derivative of the moment of inertia. All subparts of this part are rough estimates, which means you may drop numeric prefactors such as π.

   (a) The power radiated in gravitational waves by a system with moment of inertia I takes the form[1]

   $$P = kG^\alpha c^\beta \left(\frac{d^n I}{dt^n}\right)^2$$

   where k is a dimensionless constant. Determine α, β, and n.

   (b) For two black holes circularly orbiting each other in the xy plane, with center of mass at the origin, find the moment of inertia $I_y(t)$ about the y-axis in terms of M_1, M_2, R, and the angular frequency ω, defining the origin of time so that $I_y(0) = 0$.

   (c) Roughly estimate the average power radiated over an orbital period. Your final answer should be a product of powers of M_1, M_2, M, R, and fundamental constants.

   (d) The maximum power radiated occurs when $R = R_{\text{min}}$. Roughly estimate the maximum power in the case $M_1 = M_2$ in terms of fundamental constants. What is the order of magnitude of its ratio to the luminosity of the Sun?

4. The energy loss due to gravitational wave emission causes orbiting black holes to spiral towards each other, changing the orbital frequency over time. Assume the orbit is always approximately circular.

[1] Technically, the exact answer does not contain the moment of inertia, but a more complex object called the reduced quadrupole moment. However, the two are close enough for the rough estimates in this problem.

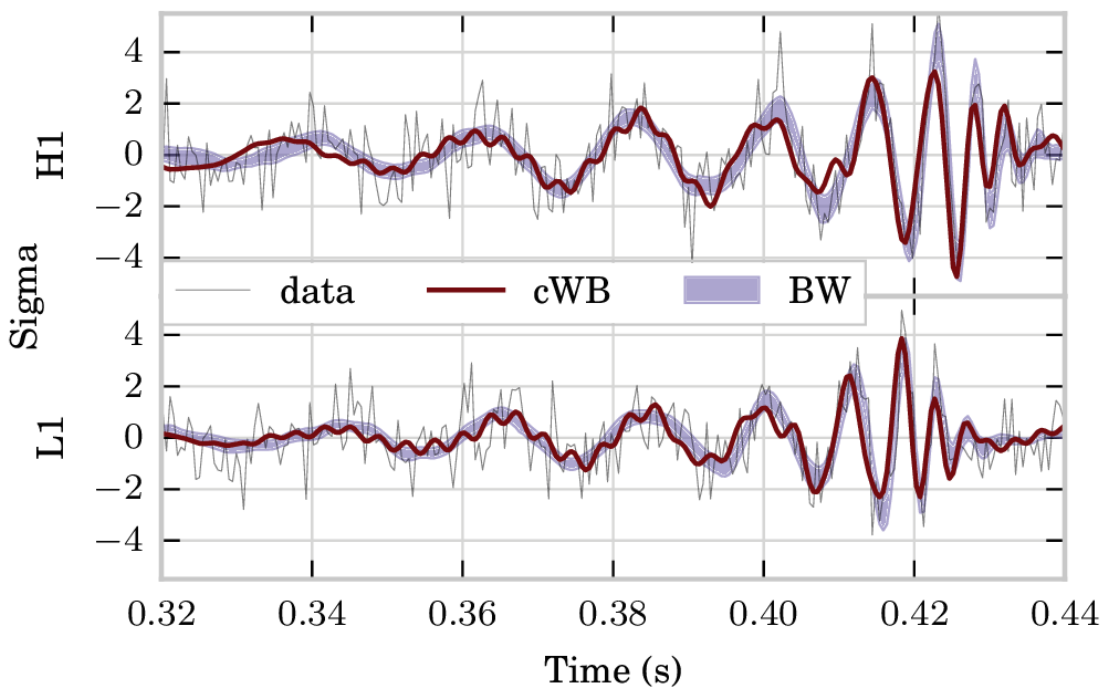
(a) Assuming the energy loss is slow, find the rate of change of the orbital frequency df/dt in terms of f, M_1 and M_2. You should find your answer is simply expressed in terms of the "chirp mass" M_c, defined as

$$M_c = \left(\frac{M_1^3 M_2^3}{M_1 + M_2}\right)^{1/5}.$$

Here you are expected to keep numeric prefactors. In particular, according to general relativity, the correct numeric prefactor to part 3(c) is 32/5.

(b) What is the frequency f_g of the gravitational waves emitted when the orbital frequency is f?

(c) The Hanford, Washington and Livingston, Louisiana LIGO detectors observed a binary black hole merger event on September 14, 2015. Their data is shown in the graphs marked H1 and L1. Use the smoothed (shaded) H1 data to answer the questions below. No detailed data analysis is expected.



Graph downloaded from LIGO Open Science Center, operated by California Institute of Technology and Massachussets Institute of Technology and supported by the U. S. National Science Foundation: losc.ligo.org

i. Estimate the maximum gravitational wave frequency, and thereby estimate the total mass M, giving your answer as a multiple of the solar mass $M_\odot$.

ii. Estimate the chirp mass M_c, giving your answer as a multiple of the solar mass $M_\odot$.

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Take Five

The five parts of this question are unrelated.

1. Consider a partially polarized light beam, containing a mix of unpolarized and linearly polarized light. The intensity of the beam is analyzed using a linear polarizer. At a particular orientation of the polarizer, the outgoing beam has maximum intensity $I_{\max}$. Turning the polarizer by a 30° angle reduces the outgoing beam's intensity by 10%. Find the degree of polarization of the beam,

$$V \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

where $I_{\min}$ is the minimum intensity for any orientation of the polarizer.

Solution

The intensity of partially polarized light that is passed through a linear polarizer is

$$\frac{I_u}{2} + I_p \cos^2\theta,$$

where I_u is the intensity of the unpolarized light, I_p is the intensity of the polarized light, and θ is the angle between the polarization of the polarized light and the linear polarizer. The second term follows directly from Malus' law, and the first term is an average of Malus' law over all angles.

The maximum intensity occurs when $\theta = 0°$ or $180°$ and is given by

$$I_{\max} = \frac{I_u}{2} + I_p.$$

Per the problem statement, a 30° rotation reduces the intensity by 10%, so

$$0.9\left(\frac{I_u}{2} + I_p\right) = \frac{I_u}{2} + I_p\cos^2 30° = \frac{I_u}{2} + \frac{3I_p}{4}.$$

Solving for I_u gives us $I_u = 3I_p$. Thus, we have

$$I_{\max} = \frac{5}{2}I_p, \quad I_{\min} = \frac{3}{2}I_p$$

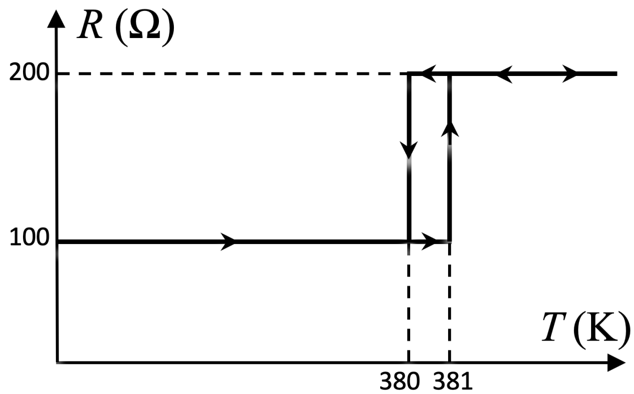
where the latter occurs at $\theta = \pm 90°$, so $\boxed{V = 1/4}$.

2. According to Newton's law of cooling, a hot object transfers heat to the environment at a rate

$$P = k(T_o - T_l),$$

where T_o is the temperature of the object, T_l is the temperature of the environment, and k is a constant.

Consider a circuit element whose resistance R depends on its temperature T as shown below (not to scale), with heat capacity $C = 2$ J/K in a lab of temperature $T_l = 270$ K. Note that the curve is multivalued, which indicates hysteresis: the resistance takes the lower value when increasing from low temperatures, and the higher value when decreasing from high temperatures.



When this component is placed in series with a voltage $V_1 = 50$ V, its temperature stabilizes at $T_1 = 350$ K. When it is instead placed in series with a voltage $V_2 = 70$ V, its temperature does not stabilize, and the current through it instead oscillates. Find the period of these oscillations.

Solution

At 50 V voltage, incoming and lost power are equal,

$$\frac{V_1^2}{R_{\min}} = k(T_1 - T_l)$$

which implies

$$k = \frac{V_1^2}{R_{\min}(T_1 - T_l)} = 0.3125\,\frac{\mathrm{W}}{\mathrm{K}}.$$

The current oscillates because the power heating up the component is larger than $k(T_r - T_l)$ for $R = 100$ ohms and smaller than $k(T_r - T_l)$ for $R = 200$ ohms. So the resistance of the component performs the cycle depicted in the diagram.

In the bottom leg of the diagram, the temperature increases from 380 K to 381 K in time τ_1. The power of dissipation almost does not change and can be approximated as $k(T_{\mathrm{av}} - T_l)$, where $T_{\mathrm{av}} \approx 380.5$ K. Balance of incoming and outgoing powers allows to evaluate τ_1:

$$C\frac{\mathrm{d}}{\mathrm{d}t}T = \frac{V_2^2}{R} - k(T_{\mathrm{ave}} - T_l)$$
$$C(T_{\max} - T_{\min}) = \left(\frac{V_2^2}{R_{\min}} - k(T_{\mathrm{av}} - T_l)\right)\tau_1$$

Similarly on the way back,

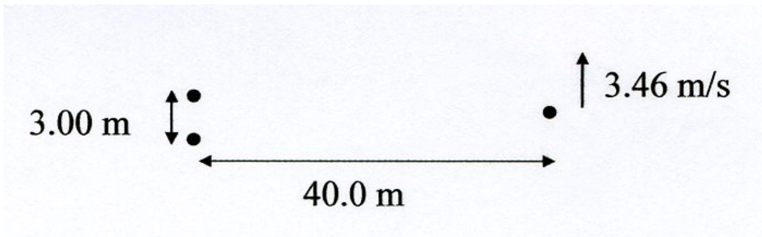
$$C(T_{\min} - T_{\max}) = \left(\frac{V_2^2}{R_{\max}} - k(T_{\mathrm{av}} - T_l)\right)\tau_2$$

The overall period of the oscillations τ is thus

$$\tau = \tau_1 + \tau_2 = C(T_{\max} - T_{\min})\left(\frac{1}{\frac{V_2^2}{R_{\min}} - k(T_{\mathrm{av}} - T_l)} + \frac{1}{k(T_{\mathrm{av}} - T_l) - \frac{V_2^2}{R_{\max}}}\right) = \boxed{0.34\ \mathrm{s}}.$$

To get an exact answer, you could solve the differential equation replacing T_{av} with T, but this would give the same result, to the number of significant figures used in the problem.

3. Two speakers are 3.00 m apart. They both emit perfect sinusoids, whose frequencies differ by 0.250 Hz. Spaceman Fred, who is standing 40.0 m away in the direction shown in the diagram, must run at 3.46 m/s to avoid hearing beats. The speed of sound in air is 343 m/s. Approximately what frequency are the speakers emitting?



Solution

Two sources of identical frequency with a phase difference $2\pi\delta$ set up an interference pattern with maxima given for small angles by $(n-\delta)\lambda = a\theta$ (where a is the source spacing). Here, we treat the problem as two sources of the same frequency with a slowly varying phase difference, where

$$\frac{d\delta}{dt} = \Delta f = 0.250\,\text{Hz}.$$

Thus the interference pattern shifts slowly; for a given maximum, *i.e.* a fixed n,

$$\lambda\frac{d\delta}{dt} = a\frac{d\theta}{dt}$$

Since Fred is at a position given approximately by $L\theta$ (where $L = 40.0\,\text{m}$), to match the movement of the interference pattern he must run at $v_0 = L\frac{d\theta}{dt}$. Putting the equations together,

$$\lambda = \frac{av_0}{L\Delta f} = 1.04\,\text{m}$$

and $f = c/\lambda = \boxed{330\text{ Hz}}$, where c is the speed of sound.

Alternative solution: This problem can be solved using Doppler shift. The frequency f_1 of the first source is perceived as

$$f_1' = \frac{c + v_{F,\parallel}}{c} f_1,$$

where $v_{F,\parallel}$ is the component of Fred's velocity parallel to his displacement with the first source (which is roughly $v_0\theta$, where $\theta \approx a/(2L)$). So,

$$f_1' \approx \frac{c + v_0\theta}{c} f_1.$$

Likewise,

$$f_2' \approx \frac{c - v_0\theta}{c} f_2.$$

(Note that when Fred moves a distance $d \ll L$, the individual angles may be different, but the difference between the angles is still 2θ by the small angle approximation.)

These two frequencies must equal each other for Fred to hear zero beats, so

$$(c + v_0\theta)f_1 = (c - v_0\theta)f_2.$$

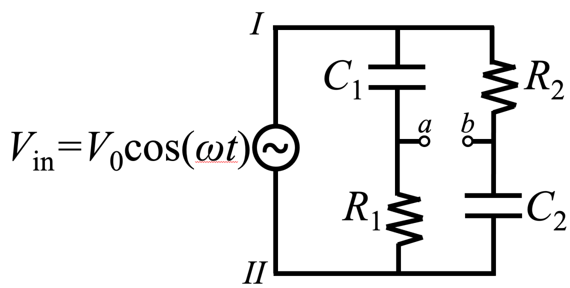
Writing $f_1 = f$ and $f_2 = f + \Delta f$, we arrive at

$$v_0\theta f \approx c\Delta f - v_0\theta f.$$

Then,

$$f \approx \frac{c\Delta f}{2\theta v_0} = \frac{cL\Delta f}{a v_0} = \boxed{330\text{ Hz}}.$$

4. The input voltage is the voltage difference between points I and II. What is the voltage difference between the points a and b in the circuit below as a function of time?



You should assume $R_1C_1 = R_2C_2$, and simplify your answer as much as possible.

Solution

We use the method of complex impedance, and write the EMF as $V = \text{Re}\left\{\tilde{V}\right\}$, where $\tilde{V} = V_0 e^{i\omega t}$.

Then, the voltage with respect to ground at points a and b are computed by the typical method of resistors in series but with complex impedances:

$$\tilde{V}_a = \frac{R_1}{R_1 + \frac{1}{i\omega C_1}}\tilde{V}, \quad \tilde{V}_b = \frac{\frac{1}{i\omega C_2}}{R_2 + \frac{1}{i\omega C_2}}\tilde{V}.$$

We define $\alpha = \omega R_1 C_1 = \omega R_2 C_2$, so that

$$\tilde{V}_a = \frac{i\alpha}{1 + i\alpha}\tilde{V}, \quad \tilde{V}_b = \frac{1}{1 + i\alpha}\tilde{V}.$$

Subtracting the two gives

$$\tilde{V}_a - \tilde{V}_b = \frac{i\alpha - 1}{i\alpha + 1}\tilde{V}.$$

We finally need to take the real part of this expression. The magnitude of the expression is just

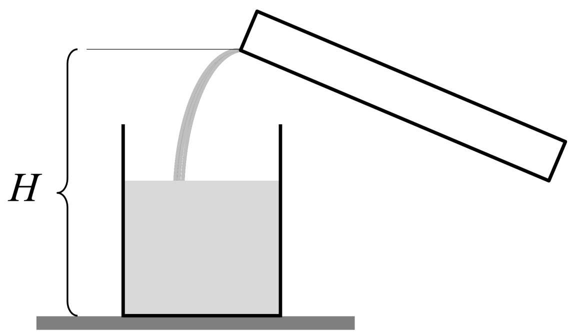
V_0 because $i\alpha$ is purely imaginary. The phase can be computed as follows by noting that

$$\frac{i\alpha - 1}{i\alpha + 1} = \frac{(i\alpha - 1)((1 - i\alpha)}{1 + |\alpha|^2} = \frac{-1 + \alpha^2}{1 + \alpha^2} + \frac{2i\alpha}{1 + \alpha^2} = -e^{-i\phi},$$

where $\phi = \arctan\left(\frac{2\omega RC}{1-\omega^2 R^2 C^2}\right) = 2\arctan(\omega RC)$. Then,

$$V_a - V_b = \boxed{V_0 \cos(\omega t - 2\arctan(\omega RC) + \pi)}.$$

5. An empty cylindrical glass of cross-sectional area A is resting on a table. Water of density ρ is slowly poured into the glass from a beaker, at a constant volume per unit time Q.



The beaker nozzle is at height H, and the water exits the beaker with negligible speed. Let $t = 0$ at the moment the water first hits the bottom of the glass. Find the force of the water on the glass as a function of time, until the glass overflows. Assume the water does not splash and the atmospheric pressure is P_0. Furthermore, assume the glass is wide, so that the rate at which the water level rises is negligible compared to the speed at which water enters the glass.

Solution

The water in the glass is acted on by four forces: its weight, the normal force from the glass, atmospheric pressure $P_0 A$, and the impact force from the poured water. The problem asks for the normal force from the glass.

The weight is $\rho g V$, where V is the volume of water in the glass. Since we have assumed the water level rises slowly, the volume increases at the same rate Q that water pours out of the beaker, so $W = \rho g Q t$.

The impact force comes from the change of momentum of the falling jet of water. Then,

$$F_j = \frac{\mathrm{d}p}{\mathrm{d}t} \approx \frac{\delta m \Delta v}{\delta t} = \rho Q \Delta v,$$

where δm is the mass of water that enters the beaker in time δt, and Δv is the change of velocity of the water entering the beaker. From elementary kinematics, $\Delta v = \sqrt{2g(H - h)}$, where $h = Qt/A$ is the water level in the beaker. Then,

$$F_j = \rho Q \sqrt{2g\left(H - \frac{Qt}{A}\right)}.$$

The total force is

$$F = F_j + W + P_0 A = \boxed{Q\rho\left[\sqrt{2g\left(H - \frac{Q}{A}t\right)} + gt\right] + P_0 A}.$$

Chain Reaction

The three parts of this question are unrelated.

