Relativity I: Kinematics

Chapter 11 of Morin is a clear, comprehensive, highly recommended introduction to relativistic kinematics. Also read sections 13.1–13.3 for four-vectors, and see appendices F, G, H, and I for enrichment. Alternatively, see chapter 12 and sections 14.1–14.3 of Kleppner and Kolenkow, chapter 11 of Wang and Ricardo, volume 2, or Morin's newer book, *Special Relativity: For the Enthusiastic Beginner*, which covers similar ground with slightly more detail. An entertaining introduction is also given in chapters I-15 through I-17 of the Feynman lectures. To learn about tests of special relativity, see *The Special Theory of Relativity* by Christodoulides. There is a total of **81** points.

1 Lorentz Transformations

Special relativity is uniquely subtle among introductory physics topics, and requires a solid, detailed introduction. This problem set assumes you've already done that, by reading chapter 11 of Morin or the equivalent in another book. (The short chapter in Halliday, Resnick, and Krane is not sufficient.)

Idea 1: Lorentz Transformation

Let S' be the frame of an observer moving to the right with velocity $v\hat{\mathbf{x}}$ with respect to the frame S. Then the coordinates in S and S' are related by the Lorentz transformation

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

This implies that the lengths of moving objects are contracted by γ, moving clocks run slow by a factor of γ, and that if two clocks are synchronized in the frame S' and separated by a distance L, then in the frame S the rear clock is ahead by Lv/c^2.

[**2**] **Problem 1** (Morin 11.2). Two planets, A and B, are at rest with respect to each other, a distance L apart, with synchronized clocks. A spaceship flies at speed v past planet A toward planet B and synchronizes its clock with A's right when it passes A (they both set their clocks to zero). The spaceship eventually flies past planet B and compares its clock with B's. We know, from working in the planets' frame, that when the spaceship reaches B, B's clock reads L/v. And the spaceship's clock reads $L/\gamma v$, because it runs slow by a factor of γ when viewed in the planets' frame.

How would someone on the spaceship quantitatively explain to you why B's clock reads L/v (which is *more* than its own $L/\gamma v$), considering that the spaceship sees B's clock running *slow*?

[**2**] **Problem 2** (Morin 11.4). A stick of (proper) length L moves past you at speed v. There is a time interval between the front end coinciding with you and the back end coinciding with you. What is this time interval in:

(a) your frame? (Calculate this by working in your frame.)

(b) the stick's frame? (Work in the stick's frame.)

(c) your frame? (Work in the stick's frame.)

(d) the stick's frame? (Work in your frame. This is the tricky one.)

[2] **Problem 3** (Morin 11.9)**.** Two balls move with speed v along a line toward two people standing along the same line. The proper distance between the balls is γL, and the proper distance between the people is L. Due to length contraction, the people measure the distance between the balls to be L, so the balls pass the people simultaneously (as measured by the people), as shown.

Assume that the people's clocks both read zero at this time. If the people catch the balls, then the resulting proper distance between the balls becomes L, which is shorter than the initial proper distance of γL. Your task is to explain how the proper distance between the balls decreases from γL to L, by working in the frame where the balls are initially at rest.

(a) Draw the beginning and ending pictures for the process. Indicate the readings on both clocks in the two pictures, and label all relevant lengths.

(b) Explain in words how the proper distance between the balls decreases from γL to L.

[3] **Problem 4.** USAPhO 2016, problem A3. Print out the custom answer sheet before starting.

[5] **Problem 5.** IPhO 2006, problem 2. A nice problem about relativistic visual effects.

2 Velocity Addition

Idea 2: Velocity Addition

Again, let frame S' moves with velocity $v\hat{\mathbf{x}}$ with respect to frame S. If an object has velocity (u'_x, u'_y) in frame S', then the velocity in S is

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}, \quad u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ as usual.

Idea 3: Relativistic Doppler Shift

If a light source with (proper) frequency f' is moving directly towards you at speed v, then naively one would have, in nonrelativistic physics,

$$f_{\text{nr}} = \frac{f'}{1 - v/c}.$$

In relativity, we also need to account for the source being time dilated, so

$$f = \frac{f_{\text{nr}}}{\gamma} = \sqrt{\frac{1 + v/c}{1 - v/c}}\, f'.$$

This additional, second-order correction was first measured by Ives and Stilwell, in the late 1930s. (The transverse Doppler effect is more subtle, and we'll come back to it in problem 21.)

Example 1

A reference frame is a formal object made of rulers and synchronized clocks. The length of an object in a given reference frame isn't necessarily the same thing as how long the object *looks*, to somebody at rest in the frame using their own eyes. That is different, because one has to account for the time the light needs to travel to the eyes.

Consider a train of rest length L moving with speed v in the ground frame. How long does the train look to somebody standing on the ground directly in front of it, or behind it?

Solution

Both ends of the train continually emit light. Suppose two flashes of light, one from each end, hit an observer's eyes simultaneously. Then the apparent length of the train L_{app} is the distance between the points where the light flashes were originally launched.

For somebody in front of the train, the pulse from the back of the train had to travel an extra distance L_{app}, so it must have been emitted a time L_{app}/c earlier. At this time, the back of the train was vL_{app}/c behind where it is when the pulse from the front of the train hits the observer. So the apparent length is

$$L_{\text{app}} = \frac{L}{\gamma} + \frac{v}{c} L_{\text{app}}.$$

Solving this for L_{app} gives

$$L_{\text{app}} = L\sqrt{\frac{1+v/c}{1-v/c}}.$$

For somebody behind the train, similar reasoning gives

$$L_{\text{app}} = \frac{L}{\gamma} - \frac{v}{c} L_{\text{app}}$$

which yields

$$L_{\text{app}} = L\sqrt{\frac{1-v/c}{1+v/c}}.$$

These expressions should look suspiciously similar to the relativistic Doppler shift. In fact, they can also be derived that way. Imagine a light source at one end of the train shoots light of wavelength $\lambda = L$ towards the other, in the train's frame. In the ground frame, we have $\lambda' = L_{\text{app}}$, because the light wave goes through one cycle by the time it gets from the back of the train to the front. But the transformation of λ can also be found using the relativistic Doppler shift and $c = f\lambda$, giving the same result.

Remark

In most of the problems below, we'll focus on how objects are measured in inertial reference frames, not on how they physically appear to an observer's eyes. This is a complicated but fascinating subject. For instance, it turns out that once one accounts for the light travel time delay, moving objects appear to be rotated. For an interactive simulation, check out the game A Slower Speed of Light (3D) and Velocity Raptor (2D only).

[2] **Problem 6** (KK 12.6)**.** A rod of proper length ℓ_0 oriented parallel to the x axis moves with velocity $u\hat{\mathbf{x}}$ in frame S. What is the length measured by an observer in frame S', which, as usual, moves with velocity $v\hat{\mathbf{x}}$ with respect to S?

[4] **Problem 7.** An object at rest at the origin in frame S' emits a flash of light uniformly in all directions.

(a) In frame S', the expanding shell of radiation is a perfect sphere. Explain why it is also a perfect sphere, at any moment, in any other frame S.

(b) Let frames S and S' be related as usual. Consider the light emitted at an angle θ_0 with respect to the x' axis in S'. Show that the angle θ it makes with respect to the x axis in S obeys

$$\cos\theta = \frac{\cos\theta_0 + v/c}{1 + (v/c)\cos\theta_0}.$$

(c) Therefore, if the object has an ultrarelativistic speed $v \approx c$ in frame S, argue that in this frame, most of its radiation comes out in a narrow cone of opening angle $1/\gamma$ along the direction of travel. This "relativistic beaming" effect is important in the Large Hadron Collider, where high-energy particles decay into lower-energy particles concentrated in narrow "jets".

Now consider the case where the object is at rest, but the light is viewed by a very distant, slowly moving observer going in a circle, with momentarily comoving frame S. Because of your result in part (b), the observer will see the object perform an apparent circular motion. When the object is a star and the observer is a telescope on the Earth, this phenomenon is known as stellar aberration.

(d) Suppose that the displacement from the sun to the distant star is perpendicular to the plane of orbit of the Earth. If the Earth performs a circular orbit with speed $v \ll c$, find the apparent angular radius θ_A of the circle the star moves in.

(e) There is another independent effect at play here, which is that the star will also seem to move in a circle due to parallax. Parallax exists even if the speed of light is taken to infinity; it is the result of the Earth moving in its orbit, and hence seeing the star from different angles. If the Earth orbits with radius r, and the star of part (d) is a distance $d \gg r$ away, find the apparent angular radius θ_P of the circle the star moves in.

(f) For a typical star in the galaxy, which is larger, θ_A or θ_P?

The fact that both aberration and parallax escaped detection over centuries of effort was a strong early piece of evidence against heliocentrism. Today we know that they are hard to observe because c and d are very large.

[2] **Problem 8.** The frequency of light reflected from a moving mirror gets a Doppler shift.

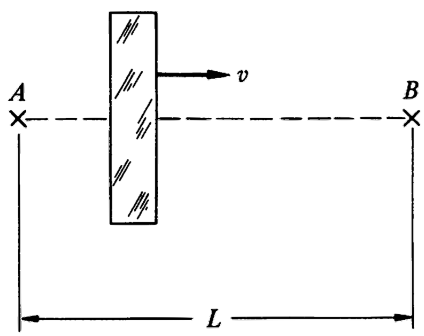
(a) Find the frequency of light reflected directly back from a mirror which is approaching the observer with speed v, if the light originally had frequency f.

(b) Show that this is the same as if the light were sourced with frequency f by an object moving at speed $2v/(1+v^2/c^2)$ towards the observer. Can you find an intuitive reason for this?

[3] **Problem 9** (Morin 11.16). In frame S', a particle moves with velocity $(0, u')$ as shown at left.

Frame S moves to the left with speed v, so the situation in S is as shown at right, with the y speed now u. Consider a series of equally spaced dotted lines, as shown. By considering the rate at which the particle crosses the dotted lines in each frame, find u in terms of u' and v, and confirm the result agrees with the velocity addition formula.

[3] **Problem 10** (KK 12.9). A slab of glass moves to the right with speed $v \ll c$. A flash of light is emitted from A and passes through the glass to arrive at B, a distance L away.



In the rest frame of the glass, it has thickness D and the speed of light in the glass is c/n.

(a) If you were a 19^{th} century physicist, who didn't know relativity but did know about the index of refraction and Galilean velocity addition, how long would you expect it to take the light to go from A to B? Keep the lowest order term in v/c.

(b) How long does it actually take the light to go from A to B, again to lowest order in v/c?

This kind of setup could be part of an interference experiment, which would allow the tiny time difference to be effectively measured. Before the advent of special relativity, experiments like these which require relativistic velocity addition were very puzzling. They were interpreted by imagining that materials that slowed down light also partially "dragged" the ether along with it.

[3] **Problem 11.** USAPhO 2021, problem A2. A simple, elegant problem with a useful punchline.

[3] **Problem 12** (Morin 11.58). A person walks very slowly at speed u from the back of a train of proper length L to the front. The total time dilation effect in the train frame can be made arbitrarily small by picking u to be sufficiently small, so that if a person's watch agrees with a clock at the back of the train when he starts, then it also agrees with a clock at the front when he finishes, to arbitrary accuracy.

Now consider this setup in the ground frame, where the train moves at speed v. The rear clock reads Lv/c^2 more than the front, so in view of the preceding paragraph, the time gained by the person's watch during the process must be Lv/c^2 less than the time gained by the front clock. By working in the ground frame, explain why this is the case. Assume $u \ll v$.

3 Paradoxes

Now you're prepared to confront some classic relativistic paradoxes. They won't appear in competitions, but your understanding of relativity will be deeper if you grapple with them. (Also, now that we've got the basics out of the way, we'll start setting $c = 1$ for most problems.)

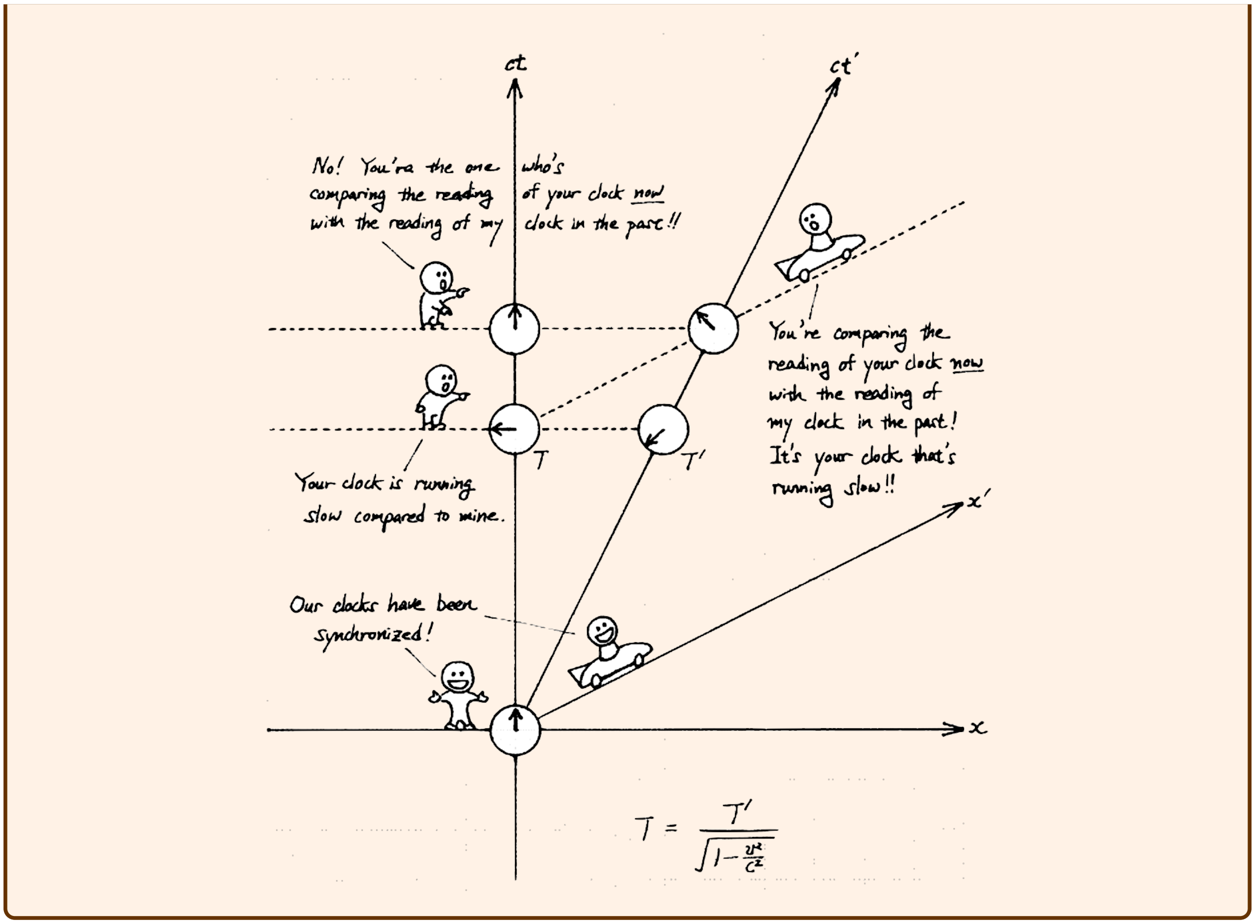
Example 2

Bob moves away from Alice at constant speed. According to special relativity, each sees the other as aging slower. (This is true both in terms of their reference frames, and in terms of what they see with their eyes.) How can that possibly be self-consistent? Shouldn't time be running slower for one or the other?