1. A frictionless circular cylinder of radius R is placed with its axis horizontal, and a flexible, inextensible string of uniform linear mass density λ is wrapped around it. When the length of the string is slightly longer than $2\pi R$, part of the string will sag below the cylinder. Now suppose the string is slowly shortened, until the entire string just touches the cylinder. At this moment, find the tension at the top of the string.

Solution

First, let's solve for the tension T_0 at the bottom of the string. By considering force balance on a small segment of angle $d\theta$ at the bottom, we have

$$\lambda R g \, d\theta = T_0 \, d\theta$$

which implies $T_0 = \lambda R g$.

Next, we need to find the tension at the top of the string. Again consider force balance on a small segment of angle $d\theta$, where the segment is at an angle θ from the top. The tangential component of the gravitational force is balanced by the difference in tension across the segment, so

$$-\lambda R g \sin\theta \, d\theta = dT.$$

Thus, the tension at the top is

$$T = T_0 + \int_{\pi}^{0} \frac{dT}{d\theta} = T_0 + \int_0^{\pi} \lambda R g \sin\theta \, d\theta = \boxed{3\lambda R g}$$

2. Model a grappling hook as a point mass m attached to the end of a uniform chain of linear mass density λ. Initially, the chain is loosely coiled on the ground. Then the mass is launched directly upward from the ground, with an initial speed v_0. The chain is flexible, so that when the mass is at a height y, a length y of the chain dangles directly beneath it, while the rest of the chain remains at rest on the ground. Find the maximum height reached by the mass, assuming this is less than the length of the chain. (Hint: if you directly compute the acceleration, you will find an intractable differential equation, but it can be solved with a clever change of variable.)

Solution

Let h be the final height. A naive application of energy conservation would yield

$$\frac{1}{2} m v_0^2 = mgh + \frac{\lambda g h^2}{2}$$

which yields

$$h = \frac{m}{\lambda} \left(\sqrt{1 + \frac{\lambda v_0^2}{mg}} - 1 \right).$$

However, this is incorrect, as the raising of the chain from the ground is an inelastic process, which dissipates energy. Instead, we consider forces. The only external forces on the entire grappling hook are gravity and the normal force from the ground. The normal force is precisely enough to support the part of the chain lying on the ground, so the net force is

$$F = -(m + \lambda y)g.$$

This is equal to the rate of change of momentum,

$$F = \frac{dp}{dt}, \quad p = (m + \lambda y)\dot{y}.$$

If you expand this out, you'll get an intractable nonlinear second-order differential equation. The trick is to consider the momentum as a function of height. We have

$$\frac{dp}{dy} = \frac{dp}{dt}\frac{dt}{dy} = -(m + \lambda y)g\,\frac{1}{\dot{y}} = -\frac{(m + \lambda y)^2 g}{p}.$$

Separating and integrating gives

$$\int p\,dp = -\int (m + \lambda y)^2 g\,dy$$

which yields

$$-\frac{(mv_0)^2}{2} = -\frac{((m + \lambda h)^3 - m^3)g}{3\lambda}$$

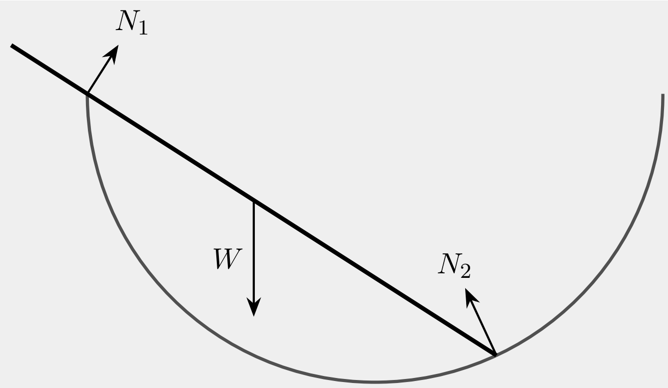
since the final momentum is zero. Solving for h gives

$$\boxed{h = \frac{m}{\lambda}\left(\sqrt[3]{1 + \frac{3\lambda v_0^2}{2mg}} - 1\right)}$$

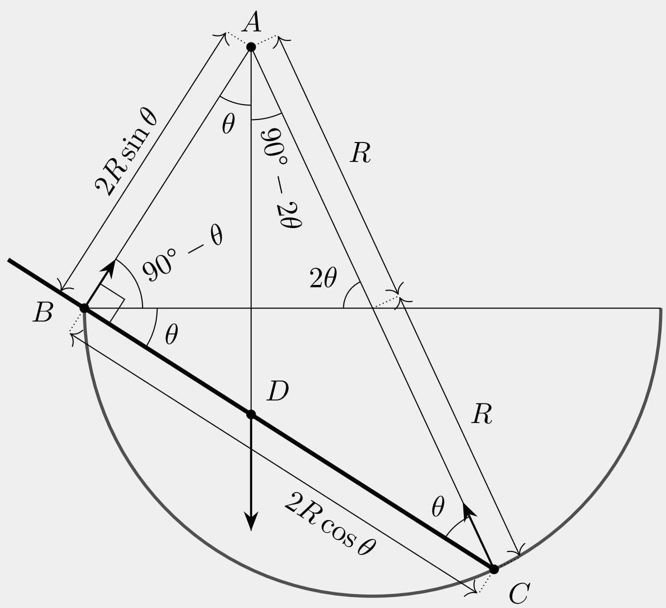
3. A uniform rod of length $2R$ is placed inside a fixed, frictionless hemispherical bowl of radius R. In equilibrium, the rod makes an angle θ with the horizontal. Assume that the rod and bowl are ideally rigid, but that the lip of the bowl and the end of the rod are both slightly rounded, so that there is a well-defined normal direction at the points they touch. Find an analytic expression for θ and evaluate it to three significant figures, in degrees.

Solution

First solution: Gravity acts downward at the midpoint of the rod, while the reaction forces are in the directions shown (perpendicular to the rod, and perpendicular to the bowl).



Now, we use the fact that when an object in static equilibrium is acted on by three forces, the lines of the three forces must intersect. (Otherwise, torque balance about the intersection point of the lines of any two of the forces couldn't be obeyed.) This turns the problem into one of pure Euclidean geometry. After some angle chasing, we arrive at the below diagram, where we found the lengths $\overline{AB}$ and $\overline{BC}$ by applying trigonometry to the right triangle ABC.



Now, by considering the right triangle ABD, we find

$$\tan\theta = \frac{\overline{BD}}{\overline{AB}} = \frac{2R\cos\theta - R}{2R\sin\theta}.$$

Simplifying this gives

$$\cos\theta = 2\cos 2\theta$$

which is equivalent to

$$4\cos^2\theta - \cos\theta - 2 = 0.$$

Applying the quadratic equation, we conclude

$$\boxed{\theta = \cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right) = 32.5^\circ}$$

This is actually a classic problem, but it usually isn't stated correctly: if you don't explicitly specify how the end of the rod and the lip of the bowl are shaped, as we did, then there aren't well-defined normal directions. The answer would then depend on how the rod and bowl deform, which in turn depends sensitively on their dimensions, and what they're made of.

Second solution: We balance forces perpendicular to the rod, forces parallel to the rod, and torques. Let W be the weight of the rod. We use the fact that N_2 makes an angle θ with the rod, and that the length $\overline{BC}$ of the rod in the bowl is $2R\cos\theta$. Force balance parallel to the rod gives

$$N_2 \cos\theta = W \sin\theta.$$

Force balance perpendicular to the rod gives

$$N_1 + N_2 \sin\theta = W\cos\theta.$$

Torque balance about the bottom end of the rod gives

$$2RN_1\cos\theta = RW\cos\theta.$$

Solving for N_1 and N_2 in the first and third equations, and plugging them into the second gives

$$\frac{1}{2} + \sin\theta\tan\theta = \cos\theta.$$

Multiplying both sides by $\cos\theta$ and rearranging gives us

$$\frac{1}{2}\cos\theta = \cos^2\theta - \sin^2\theta \implies \cos\theta = 2\cos 2\theta.$$

This can be solved in the same way as above.

Third solution: We minimize the energy of the rod. Defining the height to be $y = 0$ at the rim of the bowl, the bottom end of the rod is at height

$$-2R\cos\theta\sin\theta$$

where we again used the fact that the length of the rod in the bowl is $\overline{BC} = 2R\cos\theta$. The top end of the rod is at height

$$2R(1-\cos\theta)\sin\theta.$$

Averaging the two, the center of mass is at height

$$R(\sin\theta - 2\sin\theta\cos\theta) = R(\sin\theta - \sin 2\theta).$$

Setting the derivative to zero gives

$$\cos\theta = 2\cos 2\theta$$

which can again be solved in the same way as above.

Lego Movie

Gravitational waves are predicted by general relativity, but can be modeled with Newtonian physics and a few small assumptions. Throughout this problem, assume classical Newtonian physics, and ignore special relativistic effects. The mass of the sun is $M_\odot = 2.0 \times 10^{30}$ kg, and the luminosity of the sun is $L_\odot = 3.8 \times 10^{26}$ W. In all parts of this problem, you may use the fundamental constants c and G in your answers.

1. Consider a spherically symmetric body with mass M. Determine the Schwarzschild radius R_s of such a body so that the escape velocity would be equal to the speed of light c.

Solution

From conservation of energy,

$$R_s = \frac{2GM}{c^2}.$$

2. Two such bodies, with masses M_1 and M_2, are in circular orbits about their common center of mass. The separation of the bodies is R, and the total mass is $M = M_1 + M_2$.

(a) Find the frequency f of the orbital motion in terms of M and R.

Solution

Since the gravitational force is the centripetal force,

$$M_1\omega^2 R_1 = \frac{GM_1M_2}{R^2}, \quad M_2\omega^2 R_2 = \frac{GM_1M_2}{R^2}.$$

Adding these two expressions gives

$$\omega^2 R = \frac{GM}{R^2}.$$

Solving for the frequency gives

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{GM}{R^3}}.$$

(b) Find the total energy E of the system in terms of M_1, M_2, and R.

Solution

The potential energy is

$$U = -\frac{GM_1M_2}{R}.$$

The kinetic energy is

$$T = \frac{1}{2}\left(M_1R_1^2 + M_2R_2^2\right)\omega^2.$$

After some algebra, which is rather similar to that used in part 3(b) below, we have

$$T = \frac{1}{2}\left(\frac{M_1M_2}{M}R^2\right)\frac{GM}{R^3} = \frac{GM_1M_2}{2R}.$$

Thus, we have

$$E = -\frac{1}{2}\frac{GM_1M_2}{R}.$$

Alternatively, you could have skipped straight to this result by invoking the virial theorem.

(c) The minimum possible orbital separation is $R_{\text{min}} = R_1 + R_2$, where R_1 and R_2 are the Schwarzschild radii for masses 1 and 2. Find the maximum possible orbital frequency f_{max} in terms of M.

Solution

In terms of M, R_{min} is given by

$$R_{\text{min}} = \frac{2G}{c^2}(M_1 + M_2) = \frac{2GM}{c^2}.$$

Plugging this into the answer to part 2(a) gives

$$f_{\text{max}} = \frac{\sqrt{2}\,c^3}{8\pi GM}.$$

3. We would like to estimate the rate at which the system loses energy due to the emission of gravitational waves. In classical electromagnetism, the simplest form of radiation is dipole radiation, which results from a second time derivative of the electric dipole moment. However, for gravity the analogue of the electric dipole moment is the center of mass, which always moves at constant velocity by momentum conservation. Thus, the leading source of gravitational radiation is quadrupole radiation, which depends on a time derivative of the moment of inertia. All subparts of this part are rough estimates, which means you may drop numeric prefactors such as π.

(a) The power radiated in gravitational waves by a system with moment of inertia I takes the form[1]

$$P = kG^\alpha c^\beta \left(\frac{d^n I}{dt^n}\right)^2$$

where k is a dimensionless constant. Determine α, β, and n.

Solution

This is a dimensional analysis problem, and we have

$$[G] = [\text{L}]^3/[\text{M}][\text{T}]^2, \quad [c] = [\text{L}]/[\text{T}], \quad [I] = [\text{M}][\text{L}]^2.$$

The only way to get the dimensions to match is $\alpha = 1$, $\beta = -5$, and $n = 3$, giving

$$P = k\,\frac{G}{c^5}\left(\frac{d^3 I}{dt^3}\right)^2.$$

(b) For two black holes circularly orbiting each other in the xy plane, with center of mass at the origin, find the moment of inertia $I_y(t)$ about the y-axis in terms of M_1, M_2, R, and the angular frequency ω, defining the origin of time so that $I_y(0) = 0$.

[1] Technically, the exact answer does not contain the moment of inertia, but a more complex object called the reduced quadrupole moment. However, the two are close enough for the rough estimates in this problem.

Solution

Letting the distances of the black holes to the y-axis be x_1 and x_2, we have

$$I_y = M_1 x_1^2 + M_2 x_2^2, \quad x_1 + x_2 = x.$$

Since the center of mass is at the origin,

$$M_1 x_1 = M_2 x_2$$

which implies

$$x_1 = \frac{M_2}{M_1 + M_2}\, x, \quad x_2 = \frac{M_1}{M_1 + M_2}\, x.$$

Thus, we have

$$I_y = \frac{M_1 M_2}{M_1 + M_2} x^2.$$

Defining the origin of time as requested, we must have $x(t) = R\sin(\omega t)$, giving

$$I_y(t) = \frac{M_1 M_2}{M_1 + M_2}\, R^2 \sin^2(\omega t).$$

(c) Roughly estimate the average power radiated over an orbital period. Your final answer should be a product of powers of M_1, M_2, M, R, and fundamental constants.

Solution

The moment of inertia has period $2f$, which means that, neglecting numeric factors, every time derivative of it yields a factor of ω. Therefore,

$$\frac{d^3 I}{dt^3} \sim \omega^3 I \sim \frac{M_1 M_2}{M_1 + M_2}\, \omega^3 R^2.$$

Combining this with the results of parts 3(a) and 2(a) gives

$$P \sim \frac{G(M_1 M_2)^2 R^4}{c^5 (M_1 + M_2)^2} \omega^6 \sim \frac{G^4}{c^5} \frac{(M_1 + M_2)(M_1 M_2)^2}{R^5}.$$

(d) The maximum power radiated occurs when $R = R_{\text{min}}$. Roughly estimate the maximum power in the case $M_1 = M_2$ in terms of fundamental constants. What is the order of magnitude of its ratio to the luminosity of the Sun?

Solution

The result is quite simple, as the mass drops out,

$$P \sim \frac{G^4 M_1^5}{c^5 R_1^5} \sim \frac{c^5}{G}.$$

When we plug in the numbers, we find

$$P \sim 10^{26} L_\odot.$$

This is an enormous power, comparable to the output of all the stars in the observable universe!

4. The energy loss due to gravitational wave emission causes orbiting black holes to spiral towards each other, changing the orbital frequency over time. Assume the orbit is always approximately circular.

   (a) Assuming the energy loss is slow, find the rate of change of the orbital frequency df/dt in terms of f, M_1 and M_2. You should find your answer is simply expressed in terms of the "chirp mass" M_c, defined as

$$M_c = \left(\frac{M_1^3 M_2^3}{M_1 + M_2} \right)^{1/5}.$$

   Here you are expected to keep numeric prefactors. In particular, according to general relativity, the correct numeric prefactor to part 3(c) is 32/5.

Solution

Combining our results for 2(a) and 2(b), the energy as a function of angular frequency is

$$E = -\frac{1}{2} \frac{GM_1M_2}{(GM)^{1/3}} \left(\frac{GM}{R^3} \right)^{1/3} = -\frac{1}{2} \frac{GM_1M_2}{(GM)^{1/3}} \omega^{2/3}.$$

Taking the time derivative gives

$$\frac{dE}{dt} = -\frac{1}{3} \frac{GM_1M_2}{(GM)^{1/3}} \omega^{-1/3} \frac{d\omega}{dt}.$$

On the other hand, we have $P = -dE/dt$, where we've been told that

$$P = \frac{32}{5} \frac{G^4}{c^5} \frac{(M_1 + M_2)(M_1M_2)^2}{R^5}.$$

Combining these results, simplifying, and eliminating R gives

$$\frac{d\omega}{dt} = \frac{96}{5} \frac{G^{5/3}}{c^5} \frac{M_1M_2}{M^{1/3}} \omega^{11/3} = \frac{96}{5} \frac{(GM_c)^{5/3}}{c^5} \omega^{11/3}.$$

Converting from angular frequency to frequency gives

$$\frac{df}{dt} = \frac{96\,(2\pi)^{8/3}}{5} \frac{(GM_c)^{5/3}}{c^5} f^{11/3}.$$

   (b) What is the frequency f_g of the gravitational waves emitted when the orbital frequency is f?

Solution

Note that the power in a wave scales with the amplitude squared, so by part 3(a), the amplitude must scale with d^3I/dt^3. From part 3(b), we have

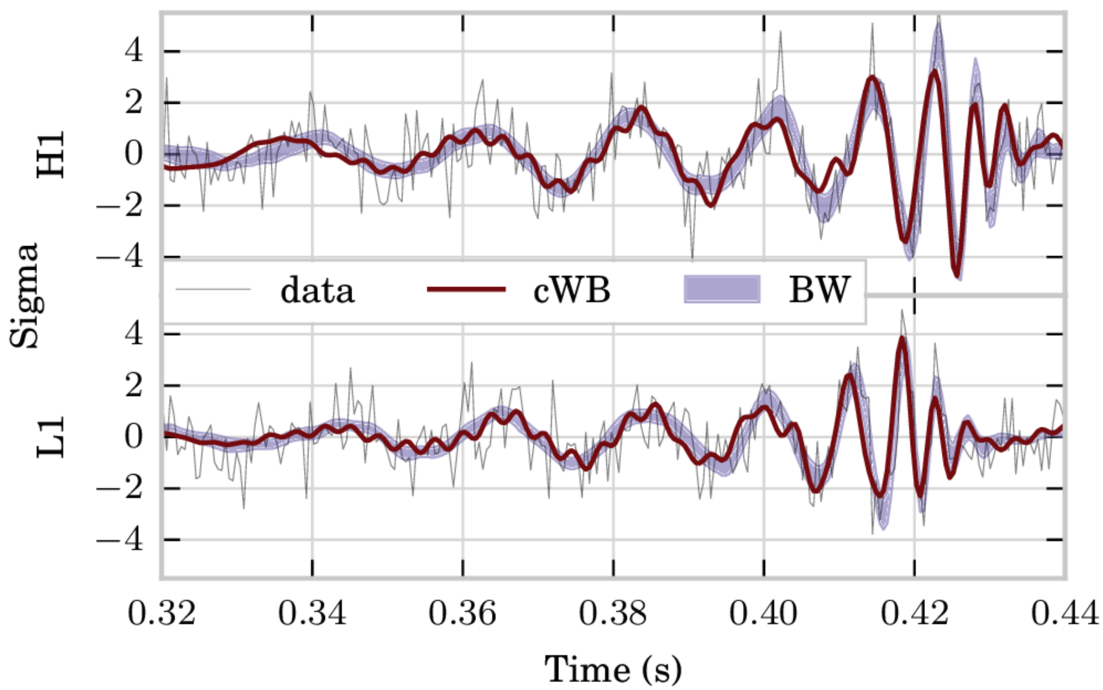
$$I \propto x^2 \propto \cos^2(\omega t) \propto 1 + \cos(2\omega t)$$

which implies that

$$\frac{d^3 I}{dt^3} \propto \sin(2\omega t).$$

The factor of 2 here implies that the gravitational waves have frequency $f_g = 2f$.

(c) The Hanford, Washington and Livingston, Louisiana LIGO detectors observed a binary black hole merger event on September 14, 2015. Their data is shown in the graphs marked H1 and L1. Use the smoothed (shaded) H1 data to answer the questions below. No detailed data analysis is expected.



Graph downloaded from LIGO Open Science Center, operated by California Institute of Technology and Massachussets Institute of Technology and supported by the U. S. National Science Foundation: losc.ligo.org

i. Estimate the maximum gravitational wave frequency, and thereby estimate the total mass M, giving your answer as a multiple of the solar mass $M_\odot$.

Solution

By looking at the period starting near $t = 0.42\,\text{s}$, we estimate a maximum gravitational wave frequency 150 Hz. By part 4(b), this implies a maximum orbital frequency 75 Hz. From part 2(c) we have

$$M = \frac{\sqrt{2}c^3}{8\pi G f_{\max}} = 150\, M_\odot.$$

Any result within 50% is acceptable.

The actual result reported by LIGO is roughly $(70 \pm 5)\, M_\odot$. Our result is of the right order of magnitude, but it's still far off because our expression for the maximum orbital frequency is itself a rough approximation. It's possible to extract M from the final stages of the merger, but it requires something more sophisticated than what we've done.

ii. Estimate the chirp mass M_c, giving your answer as a multiple of the solar mass $M_\odot$.

Solution

Solving the result of part 4(a) for M_c and reexpressing the result in terms of f_g gives

$$M_c = \frac{c^3}{G}\left(\frac{5}{96\,\pi^{8/3}}\,f_g^{-11/3}\,\frac{df_g}{dt}\right)^{3/5}.$$

We need to read off f_g and df_g/dt from the diagram. We shouldn't look at the end of the merger process, because there df_g/dt is too large, while the above formula was derived assuming df_g/dt was small. But at the beginning, df_g/dt is tiny, and thus hard to measure precisely. The best data comes from the middle.

For example, there are minima at $t = 0.373\,\mathrm{s}$, $0.392\,\mathrm{s}$, $0.408\,\mathrm{s}$. Considering these two time intervals gives frequencies $53\,\mathrm{Hz}$, $63\,\mathrm{Hz}$. We thus take

$$f_g = 58\,\mathrm{Hz}, \qquad \frac{df_g}{dt} = \frac{10\,\mathrm{Hz}}{17\,\mathrm{ms}} = 600\,\mathrm{Hz/s}.$$

Plugging these numbers in gives

$$M_c = 34\,M_\odot.$$

Any result within 50% is acceptable.

The result reported by LIGO was roughly $(30.2 \pm 2.0)\,M_\odot$, so our analysis is decent.

AAPT UNITED STATES PHYSICS TEAM
AIP 2023

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination has two parts. Each part has three questions and lasts for 90 minutes.
- For each student, print out one copy of the exam and one copy of the answer sheets. Print everything single-sided, and do not staple anything. Divide the exam into the instructions (pages 2–3), Part A questions (pages 4–8), and Part B questions (pages 9–11).
- Begin by giving students the instructions and all of the answer sheets. Let the students read the instructions and fill out their information on the answer sheets. They can keep the instructions for both parts of the exam. Also give students blank sheets of paper to use as scratch paper throughout the exam.
- Students may bring calculators, but they may not use symbolic math, programming, or graphing features of these calculators. Calculators may not be shared, and their memory must be cleared of data and programs. Cell phones or other electronics may not be used during the exam or while the exam papers are present. Students may not use books or other references.
- To start the exam, give students the Part A questions, and allow 90 minutes to complete Part A. Do not give students Part B during this time, even if they finish with time remaining. At the end of the 90 minutes, collect the Part A questions and answer sheets.
- Then give students a 5 to 10 minute break. Then give them the Part B questions, and allow 90 minutes to complete Part B. Do not let students go back to Part A.
- At the end of the exam, collect everything, including the questions, the instructions, the answer sheets, and the scratch paper. Students may *not* keep the exam questions. Everything can be returned to the students after April 19th, 2023.
- After the exam, sort each student's answer sheets by page number. Scan every answer sheet, including blank ones.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Tengiz Bibilashvili, Kellan Colburn, Samuel Gebretsadkan, Abi Krishnan, Natalie LeBaron, Kye Shi, Brian Skinner, and Kevin Zhou.

AAPT UNITED STATES PHYSICS TEAM
AIP 2023

USA Physics Olympiad Exam

Instructions for the Student

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Reference table of possibly useful information

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Possibly useful integrals

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Part A

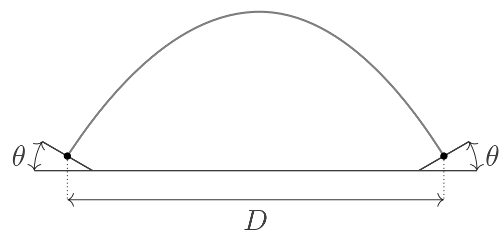
Question A1

Circus Act

In this problem we consider a small ball bouncing back and forth between two points. In all parts below, the acceleration of gravity is g, collisions are perfectly elastic, air resistance is negligible, and the impact points are at the same height. The diagrams are not drawn to scale.

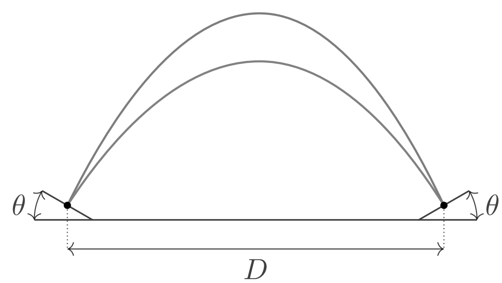
a. Consider a ball bouncing between two inclined planes, which each make an angle $\theta < 90°$ to the horizontal. The ball has speed v_0 at the impact points, which are separated by a distance D.

   i. The ball can bounce back and forth along the same path, as shown.



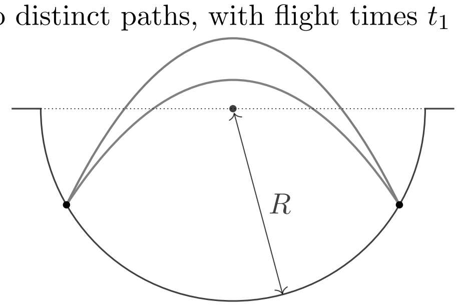
   For what values of θ is this motion possible? For these values, what is v_0?

   ii. The ball can also take one trajectory while traveling to the right, and a separate trajectory when traveling back. Let $\phi \neq 0$ be the angle between the paths at the impact points.



   For what values of θ and ϕ is this motion possible? For these values, what is v_0?

b. Now suppose the ball bounces within a hemispherical well of radius of curvature R. As in part a.ii, it alternates between two distinct paths, with flight times t_1 and $t_2 \neq t_1$.



   Find all of the possible values of R, in terms of t_1 and t_2.

c. Finally, suppose the well has a sinusoidal shape, described by $y(x) = -L\sin\big(2x/L\big)$. The ball takes two distinct paths with flight times t_1 and $t_2 \neq t_1$, and the horizontal distance between the impact points is less than πL. Find all of the possible values of L, in terms of t_1 and t_2.

Question A2

Time is a Flat Circle

A particle of mass m and negative charge $-q$ is constrained to move in a horizontal plane. In the situations described below, this particle can either oscillate back and forth in a straight line or move in a circle. (These two modes of motion are interesting because they generate linearly and circularly polarized radiation, respectively, but in this problem you may ignore any energy lost to radiation.)

a. A large positive charge $Q \gg q$ is fixed in place a distance R directly below the origin of the plane.

   i. When the particle is a distance $r \ll R$ from the origin, find an approximate expression for its potential energy due to the charge Q to second order in r/R, up to an arbitrary constant. You may use this result for the rest of the problem.

   ii. If the particle oscillates linearly with amplitude $a \ll R$, what is its angular frequency ω_ℓ?

   iii. If the particle performs circular motion with radius $r \ll R$, what is its angular frequency ω_c?

b. Now an additional negative charge $-q$ is fixed in place at the origin of the plane.

   i. What is the equilibrium distance r_0 of the particle from the origin?

   ii. If the particle oscillates linearly with amplitude $a \ll r_0$, what is its angular frequency Ω_ℓ? Is it higher or lower than ω_ℓ?

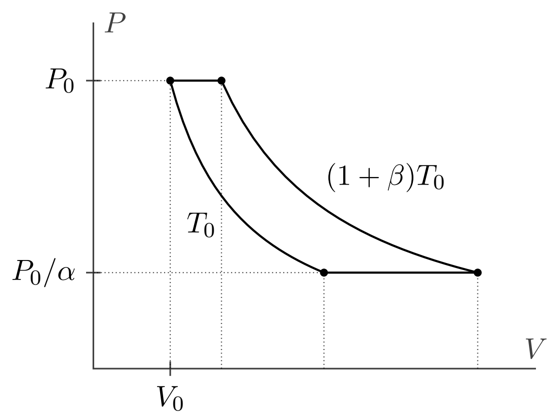
   iii. Now suppose the particle performs circular motion with radius $r = r_0 + \delta r$, where $\delta r \ll r_0$. What is its angular frequency Ω_c, in terms of ω_c, r, and δr? Is it higher or lower than ω_c?

Question A3

The Motive Power of Ice

In the Carnot cycle, a gas is heated at constant temperature T_H and cooled at constant temperature T_C. Furthermore, no other heat transfer occurs, and all other steps of the cycle are reversible. The laws of thermodynamics state that any such cycle must have efficiency $\eta = W/Q_{\text{in}} = 1 - (T_C/T_H)$. Below we will explore two other heat engines, which recover this efficiency in certain limits.

a. Consider the following heat engine involving one mole of ideal monatomic gas. The gas begins at temperature T_0, pressure P_0, and volume V_0, and undergoes four reversible steps.



1. The gas is expanded at constant pressure until its temperature rises to $(1+\beta)T_0$.
2. The gas is expanded at constant temperature until its pressure falls to P_0/α.
3. The gas is contracted at constant pressure until its temperature falls back to T_0.
4. The gas is contracted at constant temperature until its pressure rises back to P_0.

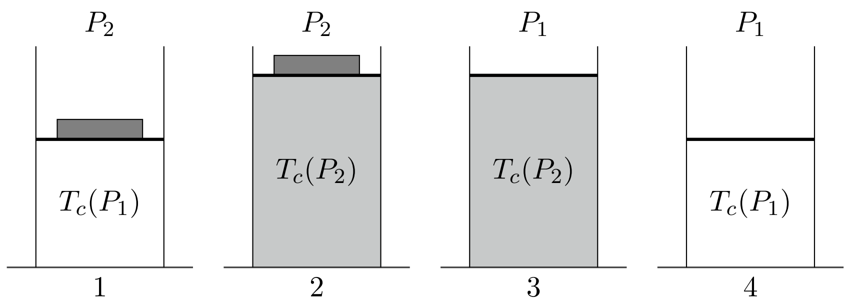
i. Which steps require heat to be transferred to the gas? For each such step, give the total heat input in terms of P_0, V_0, α, and β.

ii. Under what conditions on α and β would we expect the efficiency of this heat engine to approach that of a Carnot cycle working between the same maximum and minimum temperatures?

iii. Find the efficiency of this heat engine for general α and β.

The second half of the problem is on the next page.

b. Now consider a heat engine built around the freezing and melting of water, which occurs at a pressure-dependent temperature $T_c(P)$. Initially, a volume of V of water is squeezed underneath a piston, so that it experiences a total pressure P_1, and the water is on the edge of freezing, with temperature $T_c(P_1)$. The engine then undergoes four reversible steps.



1. A mass is slowly placed on the piston, raising the total pressure to P_2.
2. The water is cooled to temperature $T_c(P_2)$ and frozen.
3. The mass is slowly removed from the piston, lowering the pressure back to P_1.
4. The ice is heated back to temperature $T_c(P_1)$ and melted.

Assume that water and ice are incompressible, with fixed densities ρ_w and ρ_i.

i. What is the net work done by this engine, in terms of P_1, P_2, V, and the densities?

ii. Assume the latent heat per unit mass L to melt ice is large, so that freezing and melting account for essentially all of the heat transfer in the cycle. What is the efficiency of the engine, in terms of P_1, P_2, L, and the densities?

iii. Since we assumed all heat transfer occurs during melting or freezing, this cycle has the same efficiency as a Carnot cycle. In the limit where P_1 and P_2 are very close, use this fact to infer an expression for dT_c/dP in terms of T_c, L, and the densities.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you can review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

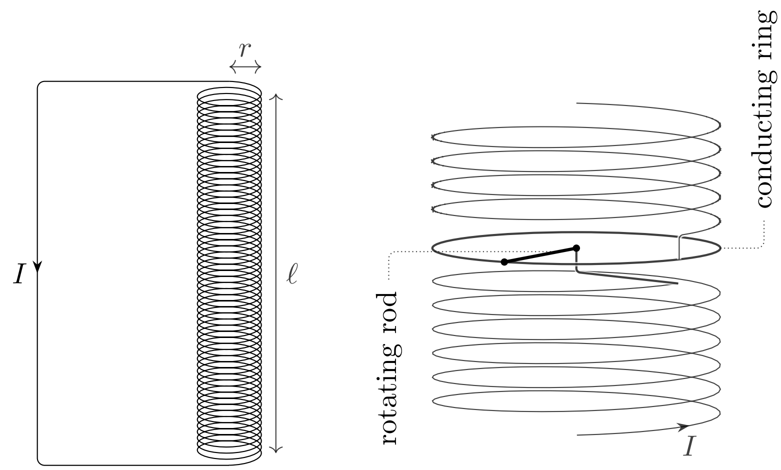
Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

Electric Roulette

Consider a cylindrical solenoid with radius r, length $\ell \gg r$, and n turns per unit length. It is made of one continuous wire, with the top connecting back to the bottom as shown at left.



In the middle of the solenoid, part of the wire is replaced with the assembly shown at right. A uniform conducting rod of mass m and radius r is connected to the bottom half of the solenoid, and is free to rotate about the solenoid's axis of symmetry. The end of the rod slides on a fixed conducting ring, which is attached to the top half of the solenoid. This assembly and the solenoid form one continuous conductor, carrying total current I.

a. What is the inductance of this system? Assume $nr \gg 1$, so that the magnetic field produced by the current in the rod and ring is negligible.

b. When the rod is within a uniform vertical magnetic field B, find the torque it experiences in terms of I, B, and r.

c. If the rod rotates with angular velocity ω, the electrons inside have a tangential velocity. Find the electromotive force across the rod in terms of ω, B, and r.

Now we will consider the dynamics of this system in some simple situations. For all the parts below, neglect energy losses due to friction, resistance, and radiation.

d. First, suppose the system initially carries no current, and the entire system is inside a uniform external magnetic field B_0 parallel to the axis of the solenoid. If the rod is given a small initial angular velocity, its angular velocity will oscillate in time. Find the period of these oscillations.

e. Next, suppose there is no external magnetic field, $B_0 = 0$, and at time $t = 0$, the system carries current I_0 and the rod has zero angular velocity.

  i. The rod's angular velocity $\omega(t)$ approaches a value ω_0 after a long time. What is ω_0?

  ii. Find $\omega(t)/\omega_0$ in terms of ω_0, t, n, and ℓ. You may use the integrals on the reference sheet.

Question B2

Fast and Furious

A space program wants to accelerate a spaceship of final mass $m = 100\,\text{kg}$ to relativistic speeds to observe distant stars. They have two proposals to evaluate.

a. Their first proposal is to use traditional rocket propulsion. A rocket of initial mass m_0 and final mass m that expels propellant with exhaust speed u relative to the rocket will reach a speed

$$v = u \ln\left(\frac{m_0}{m}\right).$$

Suppose the desired final speed is $v_f = 3c/5$. In the subparts below, neglect relativistic effects and give your answers in the form 10^n, where n has at least two significant figures.

i. If the rocket has exhaust speed $u = 3.5\,\text{km/s}$, what must its starting mass be in kilograms?

ii. If the propellant is exhausted at rate $7.0\,\text{kg/s}$, how long does the acceleration take, in centuries?

iii. If the energy density of the fuel is $2.0 \times 10^7\,\text{J/kg}$, how much total energy is required, in Joules?