Solution

The first thing to point out about this paradox, and many other relativistic paradoxes, is that they rely on slipping in nonrelativistic assumptions using tricky wording. If you're fine with the idea of time being relative, there's nothing paradoxical about people disagreeing on whose clock runs slower. It's not really more confusing than the fact that when I walk away from you, I see you getting smaller, but you also see me getting smaller.

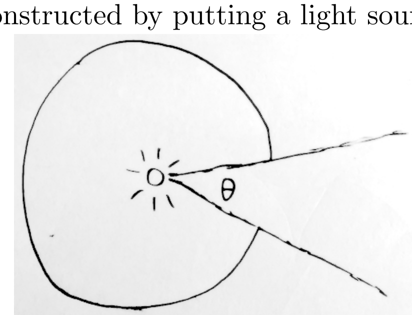
More seriously, though, the reason time dilation can be symmetric is the loss of simultaneity effect, as beautifully shown in Tatsu Takeuchi's Illustrated Guide to Relativity.



[2] **Problem 13.** The Lorentz transformations treat x and t completely symmetrically. So why is it that lengths *contract* while times *dilate*? Shouldn't both do the same thing?

[3] **Problem 14.** A scientist is trying to drill through a piece of wood of thickness $2L$, but the longest drill bit they own has a length L. The scientist decides to move the wood relativistically fast, so that it length contracts to a thickness less than L. Then the drill can be held in the path of the wood, and pulled out once it goes through. Can this really be done without harming the drill bit or ruining the wood? If not, what's wrong? If yes, then what does it look like in the rest frame of the wood?

[3] **Problem 15.** A headlight is constructed by putting a light source inside a spherical cavity.



The opening of the cavity has angular width θ, so a beam of light comes out with width θ. The headlight is mounted on the front of a car, which then moves forward at a relativistic speed. The new width of the headlight's beam is θ', in the frame of the Earth. Consider the following two arguments.

> The headlight length contracts, increasing the cavity opening angle. Therefore, $\theta' > \theta$.
>
> By relativistic velocity addition, the light must have a greater forward velocity in the Earth's frame than in the car frame, because the car is moving forward. So the light must come out at a shallower angle in the Earth's frame. Therefore, $\theta' < \theta$.

Which argument is right?

[3] **Problem 16.** In the lab frame, a horizontal stick of proper length L has horizontal speed v. There is a horizontal thin sheet which has a hole of length L. Since the stick's length is contracted to L/γ, it easily passes through the hole in the sheet, if the sheet is moved vertically. But in the frame of the stick, the sheet is moving horizontally, so the hole is length contracted instead. Qualitatively explain how the stick can still pass through the hole in this frame, in the following two cases:

(a) The sheet has a uniform vertical velocity in the lab frame.

(b) The sheet begins at rest at the lab frame, but is pushed upward a small amount when the stick passes over the hole, then ends at rest again.

[4] **Problem 17.** Here is the statement of the traditional twin paradox.

> Bob is an astronaut who leaves home on a rocket with speed v. Alice stays home. After time T in Alice's frame, Bob reverses direction and travels home with speed v. Who, if either, has aged more?

The obvious answer is that Alice has aged more by time dilation. The trouble is explaining why we can't just work in Bob's frame and conclude that Bob has aged more by time dilation.

(a) Draw a Minkowski diagram for Alice and Bob where Alice's worldline is $x = 0$.

(b) The reason that working in Bob's frame is subtle is that it is not a single inertial frame. Draw x' and t' axes for Bob at several points on Bob's worldline. Argue that when Bob turns around, thereby moving to a different inertial frame, Alice's age jumps upward. (Using the results of chapter 11 of Morin, you can even show that the amount of aging is exactly what is needed, using the Minkowski diagram alone.)

This illustrates why the situation is not symmetric between Alice and Bob. But this resolution of the twin paradox is a little unphysical. It does explain what goes wrong working in Bob's frame, but it's not related to what Bob actually physically *sees*, which is determined by when photons from Earth reach his eyes; nothing about that changes discontinuously when he turns around.

(c) More physically, let us suppose that Bob continually emits radiation of frequency f (in his frame) towards Alice. Suppose that in Alice's frame, Bob travels with speed v, reaches a maximum distance L from Alice, and accelerates quickly to return with speed v. If Alice sees N_b wave crests in total during Bob's trip, then Bob has aged by N_b/f. Use the relativistic Doppler effect to compute N_b/f, working entirely from Alice's perspective.

(d) Now suppose Alice continually emits radiation of frequency f (in her frame) towards Bob. If Bob sees N_a wave crests, use the relativistic Doppler effect to compute N_a/f, working entirely from Bob's perspective. If you're careful, this should differ from the answer to (c).

(e) **[A]** Now consider a trickier example. Suppose Alice and Bob live on a torus, i.e. a spacetime where the point (x, y, z) is the same as the point $(x + L, y, z)$. Alice stays home, while Bob leaves on a rocket with velocity $v\hat{\mathbf{x}}$. After a while, Bob returns home, without having done any acceleration along the way! It seems like the resolution above does not apply, so who, if either, has aged more? Can you explain the results from Bob's reference frame?

Remark

The above problem on the twin paradox is quite long. Every physics textbook that covers relativity mentions the twin paradox, and Morin even has a whole appendix with five different resolutions of it. But it's not *that* hard to resolve, so why spend so much energy on it?

The answer is that seemingly intelligent people really can get stuck on these things for years, or even decades. As an example, consider the case of Herbert Dingle, one of the foremost science popularizers in the mid-20$^{\text{th}}$ century. Dingle was an experimental physicist and philosopher of science, but he was best known for his eloquent, equation-free explanations of relativity, which made him the Brian Greene of his day. But soon after Einstein's death, he suddenly realized that relativity could not explain the twin paradox.

Here is one version of Dingle's argument. We write down the Lorentz transformations

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx), \quad x = \gamma(x' + vt'), \quad t = \gamma(t' + vx').$$

Then we notice that if we set $x = 0$, then $t' = \gamma t$, while if we set $x' = 0$, then $t = \gamma t'$. This "implies" that aging must always be symmetric. In fact, if we combine the equations, we conclude $\gamma = 1/\gamma$, which implies time dilation can't even happen at all! Thus, relativity collapses.

Dingle continued pushing this for the rest of his life, writing endless letters and articles, and even publishing a book, *Science at the Crossroads*, which warned of the grave societal dangers of trusting relativity. Today, it is a favorite of flat Earthers. And it's far from the only example. For instance, there was a book published in Nazi Germany called *A Hundred Authors Against Einstein*, where a vast array of philosophers argued that relativity had to be wrong, because it contradicted the metaphysical system of the native German, 18$^{\text{th}}$ century philosopher Immanuel Kant. Kant's ideas about space and time, they said, could be proven true by pure reason alone, so any theory or experiment saying otherwise had to be wrong.

If there's a lesson to be drawn from this bizarre history, it's that the ability to write or speak is not the same as the ability to think. Like GPT-3, one can churn out pages of flowing prose without having a single coherent thought. This haze of vague reasoning is a dark cave we're all born in. Physicists escape the cave by solving problems mathematically; many others never escape, and eventually grow to believe that nothing can exist outside it.

4 Four-Vectors

Idea 4

A four-vector V^μ is a set of four quantities (V^0, V^1, V^2, V^3) that transform in the same manner as (ct, x, y, z). The inner product of two four-vectors is defined as

$$V \cdot W = V^0 W^0 - V^1 W^1 - V^2 W^2 - V^3 W^3.$$

It is invariant under Lorentz transformations. By convention, $V \cdot W$ is also written as $V^\mu W_\mu$.

[2] **Problem 18.** Show explicitly that the norm of the displacement four-vector is invariant under Lorentz transformations, i.e. that

$$(\Delta s)^2 = \Delta s \cdot \Delta s = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

is Lorentz invariant. Since all four-vectors transform the same way, this proves it for all of them.

Example 3

Find a four-vector representing the velocity of a particle with position $\mathbf{x}(t)$.

Solution

Just as multiplying an ordinary vector with a rotational invariant produces another vector, multiplying or dividing a four-vector with a Lorentz invariant gives another four-vector. In this case, the appropriate four-vector is found by dividing displacement by the proper time experienced by the particle,

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma, \gamma \mathbf{v})$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the spatial velocity and $\gamma = 1/\sqrt{1 - v^2}$ as usual. Since its spatial part reduces to the spatial velocity in the limit of low speeds, it is the relativistic generalization of the spatial velocity. Finally, we define the four-momentum as $p^\mu = m u^\mu = (E, \mathbf{p})$, where $E = \gamma m$ and $\mathbf{p} = \gamma m \mathbf{v}$ are the relativistic energy and momentum.

Example 4

Give a simple interpretation of the squared norm of a particle's four-velocity, $u \cdot u$, and its four-momentum, $p \cdot p$.

Solution

The answers have to be simple, because they must be invariants that only depend on the intrinsic properties of the particle, i.e. only on the invariant mass m. For the four-velocity,

$$u \cdot u = \gamma^2 - \gamma^2 v^2 = 1$$

which is clearly invariant. For the four-momentum we have $p \cdot p = m^2$.

Remark

Pop-science books usually describe the result $u \cdot u = 1$ by saying that "particles always move with the same speed through spacetime", just like how a particle in uniform circular motion always has the same spatial speed. However, this is very misleading, because it makes people think that if $dx/d\tau$ increases in magnitude, then $dt/d\tau$ decreases. In fact it's the opposite: time dilation means more time passes for each tick of a moving clock, so $dt/d\tau$ increases. The analogy can't work, because inner products of four-vectors have terms with minus signs, while ordinary inner products of three-vectors don't. The point, as always, is that there are a lot of simple things in physics which are almost impossible to explain properly with fuzzy math-free analogies. With math, relativity can make sense to high school students. Without math, it can't really make sense to anyone.

Example 5

Give a simple interpretation of the inner product of two momentum four-vectors, $p_1 \cdot p_2$.

Solution

By definition, this is equal to $m_1 m_2 u_1 \cdot u_2$, and since the inner product is invariant, we can evaluate $u_1 \cdot u_2$ in any frame. Suppose we work in the frame of the first particle, where

$$u_1^\mu = (1, \mathbf{0}), \quad u_2^\mu = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\mathbf{v}}{\sqrt{1-v^2}} \right).$$

Carrying out the inner product, we have the relatively simple result

$$p_1 \cdot p_2 = \frac{m_1 m_2}{\sqrt{1-v^2}}$$

where v is the relative speed, meaning the speed of one particle in the frame of the other.

[2] **Problem 19.** In your inertial frame, there is a particle with four-momentum p^μ, and an observer moving with four-velocity u^μ. The observer measures the particle in *their* inertial frame.

(a) Show that the energy they measure is $p \cdot u$.

(b) Show that the momentum they measure has magnitude $\sqrt{(p \cdot u)^2 - p \cdot p}$.

(c) What is the speed that they measure?

Don't use Lorentz transformations here; everything can be done with four-vectors alone.

[3] **Problem 20.** In A's frame, B has speed u, and C has speed v.

(a) Suppose B and C have velocities in opposite directions. Find the speed of B with respect to C using four-vectors, by computing the inner product $v_B \cdot v_C$ in two different frames.

(b) The answer of part (a) should look familiar, but with four-vectors we can easily go further. Generalize part (a) to the case where B and C have velocities an angle θ apart.

[3] **Problem 21.** Four-vectors provide a quick derivation of the relativistic Doppler effect. Given a plane wave, define $k^\mu = (\omega, \mathbf{k})$. Then the plane wave is proportional to $e^{i\phi}$, where the phase is

$$\phi = \omega t - \mathbf{k} \cdot \mathbf{x} = k \cdot x.$$

Since the phase ϕ is Lorentz invariant, and we know x^μ is a four-vector, k^μ is a four-vector as well.

(a) Show that for light, $k^\mu k_\mu = 0$.

(b) Consider a light ray with angular frequency ω traveling along the x axis, and an observer moving with speed v along the x-axis. Use an explicit Lorentz transformation to find the angular frequency ω' the observer sees, thus rederiving the longitudinal Doppler shift for light.

(c) Now it's easy to go further. Repeat the previous part for a light ray traveling at an arbitrary angle θ to the x axis. You can do this using either an explicit Lorentz transformation, or just properties of four-vectors.

(d) One subtlety with the general relativistic Doppler effect is the definition of θ, because it has different values in the source's frame, and in the observer's frame. The previous two parts started in the source's frame, but the usual formula defines θ in the *observer's* frame.

To recover this formula, repeat the previous part, but now suppose we're already in the observer's frame, where the source moves with velocity $-v\hat{\mathbf{x}}$, and the light ray is traveling at an angle θ to the x-axis. Find the relationship between ω' and ω.

The result of part (d) should look familiar: it's the final result of USAPhO 2021, problem A2. For more on the general Doppler effect, see section 11.8.2 of Morin. (By the way, now that we have the four-vector formalism set up, it's not that much harder to compute the Doppler effect for waves that travel at general speeds. You probably won't need that result, but it's an example of something that's a total nightmare to derive without four-vectors.)

Example 6: Woodhouse 6.6

Four distant stars S_i are observed. Let θ_{ij} denote the observed angle between the directions to S_i and S_j. Show that the ratio

$$\frac{(1-\cos\theta_{12})(1-\cos\theta_{34})}{(1-\cos\theta_{13})(1-\cos\theta_{24})}$$

is independent of the motion of the observer.

Solution

This Oxford undergraduate exam question is too technical to be relevant to Olympiads, but it shows how four-vectors can be essential. The θ_{ij} depend on the motion of the observer because of the aberration effect in problem 7. That is, when you Lorentz transform to a moving observer's frame, it changes the direction of the incoming light. A direct attack on the question would thus require applying the full, four-dimensional Lorentz transformations to four vectors with arbitrary orientations, which would be a nightmare. Here's an alternative:

let k_i^μ be the wave vectors of an incoming photon from each star. Then

$$k_i \cdot k_j = \omega_i \omega_j - \mathbf{k}_i \cdot \mathbf{k}_j = \omega_i \omega_j (1 - \cos\theta_{ij})$$

where we used $\omega_i = |\mathbf{k}_i|$. Therefore, the ratio is

$$\frac{(k_1 \cdot k_2)(k_3 \cdot k_4)/\omega_1\omega_2\omega_3\omega_4}{(k_1 \cdot k_3)(k_2 \cdot k_4)/\omega_1\omega_2\omega_3\omega_4} = \frac{(k_1 \cdot k_2)(k_3 \cdot k_4)}{(k_1 \cdot k_3)(k_2 \cdot k_4)}$$

which is manifestly independent of frame.