For the rest of this problem, you should account for special relativity.

b. Another option is to use a spaceship with constant mass m, propelled by light produced by lasers on Earth, with total power $P = 6 \times 10^{12}\,\text{W}$. The light evenly impacts a sail on the spaceship, and reflects off the sail directly back towards the Earth. Neglect the orbital motion of the Earth, and give all your answers in the frame of the Earth.

i. What is the force on the spaceship when the spaceship has speed v?

ii. How long will it take to accelerate the spaceship to speed $v_f = 3c/5$, in seconds? You may use the integrals on the reference sheet.

iii. At the moment the spaceship reaches this speed, how much total energy has been used to power the lasers, in Joules?

The following results from relativity may be helpful:

- The Lorentz factor is defined as $\gamma = 1/\sqrt{1 - v^2/c^2}$.
- An object of mass m and velocity $\mathbf{v}$ has momentum $\mathbf{p} = \gamma m \mathbf{v}$ and energy $E = \gamma m c^2$. The force is defined by $\mathbf{F} = d\mathbf{p}/dt$.
- The momentum and energy of light are related by $E = pc$.
- In a frame S' with velocity $v\hat{\mathbf{x}}$ relative to a frame S, the energy and momentum are

$$E' = \gamma(E - v p_x), \quad p'_x = \gamma(p_x - vE/c^2).$$

Question B3

Starry Messengers

In 1987, light from supernova SN1987A was detected by telescopes on Earth. The supernova occurred in the Large Magellanic Cloud, a distance $d = 1.5 \times 10^{21}$ m away, making it the closest in centuries. Observations of this event tell us a remarkable amount about elementary particles.

a. Both light and neutrinos were produced in the core of the supernova. Neutrinos are elementary particles which interact extremely weakly with ordinary matter. Detectors on Earth saw a few dozen of these neutrinos, in a burst which occurred about $T = 3$ hours *before* the light arrived.

   i. One explanation of these observations is that the neutrinos' speed v was faster than the speed of light c, violating special relativity. If this is the case, find $v - c$ in m/s.

   ii. Another explanation is that the light was slowed down by the gas in the solar system, while the neutrinos always moved at speed c. Suppose the solar system has a uniform index of refraction n within a radius $D = 10^{13}$ m. What would n have to be to explain the time delay?

   Neither of these explanations seem plausible; the modern accepted explanation is that the light was trapped for some time inside the supernova, while the neutrinos were able to leave immediately. Therefore, for the rest of this problem you should assume special relativity holds. The results listed on the previous page may be helpful.

The neutrinos did not all arrive at once. The first arrived with an energy of about $E_1 = 40$ MeV, and the last arrived about $t = 10$ s later with an energy of about $E_2 = 20$ MeV.

b. One explanation of these observations is that neutrinos have a small mass m, so that when they have energy $E \gg mc^2$, their speed v is slightly slower than the speed of light.

   i. Find an approximate expression for $c - v$, to leading nontrivial order in mc^2/E.

   ii. Using the information above, numerically estimate the neutrino mass m, in units of eV$/c^2$.

c. Another explanation is that the neutrinos did not travel in straight lines, but rather were deflected by the intergalactic magnetic field. Suppose this field is uniform, $B = 10^{-13}$ T, and directed perpendicular to the line joining Earth and the supernova, and that neutrinos have charge $q = \epsilon e$.

   i. If a neutrino has momentum p, then in the presence of the magnetic field, it travels in a circle of radius $r = p/(qB) \gg d$, and its path to the Earth has a total length ℓ. Find an approximate expression for $\ell - d$, to leading nontrivial order in d/r.

   ii. Using the information above, and assuming the neutrino mass is very small so that the effect in part b is negligible, numerically estimate ϵ.

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USA Physics Olympiad Exam

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- For each student, print out one copy of the exam and one copy of the answer sheets. Print everything single-sided, and do not staple anything. Divide the exam into the instructions (pages 2–3), Part A questions (pages 4–14), and Part B questions (pages 15–24).
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End of Instructions for the Student

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Part A

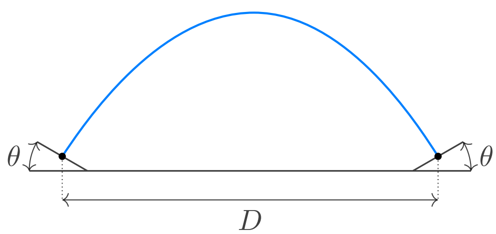
Question A1

Circus Act

In this problem we consider a small ball bouncing back and forth between two points. In all parts below, the acceleration of gravity is g, collisions are perfectly elastic, air resistance is negligible, and the impact points are at the same height. The diagrams are not drawn to scale.

a. Consider a ball bouncing between two inclined planes, which each make an angle $\theta < 90°$ to the horizontal. The ball has speed v_0 at the impact points, which are separated by a distance D.

i. The ball can bounce back and forth along the same path, as shown.



For what values of θ is this motion possible? For these values, what is v_0?

Solution

To go back and forth on the same path, the ball must impact the plane normally, which means that θ is the angle of its velocity to the vertical. The range of the ball is

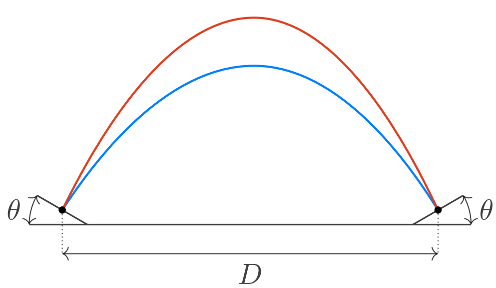
$$D = \frac{v_0^2 \sin\big(2(90° - \theta)\big)}{g} = \frac{v_0^2 \sin 2\theta}{g}$$

which implies

$$v_0 = \sqrt{\frac{gD}{\sin 2\theta}}.$$

Evidently, the motion is possible for any θ.

ii. The ball can also take one trajectory while traveling to the right, and a separate trajectory when traveling back. Let $\phi \neq 0$ be the angle between the paths at the impact points.



For what values of θ and ϕ is this motion possible? For these values, what is v_0?

Solution

The two paths have the same initial speed and the same range, which means the initial angles of the velocity to the horizontal must be $\pi/4 \pm \phi/2$. Since the initial and final angles to the normal are equal in the collision, the angle of the planes to the horizontal must be $\theta = 45^\circ$. Given this, ϕ can take any value in the range $0 < \phi < 90^\circ$.

Considering the range yields

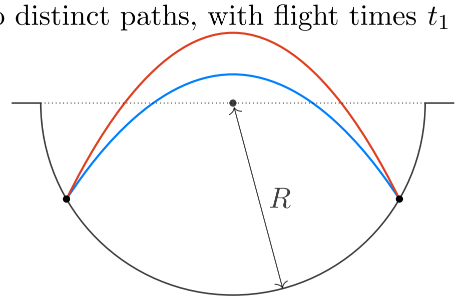
$$D = \frac{v_0^2 \sin\big(2(45^\circ + \phi/2)\big)}{g} = \frac{v_0^2 \cos\phi}{g}$$

from which we find

$$v_0 = \sqrt{\frac{gD}{\cos\phi}}.$$

As expected, this reduces to the answer above as $\phi \to 0$, and becomes infinite as $\phi \to 90^\circ$.

b. Now suppose the ball bounces within a hemispherical well of radius of curvature R. As in part a.ii, it alternates between two distinct paths, with flight times t_1 and $t_2 \neq t_1$.



Find all of the possible values of R, in terms of t_1 and t_2.

Solution

From the previous question we know that the slope of the wall $\theta = \frac{\pi}{4}$ at the collision points. So $D = R\sqrt{2}$.

x displacement with t_1 is

$$v_0 \cos\left(\frac{\pi}{4} - \phi/2\right) t_1 = R\sqrt{2} \tag{A1-1}$$

y-component of velocity in the upper point is 0

$$v_0 \sin\left(\frac{\pi}{4} + \phi/2\right) - g\frac{t_2}{2} = 0$$

transforms into

$$v_0 \cos\left(\frac{\pi}{4} - \phi/2\right) = g\frac{t_2}{2} \tag{A1-2}$$

Dividing (A1-1) by (A1-2) we get

$$R = \frac{g t_1 t_2}{2\sqrt{2}}.$$

Only one R satisfies this, so the answer is unique.

c. Finally, suppose the well has a sinusoidal shape, described by $y(x) = -L\sin\bigl(2x/L\bigr)$. The ball takes two distinct paths with flight times t_1 and $t_2 \neq t_1$, and the horizontal distance between the impact points is less than πL. Find all of the possible values of L, in terms of t_1 and t_2.

Solution

The slope of the wall $\theta = \frac{\pi}{4}$ at the collision points. So $z'_x = -1$ for one end and there is a symmetric point on the other point of collision. $-2\cos\frac{2x}{L} = -1$ has solutions $x = \pm\frac{\pi L}{6}$. So we got two possible values of L.

i. Case of $x = \frac{\pi L}{6}$ corresponds to $D = \frac{\pi L}{2} - \frac{\pi L}{3} = \frac{\pi L}{6}$

$$v_0 \cos\left(\frac{\pi}{4} - \phi/2\right) t_1 = \frac{\pi L}{6}$$

and

$$v_0 \cos\left(\frac{\pi}{4} - \phi/2\right) = g\frac{t_2}{2}.$$

Combining these two we get

$$L_a = \frac{3gt_1t_2}{\pi}$$

ii. Case of $x = -\frac{\pi L}{6}$ corresponds to $D = \frac{\pi L}{2} + \frac{\pi L}{3} = \frac{5\pi L}{6}$

$$v_0 \cos\left(\frac{\pi}{4} - \phi/2\right) t_1 = \frac{5\pi L}{6}$$

and

$$v_0 \cos\left(\frac{\pi}{4} - \phi/2\right) = g\frac{t_2}{2}.$$

Combining these two we get

$$L_b = \frac{3gt_1t_2}{5\pi}$$

Question A2

Time is a Flat Circle

A particle of mass m and negative charge $-q$ is constrained to move in a horizontal plane. In the situations described below, this particle can either oscillate back and forth in a straight line or move in a circle. (These two modes of motion are interesting because they generate linearly and circularly polarized radiation, respectively, but in this problem you may ignore any energy lost to radiation.)

a. A large positive charge $Q \gg q$ is fixed in place a distance R directly below the origin of the plane.

i. When the particle is a distance $r \ll R$ from the origin, find an approximate expression for its potential energy due to the charge Q to second order in r/R, up to an arbitrary constant. You may use this result for the rest of the problem.

Solution

The potential energy of the particle, relative to infinity, is

$$V(r) = -\frac{qQ}{4\pi\epsilon_0\sqrt{R^2+r^2}} \approx -\frac{qQ}{4\pi\epsilon_0 R} + \frac{qQr^2}{8\pi\epsilon_0 R^3}$$

where we have used the binomial theorem on the square root, since $r \ll R$. The constant doesn't matter, so we could equivalently write this as

$$V(r) \approx \frac{qQr^2}{8\pi\epsilon_0 R^3} + \text{const.}$$

ii. If the particle oscillates linearly with amplitude $a \ll R$, what is its angular frequency ω_ℓ?

Solution

For simple harmonic motion, the potential energy $V - V_0 = \frac{1}{2}kx^2$, where the "spring constant" k is related to the angular frequency of oscillation by $\omega = \sqrt{k/m}$. Reading the value of k from the expression above for $V(r)$ gives $k = qQ/(4\pi\epsilon_0 R^3)$, and therefore

$$\omega_\ell = \sqrt{\frac{qQ}{4\pi\epsilon_0 m R^3}}.$$

As usual for simple harmonic motion, the amplitude does not alter the frequency.

iii. If the particle performs circular motion with radius $r \ll R$, what is its angular frequency ω_c?

Solution

For a radially symmetric, parabolic potential, a circular orbit can be thought of as simultaneous harmonic oscillation in the x direction and the y direction. Both oscillations have the same angular frequency ω_ℓ as derived in part 1. So the answer is the same:

$$\omega_c = \sqrt{\frac{qQ}{4\pi\epsilon_0 m R^3}} = \omega_\ell.$$

Note that the orbit radius does not matter, so long as it is small enough ($r \ll R$) to still correspond to simple harmonic motion.

b. Now an additional negative charge $-q$ is fixed in place at the origin of the plane.

i. What is the equilibrium distance r_0 of the particle from the origin?

Solution

With the additional charge at the origin, the total potential energy of the particle as a function of r is

$$V(r) = V_0 + \frac{qQr^2}{8\pi\epsilon_0 R^3} + \frac{q^2}{4\pi\epsilon_0 r}.$$

Notice that the potential energy as a function of position has a ring of minima with a certain radius r_0. When the particle is not in motion, its rest position is somewhere along the ring. The value of r_0 can be found from $V'(r_0) = 0$, which gives $r_0 = R(q/Q)^{1/3}$.

ii. If the particle oscillates linearly with amplitude $a \ll r_0$, what is its angular frequency Ω_ℓ? Is it higher or lower than ω_ℓ?

Solution

A linear oscillation involves the radius r oscillating around the value r_0. Taylor expanding the expression for $V(r)$ in part b.i around $r = r_0$ gives

$$V \approx V(r_0) + \frac{1}{2}\frac{3qQ}{4\pi\epsilon_0 R^3}\delta_r^2,$$

where $\delta_r = r - r_0$. One can read from this expression the value of the spring constant k, which gives for the angular frequency $\omega = \sqrt{k/m}$ the result

$$\Omega_\ell = \sqrt{\frac{3qQ}{4\pi\epsilon_0 m R^3}} = \sqrt{3}\,\omega_\ell$$

so that $\Omega_\ell > \omega_\ell$. Evidently, adding the charge at the origin slightly stiffens the oscillation in the radial direction.

iii. Now suppose the particle performs circular motion with radius $r = r_0 + \delta r$, where $\delta r \ll r_0$. What is its angular frequency Ω_c, in terms of ω_c, r, and δr? Is it higher or lower than ω_c?

Solution

For circular motion with radius $r + \delta r$ and $\delta r \ll r_0$, the orbit is above, but is very close to the bottom of the ring of minima, $r \approx r_0$. For such small δr the force linearly depends on $r - r_0$, as in Hooke's law, with

$$k = V''(r_0) = \frac{3Qq}{4\pi\epsilon_0 R^3}$$

Now Newton's second law gives

$$-k\delta r = -m\Omega_c^2 r,$$

with $r \approx r_0$. This equation has a solution for $\delta r > 0$ (so that the centripetal force is inward):

$$\Omega_c \approx \omega_c \sqrt{\frac{3\,\delta r}{r_0}}$$

Notice that the orbit frequency now explicitly depends on the orbit radius through δr. So, unlike in the case of the usual harmonic oscillator, in this case one can generate circularly polarized light with a wide range of frequencies by exciting circular motion with different radii.

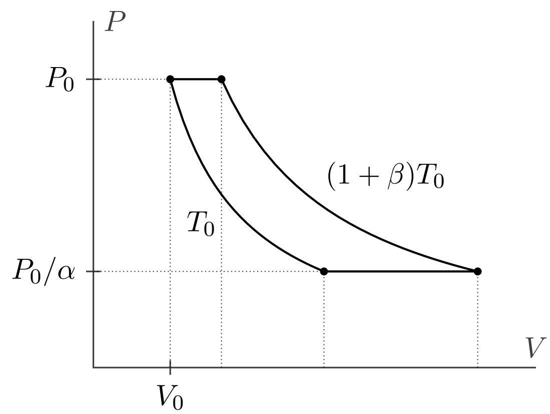
Since $\delta r \ll r_0$, the angular frequency $\Omega_c \ll \omega_c$, so that adding the charge $-q$ at the origin has made the orbit much slower. Intuitively, this is because the set of potential minima is a "flat circle," so that the orbit frequency can get arbitrarily small for arbitrarily small angular momenta. This is connected to Goldstone's theorem in quantum field theory, which states that spontaneously broken symmetries give rise to low frequency modes.

Question A3

The Motive Power of Ice

In the Carnot cycle, a gas is heated at constant temperature T_H and cooled at constant temperature T_C. Furthermore, no other heat transfer occurs, and all other steps of the cycle are reversible. The laws of thermodynamics state that any such cycle must have efficiency $\eta = W/Q_{\text{in}} = 1 - (T_C/T_H)$. Below we will explore two other heat engines, which recover this efficiency in certain limits.

a. Consider the following heat engine involving one mole of ideal monatomic gas. The gas begins at temperature T_0, pressure P_0, and volume V_0, and undergoes four reversible steps.



1. The gas is expanded at constant pressure until its temperature rises to $(1+\beta)T_0$.
2. The gas is expanded at constant temperature until its pressure falls to P_0/α.
3. The gas is contracted at constant pressure until its temperature falls back to T_0.
4. The gas is contracted at constant temperature until its pressure rises back to P_0.

i. Which steps require heat to be transferred to the gas? For each such step, give the total heat input in terms of P_0, V_0, α, and β.

Solution

Heat is added to the gas in the first two steps. In the first step, we have heating at constant pressure, which has molar heat capacity $c_p = 5R/2$, so

$$Q_1 = c_p \Delta T = \frac{5}{2} R\beta T_0 = \frac{5}{2}\beta P_0 V_0$$

where we used the ideal gas law in the final step. In the second step, the heat added to the gas compensates for the work done while it expands, so

$$Q_2 = (1+\beta)P_0 V_0 \ln\frac{V_f}{V_i} = (1+\beta)P_0 V_0 \ln\alpha.$$

ii. Under what conditions on α and β would we expect the efficiency of this heat engine to approach that of a Carnot cycle working between the same maximum and minimum temperatures?