[4] **Problem 22.** In this problem we'll construct a four-vector a^μ for the acceleration of a particle, and use it to derive the Lorentz transformation of the ordinary three-vector acceleration $\mathbf{a} = d\mathbf{v}/dt$.

(a) Explain why $a^\mu = du^\mu/d\tau$ is a four-vector, and why $u \cdot a$ is always zero.

(b) Show that when $\mathbf{v} = v\hat{\mathbf{x}}$, the components of a^μ are

$$a^\mu = (\gamma^4 v a_x, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z)$$

where $\gamma = 1/\sqrt{1-v^2}$ as usual. As a check, what is the meaning of $a \cdot a$?

(c) Now let S' be the momentary rest frame of a particle, i.e. the inertial frame that, at a given moment, is moving with the same velocity as the particle. Let the particle have three-acceleration $\mathbf{a}'$ in that frame. Show that in this frame, $a^{\mu'} = (0, a'_x, a'_y, a'_z)$.

(d) By Lorentz transforming to S and using part (b), show that the acceleration in frame S is

$$\mathbf{a} = (a'_x/\gamma^3, a'_y/\gamma^2, a'_z/\gamma^2).$$

As you can see, transformations of three-vector quantities can get quite nasty!

Remark

We can rewrite a lot of our results in terms of three-vectors. First, the Lorentz transformations for general $\mathbf{v}$ are, using the same notation as in idea 1,

$$t' = \gamma(t - \mathbf{v} \cdot \mathbf{r}), \quad \mathbf{r}' = \mathbf{r} - \gamma \mathbf{v} t + (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{r})\hat{\mathbf{v}}.$$

The velocity addition formula for general $\mathbf{v}$ and $\mathbf{u}'$ is, using the same notation as in idea 2,

$$\mathbf{u} = \frac{1}{1 + \mathbf{v} \cdot \mathbf{u}'} \left(\mathbf{v} + \frac{\mathbf{u}'}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{u}') \right).$$

The first result of problem 22 is

$$a^\mu = (\gamma^4 \mathbf{a} \cdot \mathbf{u},\, \gamma^4(\mathbf{a} + \mathbf{u} \times (\mathbf{u} \times \mathbf{a})))$$

and the second result, for the transformation of acceleration, is

$$\mathbf{a} = \frac{\mathbf{a}'}{\gamma^2} - \frac{\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{a}')(\gamma - 1)}{\gamma^3}.$$

As you can see, these aren't very enlightening, and they don't tend to be useful in solving problems. The reason is that in relativity, there's nothing special about three-vectors. For concrete problems, you'll typically either want to do everything in terms of four-vectors, or descend all the way down to individual components – in which case you would align your axes so that $\mathbf{v}$ points along one of them, rather than considering a completely general $\mathbf{v}$.

On the other hand, you can get practice with three-vectors by staring at the above expressions until you see how they reduce to the component forms we had earlier. If you do this, you'll learn how to translate just about *any* component expression into three-vector notation.

5 Acceleration and Rapidity

Idea 5

The geometry of special relativity is much like ordinary geometry, except that the dot product is replaced with an inner product, which has some minus signs. Lorentz transformations can be thought of as "generalized rotations" which mix up time and space, just as ordinary rotations mix up different spatial axes. The generalized angle is the rapidity $\phi = \tanh^{-1} v$.

[3] **Problem 23** (Morin 11.27). In this problem, we'll see the meaning of the rapidity more precisely.

(a) Show that a Lorentz transformation may be written as

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}.$$

(b) Show that the composition of Lorentz transformations with rapidity ϕ_1 and ϕ_2 is a Lorentz transformation with rapidity $\phi_1 + \phi_2$. This makes rapidity extremely useful in kinematics problems with multiple boosts, such as problems involving acceleration.

(c) An ordinary rotation of spatial axes has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Show that a Lorentz transformation is essentially an ordinary rotation between space and time, if we treat time as like "imaginary space" and the rotation as by an imaginary angle. This was one of the ways the founders of relativity thought about it.

Remark

We know from **M8** that if we do multiple rotations in a row, the final result depends on the order the rotations are performed. The analogy between boosts and rotations gives intuition for the analogous result for boosts: the order matters. For example, the difference between boosting along the x-axis and then the y-axis, or vice versa, is a rotation in the xy plane. Therefore, if an object is boosted in a circle in the xy plane, it will have an extra rotation in that plane. This subtle phenomenon is called Thomas precession.

Idea 6

The next few questions will deal with accelerating objects. In Newtonian mechanics, a common strategy is to work in the accelerating frame of the object, but that's not a good idea at this stage of your education. (There's nothing wrong with doing so, but it brings in complications that one usually needs a course in general relativity to fully appreciate.)

Instead, we will describe accelerating objects using inertial frames. In principle we could do everything in the lab frame, but it is also often useful to work in a momentarily comoving frame, i.e. the inertial frame that, at some time t, moves with the same velocity as the object.

[4] **Problem 24.** A rocket starts from rest in the lab frame at the origin, then accelerates in a straight line at constant rate a_0 as measured by an accelerometer on the ship; that is, the proper acceleration is always a_0.

(a) Show that the acceleration measured in the lab frame is a_0/γ^3. (We already proved this more generally in problem 22, but try to do this more explicitly by working in the comoving frame, then going back to the lab frame.)

(b) Find the speed of the rocket ship in the lab frame as a function of time t in the lab frame.

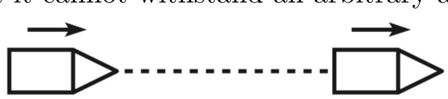
(c) Find the speed of the rocket ship in the lab frame as a function of the proper time τ elapsed on the rocket. Can you explain the simplicity of your result using rapidity?

(d) To conclude, find expressions for $t(\tau)$, $x(t)$, and $x(\tau)$, and comment on their limits.

[3] **Problem 25.** USAPhO 2020, problem A3. An unusual problem that tests your understanding of momentarily comoving frames, and higher-dimensional Lorentz transformations. As a warning, this question requires you to make an unstated assumption. The fact that uniformly moving clocks have their time dilated by a factor of γ follows directly from the postulates of special relativity. But here you'll have to assume this also holds for *accelerating* clocks, even though clocks can tell if they're accelerating, and may tick differently. This is called the clock hypothesis. For example, on a roller coaster, a pendulum clock doesn't obey the clock hypothesis, but a quartz watch does. Also, the solution is a bit misleading, so don't worry if you thought about the problem differently as long as you got the same final answers.

[3] **Problem 26** (Morin 11.26)**.** The following problem is called Bell's spaceship paradox. It caused a stir at CERN when many particle physicists could not agree on the answer.

Two spaceships float in space and are at rest relative to each other. They are connected by a string. The string is strong, but it cannot withstand an arbitrary amount of stretching.



At a given instant, the spaceships simultaneously (with respect to their initial inertial frame) start accelerating in the same direction along the line between them, with the same constant proper acceleration. In other words, assume they bought identical engines from the same store, and they put them on the same setting. Will the string eventually break?

[5] **Problem 27.** APhO 2013, problem 2. This is a challenging question that ties together everything you've learned about kinematics.

Relativity I: Kinematics

Chapter 11 of Morin is a clear, comprehensive, highly recommended introduction to relativistic kinematics. Also read sections 13.1–13.3 for four-vectors, and see appendices F, G, H, and I for enrichment. Alternatively, see chapter 12 and sections 14.1–14.3 of Kleppner and Kolenkow, chapter 11 of Wang and Ricardo, volume 2, or Morin's newer book, *Special Relativity: For the Enthusiastic Beginner*, which covers similar ground with slightly more detail. An entertaining introduction is also given in chapters I-15 through I-17 of the Feynman lectures. To learn about tests of special relativity, see *The Special Theory of Relativity* by Christodoulides. There is a total of **81** points.

1 Lorentz Transformations

Special relativity is uniquely subtle among introductory physics topics, and requires a solid, detailed introduction. This problem set assumes you've already done that, by reading chapter 11 of Morin or the equivalent in another book. (The short chapter in Halliday, Resnick, and Krane is not sufficient.)

Idea 1: Lorentz Transformation

Let S' be the frame of an observer moving to the right with velocity $v\hat{\mathbf{x}}$ with respect to the frame S. Then the coordinates in S and S' are related by the Lorentz transformation

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

This implies that the lengths of moving objects are contracted by γ, moving clocks run slow by a factor of γ, and that if two clocks are synchronized in the frame S' and separated by a distance L, then in the frame S the rear clock is ahead by Lv/c^2.

[2] **Problem 1** (Morin 11.2). Two planets, A and B, are at rest with respect to each other, a distance L apart, with synchronized clocks. A spaceship flies at speed v past planet A toward planet B and synchronizes its clock with A's right when it passes A (they both set their clocks to zero). The spaceship eventually flies past planet B and compares its clock with B's. We know, from working in the planets' frame, that when the spaceship reaches B, B's clock reads L/v. And the spaceship's clock reads $L/\gamma v$, because it runs slow by a factor of γ when viewed in the planets' frame.

How would someone on the spaceship quantitatively explain to you why B's clock reads L/v (which is *more* than its own $L/\gamma v$), considering that the spaceship sees B's clock running *slow*?

Solution. Let us work in the frame of the spaceship. Since AB is moving to the left with v, when the ship is at A, the clock at B reads Lv/c^2. Now, the time it takes for B to reach the spaceship is $(L/\gamma)/v$, so the time on B's clock is

$$\frac{L}{\gamma^2 v} + \frac{Lv}{c^2} = \frac{L}{v}\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right) = \frac{L}{v}.$$

[2] **Problem 2** (Morin 11.4). A stick of (proper) length L moves past you at speed v. There is a time interval between the front end coinciding with you and the back end coinciding with you. What is this time interval in:

(a) your frame? (Calculate this by working in your frame.)

(b) the stick's frame? (Work in the stick's frame.)

(c) your frame? (Work in the stick's frame.)

(d) the stick's frame? (Work in your frame. This is the tricky one.)

Solution. (a) The stick is length contracted to L/γ, so it takes time $L/\gamma v$ for the stick to pass.

(b) The stick has length L and you move past it at speed v, so it takes time L/v.

(c) The same reasoning as part (b) applies. But in the stick's frame, your clock is running slow by a factor of γ, so the time measured by your clock is $L/\gamma v$.

(d) The time measured in your frame is $L/\gamma v$ from part (a), but the clocks on the ends of the stick are running slow. In addition, those clocks are not synchronized in your frame. Thus, the time measured in the stick's frame is

$$\frac{1}{\gamma}\frac{L}{\gamma v} + \frac{Lv}{c^2} = \frac{L}{v}$$

just as in the previous problem.

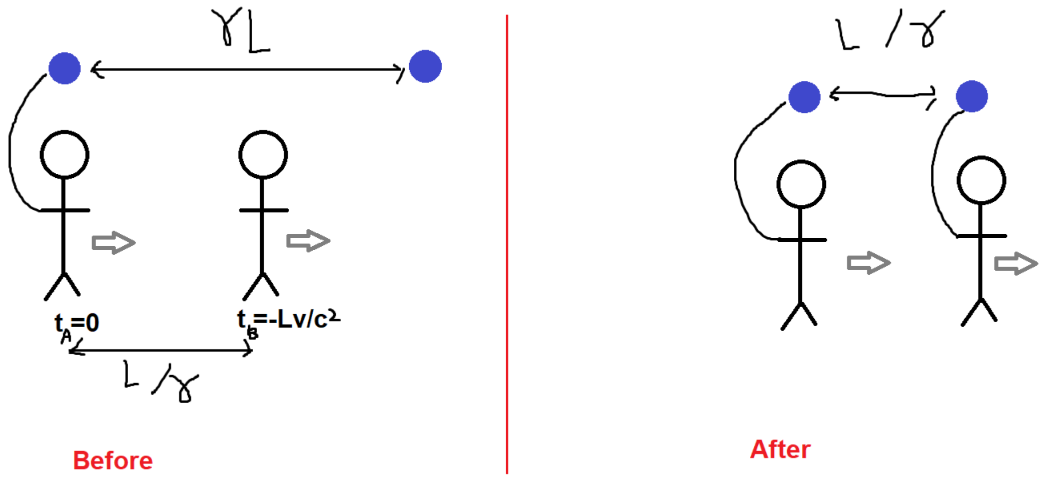
[2] **Problem 3** (Morin 11.9)**.** Two balls move with speed v along a line toward two people standing along the same line. The proper distance between the balls is γL, and the proper distance between the people is L. Due to length contraction, the people measure the distance between the balls to be L, so the balls pass the people simultaneously (as measured by the people), as shown.

Assume that the people's clocks both read zero at this time. If the people catch the balls, then the resulting proper distance between the balls becomes L, which is shorter than the initial proper distance of γL. Your task is to explain how the proper distance between the balls decreases from γL to L, by working in the frame where the balls are initially at rest.

(a) Draw the beginning and ending pictures for the process. Indicate the readings on both clocks in the two pictures, and label all relevant lengths.

(b) Explain in words how the proper distance between the balls decreases from γL to L.

Solution. (a) Call the person on the left A, and the person on the right B. Here's the diagram.



(b) The amount of time it takes B to get to his ball is $(\gamma L - L/\gamma)/v = L\gamma(1-1/\gamma^2)/v = L\gamma v/c^2$. This means that B catches his ball when his clock reads 0, since time for him runs a factor of γ slower, which makes sense. Therefore, at the end, the distance between the balls is L/γ, but since they are moving at v, their proper distance is L.

The point is that since we lose simultaneity, by the time B catches his ball in his frame, A has dragged his ball closer to B's ball, reducing their distance in the process.

[3] **Problem 4.** USAPhO 2016, problem A3. Print out the custom answer sheet before starting.

[5] **Problem 5.** IPhO 2006, problem 2. A nice problem about relativistic visual effects.

2 Velocity Addition

Idea 2: Velocity Addition

Again, let frame S' moves with velocity $v\hat{\mathbf{x}}$ with respect to frame S. If an object has velocity (u'_x, u'_y) in frame S', then the velocity in S is

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}, \quad u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ as usual.

Idea 3: Relativistic Doppler Shift

If a light source with (proper) frequency f' is moving directly towards you at speed v, then naively one would have, in nonrelativistic physics,

$$f_{\rm nr} = \frac{f'}{1 - v/c}.$$

In relativity, we also need to account for the source being time dilated, so

$$f = \frac{f_{\text{nr}}}{\gamma} = \sqrt{\frac{1+v/c}{1-v/c}}\, f'.$$

This additional, second-order correction was first measured by Ives and Stilwell, in the late 1930s. (The transverse Doppler effect is more subtle, and we'll come back to it in problem 21.)

Example 1

A reference frame is a formal object made of rulers and synchronized clocks. The length of an object in a given reference frame isn't necessarily the same thing as how long the object *looks*, to somebody at rest in the frame using their own eyes. That is different, because one has to account for the time the light needs to travel to the eyes.

Consider a train of rest length L moving with speed v in the ground frame. How long does the train look to somebody standing on the ground directly in front of it, or behind it?

Solution

Both ends of the train continually emit light. Suppose two flashes of light, one from each end, hit an observer's eyes simultaneously. Then the apparent length of the train L_{app} is the distance between the points where the light flashes were originally launched.

For somebody in front of the train, the pulse from the back of the train had to travel an extra distance L_{app}, so it must have been emitted a time L_{app}/c earlier. At this time, the back of the train was vL_{app}/c behind where it is when the pulse from the front of the train hits the observer. So the apparent length is

$$L_{\text{app}} = \frac{L}{\gamma} + \frac{v}{c} L_{\text{app}}.$$

Solving this for L_{app} gives

$$L_{\text{app}} = L\sqrt{\frac{1+v/c}{1-v/c}}.$$

For somebody behind the train, similar reasoning gives

$$L_{\text{app}} = \frac{L}{\gamma} - \frac{v}{c} L_{\text{app}}$$

which yields

$$L_{\text{app}} = L\sqrt{\frac{1-v/c}{1+v/c}}.$$

These expressions should look suspiciously similar to the relativistic Doppler shift. In fact, they can also be derived that way. Imagine a light source at one end of the train shoots light of wavelength $\lambda = L$ towards the other, in the train's frame. In the ground frame, we have

$\lambda' = L_{\text{app}}$, because the light wave goes through one cycle by the time it gets from the back of the train to the front. But the transformation of λ can also be found using the relativistic Doppler shift and $c = f\lambda$, giving the same result.

Remark

In most of the problems below, we'll focus on how objects are measured in inertial reference frames, not on how they physically appear to an observer's eyes. This is a complicated but fascinating subject. For instance, it turns out that once one accounts for the light travel time delay, moving objects appear to be rotated. For an interactive simulation, check out the game A Slower Speed of Light (3D) and Velocity Raptor (2D only).

[2] **Problem 6** (KK 12.6)**.** A rod of proper length ℓ_0 oriented parallel to the x axis moves with velocity $u\hat{\mathbf{x}}$ in frame S. What is the length measured by an observer in frame S', which, as usual, moves with velocity $v\hat{\mathbf{x}}$ with respect to S?

Solution. The speed of the rod measured by an observer in S' is

$$u' = \frac{u - v}{1 - uv/c^2}.$$

The length contraction will result in an observed length of

$$\ell' = \ell\sqrt{1 - \left(\frac{u - v}{c - uv/c}\right)^2}.$$

[4] **Problem 7.** An object at rest at the origin in frame S' emits a flash of light uniformly in all directions.

(a) In frame S', the expanding shell of radiation is a perfect sphere. Explain why it is also a perfect sphere, at any moment, in any other frame S.

(b) Let frames S and S' be related as usual. Consider the light emitted at an angle θ_0 with respect to the x' axis in S'. Show that the angle θ it makes with respect to the x axis in S obeys

$$\cos\theta = \frac{\cos\theta_0 + v/c}{1 + (v/c)\cos\theta_0}.$$

(c) Therefore, if the object has an ultrarelativistic speed $v \approx c$ in frame S, argue that in this frame, most of its radiation comes out in a narrow cone of opening angle $1/\gamma$ along the direction of travel. This "relativistic beaming" effect is important in the Large Hadron Collider, where high-energy particles decay into lower-energy particles concentrated in narrow "jets".

Now consider the case where the object is at rest, but the light is viewed by a very distant, slowly moving observer going in a circle, with momentarily comoving frame S. Because of your result in part (b), the observer will see the object perform an apparent circular motion. When the object is a star and the observer is a telescope on the Earth, this phenomenon is known as stellar aberration.

(d) Suppose that the displacement from the sun to the distant star is perpendicular to the plane of orbit of the Earth. If the Earth performs a circular orbit with speed $v \ll c$, find the apparent angular radius θ_A of the circle the star moves in.

(e) There is another independent effect at play here, which is that the star will also seem to move in a circle due to parallax. Parallax exists even if the speed of light is taken to infinity; it is the result of the Earth moving in its orbit, and hence seeing the star from different angles. If the Earth orbits with radius r, and the star of part (d) is a distance $d \gg r$ away, find the apparent angular radius θ_P of the circle the star moves in.

(f) For a typical star in the galaxy, which is larger, θ_A or θ_P?

The fact that both aberration and parallax escaped detection over centuries of effort was a strong early piece of evidence against heliocentrism. Today we know that they are hard to observe because c and d are very large.

Solution. (a) Since the radiation is emitted from a single point, all the light is emitted at the same time in any frame. From that point on, the shell of radiation is a sphere because the speed of light is the same in all frames.

(b) In S', the end of the light beam is described by $x' = ct' \cos\theta_0$. Lorentz transforming to S, we see that

$$(ct, x) = \gamma ct'(1 + (v/c)\cos\theta_0, v/c + \cos\theta_0).$$

Therefore, the angle is

$$\cos\theta = \frac{x}{ct} = \frac{\cos\theta_0 + v/c}{1 + (v/c)\cos\theta_0}.$$

This conclusion can also be reached using relativistic velocity addition.

(c) In frame S', half of the radiation comes out at an angle $|\theta_0| \leq 90°$. So let's consider how the radiation at $\theta_0 = 90°$ comes out, in frame S. Plugging in $\cos\theta_0 = 0$, we find

$$\cos\theta = \frac{v}{c} = \sqrt{1 - 1/\gamma^2}.$$

Using the usual right triangle trick, these corresponds to

$$\sin\theta = \frac{1}{\gamma}$$

which is a small angle! (In fact, more than half the radiation power comes out within this small angle, because the radiation going forward in S is blueshifted, while the radiation going backwards is redshifted, as one can see with the relativistic Doppler effect.)

(d) Let the star by displaced relative to the Earth along the z axis, and let the Earth's velocity be along its x axis. Then the formula in part (b) applies, where $\theta_0 = \pi/2$. We thus have $\cos\theta = v/c$, where $\theta = \theta_0 + \theta_A$, and applying the small angle approximation gives $\theta_A = v/c$. (If you find the geometry of the effect confusing, see this diagram.)

(e) Using the small angle approximation, the answer is straightforwardly $\theta_P = r/d$.

(f) Earth's orbit speed is about $30\,\mathrm{km/s}$, so $v/c \sim 10^{-4}$. By contrast, r is a few light-minutes, while d is at the minimum a few light-years, so $r/d \lesssim 10^{-6}$ even for the closest stars. So the aberration effect is much larger. Aberration was first seen by Bradley in 1725, while parallax was not seen until the mid 1800s. (By the way, aberration applies to the Sun too; the actual position of the Sun, in an inertial frame on Earth, is an angle 10^{-4} away from where it appears in the sky. But this deflection isn't so practical to measure.)

[2] **Problem 8.** The frequency of light reflected from a moving mirror gets a Doppler shift.

(a) Find the frequency of light reflected directly back from a mirror which is approaching the observer with speed v, if the light originally had frequency f.

(b) Show that this is the same as if the light were sourced with frequency f by an object moving at speed $2v/(1+v^2/c^2)$ towards the observer. Can you find an intuitive reason for this?

Solution. (a) Consider the frame of the mirror. In this frame, the light comes in with frequency

$$f_1 = \sqrt{\frac{1+v/c}{1-v/c}}\, f$$

by the Doppler shift, and it bounces off with the same frequency f_1. Now go back to the frame of the observer. By using the Doppler shift formula again, the observer sees a frequency

$$f_2 = \frac{1+v/c}{1-v/c}\, f.$$

That is, a moving mirror causes a double Doppler shift.

(b) Let $u = 2v/(1+v^2/c^2)$. Then verifying the claim boils down to showing that

$$f_2 = \sqrt{\frac{1+u/c}{1-u/c}}\, f$$

which is equivalent to (setting $c=1$),

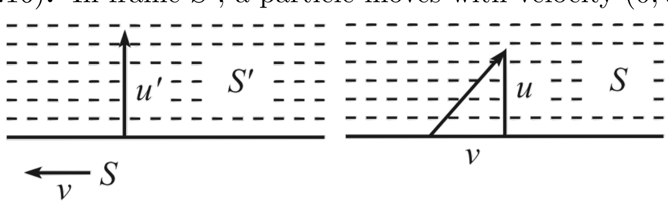
$$\frac{(1+v)^2}{(1-v)^2} = \frac{1+u}{1-u}.$$

This holds because

$$\frac{1+u}{1-u} = \frac{1+\frac{2v}{1+v^2}}{1-\frac{2v}{1+v^2}} = \frac{1+v^2+2v}{1+v^2-2v}.$$

The intuition is that we can think of the reflected wave as being sourced by an image. Both the source and the image have speed v relative to the mirror, so by relativistic velocity addition, the image has speed $2v/(1+v^2)$ relative to the source.

[3] **Problem 9** (Morin 11.16). In frame S', a particle moves with velocity $(0, u')$ as shown at left.



Frame S moves to the left with speed v, so the situation in S is as shown at right, with the y speed now u. Consider a series of equally spaced dotted lines, as shown. By considering the rate at which the particle crosses the dotted lines in each frame, find u in terms of u' and v, and confirm the result agrees with the velocity addition formula.

Solution. Before starting, let's recall how the time dilation formula works. Suppose we have two events with the same x coordinate (such as the ticking of a clock at rest in frame S), separated by time Δt. Then applying the Lorentz transformation yields $\Delta t' = \gamma \Delta t$ for the time separation in the primed frame. Conversely, if we had two events with the same x' coordinate (such as the ticking of a clock at rest in frame S'), then $\Delta t = \gamma \Delta t'$.

In this problem, the particle isn't at rest in either frame S or S'. But the Lorentz transformations don't do anything to the y coordinate, so the motion in the y-direction doesn't matter for the purposes of the above argument. Suppose that in frames S and S', there is an interval Δt and $\Delta t'$ between crossing adjacent dotted lines, respectively. Since these occur at the same x' coordinate in frame S', we have

$$\Delta t = \gamma \Delta t'.$$

Moreover, length in the y-direction isn't contracted at all, so

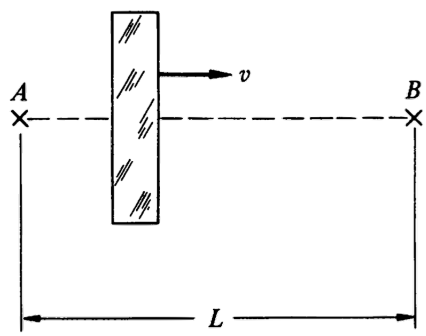
$$\gamma = \frac{\Delta t}{\Delta t'} = \frac{u'}{u}.$$

Thus, we have

$$u_y = \frac{u'_y}{\gamma}$$

which agrees with the velocity addition formula, when we plug in $u'_x = 0$.

[3] **Problem 10** (KK 12.9). A slab of glass moves to the right with speed $v \ll c$. A flash of light is emitted from A and passes through the glass to arrive at B, a distance L away.



In the rest frame of the glass, it has thickness D and the speed of light in the glass is c/n.

(a) If you were a 19^{th} century physicist, who didn't know relativity but did know about the index of refraction and Galilean velocity addition, how long would you expect it to take the light to go from A to B? Keep the lowest order term in v/c.

(b) How long does it actually take the light to go from A to B, again to lowest order in v/c?

This kind of setup could be part of an interference experiment, which would allow the tiny time difference to be effectively measured. Before the advent of special relativity, experiments like these which require relativistic velocity addition were very puzzling. They were interpreted by imagining that materials that slowed down light also partially "dragged" the ether along with it.

Solution. (a) Naively, the light moves with speed c in free space, and speed $v_{\text{in}} = c/n + v$ inside the slab, by Galilean velocity addition. So when the light is in the slab, the relative speed of the light and slab is exactly

$$v_{\text{rel}} = \frac{c}{n}.$$

Therefore, by routine kinematics, the time spent in the slab is

$$t_{\text{in}} = \frac{D}{v_{\text{rel}}}$$

during which the light moves forward by $D + vt_{\text{in}}$. The rest of the time is

$$t_{\text{out}} = \frac{L - D - vt_{\text{in}}}{c}.$$

Adding these together gives a total time of

$$T = \frac{L}{c} + D\left(\frac{1}{v_{\text{rel}}} - \frac{1}{c} - \frac{v}{cv_{\text{rel}}}\right) = \frac{L}{c} + \frac{D}{c}\left(n - 1 - \frac{vn}{c}\right).$$

(b) The slab length contracts, but this is second order in v/c, while we're just interested in the first order effect. The key difference is that because of relativistic velocity addition, the light in the slab moves with speed

$$v_{\text{in}} = \frac{c/n + v}{1 + v/nc} = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)v + O(v^2/c).$$

Thus, to leading order in v/c, when the light is in the slab, the relative speed of the light and slab is, in the lab frame,

$$v_{\text{rel}} \approx \frac{c}{n} - \frac{v}{n^2}.$$

The rest of the above derivation goes through unchanged, giving

$$T = \frac{L}{c} + D\left(\frac{1}{v_{\text{rel}}} - \frac{1}{c} - \frac{v}{cv_{\text{rel}}}\right) \approx \frac{L}{c} + \frac{D}{c}\left(n - 1 - \frac{v(n-1)}{c}\right)$$

again to first order in v/c. (Before the advent of relativity, this result was explained by an "ether drag" coefficient of $1 - 1/n^2$.)

[3] **Problem 11.** USAPhO 2021, problem A2. A simple, elegant problem with a useful punchline.

[3] **Problem 12** (Morin 11.58)**.** A person walks very slowly at speed u from the back of a train of proper length L to the front. The total time dilation effect in the train frame can be made arbitrarily small by picking u to be sufficiently small, so that if a person's watch agrees with a clock at the back of the train when he starts, then it also agrees with a clock at the front when he finishes, to arbitrary accuracy.

Now consider this setup in the ground frame, where the train moves at speed v. The rear clock reads Lv/c^2 more than the front, so in view of the preceding paragraph, the time gained by the person's watch during the process must be Lv/c^2 less than the time gained by the front clock. By working in the ground frame, explain why this is the case. Assume $u \ll v$.

Solution. This is a tricky issue: even though the extra time dilation effect can be made arbitrarily small by making u smaller, doing so would make the effect last for a longer time. In this particular situation, that means the effect doesn't go away even as $u \to 0$! In this respect, it has something in common with the more subtle approximation problems in **P1**.