Solution

Since all the steps are reversible, we recover the Carnot efficiency when almost all the heat transfer happens at the same temperature, i.e. when $Q_1 \ll Q_2$. This holds when

$$\frac{\beta}{\beta+1} \ll \ln \alpha.$$

We should also consider the other two steps, which have heat transfer

$$|Q_3| = c_p|\Delta T| = \frac{5}{2}R\beta T_0 = \frac{5}{2}\beta P_0 V_0$$

and

$$|Q_4| = P_0 V_0 \ln \frac{V_f}{V_i} = P_0 V_0 \ln \alpha.$$

We have $|Q_3| \ll |Q_4|$ when

$$\beta \ll \ln \alpha$$

which is stricter than the previous condition. Thus, we recover the Carnot efficiency when $\beta \ll \ln \alpha$.

iii. Find the efficiency of this heat engine for general α and β.

Solution

We calculate the net work throughout all four steps,

$$W = \beta P_0 V_0 + (1+\beta) P_0 V_0 \ln \alpha - \beta P_0 V_0 - P_0 V_0 \ln \alpha$$

where the first and third terms are just the usual $P\Delta V$ work, and the second and fourth use the form done in an isothermal process. Simplifying gives

$$W = P_0 V_0 \beta \ln \alpha.$$

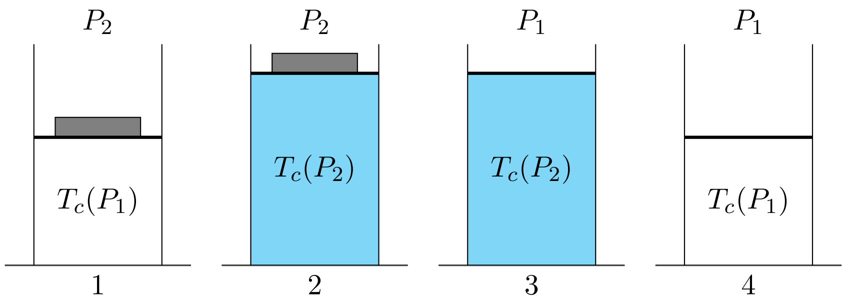
Therefore, the efficiency of the process is

$$\eta = \frac{\beta \ln \alpha}{(1+\beta)\ln \alpha + 5\beta/2}.$$

One way to check the answer is to note that the Carnot efficiency would be $\beta/(1+\beta)$. Our general result reduces to this efficiency when the second term in the denominator is negligible, which is precisely the condition we identified in part (b).

The second half of the problem is on the next page.

b. Now consider a heat engine built around the freezing and melting of water, which occurs at a pressure-dependent temperature $T_c(P)$. Initially, a volume of V of water is squeezed underneath a piston, so that it experiences a total pressure P_1, and the water is on the edge of freezing, with temperature $T_c(P_1)$. The engine then undergoes four reversible steps.



1. A mass is slowly placed on the piston, raising the total pressure to P_2.
2. The water is cooled to temperature $T_c(P_2)$ and frozen.
3. The mass is slowly removed from the piston, lowering the pressure back to P_1.
4. The ice is heated back to temperature $T_c(P_1)$ and melted.

Assume that water and ice are incompressible, with fixed densities ρ_w and ρ_i.

i. What is the net work done by this engine, in terms of P_1, P_2, V, and the densities?

Solution

The system expands at a pressure P_2 and contracts at a pressure P_1, so the net work done is

$$W = (P_2 - P_1)\Delta V = (P_2 - P_1)V\,\frac{\rho_w - \rho_i}{\rho_i}.$$

Concretely, this work used to raise the mass, so it could also be calculated as $Mg\Delta H$.

ii. Assume the latent heat per unit mass L to melt ice is large, so that freezing and melting account for essentially all of the heat transfer in the cycle. What is the efficiency of the engine, in terms of P_1, P_2, L, and the densities?

Solution

The heat transferred in is used to melt the ice, $Q_{\text{in}} = \rho_w V L$, so

$$\eta = \frac{W}{Q_{\text{in}}} = \frac{P_2 - P_1}{L}\frac{\rho_w - \rho_i}{\rho_w \rho_i}.$$

Note that the efficiency can also be expressed as $\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}}$, but in order to find Q_{out} directly, we would need to know how to latent heat varies with pressure, which isn't given in the problem.

iii. Since we assumed all heat transfer occurs during melting or freezing, this cycle has the same efficiency as a Carnot cycle. In the limit where P_1 and P_2 are very close, use this fact to infer an expression for dT_c/dP in terms of T_c, L, and the densities.

Solution

The Carnot efficiency, for the same high and low temperatures, is

$$\eta = \frac{T_c(P_1) - T_c(P_2)}{T_c(P_1)}.$$

Combining this with the result of part (b) gives

$$\frac{T_c(P_2) - T_c(P_1)}{P_2 - P_1} = -\frac{T_c(P_1)}{L}\frac{\rho_w - \rho_i}{\rho_w \rho_i}$$

and taking the limit of $P_1 \approx P_2$, the left-hand side becomes a derivative, so

$$\frac{dT_c}{dP} = -\frac{T_c}{L}\frac{\rho_w - \rho_i}{\rho_w \rho_i}.$$

This is equivalent to the well-known Clausius–Clapeyron equation, and this heat engine was first devised by the brothers Thomson and Kelvin.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you can review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

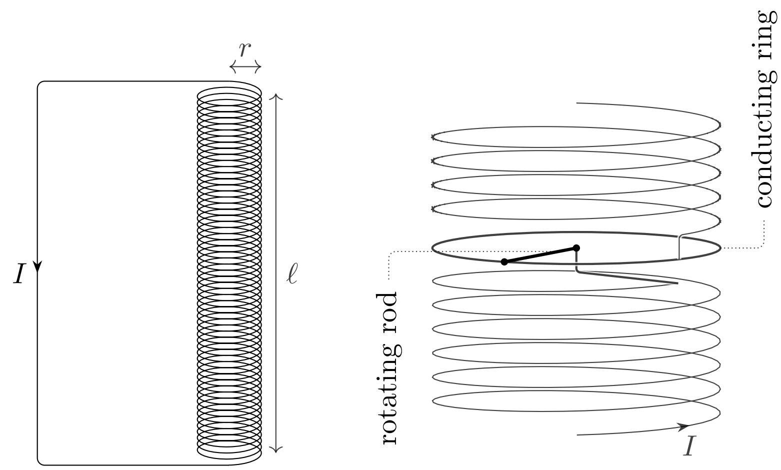
Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

Electric Roulette

Consider a cylindrical solenoid with radius r, length $\ell \gg r$, and n turns per unit length. It is made of one continuous wire, with the top connecting back to the bottom as shown at left.



In the middle of the solenoid, part of the wire is replaced with the assembly shown at right. A uniform conducting rod of mass m and radius r is connected to the bottom half of the solenoid, and is free to rotate about the solenoid's axis of symmetry. The end of the rod slides on a fixed conducting ring, which is attached to the top half of the solenoid. This assembly and the solenoid form one continuous conductor, carrying total current I.

a. What is the inductance of this system? Assume $nr \gg 1$, so that the magnetic field produced by the current in the rod and ring is negligible.

Solution

The magnetic field produced is $B = \mu_0 n I$, so the flux through one turn of the solenoid is $\mu_0 n I(\pi r^2)$. There are $n\ell$ turns in total, so the inductance is

$$L = \frac{\Phi}{I} = \mu_0 n^2 \pi r^2 \ell.$$

b. When the rod is within a uniform vertical magnetic field B, find the torque it experiences in terms of I, B, and r.

Solution

The torque is due to the Lorentz force on the current as it travels radially outward, from the center of the disc towards its edge. Note that only the radial motion of the current contributes to the torque, so we would get the same torque if the current traveled in a straight line from the center to the rim. In this case, the total torque is

$$\tau = -\int_0^r IBs\,ds = -\frac{IBr^2}{2}$$

where the minus sign indicates that the torque tends to slow down the rotation, when I is defined in the direction shown in the figure. (This is consistent with defining positive angular velocity and torque by the right-hand rule; if you defined it the other way around, this equation and several others below would pick up a sign flip. Since we didn't specify the sign convention in the problem text, either set of signs received full credit.)

c. If the rod rotates with angular velocity ω, the electrons inside have a tangential velocity. Find the electromotive force across the rod in terms of ω, B, and r.

Solution

At a radius s, the tangential speed is ωs, leading to a radially outward Lorentz force per charge of $\omega s B$. (The electrons also have radial motion, but this does not contribute to the electromotive force; it instead contributes to the torque found in the previous subpart.) The emf is therefore

$$\mathcal{E} = \int_0^r vB\,ds = \int_0^r \omega s B\,ds = \frac{\omega B r^2}{2}.$$

Now we will consider the dynamics of this system in some simple situations. For all the parts below, neglect energy losses due to friction, resistance, and radiation.

d. First, suppose the system initially carries no current, and the entire system is inside a uniform external magnetic field B_0 parallel to the axis of the solenoid. If the rod is given a small initial angular velocity, its angular velocity will oscillate in time. Find the period of these oscillations.

Solution

Using our results from above, the angular acceleration of the disc is is

$$\frac{d\omega}{dt} = \frac{\tau}{mr^2/3} = -\frac{3IB_0}{2m}.$$

Kirchoff's loop rule is $\mathcal{E} - L\,dI/dt = 0$, which implies that

$$\frac{dI}{dt} = \frac{B_0}{2\pi\mu_0 n^2 \ell}\,\omega.$$

Taking the time derivative of our expression for $d\omega/dt$ gives

$$\frac{d^2\omega}{dt^2} = -\frac{3}{4\pi}\frac{B_0^2}{\mu_0 n^2 \ell m}\,\omega$$

which is a simple harmonic motion equation. The period is therefore

$$T = 2\pi\sqrt{\frac{4\pi\mu_0 n^2 \ell m}{3B_0^2}}.$$

e. Next, suppose there is no external magnetic field, $B_0 = 0$, and at time $t = 0$, the system carries current I_0 and the rod has zero angular velocity.

i. The rod's angular velocity $\omega(t)$ approaches a value ω_0 after a long time. What is ω_0?

Solution

The basic equations are similar, except now the magnetic field is sourced by the solenoid itself, so instead of $B = B_0$ we now have $B = \mu_0 n I$. The results are

$$\frac{d\omega}{dt} = -\frac{3\mu_0 n}{2m} I^2, \quad \frac{dI}{dt} = \frac{\omega I}{2\pi n \ell}.$$

Initially I is positive, so $d\omega/dt$ is negative, which then causes dI/dt to be negative. This remains true until I falls to zero, at which point dI/dt and $d\omega/dt$ both remain zero. In other words, the solenoid speeds up the rod until it has given all of its energy to it.

Now that we know this, we can find the answer just using energy conservation,

$$\frac{1}{2}\frac{mr^2}{3}\omega^2 + \frac{1}{2}LI^2 = \frac{1}{2}LI_0^2.$$

This equation is equivalent to

$$\frac{I^2}{I_0^2} + \frac{\omega^2}{\omega_0^2} = 1$$

where

$$\omega_0 = -\sqrt{\frac{3\pi\mu_0\ell}{m}}\, nI_0.$$

This is the angular velocity attained after a long time, when I approaches zero. Again, the opposite sign would also receive full credit.

ii. Find $\omega(t)/\omega_0$ in terms of ω_0, t, n, and ℓ. You may use the integrals on the reference sheet.

Solution

Plugging the energy conservation equation into our equation for $d\omega/dt$, and writing it in terms of ω_0, we have

$$\frac{d\omega}{dt} = -\frac{\omega_0^2 - \omega^2}{2\pi n \ell}.$$

Separating and integrating yields

$$\frac{t}{2\pi n\ell} = -\int_0^\omega \frac{d\omega'}{\omega_0^2 - \omega'^2} = -\frac{1}{\omega_0}\tanh^{-1}\left(\frac{\omega}{\omega_0}\right).$$

Solving for ω gives

$$\frac{\omega(t)}{\omega_0} = \tanh\left(\frac{-\omega_0 t}{2\pi n\ell}\right)$$

which has the right limiting behavior.

Question B2

Fast and Furious

A space program wants to accelerate a spaceship of final mass $m = 100\,\text{kg}$ to relativistic speeds to observe distant stars. They have two proposals to evaluate.

a. Their first proposal is to use traditional rocket propulsion. A rocket of initial mass m_0 and final mass m that expels propellant with exhaust speed u relative to the rocket will reach a speed

$$v = u \ln\left(\frac{m_0}{m}\right).$$

Suppose the desired final speed is $v_f = 3c/5$. In the subparts below, neglect relativistic effects and give your answers in the form 10^n, where n has at least two significant figures.

i. If the rocket has exhaust speed $u = 3.5\,\text{km/s}$, what must its starting mass be in kilograms?

Solution

Plugging in the numbers gives

$$m_0 = me^{v_f/u} = 100\,\text{kg} \cdot e^{0.6\times3\times10^8/3.5\times10^3} \approx 10^{22337}\ \text{kg}$$

Equivalently, the exponent is $n = 2.2 \times 10^4$.

ii. If the propellant is exhausted at rate $7.0\,\text{kg/s}$, how long does the acceleration take, in centuries?

Solution

The result is

$$t = \frac{m_0 - m}{7.0\,\text{kg/s}} \approx 10^{22336}\,\text{s} \approx 10^{22326}\,\text{centuries}.$$

Equivalently, the exponent is $n = 2.2 \times 10^4$.

iii. If the energy density of the fuel is $2.0 \times 10^7\ \text{J/kg}$, how much total energy is required, in Joules?

Solution

The energy required is

$$E = (m_0 - m)(2.0 \times 10^7\,\text{J}) = 10^{22344}J.$$

Equivalently, the exponent is $n = 2.2 \times 10^4$. When working with such absurdly large numbers, exponents essentially always stay unchanged.

For the rest of this problem, you should account for special relativity.

b. Another option is to use a spaceship with constant mass m, propelled by light produced by lasers on Earth, with total power $P = 6 \times 10^{12}\,\text{W}$. The light evenly impacts a sail on the spaceship, and reflects off the sail directly back towards the Earth. Neglect the orbital motion of the Earth, and give all your answers in the frame of the Earth.

i. What is the force on the spaceship when the spaceship has speed v?

Solution

Let $\beta = v/c$, where v is the ship's speed, and consider a piece of the beam with total momentum dp_x, in the Earth's frame. In the ship's frame, this momentum is $dp'_x = \gamma(1-\beta)dp_x$, and after the collision the momentum simply flips sign, $dp'_{xf} = -dp'_x$. Thus, transforming back to the Earth's frame, the final momentum is $dp_{xf} = -\gamma^2(1-\beta)^2 dp_x$. The change of the spaceship's momentum, still in the Earth's frame, is the difference

$$dP_x = -(dp_{xf} - dp_x) = \frac{2}{1+\beta}\, dp_x.$$

To find the force on the spaceship, we need to find the rate at which the beam impacts the spaceship. Accounting for the spaceship's motion, it is $dp_x = \frac{P}{c}(1-\beta)dt$, so

$$F = \frac{dP_x}{dt} = \frac{2P}{c}\frac{1-\beta}{1+\beta}.$$

Alternative solution: In the Earth's frame, if the photons in the incident beam have frequency f_i, then they are reflected with frequency

$$f_f = \sqrt{\frac{1-\beta}{1+\beta}}\sqrt{\frac{1-\beta}{1+\beta}} f_i = \frac{1-\beta}{1+\beta}\, f_i$$

where we applied the relativistic Doppler shift formula twice, since the photons are first absorbed by the moving rocket and then reemitted by it. Applying conservation of momentum, and using the fact that the momentum of a photon is related to its energy by $E = hf = pc$, we have

$$F = \frac{dN}{dt}\frac{h}{c}(f_i + f_f) = \frac{dN}{dt}\frac{hf_i}{c}\frac{2}{1+\beta}$$

where dN/dt is the rate at which photons collide with the sail. It is related to the rate at which photons are emitted from the source on Earth, dN_{em}/dt, by

$$\frac{dN}{dt} = (1-\beta)\frac{dN_{\text{em}}}{dt}.$$

Finally, since the power of the laser is $P = (dN_{\text{em}}/dt)(hf_i)$, we have

$$F = \frac{2P}{c}\frac{1-\beta}{1+\beta}.$$

ii. How long will it take to accelerate the spaceship to speed $v_f = 3c/5$, in seconds? You may use the integrals on the reference sheet.

Solution

In the Earth's frame, the relativistic momentum of the spaceship obeys

$$dP_x = mcd\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = mc\frac{d\beta}{(1-\beta^2)^{1.5}}$$

Combining this with our expression for the force gives

$$\frac{2P}{mc^2}dt = \frac{d\beta}{(1-\beta)^2\sqrt{1-\beta^2}}.$$

Integrating both sides and using an integral on the reference sheet gives

$$\frac{2P}{mc^2}t = \int_0^{0.6} \frac{d\beta}{(1-\beta)^2\sqrt{1-\beta^2}} = \frac{5}{3}.$$

Therefore, the time is

$$t = \frac{5mc^2}{6P} = \frac{4.5\times 10^{19}}{3.6\times 10^{13}} = 1.3\times 10^6 \text{ s}$$

iii. At the moment the spaceship reaches this speed, how much total energy has been used to power the lasers, in Joules?

Solution

The energy is just $5mc^2/6$ from the result in the previous problem, so we get 7.5×10^{18} J. For reference, the US consumes roughly 10^{16} J of electricity per day. For more discussion of this propulsion method, see this paper.