Now we do the analysis, taking care to expand at the lowest relevant order in u everywhere, and setting $c = 1$. Velocity addition gives a velocity of the person in the ground frame of

$$u_0 = \frac{v+u}{1+uv} \approx (u+v)(1-uv) \approx u + v - v^2 u = v + u/\gamma_v^2.$$

This gives a Lorentz factor of

$$\gamma_0 = \frac{1}{\sqrt{1-u_0^2}} \approx \frac{1}{\sqrt{1-(v^2+2vu/\gamma_v^2)}} \approx \gamma_v + \frac{vu}{\gamma_v^2(1-v^2)} = \gamma_v + vu.$$

Thus the time of crossing can be found with the relative velocity of u/γ_v^2 and a contracted length of L/γ_v to get $t = \gamma_v L/u$. In this time, the clocks on the train gain a time of t/γ_v from time dilation, and the time that passes on the persons clock is t/γ_0, giving a difference (in the ground frame) of

$$\Delta t_0 = t/(\gamma_v + vu) - t/\gamma_v \approx t/\gamma_v - tvu - t/\gamma_v = -vu\,\frac{\gamma_v L}{u} = -\gamma_v L v.$$

Due to the time dilation of the person's clock, the time difference is $\Delta t_0/\gamma_0 \approx \Delta t_0/\gamma_v = -Lv$, just as desired.

3 Paradoxes

Now you're prepared to confront some classic relativistic paradoxes. They won't appear in competitions, but your understanding of relativity will be deeper if you grapple with them. (Also, now that we've got the basics out of the way, we'll start setting $c = 1$ for most problems.)

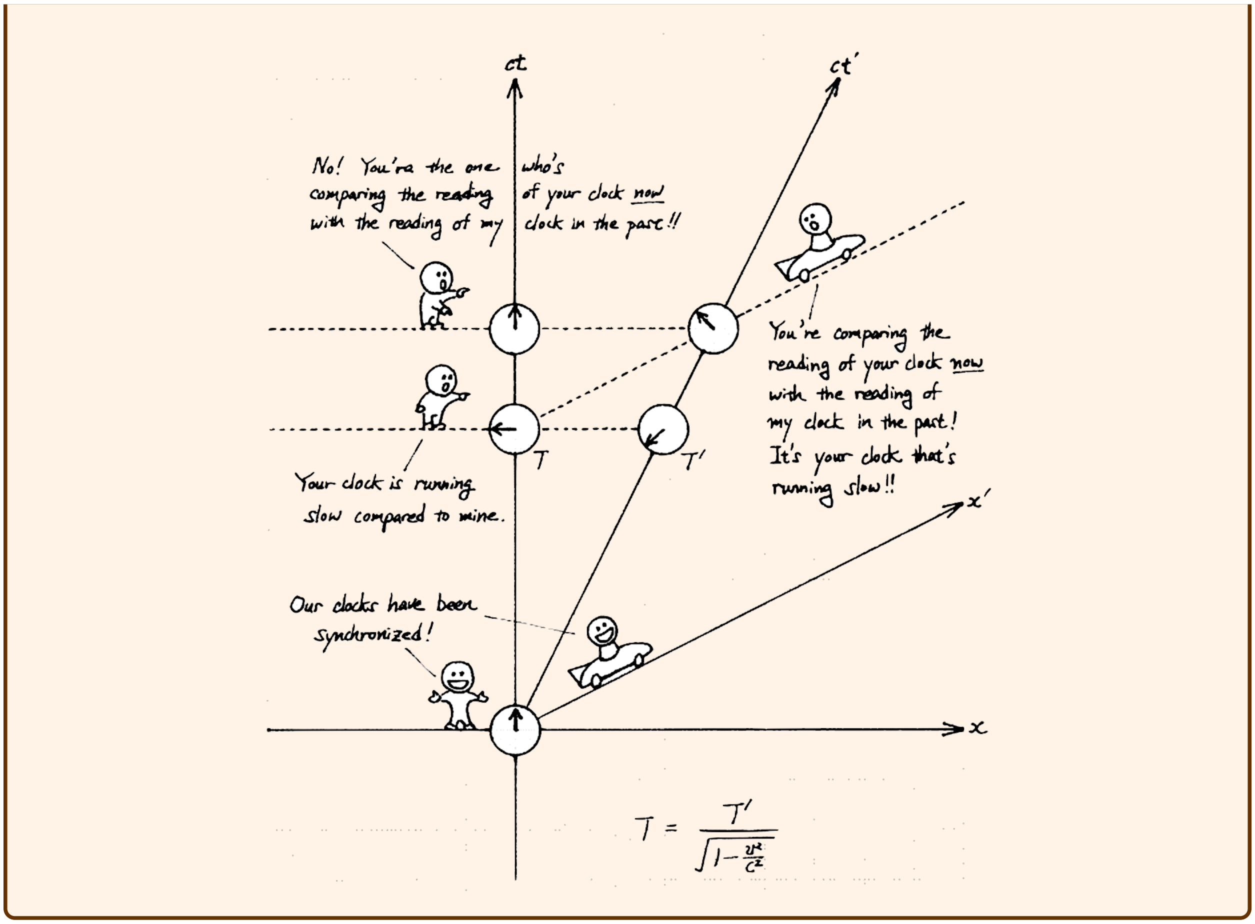
Example 2

Bob moves away from Alice at constant speed. According to special relativity, each sees the other as aging slower. (This is true both in terms of their reference frames, and in terms of what they see with their eyes.) How can that possibly be self-consistent? Shouldn't time be running slower for one or the other?

Solution

The first thing to point out about this paradox, and many other relativistic paradoxes, is that they rely on slipping in nonrelativistic assumptions using tricky wording. If you're fine with the idea of time being relative, there's nothing paradoxical about people disagreeing on whose clock runs slower. It's not really more confusing than the fact that when I walk away from you, I see you getting smaller, but you also see me getting smaller.

More seriously, though, the reason time dilation can be symmetric is the loss of simultaneity effect, as beautifully shown in Tatsu Takeuchi's Illustrated Guide to Relativity.



[2] **Problem 13.** The Lorentz transformations treat x and t completely symmetrically. So why is it that lengths *contract* while times *dilate*? Shouldn't both do the same thing?

Solution. This comes down to a difference in how lengths and times are measured. Let S be the lab frame and let S' be the frame of a moving rod and clock, and watch the primes below carefully!

- In frame S', consider two events occupied by the clock. Then by definition $\Delta x' = 0$ and the proper time read by the clock is $\Delta t'$. In our frame, for these same two events, we have $\Delta t = \gamma \Delta t'$, so a greater amount of time passes on the lab clock; we interpret this as time dilating for the moving clock.
- In frame S', consider the opposite ends of the ruler at the same time. Then by definition $\Delta t' = 0$ and the proper time is $\Delta x'$. In our frame, for these same two events, we have $\Delta x = \gamma \Delta x'$. So naively it looks like the story is the same.
- The difference comes down to how we define "time measured" and "length measured" in the lab frame S. The time measured on the clock is Δt, where the events must have $\Delta x' = 0$ so that we follow the clock. But by contrast, the length measured in the lab frame is Δx, where we must have $\Delta t = 0$ so that we measure the locations of both ends at the same time. The fundamental difference is that in frame S, time measurements can be done in different places (since we have, conceptually, a network of synchronized clocks) but length measurements must be done at the same time.

- Therefore, if we impose $\Delta t = 0$, we can use an inverse Lorentz transform to yield $\Delta x' = \gamma \Delta x$, which is length contraction.

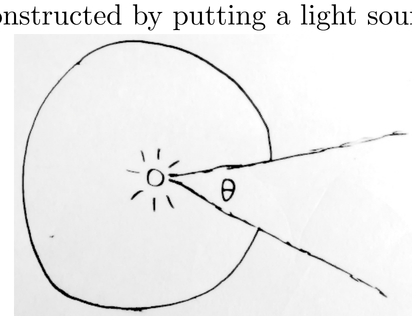
[3] **Problem 14.** A scientist is trying to drill through a piece of wood of thickness $2L$, but the longest drill bit they own has a length L. The scientist decides to move the wood relativistically fast, so that it length contracts to a thickness less than L. Then the drill can be held in the path of the wood, and pulled out once it goes through. Can this really be done without harming the drill bit or ruining the wood? If not, what's wrong? If yes, then what does it look like in the rest frame of the wood?

Solution. This is a harder version of the ladder, or barn-pole paradox. The answer is that it's not possible to pull the drill bit out in time without destroying it. The point is that, as you saw in problem 3, length contraction happens to fast-moving objects because of loss of simultaneity. In other words, if we pull the drill out without changing the proper length of any piece of it, then the *tip* of the drill bit has to start moving backwards first. The backward velocity propagates back through the drill bit, but it can't get to the back of the drill bit before it collides with the wood.

Here's an extreme example: suppose we try to pull the drill bit out at the speed of light. This means it has to length contract to zero, which means the tip of the drill bit and the motion both propagate backward at speed c. Suppose for concreteness that $\gamma = 4$, so that the piece of wood has thickness $L/2$ and speed $v = (\sqrt{15}/4)c$. At the moment the tip of the drill bit goes through the wood, we start moving the tip backwards. It takes a time L/c for this backwards motion to propagate to the back of the drill bit. During this time, the piece of wood has moved backwards a distance $vL/c > L/2$, which means it has already smashed into the back of the drill bit, ruining the wood.

For a more detailed discussion, with many nice diagrams, see section 6.3 of *Understanding Relativity* by Sartori.

[3] **Problem 15.** A headlight is constructed by putting a light source inside a spherical cavity.



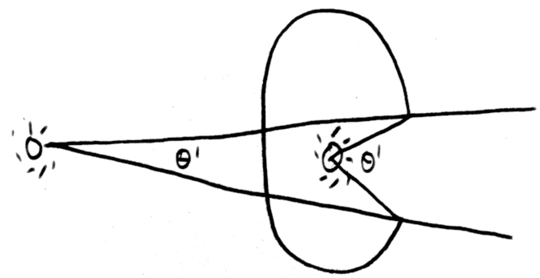
The opening of the cavity has angular width θ, so a beam of light comes out with width θ. The headlight is mounted on the front of a car, which then moves forward at a relativistic speed. The new width of the headlight's beam is θ', in the frame of the Earth. Consider the following two arguments.

> The headlight length contracts, increasing the cavity opening angle. Therefore, $\theta' > \theta$.
>
> By relativistic velocity addition, the light must have a greater forward velocity in the Earth's frame than in the car frame, because the car is moving forward. So the light must come out at a shallower angle in the Earth's frame. Therefore, $\theta' < \theta$.

Which argument is right?

Solution. The second argument is right; it's just the relativistic beaming effect from problem 7. The problem with the first argument is that, even though the cavity opening angle is bigger, that isn't the same thing as the beam's opening angle, because the light that gets to the cavity opening was emitted when the light source was further back.

Visually, here's a snapshot of the situation.



The two definitions of θ' correspond to the two arguments, but it's the smaller θ' that corresponds to the beam angle, as measured from far away. (Of course, if you take the first argument and account for the fact that the relevant light is further back, you can get precisely the same θ' as in the second argument, though the algebra to show this is a bit messy.)

[3] **Problem 16.** In the lab frame, a horizontal stick of proper length L has horizontal speed v. There is a horizontal thin sheet which has a hole of length L. Since the stick's length is contracted to L/γ, it easily passes through the hole in the sheet, if the sheet is moved vertically. But in the frame of the stick, the sheet is moving horizontally, so the hole is length contracted instead. Qualitatively explain how the stick can still pass through the hole in this frame, in the following two cases:

(a) The sheet has a uniform vertical velocity in the lab frame.

(b) The sheet begins at rest at the lab frame, but is pushed upward a small amount when the stick passes over the hole, then ends at rest again.

Solution. The idea behind this classic problem was first proposed by Rindler in 1961, then refined by Shaw in 1962, and incorporated into many textbooks. A detailed solution of Rindler's original version, with illustrations, is given in section 6.4 of *Understanding Relativity* by Sartori.

(a) The resolution is that the sheet is not horizontal in the stick's frame. The simplest way to see this is to let the z-axis be vertical, and consider when different points in the sheet cross the point $z = 0$. In the lab frame, these events are all simultaneous, so they're not simultaneous in the stick's frame, which means that in the stick's frame the sheet is rotated; you can calculate the angle with the Lorentz transformation. This is an example of a "Wigner rotation", a subtle relativistic kinematic effect. Since the sheet isn't horizontal, the hole passes around the stick at an angle, so it fits. For a detailed quantiative solution, see this paper.

(b) The resolution is that the sheet is not *straight* in the stick's frame. Again, the simplest way to see this is to consider when different points in the sheet start to be raised. In the lab frame, these events are all simultaneous, so they're not simultaneous in the stick's frame. At any given moment in the stick's frame, part of the sheet is still at the lower position, part of the sheet is already at the upper position, and part in between is moving upward while slanted, as in part (a). So, as the sheet moves horizontally, the hole appears to "bend around" the stick, letting it pass through.

This might seem very disturbing. The sheet is always perfectly straight in the lab frame, but it has two kinks in the stick's frame! But this is no more paradoxical than length contraction is. To decide whether a piece of an object is actually deformed, we need to look at it in its rest frame. An rod that's severely length contracted in one frame is in no danger of breaking, and this sheet, which is severely kinked in some frames, is in no danger of tearing.

Here's another example: suppose you have a uniformly rotating cylinder. Then in the frame of an object moving along the axis of the cylinder, the cylinder is twisted because of the loss of simultaneity effect. (But it's not *really* twisted, in the sense that if you work in the frame locally moving with any piece of the cylinder, it will have no shear stress.)

The lesson is that the classical definition of a rigid body from **M8**, i.e. that angles and lengths between points on the body always remain the same, doesn't work in special relativity; even if those conditions hold in one frame, they won't necessarily in another.

[4] **Problem 17.** Here is the statement of the traditional twin paradox.

> Bob is an astronaut who leaves home on a rocket with speed v. Alice stays home. After time T in Alice's frame, Bob reverses direction and travels home with speed v. Who, if either, has aged more?

The obvious answer is that Alice has aged more by time dilation. The trouble is explaining why we can't just work in Bob's frame and conclude that Bob has aged more by time dilation.

(a) Draw a Minkowski diagram for Alice and Bob where Alice's worldline is $x = 0$.

(b) The reason that working in Bob's frame is subtle is that it is not a single inertial frame. Draw x' and t' axes for Bob at several points on Bob's worldline. Argue that when Bob turns around, thereby moving to a different inertial frame, Alice's age jumps upward. (Using the results of chapter 11 of Morin, you can even show that the amount of aging is exactly what is needed, using the Minkowski diagram alone.)

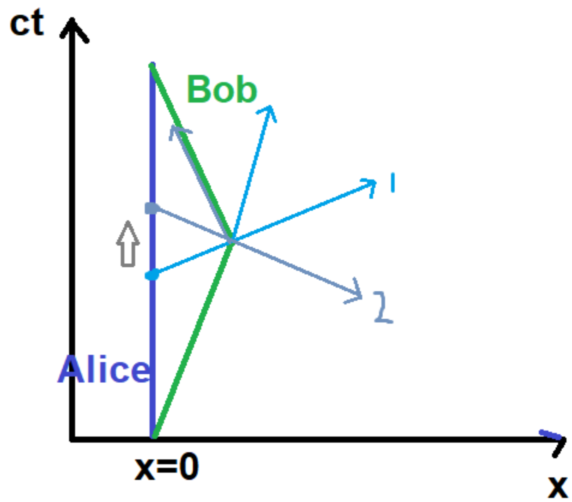
This illustrates why the situation is not symmetric between Alice and Bob. But this resolution of the twin paradox is a little unphysical. It does explain what goes wrong working in Bob's frame, but it's not related to what Bob actually physically *sees*, which is determined by when photons from Earth reach his eyes; nothing about that changes discontinuously when he turns around.

(c) More physically, let us suppose that Bob continually emits radiation of frequency f (in his frame) towards Alice. Suppose that in Alice's frame, Bob travels with speed v, reaches a maximum distance L from Alice, and accelerates quickly to return with speed v. If Alice sees N_b wave crests in total during Bob's trip, then Bob has aged by N_b/f. Use the relativistic Doppler effect to compute N_b/f, working entirely from Alice's perspective.

(d) Now suppose Alice continually emits radiation of frequency f (in her frame) towards Bob. If Bob sees N_a wave crests, use the relativistic Doppler effect to compute N_a/f, working entirely from Bob's perspective. If you're careful, this should differ from the answer to (c).

(e) [A] Now consider a trickier example. Suppose Alice and Bob live on a torus, i.e. a spacetime where the point (x, y, z) is the same as the point $(x + L, y, z)$. Alice stays home, while Bob leaves on a rocket with velocity $v\hat{\mathbf{x}}$. After a while, Bob returns home, without having done any acceleration along the way! It seems like the resolution above does not apply, so who, if either, has aged more? Can you explain the results from Bob's reference frame?

Solution. (a) Here's the result.



(b) Bob's x' and ct' axes before and after the acceleration are also displayed. We see that as these axes rotate during the acceleration, Alice's age changes extremely quickly.

This might feel strange, but it's really just an artifact of changing reference frames. As a simpler example, suppose you were a surveyor trying to measure the height of a mountain, which can be done by measuring the angle to its summit with respect to a horizontal level. If the surveyor then gets on an accelerating car, their horizontal level will tilt, causing the height reading to change extremely quickly. But that doesn't mean people living on the mountain will be flung off! They don't feel anything; it's just the surveyor's notion of horizontal that changed. Similarly, when Bob turns around, his definition of time changes, so that Alice's age "right now" (according to Bob) suddenly changes.

(c) The radiation that was emitted while Bob was moving away from Alice is received by Alice with redshifted frequency

$$f_r = \sqrt{\frac{1-v}{1+v}}\, f.$$

The radiation that was emitted while Bob was moving towards Alice is received by Alice with blueshifted frequency

$$f_b = \sqrt{\frac{1+v}{1-v}}\, f.$$

Suppose that Alice sees these frequencies for times t_r and t_b. Then the answer is

$$N_b = t_r f_r + t_b f_b.$$

It remains to compute t_r and t_b. Naively we would say $t_r = t_b = L/v$, because that's how long Bob spends moving towards and away from Alice respectively, but this question is about what Alice sees with her eyeballs. The transition point between the two phases is when the radiation that Bob emitted while turning around gets to Alice. In other words,

$$t_r = \frac{L}{v} + \frac{L}{c}, \quad t_b = \frac{L}{v} - \frac{L}{c}.$$

Then we have

$$N_b = \frac{Lf}{v}\left((v+1)\sqrt{\frac{1-v}{1+v}} + (1-v)\sqrt{\frac{1+v}{1-v}}\right) = \frac{2Lf}{v}\sqrt{1-v^2}$$

so Bob has aged by $2L/\gamma v$, exactly as expected. Physically, Alice sees Bob aging in slow motion for more than half the time, and aging in fast motion for less than half the time, with the overall effect of Alice aging more.

(d) Let's define all the terms as in the previous part. Bob turns around when he is a distance L/γ (according to him) from Alice. The fundamental difference is that Bob starts seeing the higher frequency the instant he turns around, so

$$t_r = t_b = \frac{L}{\gamma v}.$$

Therefore, we have

$$N_a = \frac{Lf}{\gamma v}\left(\sqrt{\frac{1-v}{1+v}} + \sqrt{\frac{1+v}{1-v}}\right) = \frac{2Lf}{\gamma v}\frac{1}{\sqrt{1-v^2}}$$

so Alice has aged by $2L/v$, exactly as expected. Physically, Bob sees Alice aging in slow motion for half the time, and aging in fast motion for half the time, with the overall effect of Alice aging more. Again, note that the fundamental asymmetry is due to Bob being the one accelerating, which is baked into how we computed the t_r and t_b.

(e) In this exotic spacetime, there really *is* a notion of absolute rest: we can unambiguously say that Bob moved and Alice didn't, so Alice has aged more. The reason is that the torus itself picks out a special frame. Only in Alice's frame is it true that when you wrap around the edge of the torus, you emerge on the other end *at the same time*. In Bob's frame, this isn't true, by loss of simultaneity. In Bob's frame, Alice gets to the edge of the torus, then emerges out the other edge at a later time, which ultimately makes her older than Bob when she returns.

The more general lesson is that while special relativity restricts the forms of physical laws to have certain symmetries, it doesn't mean that the *solutions* of the corresponding equations must always have the same symmetry. The dynamics of salt molecules in solution obey perfect rotational symmetry, but when they crystallize, the faces of the crystal pick out special directions. Likewise, as far as we've ever measured, all of the dynamics in our universe perfectly obey the Lorentz symmetry of special relativity, but the cosmic microwave background radiation does provide an absolute rest frame.

Remark

The above problem on the twin paradox is quite long. Every physics textbook that covers relativity mentions the twin paradox, and Morin even has a whole appendix with five different resolutions of it. But it's not *that* hard to resolve, so why spend so much energy on it?

The answer is that seemingly intelligent people really can get stuck on these things for years, or even decades. As an example, consider the case of Herbert Dingle, one of the foremost science popularizers in the mid-20$^{\text{th}}$ century. Dingle was an experimental physicist and philosopher of science, but he was best known for his eloquent, equation-free explanations of relativity, which made him the Brian Greene of his day. But soon after Einstein's death, he suddenly realized that relativity could not explain the twin paradox.

Here is one version of Dingle's argument. We write down the Lorentz transformations

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx), \quad x = \gamma(x' + vt'), \quad t = \gamma(t' + vx').$$

Then we notice that if we set $x = 0$, then $t' = \gamma t$, while if we set $x' = 0$, then $t = \gamma t'$. This "implies" that aging must always be symmetric. In fact, if we combine the equations, we conclude $\gamma = 1/\gamma$, which implies time dilation can't even happen at all! Thus, relativity collapses.

Dingle continued pushing this for the rest of his life, writing endless letters and articles, and even publishing a book, *Science at the Crossroads*, which warned of the grave societal dangers of trusting relativity. Today, it is a favorite of flat Earthers. And it's far from the only example. For instance, there was a book published in Nazi Germany called *A Hundred Authors Against Einstein*, where a vast array of philosophers argued that relativity had to be wrong, because it contradicted the metaphysical system of the native German, 18th century philosopher Immanuel Kant. Kant's ideas about space and time, they said, could be proven true by pure reason alone, so any theory or experiment saying otherwise had to be wrong.

If there's a lesson to be drawn from this bizarre history, it's that the ability to write or speak is not the same as the ability to think. Like GPT-3, one can churn out pages of flowing prose without having a single coherent thought. This haze of vague reasoning is a dark cave we're all born in. Physicists escape the cave by solving problems mathematically; many others never escape, and eventually grow to believe that nothing can exist outside it.

4 Four-Vectors

Idea 4

A four-vector V^μ is a set of four quantities (V^0, V^1, V^2, V^3) that transform in the same manner as (ct, x, y, z). The inner product of two four-vectors is defined as

$$V \cdot W = V^0 W^0 - V^1 W^1 - V^2 W^2 - V^3 W^3.$$

It is invariant under Lorentz transformations. By convention, $V \cdot W$ is also written as $V^\mu W_\mu$.

[2] **Problem 18.** Show explicitly that the norm of the displacement four-vector is invariant under Lorentz transformations, i.e. that

$$(\Delta s)^2 = \Delta s \cdot \Delta s = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

is Lorentz invariant. Since all four-vectors transform the same way, this proves it for all of them.

Solution. Plugging in the Lorentz transformations, we have

$$\begin{aligned}(\Delta s')^2 &= (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\ &= \gamma^2(\Delta t - v\Delta x)^2 - \gamma^2(\Delta x - v\Delta t)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= \gamma^2(1-v^2)(\Delta t)^2 - \gamma^2(1-v^2)(\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\end{aligned}$$

as desired.

Example 3

Find a four-vector representing the velocity of a particle with position $\mathbf{x}(t)$.

Solution

Just as multiplying an ordinary vector with a rotational invariant produces another vector, multiplying or dividing a four-vector with a Lorentz invariant gives another four-vector. In this case, the appropriate four-vector is found by dividing displacement by the proper time experienced by the particle,

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = (\gamma, \gamma\mathbf{v})$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the spatial velocity and $\gamma = 1/\sqrt{1-v^2}$ as usual. Since its spatial part reduces to the spatial velocity in the limit of low speeds, it is the relativistic generalization of the spatial velocity. Finally, we define the four-momentum as $p^\mu = mu^\mu = (E, \mathbf{p})$, where $E = \gamma m$ and $\mathbf{p} = \gamma m\mathbf{v}$ are the relativistic energy and momentum.

Example 4

Give a simple interpretation of the squared norm of a particle's four-velocity, $u \cdot u$, and its four-momentum, $p \cdot p$.

Solution

The answers have to be simple, because they must be invariants that only depend on the intrinsic properties of the particle, i.e. only on the invariant mass m. For the four-velocity,

$$u \cdot u = \gamma^2 - \gamma^2 v^2 = 1$$

which is clearly invariant. For the four-momentum we have $p \cdot p = m^2$.

Remark

Pop-science books usually describe the result $u \cdot u = 1$ by saying that "particles always move with the same speed through spacetime", just like how a particle in uniform circular motion always has the same spatial speed. However, this is very misleading, because it makes people think that if $dx/d\tau$ increases in magnitude, then $dt/d\tau$ decreases. In fact it's the opposite: time dilation means more time passes for each tick of a moving clock, so $dt/d\tau$ increases.

The analogy can't work, because inner products of four-vectors have terms with minus signs, while ordinary inner products of three-vectors don't. The point, as always, is that there are a lot of simple things in physics which are almost impossible to explain properly with fuzzy math-free analogies. With math, relativity can make sense to high school students. Without math, it can't really make sense to anyone.

Example 5

Give a simple interpretation of the inner product of two momentum four-vectors, $p_1 \cdot p_2$.

Solution

By definition, this is equal to $m_1 m_2 u_1 \cdot u_2$, and since the inner product is invariant, we can evaluate $u_1 \cdot u_2$ in any frame. Suppose we work in the frame of the first particle, where

$$u_1^\mu = (1, \mathbf{0}), \quad u_2^\mu = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\mathbf{v}}{\sqrt{1-v^2}} \right).$$

Carrying out the inner product, we have the relatively simple result

$$p_1 \cdot p_2 = \frac{m_1 m_2}{\sqrt{1-v^2}}$$

where v is the relative speed, meaning the speed of one particle in the frame of the other.

[2] **Problem 19.** In your inertial frame, there is a particle with four-momentum p^μ, and an observer moving with four-velocity u^μ. The observer measures the particle in *their* inertial frame.

(a) Show that the energy they measure is $p \cdot u$.

(b) Show that the momentum they measure has magnitude $\sqrt{(p \cdot u)^2 - p \cdot p}$.

(c) What is the speed that they measure?

Don't use Lorentz transformations here; everything can be done with four-vectors alone.

Solution. (a) We can evaluate $p \cdot u$ in the observer's frame. In that case, $u^\mu = (1, 0, 0, 0)$, so $p \cdot u$ just picks out the first component of p^μ in that frame, which is by definition the energy the observer measures.

(b) Continuing to work in the observer's frame, and writing $p^\mu = (E, \mathbf{p})$, where E and $\mathbf{p}$ are the energy and momentum in the observer's frame, we have

$$(p \cdot u)^2 - p \cdot p = E^2 - (E^2 - |\mathbf{p}|^2) = |\mathbf{p}|^2$$

which gives the desired result.

(c) Note that $E = \gamma m$ and $\mathbf{p} = \gamma m v$, so the speed they measure is the ratio

$$|\mathbf{v}| = \frac{|\mathbf{p}|}{E} = \frac{\sqrt{(p \cdot u)^2 - p \cdot p}}{p \cdot u} = \sqrt{1 - \frac{p \cdot p}{(p \cdot u)^2}}.$$

A nice feature of this result is that it's immediately clear that $|\mathbf{v}| \leq 1$.

[3] **Problem 20.** In A's frame, B has speed u, and C has speed v.

(a) Suppose B and C have velocities in opposite directions. Find the speed of B with respect to C using four-vectors, by computing the inner product $v_B \cdot v_C$ in two different frames.

(b) The answer of part (a) should look familiar, but with four-vectors we can easily go further. Generalize part (a) to the case where B and C have velocities an angle θ apart.

Solution. (a) In A's frame, the four-velocities are

$$v_B = (\gamma_u, \gamma_u u), \quad v_C = (\gamma_v, -\gamma_v v).$$

Let w be the desired answer. Then in C's frame,

$$v_B = (\gamma_w, \gamma_w w), \quad v_C = (1, 0).$$

The inner product of v_B and v_C should be independent of frame, so

$$\gamma_u \gamma_v (1 + uv) = \gamma_w$$

or equivalently

$$\frac{1 + uv}{\sqrt{1 - u^2}\sqrt{1 - v^2}} = \frac{1}{\sqrt{1 - w^2}}.$$

Solving for w gives the expected result,

$$w = \frac{u + v}{1 + uv}.$$

(b) Taking $\mathbf{v}_B$ to be along the x-axis for concreteness,

$$v_B = (\gamma_u, \gamma_u u, 0), \quad v_C = (\gamma_v, \gamma_v v \cos\theta, \gamma_v v \sin\theta).$$

By the same logic as in part (a), we have

$$\gamma_u \gamma_v (1 - uv\cos\theta) = \gamma_w$$

and solving for w gives the complicated result

$$w = \frac{\sqrt{u^2 + v^2 - 2uv\cos\theta - u^2 v^2 \sin^2\theta}}{1 - uv\cos\theta}.$$

This reduces to the usual velocity addition formula for $\theta = 0$ and $\theta = \pi$. If we didn't use the tool of four-vectors and just applied the Lorentz transformations directly, this could have been quite a mess, but instead it wasn't much harder than part (a)!