The following results from relativity may be helpful:

- The Lorentz factor is defined as $\gamma = 1/\sqrt{1-v^2/c^2}$.
- An object of mass m and velocity $\mathbf{v}$ has momentum $\mathbf{p} = \gamma m\mathbf{v}$ and energy $E = \gamma mc^2$. The force is defined by $\mathbf{F} = d\mathbf{p}/dt$.
- The momentum and energy of light are related by $E = pc$.
- In a frame S' with velocity $v\hat{\mathbf{x}}$ relative to a frame S, the energy and momentum are

$$E' = \gamma(E - vp_x), \quad p'_x = \gamma(p_x - vE/c^2).$$

Question B3

Starry Messengers

In 1987, light from supernova SN1987A was detected by telescopes on Earth. The supernova occurred in the Large Magellanic Cloud, a distance $d = 1.5 \times 10^{21}$ m away, making it the closest in centuries. Observations of this event tell us a remarkable amount about elementary particles.

a. Both light and neutrinos were produced in the core of the supernova. Neutrinos are elementary particles which interact extremely weakly with ordinary matter. Detectors on Earth saw a few dozen of these neutrinos, in a burst which occurred about $T = 3$ hours *before* the light arrived.

   i. One explanation of these observations is that the neutrinos' speed v was faster than the speed of light c, violating special relativity. If this is the case, find $v - c$ in m/s.

   Solution

   If the light took a time t to arrive at the Earth, then $ct = v(t - T) = d$. Approximately solving for $v - c$, using the fact that v is very close to c, gives

   $$v - c = \frac{c^2 T}{d} = 0.65\,\mathrm{m/s}.$$

   ii. Another explanation is that the light was slowed down by the gas in the solar system, while the neutrinos always moved at speed c. Suppose the solar system has a uniform index of refraction n within a radius $D = 10^{13}$ m. What would n have to be to explain the time delay?

   Solution

   The time delay is $T = (n - 1)D/c$, and plugging in the numbers gives $n = 1.3$. Given how thin the gas in the solar system is, such a large value is implausible.

   Neither of these explanations seem plausible; the modern accepted explanation is that the light was trapped for some time inside the supernova, while the neutrinos were able to leave immediately. Therefore, for the rest of this problem you should assume special relativity holds. The results listed on the previous page may be helpful.

The neutrinos did not all arrive at once. The first arrived with an energy of about $E_1 = 40\,\mathrm{MeV}$, and the last arrived about $t = 10\,\mathrm{s}$ later with an energy of about $E_2 = 20\,\mathrm{MeV}$.

b. One explanation of these observations is that neutrinos have a small mass m, so that when they have energy $E \gg mc^2$, their speed v is slightly slower than the speed of light.

   i. Find an approximate expression for $c - v$, to leading nontrivial order in mc^2/E.

   Solution

   We start with the equation for relativistic energy:

   $$E = \gamma mc^2$$

   where γ is the Lorentz factor defined as: $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$ where $\beta \equiv \frac{v}{c}$.

Let $\beta = 1 - \delta$, where $\delta \ll 1$. Then,

$$\gamma \approx \frac{1}{\sqrt{1-1+2\delta}} \implies \delta \approx \frac{1}{2\gamma^2}.$$

Note that $\delta = 1 - v/c$, so

$$c - v = c\delta \approx \frac{c}{2\gamma^2} = \frac{m^2c^5}{2E^2}.$$

ii. Using the information above, numerically estimate the neutrino mass m, in units of eV/c^2.

Solution

We compute a relationship between v_1 and v_2, the velocities of the first and second set of neutrinos, using the time delay. From the same equation as 1(a),

$$v_1 - v_2 = \frac{10\mathrm{s} \times c^2}{d} = 6 \times 10^{-4}\ \mathrm{m/s}.$$

We have

$$c - v_1 \approx \frac{m^2c^5}{2E_1^2}, \quad c - v_2 \approx \frac{2m^2c^5}{E_1^2}.$$

Then,

$$v_1 - v_2 = \frac{3m^2c^5}{2E_1^2}.$$

Solving for m gives

$$m = \sqrt{\frac{2(v_1 - v_2)}{3c}}\frac{E_1}{c^2} = 46\ \mathrm{eV/c^2}.$$

c. Another explanation is that the neutrinos did not travel in straight lines, but rather were deflected by the intergalactic magnetic field. Suppose this field is uniform, $B = 10^{-13}\,\mathrm{T}$, and directed perpendicular to the line joining Earth and the supernova, and that neutrinos have charge $q = \epsilon e$.

i. If a neutrino has momentum p, then in the presence of the magnetic field, it travels in a circle of radius $r = p/(qB) \gg d$, and its path to the Earth has a total length ℓ. Find an approximate expression for $\ell - d$, to leading nontrivial order in d/r.

Solution

We are computing the difference between the arc length of a small arc and the distance connecting the endpoints. If the arc length is ℓ, the angle subtended is $\theta = \ell/r$. The distance connecting the end points is

$$d = 2r\sin\bigl(\theta/2\bigr) = 2r\sin\left(\frac{\ell}{2r}\right).$$

Then, Taylor expanding the sine gives

$$\ell - d \approx \ell - \ell + \frac{\ell^3}{24r^2} = \frac{\ell^3}{24r^2} \approx \frac{d^3}{24r^2}.$$

ii. Using the information above, and assuming the neutrino mass is very small so that the effect in part b is negligible, numerically estimate ϵ.

Solution

The radius of the path is

$$r \approx \frac{E}{qBc}.$$

Substituting gives

$$\ell_2 - \ell_1 = \frac{d^3 q^2 B^2 c^2}{8E_1^2}.$$

Then,

$$q = \frac{2\sqrt{2}\,E_1}{Bcd}\left(\frac{(\ell_2 - \ell_1)}{d}\right)^{1/2} = \frac{2\sqrt{2}\,E_1}{Bcd}\left(\frac{c\Delta t}{d}\right)^{1/2} \approx 3.6 \times 10^{-15}\,e.$$

Then,

$$\epsilon \approx 3.6 \times 10^{-15}.$$

Since the effects of a neutrino mass and charge add, and we know neutrinos have mass, this result yields a (very strong) upper bound on the possible charge of a neutrino, which as far as we know could be exactly zero. For more about the physics of SN1987A, see this paper.

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USA Physics Olympiad Exam

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Fundamental Constants

$g = 9.8$ N/kg

$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2

$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2

$k_\text{m} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A

$c = 3.00 \times 10^8$ m/s

$k_\text{B} = 1.38 \times 10^{-23}$ J/K

$N_\text{A} = 6.02 \times 10^{23}$ (mol)$^{-1}$

$R = N_\text{A} k_\text{B} = 8.31$ J/(mol $\cdot$ K)

$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)

$e = 1.602 \times 10^{-19}$ C

1 eV $= 1.602 \times 10^{-19}$ J

$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s

$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2

Useful Approximations

$$(1+x)^n \approx 1 + nx + n(n-1)x^2/2 \text{ for } |nx| \ll 1$$
$$e^x \approx 1 + x + x^2/2 + x^3/6 \text{ for } |x| \ll 1$$
$$\sin\theta \approx \theta - \theta^3/6 \text{ for } |\theta| \ll 1$$
$$\cos\theta \approx 1 - \theta^2/2 \text{ for } |\theta| \ll 1$$

You may use this sheet for both parts of the exam.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

Question A1

Ping Pong

A thin wire of negligible resistance and total length D is wound to form a thin cylindrical solenoid of length $\ell \ll D$. A conducting sphere of radius $R \ll \ell$ is attached to each end of the solenoid. Initially there is no current in the wire, and the spheres have charges Q and $-Q$. Give all your answers in terms of R, ℓ, D, and the speed of light $c = 1/\sqrt{\mu_0 \epsilon_0}$.

a. Assume this system can be modeled as an LC circuit. What is the angular frequency of its oscillations?

This system loses energy because it emits electromagnetic radiation. Consider an electric dipole consisting of charges $\pm q_0 \cos(\omega t)$ separated by distance d, whose dipole moment oscillates with amplitude $p_0 = q_0 d$. If d is much smaller than the wavelength λ of the radiation produced, then it can be shown that the power radiated is roughly (i.e. up to an order-one dimensionless factor)

$$P \sim \frac{\omega^4 p_0^2}{\epsilon_0 c^3}.$$

For the rest of the problem, your answers only need to be similarly rough estimates.

b. For this setup, the above formula applies if $D \gg D_0$. Find a rough estimate for D_0.

c. Assuming $D \gg D_0$, estimate the number of oscillations that occurs until half the energy is lost.

Question A2

Stellar Stability

A star in hydrostatic equilibrium has inward gravitational forces balanced by pressure gradients. Though the material in a star is not simply an ideal gas, in many cases its pressure P and density ρ are simply related by $P = K\rho^\gamma$ for constants K and γ. Throughout this problem, assume the star is spherically symmetric, its mass is conserved, and relativistic effects can be neglected.

a. A thin shell of the star at radius r_0 has density ρ_0 and thickness Δr, and experiences an inward gravitational field of magnitude g_0.

   i. What is the pressure difference ΔP_0 across the shell in equilibrium?

   ii. Suppose the entire star expands uniformly by a factor $1 + x$, so that the shell now has radius $r = r_0(1 + x)$. In terms of ΔP_0, x, and γ, what is the new pressure difference across it?

   iii. By considering the forces on the shell, write an expression for d^2r/dt^2 valid when x is small, in terms of g_0, γ, and x. For what values of γ will the star be stable?

Next, we consider some simple models of stars, where γ can be computed.

b. In a giant star, the pressure is $P = P_{\text{gas}} + P_{\text{rad}}$, where P_{gas} is due to the gas, which obeys the ideal gas law, and $P_{\text{rad}} \propto T^4$ is due to blackbody radiation. In the star's "radiation zone", P_{rad} is much larger than P_{gas}, but the two have a constant ratio. In this case, what is the value of γ?

c. A white dwarf is composed of electrons and nuclei. The electrons provide the outward pressure, while the nuclei cancel the electrons' charge, and are responsible for most of the mass density. Consider a region of a white dwarf where the number density of electrons is n_e.

   i. The electrons obey the Heisenberg uncertainty principle, $\Delta p\, \Delta x \gtrsim \hbar$, where Δx is the spacing between them, and Δp is the typical momentum that quantum mechanics implies they must have. Find a rough estimate for Δp in terms of n_e and $\hbar$.

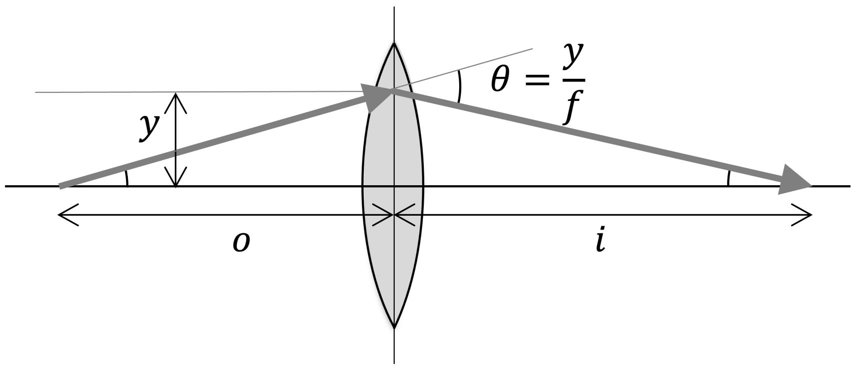
   ii. Using this result, find γ for a white dwarf.

   iii. A white dwarf has total mass M, radius R, and a relatively uniform density of order $\rho \sim M/R^3$. The radius is related to the mass by $R \propto M^n$ for a constant n. Find the value of n.

Question A3

Tilt Shift

An ideal converging lens of focal length f is centered at $x = y = 0$ with its axis of symmetry aligned with the x-axis. A light ray incident at height $y \ll f$ will be tilted inward by an angle $\theta = y/f$. In this problem, we will consider objects at $x = -o$, where $o > f$. The lens will produce a real image of the object at $x = i$, where $1/o + 1/i = 1/f$.



Even for an ideal lens, the image of a finite-sized object will generally be distorted.

a. Consider a pointlike object at $x = -o$ and $y = 0$. If it moves to the right a small distance δ_x, its image moves to the right a distance $m_x\,\delta_x$. If it moves up a small distance δ_y, its image moves up a distance $m_y\,\delta_y$. Find m_x and m_y in terms of i and o.

b. Suppose the object is a short stick, tilted an angle θ_o to the x-axis. In terms of i, o, and θ_o, what is the angle θ_i its image makes with the x-axis?

To produce a simple camera, we put the lens right next to a circular aperture of diameter $D \ll f$, and place a movable screen behind the lens. Suppose the location of the screen is chosen so that light from very distant objects will be focused to a point on the screen.

c. The light from a pointlike object at finite distance o will produce a finite-sized spot of radius r on the screen. Find r in terms of f, D, and o, assuming $o \gg f$.

d. If the camera primarily sees light of wavelength $\lambda \ll f, D$, find a rough estimate for the additional spread r_d of any image on the screen due to diffraction, in terms of f, D, and o.

e. Assuming the typical numbers $f = 5.0\,\mathrm{cm}$, $D = 5.0\,\mathrm{mm}$, and $\lambda = 500\,\mathrm{nm}$, find the numeric values of o for which the blurring due to geometric effects exceeds the blurring due to diffraction.

Real photos are noisy because light is made of discrete photons, with energy $E = hc/\lambda$. Suppose the camera is illuminated uniformly with light of intensity $I = 1\,\mathrm{W/m^2}$, its sensor has $N = 10^7$ pixels, and every photon passing through the aperture is detected, with equal probability, by one pixel in the sensor. This implies that if the expected number of photons arriving at a pixel on the sensor is n, the standard deviation of that number is $\sqrt{n}$.

f. If the aperture opens for time τ to take a photo, find the numeric value of τ for which the standard deviation of the brightness of each pixel is 1% of the mean.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you can review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

The Muon Shot

In 2023, American particle physicists recommended developing a muon collider to investigate the nature of fundamental particles. Such a collider requires much less space then other options because of the muon's high mass m, which makes it easier to accelerate to a very high energy $E \gg mc^2$.

a. When a muon colides head-on with an antimuon, which has the same energy and mass, a new particle of mass $2E/c^2$ can be produced. If the antimuon was instead at rest, what energy would the muon need to produce such a particle?

Unfortunately, muons and antimuons are unstable, with lifetime τ. That is, if one such particle exists at time $t = 0$, then in its rest frame, the probability it has not decayed by time t is $e^{-t/\tau}$.

b. Suppose the muons begin at rest, and are accelerated so that each muon's energy increases at a very large, constant rate α in the lab frame. Find the fraction f of muons that have decayed by the time each muon has energy E, assuming f is small.

The collider produces a "bunch" of muons with energy E, uniformly distributed in a thin disc of radius $R = 10^{-6}$ m. It also simultaneously produces a similar "antibunch" of antimuons. For simplicity, model each muon and antimuon as a sphere of radius $r = 10^{-21}$ m, and suppose a muon-antimuon collision occurs whenever two such spheres touch.

c. Initially, the bunch and antibunch each contain $N = 10^{14}$ particles. If they immediately collide head-on, what is the average number of muon-antimuon collisions, to one significant figure?

d. The bunch travels clockwise along a ring of circumference $\ell = 10$ km, while the antibunch travels along the same path in the opposite direction. Assume all particles maintain a constant energy $E = 10^5\, mc^2$, and that the muon lifetime is $\tau = 2.2 \times 10^{-6}$ s. To one significant figure, what is the average number of muon-antimuon collisions that occur before all of the particles decay?

Question B2

Solid Heat

In classical thermodynamics, a solid containing N atoms has a heat capacity $C_V = 3Nk_B$. The two parts of this question are independent. In both parts, we assume the solid has constant volume.

a. In a simple quantum model of a solid, the energy is $E = \hbar\omega m$, where m is the number of quanta and ω is a constant. Einstein showed that the entropy of such a solid is

$$\frac{S}{k_B} = (3N + m)\ln(3N + m) - m\ln(m)$$

up to a constant. According to the first law of thermodynamics, $dE = T\,dS$ for this system.

i. Find an expression for m in terms of N and the quantity $\alpha = \hbar\omega/k_BT$.

ii. We want to see how quantum effects modify the familiar classical result in the limit $\alpha \ll 1$, where the quantum corrections are small. Write an approximate expression for m, including terms of order α but neglecting terms of order α^2 or higher.

iii. The heat capacity, with its leading quantum correction, is $C_V \approx 3Nk_B(1 + b\alpha^n)$ for some constants b and n. Find the values of b and n.

b. A vertical cylinder is filled with a monatomic ideal gas, and capped by a movable piston. The temperature is high enough for the piston to be modeled as a classical solid. The gas and piston contain the same number of atoms, but the mass of the gas is negligible compared to that of the piston. Assume the entire cylinder is in vacuum, and that the gas and piston do not transfer heat to their environment, but always remain in thermal equilibrium with each other.

i. When the piston is in mechanical equilibrium, the column of gas has height h and pressure $P = P_0$. At this point, find dP/dh in terms of P_0 and h.

ii. If the piston is given a small vertical impulse, what is the angular frequency of its subsequent oscillations? Give your answer solely in terms of h and the gravitational acceleration g.

Question B3

Quality Quest

The quality factor is a dimensionless number which quantifies how efficiently a system stores energy and how strongly it responds on resonance. For a circuit consisting of a capacitor C, an inductor L, and a small resistance R in series, the resonant frequency is approximately $\omega_0 = 1/\sqrt{LC}$, and the quality factor, assumed to be large throughout this problem, is

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

In this problem, we explore several ways to measure Q. Uncertainty analysis is not required.

a. Alice measures Q by seeing how oscillations in the circuit damp over time. Suppose that initially, the charge on the capacitor is q and the current is zero. The next time the current is zero, the charge is $-q(1-\delta)$. Find an approximate expression for δ, in terms of ω_0 and Q.

b. Bob and Charles drive their circuits with a sinusoidal voltage $V(t) = V_0 \cos \omega t$. It can be shown that in the steady state, the voltage across the capacitor oscillates with amplitude

$$V_c = \frac{V_0}{\sqrt{(1-\omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2}}.$$

The circuits Bob and Charles have are similar, but are not precisely the same.

i. Bob fixes the value of V_0 so that the highest value of V_c at any frequency is precisely 10.00 V. His equipment can precisely compare the amplitudes of a small DC and AC voltage. He thus performs two very accurate voltage measurements.

ω (rad/s)	0.0	183.3
V_c (Volts)	0.1219	0.1219

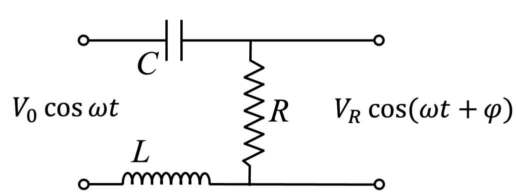
Using this data, find the numeric values of Q and ω_0 as accurately as possible.

ii. Charles can precisely tune ω, but cannot precisely measure small voltages. He thus fixes V_0 to some other value and takes data near the resonance, where V_c is relatively large.

ω (rad/s)	133.0	133.5	134.0	134.5	135.0	135.5	136.0	136.5	137.0
V_c (Volts)	3.64	4.76	6.52	8.53	8.18	6.06	4.44	3.42	2.75

Using this data, find the numeric values of Q and ω_0 as accurately as possible. (Hint: you may use the graph paper in the answer sheets, but full credit is attainable without graphing. To find Q, you should first find ω_0, then simplify the equation above using $\omega \approx \omega_0$.)

c. The gain function of this circuit is defined as $G = V_R/V_0$, where V_R is the amplitude of the voltage across the resistor, as shown below.



i. Find an expression for G in terms of ω, ω_0, and Q.

ii. This setup can be used to reject voltages at certain frequencies. Qualitatively describe the range(s) of frequencies for which G is small.

Page **Problem** Student Name: Student AAPT number:

Additional page: start additional page numbers from 14

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Fundamental Constants

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A
$c = 3.00 \times 10^8$ m/s	$k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K
$N_\mathrm{A} = 6.02 \times 10^{23}$ (mol)$^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31$ J/(mol $\cdot$ K)
$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)	$e = 1.602 \times 10^{-19}$ C
1 eV $= 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s
$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2	

Useful Approximations

$$(1+x)^n \approx 1 + nx + n(n-1)x^2/2 \text{ for } |nx| \ll 1$$
$$e^x \approx 1 + x + x^2/2 + x^3/6 \text{ for } |x| \ll 1$$
$$\sin\theta \approx \theta - \theta^3/6 \text{ for } |\theta| \ll 1$$
$$\cos\theta \approx 1 - \theta^2/2 \text{ for } |\theta| \ll 1$$

You may use this sheet for both parts of the exam.

End of Instructions for the Student

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

Part A

Question A1

Ping Pong

A thin wire of negligible resistance and total length D is wound to form a thin cylindrical solenoid of length $\ell \ll D$. A conducting sphere of radius $R \ll \ell$ is attached to each end of the solenoid. Initially there is no current in the wire, and the spheres have charges Q and $-Q$. Give all your answers in terms of R, ℓ, D, and the speed of light $c = 1/\sqrt{\mu_0 \epsilon_0}$.

a. Assume this system can be modeled as an LC circuit. What is the angular frequency of its oscillations?

Solution

The capacitance of this system is

$$C = \frac{Q}{\Delta V} = \frac{Q}{Q/(2\pi\epsilon_0 R)} \approx 2\pi\epsilon_0 R$$

since the potential on the spheres is approximately $\pm Q/4\pi\epsilon_0 R$. Its inductance can be found by considering the magnetic field energy,

$$\frac{1}{2}LI^2 = \frac{1}{2\mu_0}\int B^2\,dV.$$

If the radius of the solenoid is a, then $D = 2\pi a n \ell$, where n is the number of turns per length. Then we have

$$\int B^2\,dV = (\mu_0 n I)^2 \pi a^2 \ell = \frac{\mu_0 I^2 D^2}{4\pi\ell}$$

which implies an inductance

$$L = \frac{\mu_0 D^2}{4\pi\ell}.$$

The angular frequency of LC oscillations is

$$\omega = \frac{1}{\sqrt{LC}} = \frac{c}{D}\sqrt{\frac{2\ell}{R}}.$$

We weren't told the values of a and n, but the dependence on them simply dropped out.

This system loses energy because it emits electromagnetic radiation. Consider an electric dipole consisting of charges $\pm q_0 \cos(\omega t)$ separated by distance d, whose dipole moment oscillates with amplitude $p_0 = q_0 d$. If d is much smaller than the wavelength λ of the radiation produced, then it can be shown that the power radiated is roughly (i.e. up to an order-one dimensionless factor)

$$P \sim \frac{\omega^4 p_0^2}{\epsilon_0 c^3}.$$

For the rest of the problem, your answers only need to be similarly rough estimates.

b. For this setup, the above formula applies if $D \gg D_0$. Find a rough estimate for D_0.

Solution

The radiation produced has angular frequency ω, so wavelength

$$\lambda \sim \frac{c}{\omega} \sim D\sqrt{R/\ell}.$$

The Larmor formula works when $\ell \ll \lambda$, which corresponds to $D \gg D_0 \sim \sqrt{\ell^3/R}$.

c. Assuming $D \gg D_0$, estimate the number of oscillations that occurs until half the energy is lost.

Solution

The total stored energy is of order

$$E \sim \frac{Q^2}{C} \sim \frac{Q^2}{\epsilon_0 R}.$$

Thus, the typical number N of cycles for the energy to decay is approximately the inverse of the fraction of the energy that is radiated away in each cycle, so

$$N \sim \frac{E}{P/\omega} \sim \frac{Q^2/\epsilon_0 R}{\omega^3 (Q\ell)^2/\epsilon_0 c^3} \sim \frac{(c/\omega)^3}{R\ell^2} \sim \sqrt{\frac{RD^6}{\ell^7}}.$$

This is roughly the quality factor of the LC circuit. Note that we implicitly assumed above that many oscillations occur before half the energy is lost. This assumption made sense because we know $N \gg \sqrt{RD_0^6/\ell^7} \sim \ell/R \gg 1$.

Part (a) of this problem was inspired by problem 19.16 of Zangwill's *Modern Electrodynamics*. Compared to that problem, we replaced a straight connecting wire with a solenoid, which makes the problem a bit easier, and allows the LC circuit description to work for a broader range of parameters.

Question A2

Stellar Stability

A star in hydrostatic equilibrium has inward gravitational forces balanced by pressure gradients. Though the material in a star is not simply an ideal gas, in many cases its pressure P and density ρ are simply related by $P = K\rho^\gamma$ for constants K and γ. Throughout this problem, assume the star is spherically symmetric, its mass is conserved, and relativistic effects can be neglected.

a. A thin shell of the star at radius r_0 has density ρ_0 and thickness Δr, and experiences an inward gravitational field of magnitude g_0.

i. What is the pressure difference ΔP_0 across the shell in equilibrium?

Solution

The mass of the shell is $m = \rho_0 A_0\, \Delta r$, where $A_0 = 4\pi r_0^2$ is its surface area. The total inward gravitational force is $g_0 m$, and the total outward pressure force is $A_0\, \Delta P_0$. Equating them yields

$$\Delta P_0 = g_0 \rho_0\, \Delta r.$$

ii. Suppose the entire star expands uniformly by a factor $1 + x$, so that the shell now has radius $r = r_0(1 + x)$. In terms of ΔP_0, x, and γ, what is the new pressure difference across it?

Solution

The density scales as $\rho \propto 1/(1 + x)^3$, so the pressure everywhere in the star is scaled by the factor $\rho^\gamma \propto 1/(1 + x)^{3\gamma}$. Thus, the new pressure difference is $\Delta P_0/(1 + x)^{3\gamma}$.

iii. By considering the forces on the shell, write an expression for d^2r/dt^2 valid when x is small, in terms of g_0, γ, and x. For what values of γ will the star be stable?

Solution

The radial form of Newton's second law for the shell is

$$m\frac{d^2r}{dt^2} = F_{\text{pr}} - F_{\text{gr}}$$

where the outward pressure force is

$$F_{\text{pr}} = A\, \Delta P = A_0\, \Delta P_0\, \frac{(1 + x)^2}{(1 + x)^{3\gamma}}$$

since area is proportional to r^2, and the inward gravitational force is

$$F_{\text{gr}} = gm = \frac{g_0 m}{(1 + x)^2}$$

since gravity obeys an inverse-square law. Applying part 1(a) and simplifying gives

$$\frac{d^2r}{dt^2} = g_0\left(\frac{(1+x)^2}{(1+x)^{3\gamma}} - \frac{1}{(1+x)^2}\right) \approx g_0(4-3\gamma)x.$$

The star is stable when this is a restoring force, which requires $\gamma > 4/3$.

Next, we consider some simple models of stars, where γ can be computed.

b. In a giant star, the pressure is $P = P_{\text{gas}} + P_{\text{rad}}$, where P_{gas} is due to the gas, which obeys the ideal gas law, and $P_{\text{rad}} \propto T^4$ is due to blackbody radiation. In the star's "radiation zone", P_{rad} is much larger than P_{gas}, but the two have a constant ratio. In this case, what is the value of γ?

Solution

The density ρ of the star is due to the gas, and the ideal gas law states that $P_{\text{gas}} \propto \rho T$. On the other hand, if the two pressure contributions have a fixed ratio, then $P_{\text{gas}} \propto P_{\text{rad}} \propto T^4$. Combining these results gives $\rho \propto T^3$ and $P \propto T^4$, so that $\gamma = 4/3$.

c. A white dwarf is composed of electrons and nuclei. The electrons provide the outward pressure, while the nuclei cancel the electrons' charge, and are responsible for most of the mass density. Consider a region of a white dwarf where the number density of electrons is n_e.

i. The electrons obey the Heisenberg uncertainty principle, $\Delta p\, \Delta x \gtrsim \hbar$, where Δx is the spacing between them, and Δp is the typical momentum that quantum mechanics implies they must have. Find a rough estimate for Δp in terms of n_e and $\hbar$.

Solution

The spacing between electrons is $\Delta x \sim n_e^{-1/3}$, so $\Delta p \sim \hbar\, n_e^{1/3}$.

ii. Using this result, find γ for a white dwarf.

Solution

Since the white dwarf is electrically neutral, the density of nuclei is proportional to the density of electrons, so the mass density ρ is proportional to n_e. As for the pressure, basic kinetic theory shows that it scales as the product of the number density n_e, the momentum $\Delta p \sim n_e^{1/3}$, and the typical speed $\Delta v \sim \Delta p/m \sim n_e^{1/3}$. Thus, $P \propto n_e^{5/3}$, so $\gamma = 5/3$.

iii. A white dwarf has total mass M, radius R, and a relatively uniform density of order $\rho \sim M/R^3$. The radius is related to the mass by $R \propto M^n$ for a constant n. Find the value of n.

Solution

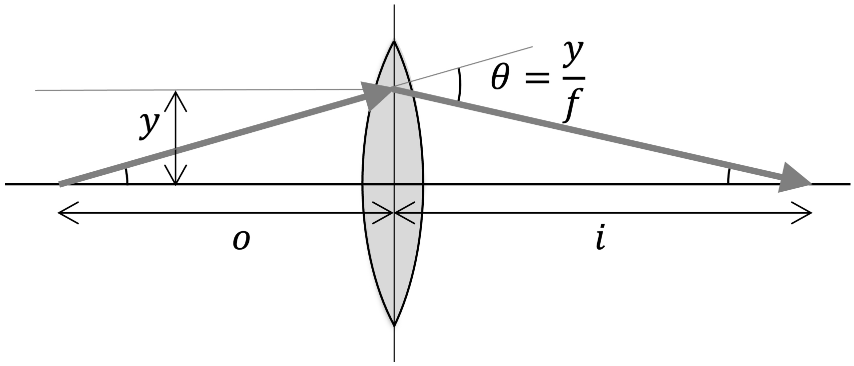
The pressure at the center is proportional to G and otherwise depends only on M and R. Thus, by dimensional analysis, $P \sim GM^2/R^4$. (This argument assumes the density is relatively uniform. By contrast, a typical star has a very large, low-density envelope surrounding a dense core of radius $R_{\text{core}} \ll R$, so the correct estimate would be $P \sim GM^2/R_{\text{core}}^4$. That is, the assumption of relatively uniform density just means that in the dimensional analysis, the only relevant length is the total radius R.)

We just showed that the electrons provide pressure $P \propto \rho^{5/3} \sim M^{5/3}/R^5$. Equating these two yields $R \propto M^{-1/3}$, so $n = -1/3$. Perhaps surprisingly, more massive white dwarfs are smaller.

Question A3

Tilt Shift

An ideal converging lens of focal length f is centered at $x = y = 0$ with its axis of symmetry aligned with the x-axis. A light ray incident at height $y \ll f$ will be tilted inward by an angle $\theta = y/f$. In this problem, we will consider objects at $x = -o$, where $o > f$. The lens will produce a real image of the object at $x = i$, where $1/o + 1/i = 1/f$.



Even for an ideal lens, the image of a finite-sized object will generally be distorted.

a. Consider a pointlike object at $x = -o$ and $y = 0$. If it moves to the right a small distance δ_x, its image moves to the right a distance $m_x\,\delta_x$. If it moves up a small distance δ_y, its image moves up a distance $m_y\,\delta_y$. Find m_x and m_y in terms of i and o.

Solution

To compute m_y, it suffices to draw a ray going from $(-o, \delta_y)$ through the center of the lens until it hits the image plane at $x = i$. From similar triangles, we immediately conclude $m_y = -i/o$. To find m_x, we take the differential of the lens equation, giving

$$\frac{do}{o^2} + \frac{di}{i^2} = 0.$$

The quantity m_x is just $-di/do$, so we read off $m_x = i^2/o^2$.

b. Suppose the object is a short stick, tilted an angle θ_o to the x-axis. In terms of i, o, and θ_o, what is the angle θ_i its image makes with the x-axis?

Solution

This follows immediately from the previous part. We have

$$\tan\theta_o = \frac{\delta_y}{\delta_x}, \qquad \tan\theta_i = \frac{m_y\delta_y}{m_x\delta_x}$$

from which we conclude

$$\theta_i = \tan^{-1}\left(-\frac{o}{i}\,\theta_o\right).$$

This is equivalent to the statement that if you extend the stick and its image, then they will meet at the lens plane $x = 0$, which is called the Scheimpflug principle. It is used in "tilt shift" photography to produce focused images of objects tilted relative to the camera's plane, by tilting the camera's screen.

To produce a simple camera, we put the lens right next to a circular aperture of diameter $D \ll f$, and place a movable screen behind the lens. Suppose the location of the screen is chosen so that light from very distant objects will be focused to a point on the screen.

c. The light from a pointlike object at finite distance o will produce a finite-sized spot of radius r on the screen. Find r in terms of f, D, and o, assuming $o \gg f$.

Solution

Such an object produces an image at

$$i = \frac{of}{o - f} \approx f + f^2/o.$$

Since we are assuming the camera is focused on infinitely distant objects, the screen is a distance f from the lens, so the image of this object is f^2/o behind the lens. By drawing similar triangles, we conclude $r = Df/(2o)$.

Alternatively, by drawing similar triangles (or by doing a bit more algebra), you can show that this is the exact answer: $r = D/2 \cdot (i - f)/i = Df/(2o)$, without approximating $o \gg f$.

d. If the camera primarily sees light of wavelength $\lambda \ll f, D$, find a rough estimate for the additional spread r_d of any image on the screen due to diffraction, in terms of f, D, and o.

Solution

In general, diffraction will spread out light in an angle $\theta \sim \lambda/D$. Thus, it will arrive at the screen spread out by $r_d \sim f\lambda/D$. Any answer within an order of magnitude is acceptable.

e. Assuming the typical numbers $f = 5.0\,\text{cm}$, $D = 5.0\,\text{mm}$, and $\lambda = 500\,\text{nm}$, find the numeric values of o for which the blurring due to geometric effects exceeds the blurring due to diffraction.

Solution

Setting our previous two expressions equal gives $o \sim D^2/(2\lambda) = 25\,\text{m}$. So for objects at distance $o < 25\,\text{m}$, the geometric blurring dominates. Any answer within an order of magnitude is acceptable.

Real photos are noisy because light is made of discrete photons, with energy $E = hc/\lambda$. Suppose the camera is illuminated uniformly with light of intensity $I = 1\,\text{W/m}^2$, its sensor has $N = 10^7$ pixels, and every photon passing through the aperture is detected, with equal probability, by one pixel in the sensor. This implies that if the expected number of photons arriving at a pixel on the sensor is n, the standard deviation of that number is $\sqrt{n}$.

f. If the aperture opens for time τ to take a photo, find the numeric value of τ for which the standard deviation of the brightness of each pixel is 1% of the mean.

Solution

On average, the number of photons hitting each pixel is

$$N_\gamma = \frac{I\tau(\pi D^2/4)}{NE}.$$

For the standard deviation to be 1% of the mean, we need N_γ to be at least 10^4. Plugging in the numbers yields $\tau = 2\,\text{ms}$, which is a typical camera shutter speed in good lighting.

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you can review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Once you start Part B, you will not be able to return to Part A.