[3] **Problem 21.** Four-vectors provide a quick derivation of the relativistic Doppler effect. Given a plane wave, define $k^\mu = (\omega, \mathbf{k})$. Then the plane wave is proportional to $e^{i\phi}$, where the phase is

$$\phi = \omega t - \mathbf{k} \cdot \mathbf{x} = k \cdot x.$$

Since the phase ϕ is Lorentz invariant, and we know x^μ is a four-vector, k^μ is a four-vector as well.

(a) Show that for light, $k^\mu k_\mu = 0$.

(b) Consider a light ray with angular frequency ω traveling along the x axis, and an observer moving with speed v along the x-axis. Use an explicit Lorentz transformation to find the angular frequency ω' the observer sees, thus rederiving the longitudinal Doppler shift for light.

(c) Now it's easy to go further. Repeat the previous part for a light ray traveling at an arbitrary angle θ to the x axis. You can do this using either an explicit Lorentz transformation, or just properties of four-vectors.

(d) One subtlety with the general relativistic Doppler effect is the definition of θ, because it has different values in the source's frame, and in the observer's frame. The previous two parts started in the source's frame, but the usual formula defines θ in the *observer's* frame.

To recover this formula, repeat the previous part, but now suppose we're already in the observer's frame, where the source moves with velocity $-v\hat{\mathbf{x}}$, and the light ray is traveling at an angle θ to the x-axis. Find the relationship between ω' and ω.

The result of part (d) should look familiar: it's the final result of USAPhO 2021, problem A2. For more on the general Doppler effect, see section 11.8.2 of Morin. (By the way, now that we have the four-vector formalism set up, it's not that much harder to compute the Doppler effect for waves that travel at general speeds. You probably won't need that result, but it's an example of something that's a total nightmare to derive without four-vectors.)

Solution. (a) For plane waves $\omega = vk$, where $v = c$ for light. Thus the norm is $\omega^2 - k^2 = 0$.

(b) Setting $c = 1$ now, a light ray traveling along the x axis has $k^\mu = (\omega, \omega, 0, 0)$. Applying a boost along the x axis, the new angular frequency is

$$\omega' = (k')^0 = \gamma(\omega - v\omega) = \sqrt{\frac{1-v}{1+v}}\,\omega$$

which is precisely the longitudinal Doppler effect. The $v\omega$ term above is just what we would expect from Galilean physics, while the relativistic factor of γ modifies the effect to second order in v.

(c) For variety, we'll do this part with four-vectors. We have

$$k^\mu = (\omega, \omega\cos\theta, \omega\sin\theta, 0), \quad v^\mu = (\gamma, \gamma v, 0, 0)$$

and by slightly modifying part (a) of problem 19, we have

$$\omega' = k \cdot v = \gamma\omega - \gamma\omega v\cos\theta = \frac{1 - v\cos\theta}{\sqrt{1-v^2}}\,\omega.$$

(d) In this case, let v^μ be the four-velocity of the source. In the observer's frame,

$$k^\mu = (\omega', \omega'\cos\theta, \omega'\sin\theta, 0), \quad v^\mu = (\gamma, -\gamma v, 0, 0)$$

and the angular frequency measured in the source's frame is

$$\omega = k \cdot v = \gamma\omega' + \gamma\omega' v\cos\theta = \frac{1 + v\cos\theta}{\sqrt{1-v^2}}\,\omega'.$$

Rearranging, we conclude that

$$\omega' = \frac{\sqrt{1-v^2}}{1 + v\cos\theta}\,\omega$$

which differs from the result of part (c) by second-order terms.

Example 6: Woodhouse 6.6

Four distant stars S_i are observed. Let θ_{ij} denote the observed angle between the directions to S_i and S_j. Show that the ratio

$$\frac{(1-\cos\theta_{12})(1-\cos\theta_{34})}{(1-\cos\theta_{13})(1-\cos\theta_{24})}$$

is independent of the motion of the observer.

Solution

This Oxford undergraduate exam question is too technical to be relevant to Olympiads, but it shows how four-vectors can be essential. The θ_{ij} depend on the motion of the observer because of the aberration effect in problem 7. That is, when you Lorentz transform to a moving observer's frame, it changes the direction of the incoming light. A direct attack on the question would thus require applying the full, four-dimensional Lorentz transformations to four vectors with arbitrary orientations, which would be a nightmare. Here's an alternative: let k_i^μ be the wave vectors of an incoming photon from each star. Then

$$k_i \cdot k_j = \omega_i\omega_j - \mathbf{k}_i \cdot \mathbf{k}_j = \omega_i\omega_j(1-\cos\theta_{ij})$$

where we used $\omega_i = |\mathbf{k}_i|$. Therefore, the ratio is

$$\frac{(k_1\cdot k_2)(k_3\cdot k_4)/\omega_1\omega_2\omega_3\omega_4}{(k_1\cdot k_3)(k_2\cdot k_4)/\omega_1\omega_2\omega_3\omega_4} = \frac{(k_1\cdot k_2)(k_3\cdot k_4)}{(k_1\cdot k_3)(k_2\cdot k_4)}$$

which is manifestly independent of frame.

[4] **Problem 22.** In this problem we'll construct a four-vector a^μ for the acceleration of a particle, and use it to derive the Lorentz transformation of the ordinary three-vector acceleration $\mathbf{a} = d\mathbf{v}/dt$.

(a) Explain why $a^\mu = du^\mu/d\tau$ is a four-vector, and why $u \cdot a$ is always zero.

(b) Show that when $\mathbf{v} = v\hat{\mathbf{x}}$, the components of a^μ are

$$a^\mu = (\gamma^4 v a_x, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z)$$

where $\gamma = 1/\sqrt{1-v^2}$ as usual. As a check, what is the meaning of $a \cdot a$?

(c) Now let S' be the momentary rest frame of a particle, i.e. the inertial frame that, at a given moment, is moving with the same velocity as the particle. Let the particle have three-acceleration $\mathbf{a}'$ in that frame. Show that in this frame, $a^{\mu'} = (0, a'_x, a'_y, a'_z)$.

(d) By Lorentz transforming to S and using part (b), show that the acceleration in frame S is

$$\mathbf{a} = (a'_x/\gamma^3, a'_y/\gamma^2, a'_z/\gamma^2).$$

As you can see, transformations of three-vector quantities can get quite nasty!

Solution. (a) We know that u^μ is a four-vector, and $d\tau$ is Lorentz invariant, so $du^\mu/d\tau = a^\mu$ is a four-vector. Next, we know from an example that $u \cdot u$ is constant, so

$$\frac{d}{d\tau}(u\cdot u) = 2u\cdot a = 0.$$

(b) The four-velocity will be $(\gamma, \gamma\mathbf{v})$. Since $d\tau = dt/\gamma$, we have $a^\mu = du^\mu/d\tau = \gamma du^\mu/dt$. Note that $d\gamma/dt = (1 - v^2/c^2)^{-3/2}(-1/2)(-2va_x/c^2) = \gamma^3 va_x/c^2$, and acceleration in the y and z components do not change the magnitude of the speed (first order), thus won't change γ.

$$a^\mu = \gamma \frac{d}{dt}(\gamma c, \gamma\mathbf{v}) = \gamma(\gamma^3 va_x/c, \gamma^3 a_x v^2/c^2 + \gamma a_x, \gamma a_y, \gamma a_z)$$
$$= (\gamma^4 va_x, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z).$$

To understand $a \cdot a$, we evaluate it in the momentary rest frame,

$$a \cdot a = (0, a_x, a_y, a_z) \cdot (0, a_x, a_y, a_z) = -|\mathbf{a}|^2.$$

That is, it indicates the magnitude of the three-acceleration in that frame.

(c) Since $v = 0$, $\gamma = 1$ and we get $a^{\mu'} = (0, a_x, a_y, a_z)$.

(d) Lorentz transforming the acceleration components of a^μ,

$$a^\mu = (\gamma(0 + va_x), \gamma(a'_x + 0), a'_y, a'_z) = (\gamma^4 va_x, \gamma^4 a_x, \gamma^2 a_y, \gamma^2 a_z)$$

Equating a_x, a_y, a_z will yield

$$\mathbf{a} = (a'_x/\gamma^3, a'_y/\gamma^2, a'_z/\gamma^2).$$

as desired. Note that in the low velocity limit, the acceleration components stay the same, $a'_i = a_i$, as expected from Galilean relativity.

Remark

We can rewrite a lot of our results in terms of three-vectors. First, the Lorentz transformations for general $\mathbf{v}$ are, using the same notation as in idea 1,

$$t' = \gamma(t - \mathbf{v} \cdot \mathbf{r}), \quad \mathbf{r}' = \mathbf{r} - \gamma\mathbf{v}t + (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{r})\hat{\mathbf{v}}.$$

The velocity addition formula for general $\mathbf{v}$ and $\mathbf{u}'$ is, using the same notation as in idea 2,

$$\mathbf{u} = \frac{1}{1 + \mathbf{v} \cdot \mathbf{u}'} \left(\mathbf{v} + \frac{\mathbf{u}'}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{u}') \right).$$

The first result of problem 22 is

$$a^\mu = (\gamma^4 \mathbf{a} \cdot \mathbf{u},\, \gamma^4(\mathbf{a} + \mathbf{u} \times (\mathbf{u} \times \mathbf{a})))$$

and the second result, for the transformation of acceleration, is

$$\mathbf{a} = \frac{\mathbf{a}'}{\gamma^2} - \frac{\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{a}')(\gamma - 1)}{\gamma^3}.$$

As you can see, these aren't very enlightening, and they don't tend to be useful in solving problems. The reason is that in relativity, there's nothing special about three-vectors. For concrete problems, you'll typically either want to do everything in terms of four-vectors, or descend all the way down to individual components – in which case you would align your axes so that $\mathbf{v}$ points along one of them, rather than considering a completely general $\mathbf{v}$.

On the other hand, you can get practice with three-vectors by staring at the above expressions until you see how they reduce to the component forms we had earlier. If you do this, you'll learn how to translate just about *any* component expression into three-vector notation.

5 Acceleration and Rapidity

Idea 5

The geometry of special relativity is much like ordinary geometry, except that the dot product is replaced with an inner product, which has some minus signs. Lorentz transformations can be thought of as "generalized rotations" which mix up time and space, just as ordinary rotations mix up different spatial axes. The generalized angle is the rapidity $\phi = \tanh^{-1} v$.

[3] **Problem 23** (Morin 11.27)**.** In this problem, we'll see the meaning of the rapidity more precisely.

(a) Show that a Lorentz transformation may be written as

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}.$$

(b) Show that the composition of Lorentz transformations with rapidity ϕ_1 and ϕ_2 is a Lorentz transformation with rapidity $\phi_1 + \phi_2$. This makes rapidity extremely useful in kinematics problems with multiple boosts, such as problems involving acceleration.

(c) An ordinary rotation of spatial axes has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Show that a Lorentz transformation is essentially an ordinary rotation between space and time, if we treat time as like "imaginary space" and the rotation as by an imaginary angle. This was one of the ways the founders of relativity thought about it.

Solution. (a) The rapidity ϕ is defined by $\tanh\phi = v$. Then using $\tanh\phi = \sinh\phi/\cosh\phi$ and $\cosh^2\phi - \sinh^2\phi = 1$, we have

$$\sinh\phi = \gamma v, \quad \cosh\phi = \gamma.$$

On the other hand, the Lorentz transformations are

$$t = \gamma(t' + vx'), \quad x = \gamma(x' + vt')$$

which are exactly of the desired form.

(b) Explicitly, we have

$$\begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where

$$A = \cosh\phi_1 \cosh\phi_2 + \sinh\phi_1 \sinh\phi_2, \quad B = \cosh\phi_1 \sinh\phi_2 + \sinh\phi_1 \cosh\phi_2.$$

By using the hyperbolic trig sum rules, we have

$$A = \cosh(\phi_1 + \phi_2), \quad B = \sinh(\phi_1 + \phi_2)$$

as desired.

(c) Let us substitute $\theta = i\phi$ and $y = it$. Then the rotation becomes

$$\begin{pmatrix} x \\ it \end{pmatrix} = \begin{pmatrix} \cos(i\phi) & -\sin(i\phi) \\ \sin(i\phi) & \cos(i\phi) \end{pmatrix} \begin{pmatrix} x' \\ it' \end{pmatrix}.$$

This can be converted to a transformation between (x, t) and (x', t'),

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \cos(i\phi) & -i\sin(i\phi) \\ -i\sin(i\phi) & \cos(i\phi) \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}.$$

However, the quantities on and off the diagonal are simply the definitions of $\cosh(\phi)$ and $\sinh(-\phi)$, so we're done! (Up to an annoying overall sign on v, which was just up to our conventions for rotation matrices.)

Remark

We know from **M8** that if we do multiple rotations in a row, the final result depends on the order the rotations are performed. The analogy between boosts and rotations gives intuition for the analogous result for boosts: the order matters. For example, the difference between boosting along the x-axis and then the y-axis, or vice versa, is a rotation in the xy plane. Therefore, if an object is boosted in a circle in the xy plane, it will have an extra rotation in that plane. This subtle phenomenon is called Thomas precession.

Idea 6

The next few questions will deal with accelerating objects. In Newtonian mechanics, a common strategy is to work in the accelerating frame of the object, but that's not a good idea at this stage of your education. (There's nothing wrong with doing so, but it brings in complications that one usually needs a course in general relativity to fully appreciate.)

Instead, we will describe accelerating objects using inertial frames. In principle we could do everything in the lab frame, but it is also often useful to work in a momentarily comoving frame, i.e. the inertial frame that, at some time t, moves with the same velocity as the object.

[4] **Problem 24.** A rocket starts from rest in the lab frame at the origin, then accelerates in a straight line at constant rate a_0 as measured by an accelerometer on the ship; that is, the proper acceleration is always a_0.

(a) Show that the acceleration measured in the lab frame is a_0/γ^3. (We already proved this more generally in problem 22, but try to do this more explicitly by working in the comoving frame, then going back to the lab frame.)

(b) Find the speed of the rocket ship in the lab frame as a function of time t in the lab frame.

(c) Find the speed of the rocket ship in the lab frame as a function of the proper time τ elapsed on the rocket. Can you explain the simplicity of your result using rapidity?

(d) To conclude, find expressions for $t(\tau)$, $x(t)$, and $x(\tau)$, and comment on their limits.

Solution. (a) Suppose the rocket has speed v in the lab frame, and now consider the momentarily comoving frame S' moving with speed v. In that frame, in time dt', the rocket accelerates from zero speed to speed $a_0\, dt'$. Then the new speed in the lab frame is the sum of v and $a_0\, dt$, which is

$$\frac{v + a_0\, dt'}{1 + v a_0\, dt'} = v + a_0\, dt' - v^2 a_0\, dt' + O(dt^2).$$

Therefore, we have

$$\frac{dv}{dt} = a_0(1 - v^2)\frac{dt'}{dt} = a_0(1 - v^2)^{3/2} = \frac{a_0}{\gamma^3}$$

as desired.