Part B

Question B1

The Muon Shot

In 2023, American particle physicists recommended developing a muon collider to investigate the nature of fundamental particles. Such a collider requires much less space then other options because of the muon's high mass m, which makes it easier to accelerate to a very high energy $E \gg mc^2$.

a. When a muon colides head-on with an antimuon, which has the same energy and mass, a new particle of mass $2E/c^2$ can be produced. If the antimuon was instead at rest, what energy would the muon need to produce such a particle?

Solution

The shortest solution involves setting $c = 1$ and using four-vectors. In the original situation, the muon and antimuon have four-momenta p_1^μ and p_2^μ, where

$$p_1 \cdot p_2 = (E, p) \cdot (E, -p) = E^2 + p^2.$$

To have an equivalent collision where the antimuon is at rest, we boost this configuration to the antimuon's rest frame, where $p_1'^\mu = (E', p')$ and $p_2'^\mu = (m, 0)$. The inner product of the four-vectors stays the same, so

$$p_1' \cdot p_2' = E'm = p_1 \cdot p_2.$$

Since $E \gg m$, we have $p^2 \approx E^2$, so solving for E' and restoring the factors of c gives

$$E' = \frac{2E^2}{mc^2}.$$

This is much greater than E, so it is practical to accelerate the antimuon as well. Incidentally, you can also solve the problem exactly, in which case you'll get

$$E' = \frac{2E^2}{mc^2} - mc^2 \approx \frac{2E^2}{mc^2}.$$

Either answer is acceptable.

Unfortunately, muons and antimuons are unstable, with lifetime τ. That is, if one such particle exists at time $t = 0$, then in its rest frame, the probability it has not decayed by time t is $e^{-t/\tau}$.

b. Suppose the muons begin at rest, and are accelerated so that each muon's energy increases at a very large, constant rate α in the lab frame. Find the fraction f of muons that have decayed by the time each muon has energy E, assuming f is small.

Solution

Continuing to set $c = 1$, the energy of the muon in the lab frame is $E(t) = \alpha t + m$, but we also know that $E(t) = \gamma(t)m$, so that

$$\gamma = 1 + \frac{\alpha t}{m}.$$

The acceleration process begins at time $t = 0$, and ends at time $t_f = (E - m)/\alpha$. Accounting for time dilation, the amount of time that elapses in the muon's frame is

$$\tau_f = \int_0^{t_f} \frac{dt}{\gamma} = \frac{m}{\alpha} \int_0^{t_f} \frac{1}{t + m/\alpha} = \frac{m}{\alpha} \ln \frac{E}{m}.$$

To relate this to f, we note that

$$f = 1 - e^{-\tau_f/\tau} \approx \frac{\tau_f}{\tau}$$

where the second step uses the assumption that f is small. Therefore,

$$f = \frac{mc^2}{\alpha\tau} \ln \frac{E}{mc^2}$$

where we restored the factors of c.

The collider produces a "bunch" of muons with energy E, uniformly distributed in a thin disc of radius $R = 10^{-6}$ m. It also simultaneously produces a similar "antibunch" of antimuons. For simplicity, model each muon and antimuon as a sphere of radius $r = 10^{-21}$ m, and suppose a muon-antimuon collision occurs whenever two such spheres touch.

c. Initially, the bunch and antibunch each contain $N = 10^{14}$ particles. If they immediately collide head-on, what is the average number of muon-antimuon collisions, to one significant figure?

Solution

Consider one muon and antimuon. A collision occurs when their centers are separated by less than $2r$. Fixing the location of the muon, the probability that the antimuon is within the appropriate area is approximately $\pi(2r)^2/(\pi R^2) = (2r/R)^2$, since $r \ll R$. Each pair of muons and antimuons has the same chance to collide, so the expected number of collision events is

$$\left(\frac{2rN}{R}\right)^2 = 0.04.$$

This is much smaller than N, which justifies our assumption that the collision events are independent. It might seem odd for the answer to be less than 1, but this is desired, as having many collisions occur at once would make it hard to see what happens in each collision.

d. The bunch travels clockwise along a ring of circumference $\ell = 10$ km, while the antibunch travels along the same path in the opposite direction. Assume all particles maintain a constant energy $E = 10^5\, mc^2$, and that the muon lifetime is $\tau = 2.2 \times 10^{-6}$ s. To one significant figure, what is the average number of muon-antimuon collisions that occur before all of the particles decay?

Solution

If the number of particles remaining in the bunch and the antibunch is N_k, where $k = 0$ for the first collision, then the expected number of collisions is

$$\left(\frac{2rN}{R}\right)^2 \sum_{k=0}^{\infty} \left(\frac{N_k}{N}\right)^2.$$

We found above that only a small number of collisions occurs per bunch-antibunch crossing, so the decrease in N_k over time is almost entirely due to decay. A crossing occurs every time each bunch or antibunch traverses half of the ring, corresponding to a proper time increment

$$\Delta\tau = \frac{\ell/(2c)}{10^5} = 1.67 \times 10^{-10}\,\text{s}.$$

Therefore, since $\tau \gg \Delta\tau$, we have

$$\sum_{k=0}^{\infty} \left(\frac{N_k}{N}\right)^2 = \sum_{k=0}^{\infty} e^{-2k\,\Delta\tau/\tau} = \frac{1}{1 - e^{-2\,\Delta\tau/\tau}} \approx \frac{\tau}{2\,\Delta\tau} = 6600.$$

The expected total number of collisions is 260, which rounds to 300.

The rough numbers given here correspond to a muon collider which would be able to probe new particles 10 times as heavy as those probed at the existing Large Hadron Collider. Roughly one in a million muon-antimuon collisions yield a Higgs boson, so that an enormous number of them can be produced for detailed study. Remarkably, such a muon collider could be *smaller* in size than the LHC, while other proposals involving electrons or protons would need to be about 10 times longer. However, given the muon's short lifetime, it may be very hard to create the required focused muon beams. The feasibility of this "muon shot" is currently being investigated by particle physicists around the world.

One of us (TB) thanks Nathaniel Craig, Andrew Fee and Sergo Jindariani for discussions during our work on this problem.

Question B2

Solid Heat

In classical thermodynamics, a solid containing N atoms has a heat capacity $C_V = 3Nk_B$. The two parts of this question are independent. In both parts, we assume the solid has constant volume.

a. In a simple quantum model of a solid, the energy is $E = \hbar\omega m$, where m is the number of quanta and ω is a constant. Einstein showed that the entropy of such a solid is

$$\frac{S}{k_B} = (3N+m)\ln(3N+m) - m\ln(m)$$

up to a constant. According to the first law of thermodynamics, $dE = T\,dS$ for this system.

i. Find an expression for m in terms of N and the quantity $\alpha = \hbar\omega/k_BT$.

Solution

Writing both sides of $dE = T\,dS$ in terms of dm gives

$$dE = \hbar\omega\,dm$$

and

$$T\,dS = k_BT\ln\left(\frac{3N+m}{m}\right)dm.$$

Equating these and solving for m gives

$$m = \frac{3N}{e^\alpha - 1}.$$

ii. We want to see how quantum effects modify the familiar classical result in the limit $\alpha \ll 1$, where the quantum corrections are small. Write an approximate expression for m, including terms of order α but neglecting terms of order α^2 or higher.

Solution

We Taylor expand the exponential, for

$$m = \frac{3N}{\alpha + \alpha^2/2 + \alpha^3/6 + \ldots} = \frac{3N}{\alpha}\frac{1}{1 + \alpha/2 + \alpha^2/6 + \ldots}.$$

Since there's a $1/\alpha$ in front, we need to expand the fraction to order α^2, which means we need to use the geometric series formula to *second* order,

$$\frac{1}{1+x} = 1 - x + x^2 + \ldots$$

where here $x = \alpha/2 + \alpha^2/6$. This gives the final answer,

$$m = 3N\left(\frac{1}{\alpha} - \frac{1}{2} + \frac{\alpha}{12}\right)$$

iii. The heat capacity, with its leading quantum correction, is $C_V \approx 3Nk_B(1 + b\alpha^n)$ for some constants b and n. Find the values of b and n.

Solution

The heat capacity is

$$C_V = \frac{dE}{dT} = \hbar\omega \frac{dm}{dT} \approx 3N\hbar\omega \frac{d}{dT}\left(\frac{k_B T}{\hbar\omega} - \frac{1}{2} + \frac{\hbar\omega}{12k_B T}\right).$$

Carrying out the derivative yields

$$C_V = 3Nk_B\left(1 - \frac{\alpha^2}{12}\right)$$

from which we read off $b = -1/12$ and $n = 2$.

b. A vertical cylinder is filled with a monatomic ideal gas, and capped by a movable piston. The temperature is high enough for the piston to be modeled as a classical solid. The gas and piston contain the same number of atoms, but the mass of the gas is negligible compared to that of the piston. Assume the entire cylinder is in vacuum, and that the gas and piston do not transfer heat to their environment, but always remain in thermal equilibrium with each other.

i. When the piston is in mechanical equilibrium, the column of gas has height h and pressure $P = P_0$. At this point, find dP/dh in terms of P_0 and h.

Solution

If n is the number of moles of gas and solid, then the heat capacity of the gas-piston system, at constant volume and pressure, is

$$C_V = \frac{3}{2}nR + 3nR, \quad C_P = \frac{5}{2}nR + 3nR$$

because the solid does not expand, and hence its contribution to C_P is equal to its contribution to C_V. Thus, the adiabatic index is

$$\gamma = \frac{C_P}{C_V} = \frac{11}{9}.$$

Since the system doesn't transfer heat to its environment, the quantity PV^γ remains the same throughout the oscillations, where V is the volume of the gas. Taking the differential of this relation, the pressure in the gas varies as

$$\frac{dP}{dh} = -\frac{\gamma P_0}{h} = -\frac{11P_0}{9h}.$$

ii. If the piston is given a small vertical impulse, what is the angular frequency of its subsequent oscillations? Give your answer solely in terms of h and the gravitational acceleration g.

Solution

The net force on the piston, as a function of vertical displacement z, is

$$F_z = -\frac{\gamma P_0 A}{h}\, z$$

where A is the piston's cross-sectional area. Newton's second law for the piston is

$$F_z = m\frac{d^2 z}{dt^2}$$

where m is the mass of the piston. Combining the two previous equations yields simple harmonic motion with

$$\omega^2 = \frac{\gamma P_0 A}{mh}.$$

To eliminate the unwanted parameters, we note that since the piston was originally in mechanical equilibrium, we have $P_0 A = mg$, from which we conclude

$$\omega = \sqrt{\frac{\gamma g}{h}} = \sqrt{\frac{11g}{9h}}.$$

Question B3

Quality Quest

The quality factor is a dimensionless number which quantifies how efficiently a system stores energy and how strongly it responds on resonance. For a circuit consisting of a capacitor C, an inductor L, and a small resistance R in series, the resonant frequency is approximately $\omega_0 = 1/\sqrt{LC}$, and the quality factor, assumed to be large throughout this problem, is

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

In this problem, we explore several ways to measure Q. Uncertainty analysis is not required.

a. Alice measures Q by seeing how oscillations in the circuit damp over time. Suppose that initially, the charge on the capacitor is q and the current is zero. The next time the current is zero, the charge is $-q(1-\delta)$. Find an approximate expression for δ, in terms of ω_0 and Q.

Solution

In the absence of resistance, the charge on the capacitor and the current are

$$q_C(t) = q\cos(\omega_0 t), \quad I(t) = \frac{dq_C(t)}{dt} = -q\omega_0 \sin(\omega_0 t).$$

The total energy dissipated in the resistor is approximately

$$\Delta E = \int_0^{\pi/\omega_0} I(t)^2 R\, dt = q^2\omega_0^2 R \int_0^{\pi/\omega_0} \sin^2(\omega_0 t)\, dt = \frac{\pi}{2} q^2 \omega_0 R.$$

On the other hand, we also have

$$\Delta E = \frac{q^2 - (q(1-\delta))^2}{2C} \approx \frac{q^2\delta}{C}.$$

Equating the two yields

$$\delta = \frac{\pi}{2}\omega_0 RC = \frac{\pi}{2Q}.$$

b. Bob and Charles drive their circuits with a sinusoidal voltage $V(t) = V_0 \cos\omega t$. It can be shown that in the steady state, the voltage across the capacitor oscillates with amplitude

$$V_c = \frac{V_0}{\sqrt{(1-\omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2}}.$$

The circuits Bob and Charles have are similar, but are not precisely the same.

i. Bob fixes the value of V_0 so that the highest value of V_c at any frequency is precisely 10.00 V. His equipment can precisely compare the amplitudes of a small DC and AC voltage. He thus performs two very accurate voltage measurements.

ω (rad/s)	0.0	183.3
V_c (Volts)	0.1219	0.1219

Using this data, find the numeric values of Q and ω_0 as accurately as possible.

Solution

Note that at $\omega = 0$, we simply have $V_c = V_0$, while on resonance, we have $V_c = QV_0$. Thus, in this case we can directly read off the quality factor as

$$Q = \frac{10.00}{0.1219} = 82.0.$$

As for the other data point, it also has $V_c = V_0$, which implies

$$1 = (1 - \omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2.$$

This can be simplified by recognizing a difference of squares, giving

$$(\omega^2/\omega_0^2)(2 - \omega^2/\omega_0^2) = \omega^2/\omega_0^2 Q^2$$

and solving for ω_0 gives

$$\omega_0 = \frac{\omega}{\sqrt{2 - 1/Q^2}} = 129.6\,\text{rad/s}.$$

Alternatively, since we know Q, we know that the $(\omega/\omega_0 Q)^2$ term is negligible, to the precision at which we're working. Simply dropping that term gives $\omega_0 = \omega/\sqrt{2}$, which gives the same numeric answer, up to the four significant figures used in this part.

ii. Charles can precisely tune ω, but cannot precisely measure small voltages. He thus fixes V_0 to some other value and takes data near the resonance, where V_c is relatively large.

ω (rad/s)	133.0	133.5	134.0	134.5	135.0	135.5	136.0	136.5	137.0
V_c (Volts)	3.64	4.76	6.52	8.53	8.18	6.06	4.44	3.42	2.75

Using this data, find the numeric values of Q and ω_0 as accurately as possible. (Hint: you may use the graph paper in the answer sheets, but full credit is attainable without graphing. To find Q, you should first find ω_0, then simplify the equation above using $\omega \approx \omega_0$.)

Solution

The maximum value of V_c is attained at ω_0. Looking at the data, we can see that ω_0 is between 134.5 and 135.0 rad/s, and slightly closer to the former; we therefore take $\omega_0 = 134.7\,\text{rad/s}$. This result is already as precise as the precision of the data allows, so there's no point in trying to improve it further. Any answer within 0.1 rad/s is acceptable.

The next step is to extract Q. Since all the data is taken near resonance, $\omega \approx \omega_0$, we have

$$\frac{V_0^2}{V_c^2} = (1 - \omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2 \approx 4(1 - \omega/\omega_0)^2 + \frac{1}{Q^2}.$$

We don't know V_0^2, so we divide through to get

$$\frac{1}{V_c^2} = \frac{4}{V_0^2}(1 - \omega/\omega_0)^2 + \frac{1}{V_0^2 Q^2}.$$

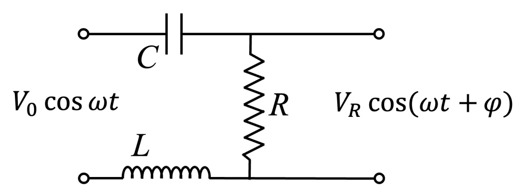
So plotting $1/V_c^2$ vs. $(1-\omega/\omega_0)^2$ gives a line with slope $m = 4/V_0^2$ and intercept $b = 1/V_0^2 Q^2$, from which we find $Q = \sqrt{m/4b}$. It would take too long to apply this to all of the data points, so let's just select one near the resonance and one at the ends, as such a pair yields maximal sensitivity to the slope and intercept.

ω (rad/s)	V_c (V)	$(1-\omega/\omega_0)^2$	$1/V_c^2$ (V^{-2})
134.5	8.53	2.205×10^{-6}	0.01374
137.0	2.75	2.916×10^{-4}	0.1322

The slope of this line is $m = 409.3\,\mathrm{V}^{-2}$, from which we compute an intercept $b = 0.01284\,\mathrm{V}^{-2}$, and a quality factor $Q = 89.3$. This is a very rough analysis, but it gets pretty close to the true answer of $Q = 88.0$.

There are many ways to do this problem. You can get decent accuracy (i.e. within ± 10 of the true answer) just by eyeballing the graph. A good result will be within ± 3 of the true answer. The analysis above is more than twice as precise as that, and you can do even better by repeating the analysis using the point at the other end, $\omega = 133.0\,\mathrm{rad/s}$, and averaging the results. Ideally, this will give you the same answer, but in practice there are errors on ω_0 and V_c at $\omega = 134.5\,\mathrm{rad/s}$, which averaging would partially cancel out.

c. The gain function of this circuit is defined as $G = V_R/V_0$, where V_R is the amplitude of the voltage across the resistor, as shown below.



i. Find an expression for G in terms of ω, ω_0, and Q.

Solution

This is simplest using complex impedances. We note that

$$G = \frac{V_R}{V_C}\frac{V_C}{V_0} = \left|\frac{Z_R}{Z_C}\right|\frac{V_C}{V_0} = \omega RC\,\frac{V_C}{V_0} = \frac{\omega/(\omega_0 Q)}{\sqrt{(1-\omega^2/\omega_0^2)^2 + (\omega/\omega_0 Q)^2}}.$$

Simplifying a bit gives the final answer,

$$G = \frac{1}{\sqrt{1 + ((\omega_0 Q/\omega)(1-\omega^2/\omega_0^2))^2}}$$

ii. This setup can be used to reject voltages at certain frequencies. Qualitatively describe the range(s) of frequencies for which G is small.

Solution

This setup is a frequency filter. The gain G is small everywhere except for a narrow band of frequencies around ω_0, whose width is of order ω_0/Q.