(b) Separating and integrating, we have

$$a_0 t = \int \frac{dv}{(1 - v^2)^{3/2}} = \frac{v}{\sqrt{1 - v^2}}$$

by a trigonometric substitution, which gives

$$v(t) = \frac{a_0 t}{\sqrt{1 + (a_0 t)^2}}.$$

(c) Note that the increment of proper time measured by the rocket is $d\tau = dt'$, because dt' is always defined in the frame momentarily moving with the rocket. Therefore

$$\frac{dv}{d\tau} = a_0(1 - v^2), \quad a_0 \tau = \int \frac{dv}{1 - v^2}.$$

Using hyperbolic trig substitution, we have

$$v(\tau) = \tanh(a_0 \tau)$$

which just tells us that the rapidity changes at rate a_0.

(d) It's easy to crank these out given the above results. First, we have

$$\int dt = \int \gamma\, d\tau = \int \frac{d\tau}{\sqrt{1 - \tanh^2(a_0\tau)}} = \int \cosh(a_0\tau)\, d\tau$$

from which we read off

$$t(\tau) = \frac{\sinh(a_0 \tau)}{a_0}.$$

Next, we integrating the answer to part (b),

$$x(t) = \int_0^t \frac{a_0 t'}{\sqrt{1 + (a_0 t')^2}}\, dt' = \frac{\sqrt{1 + (a_0 t)^2} - 1}{a_0}.$$

Finally, plugging in our expression for $t(\tau)$ gives

$$x(\tau) = \frac{\sqrt{1+\sinh^2(a_0\tau)} - 1}{a_0} = \frac{\cosh(a_0\tau) - 1}{a_0}.$$

This results make sense. At small t and τ, they just reduce to the familiar results $t = \tau$ and $x = a_0 t^2/2$. At large t, $x(t)$ increases linearly, since the speed of the rocket approaches the speed of light. What is perhaps most interesting is that at large τ, $x(\tau)$ increases *exponentially*, because of how quickly the time dilation effect increases. If it's possible to make it to another star in a human lifetime, it's actually not that much harder to cross the whole galaxy! This is a neat result, recently highlighted in the 2022 IPhO and the popular book *Project Hail Mary*.

[3] **Problem 25.** USAPhO 2020, problem A3. An unusual problem that tests your understanding of momentarily comoving frames, and higher-dimensional Lorentz transformations. As a warning, this question requires you to make an unstated assumption. The fact that uniformly moving clocks have their time dilated by a factor of γ follows directly from the postulates of special relativity. But here you'll have to assume this also holds for *accelerating* clocks, even though clocks can tell if they're accelerating, and may tick differently. This is called the clock hypothesis. For example, on a roller coaster, a pendulum clock doesn't obey the clock hypothesis, but a quartz watch does. Also, the solution is a bit misleading, so don't worry if you thought about the problem differently as long as you got the same final answers.

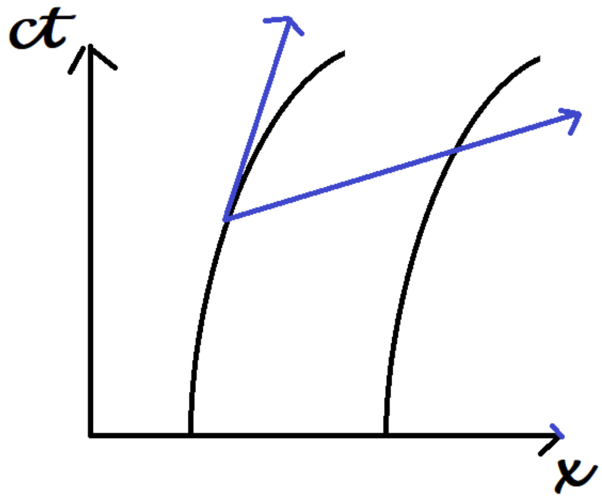
[3] **Problem 26** (Morin 11.26)**.** The following problem is called Bell's spaceship paradox. It caused a stir at CERN when many particle physicists could not agree on the answer.

Two spaceships float in space and are at rest relative to each other. They are connected by a string. The string is strong, but it cannot withstand an arbitrary amount of stretching.

At a given instant, the spaceships simultaneously (with respect to their initial inertial frame) start accelerating in the same direction along the line between them, with the same constant proper acceleration. In other words, assume they bought identical engines from the same store, and they put them on the same setting. Will the string eventually break?

Solution. The two conflicting ideas is that in the lab frame, length contraction would indicate that the distance between the two rockets is γL in the co-moving frame, indicating that the string stretches and breaks. The other idea is that in the rocket frame, they both accelerate with the same proper acceleration, and the relative velocity stays as 0.

The correct answer is that the string will eventually break. The second line of reasoning fails because of the relativity of simultaneity. We can consider, at each moment, the inertial frame comoving with the rear rocket, whose axes are as shown.



Over time, the x' axis of this set of inertial frames tilts upward, so the rear rocket sees the front rocket accelerating faster, and hence moving away.

If this isn't clear from the diagram, one can consider discretizing the acceleration, i.e. converting it into a series of rocket pulses. Since the clocks of the rockets are synchronized, the first pulses are simultaneous according to both rockets. But because of the relativity of simultaneity, within the frame moving with the rear rocket after the first pulse, the front rocket does the second pulse *earlier*, and hence starts to move away. (For a neat visual explanation, see this video.)

[5] **Problem 27.** APhO 2013, problem 2. This is a challenging question that ties together everything you've learned about kinematics.

Relativity II: Dynamics

Chapter 12 of Morin covers relativistic dynamics, as does chapter 13 of Kleppner, or chapter 12 of Wang and Ricardo, volume 2. For four-vectors in relativistic dynamics, finish chapter 13 of Morin, or chapter 14 of Kleppner. For a deeper explanation of four-vectors, see chapter 2 of Schutz. There is a total of **83** points.

1 Energy and Momentum

Idea 1

The relativistic generalizations of energy and momentum are

$$E = \gamma mc^2, \quad \mathbf{p} = \gamma m\mathbf{v}.$$

These quantities are conserved, and m is defined as the rest mass. Note that m is *not* conserved in inelastic processes, while E *is* conserved; this is precisely the opposite of what happens nonrelativistically. The relativistic energy E automatically counts all contributions to the energy, including internal energy and rest energy mc^2.

[5] **Problem 1.** A few useful facts about energy and momentum, for future reference.

(a) Recalling the definition of the four-velocity from **R1**, show that

$$(E/c, \mathbf{p}) = mu^\mu$$

where u^μ is the four-velocity. Setting $c = 1$, this establishes $p^\mu = (E, \mathbf{p})$ is a four-vector.

(b) Suppose a particle is at rest in frame S'. Confirm explicitly that the components of the four-momentum p^μ transform as expected when going to frame S.

(c) Setting $c = 1$ for all future parts, show that the norm of the four-momentum is

$$p^\mu p_\mu = E^2 - |\mathbf{p}|^2 = m^2.$$

This is a very useful result that can simplify the solutions to many problems below, especially ones that simply ask for a final mass m. In this case one can often compute a single four-momentum and find its norm to get the answer.

(d) The expressions in idea 1 for E and $\mathbf{p}$ don't work for photons, since γ is infinite and m is zero. Instead, show that for a photon we have $p^\mu = \hbar k^\mu$.

(e) A system's center of mass frame is the one where its momentum is zero. For a system with total energy E and momentum $\mathbf{p}$, show that the center of mass has velocity $\mathbf{v} = \mathbf{p}/E$.

(f) In Newtonian mechanics, the kinetic energy K of an object with fixed mass m satisfies $dK = \mathbf{v} \cdot d\mathbf{p}$. Show that this also holds in relativity, assuming the rest mass m is fixed.

(g) As we'll discuss in more detail below, the force three-vector is defined as $\mathbf{F} = d\mathbf{p}/dt$ in relativistic mechanics. Show that $dK = \mathbf{F} \cdot d\mathbf{x}$, continuing to assume that m is fixed.

Idea 2

In relativistic dynamics problems, it is almost always better to work with energy and momentum than velocity; one typically shouldn't even mention velocities unless the problem asks for or gives them.

We'll start with some very simple problems to warm up.

Example 1: KK 13.5

A particle of mass m and speed v collides and sticks to a stationary particle of mass M. Find the final speed of the composite particle.

Solution

The total four momentum is $(\gamma m + M, \gamma m v)$, so the speed is

$$v = \frac{p}{E} = \frac{\gamma m v}{\gamma m + M}.$$

Example 2: Morin 12.2

Two photons of energy E collide at an angle θ and create a particle of mass M. What is M?

Solution

The total four-momentum is

$$p^\mu = (2E, E(1+\cos\theta), E\sin\theta).$$

The mass is just the norm of the four-momentum, so

$$M = \sqrt{4E^2 - E^2(1+\cos\theta)^2 - E^2\sin^2\theta} = E\sqrt{2-2\cos\theta} = 2E\sin(\theta/2)/c^2$$

where we restored the factors of c at the end.

[1] **Problem 2** (Morin 12.4)**.** A stationary mass M_A decays into masses M_B and M_C. What are the energies of these two masses?

[2] **Problem 3.** USAPhO 2012, problem A1.

[2] **Problem 4.** A particle with mass M and energy E moves towards a detector when it suddenly decays and emits a photon in its direction of motion. The detector measures a photon angular frequency of ω. What was the photon's angular frequency in the rest frame of the decaying particle?

[3] **Problem 5.** USAPhO 2002, problem A2.

Now let's try some more involved problems.

Example 3: Woodhouse 7.5

A particle of rest mass m moves with velocity $\mathbf{u}$ and collides elastically with a second particle, also of rest mass m, which is initially at rest. After the collision, the particles have velocities $\mathbf{v}$ and $\mathbf{w}$. Show that if θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\cos\theta = \frac{(1-\sqrt{1-v^2})(1-\sqrt{1-w^2})}{vw}.$$

Solution

First, a remark: in Newtonian mechanics, you learn that in an inelastic collision, the kinetic energy is dissipated into microscopic thermal motion. This often leads students to ask: if we keep track of the motion of all particles in detail, then are all collisions actually perfectly elastic? According to particle physics, the answer is no. You really can lose kinetic energy by converting it to mass-energy, in collisions which change the identity of the particles or produce new particles. Therefore, at particle colliders, we say a collision is elastic if the particles that come out are precisely the same as the ones that came in. For this example, that means the final particles still have rest mass m.

Conservation of energy and momentum imply

$$1+\gamma_u = \gamma_v + \gamma_w, \quad \gamma_u \mathbf{u} = \gamma_v \mathbf{v} + \gamma_w \mathbf{w}.$$

To get an expression with $\cos\theta$, we take the norm squared of the momentum equation,

$$\gamma_u^2 u^2 = \gamma_v^2 v^2 + \gamma_w^2 w^2 + 2\gamma_v\gamma_w vw\cos\theta.$$

This can be substantially simplified by noting that $\gamma_u^2 u^2 = \gamma_u^2 - 1$, giving

$$2vw\gamma_v\gamma_w\cos\theta = \gamma_u^2 - \gamma_v^2 - \gamma_w^2 + 1.$$

The appearance of so many squares motivates us to square both sides of the energy equation,

$$1 + 2\gamma_u + \gamma_u^2 = \gamma_v^2 + \gamma_w^2 + 2\gamma_v\gamma_w.$$

Using this to simplify the right-hand side of the previous equation,

$$2vw\gamma_v\gamma_w\cos\theta = 2\gamma_v\gamma_w - 2\gamma_u = 2(\gamma_v\gamma_w - \gamma_v - \gamma_w + 1) = 2(\gamma_v - 1)(\gamma_w - 1)$$

where in the second step we used conservation of energy. After solving for $\cos\theta$, we get the desired result. This was a bit of a slog, but it's representative of the hardest calculations you'll ever have to do for special relativity problems.

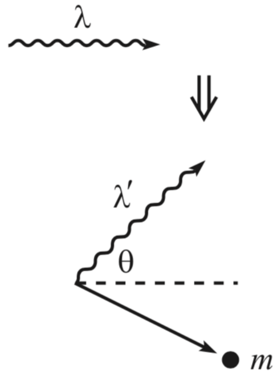
As a check on that result, note that in the nonrelativistic limit we get $\cos\theta = 0$, indicating a 90° angle, which you saw in **M3**. At relativistic speeds, the opening angle gets smaller, which is a manifestation of the "beaming" effect you saw in **R1**. This is a familiar effect, commonly observed in particle physics experiments.

[3] **Problem 6** (Morin 12.6). A ball of mass M and energy E collides head-on elastically with a stationary ball of mass m. Show that the final energy of mass M is

$$E' = \frac{2mM^2 + E(m^2 + M^2)}{2Em + m^2 + M^2}.$$

This problem is a little messy, but you can save yourself some trouble by noting that $E' = E$ must be a root of the equation you get for E'.

[3] **Problem 7** (Morin 12.7). In Compton scattering, a photon collides with a stationary electron.



(a) If the photon scatters at an angle θ, show that the resulting wavelength λ' is given in terms of the original wavelength λ by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

where m is the mass of the electron.

(b) While Compton scattering can occur for photons of any frequency, it is usually used in reference to X-rays, which have very high frequencies. Why?

[3] **Problem 8.** USAPhO 2017, problem A4. However, to make it a little harder, solve part (a) without assuming E_b is small.

2 Optimal Collisions

These collision problems are conceptually simple, but somewhat more mathematically challenging.

Idea 3

The minimum energy configuration of a system of particles with fixed total momentum is the one where they all move with the same velocity. This is easiest to show by boosting to the center of mass frame (i.e. the frame with zero total momentum) and then boosting back.

Example 4: KK 14.3

A high energy photon (γ ray) collides with a proton at rest. A neutral pi meson is produced according to the reaction

$$\gamma + p \to p + \pi^0.$$

What is the minimum energy the γ ray must have for this reaction to occur? The rest mass of a proton is 938 MeV and the rest mass of a neutral pion is 135 MeV.

Solution

The total four-momentum is $(E + m_p, E)$ where E is the energy of the γ ray in the lab frame. This four-momentum has norm $2Em_p + m_p^2$. Crucially, the norms of four-momenta don't change upon changing frames, so the total four-momentum in the center of mass frame is

$$\left(\sqrt{2Em_p + m_p^2}\,, 0\right)$$

because the total spatial momentum vanishes by definition. On the other hand, we also know that the reaction can just barely happen when both the proton and pion are produced at rest in the center of mass frame, with a final four-momentum of $(m_p + m_\pi, 0)$. Hence we have

$$\sqrt{2Em_p + m_p^2} = m_p + m_\pi$$

and plugging in the numbers gives $E = 145\,\text{MeV}$. As expected, this is a little bit more than the mass-energy of the pion, because the final system inevitably has some kinetic energy too.

Example 5

Two photons of angular frequencies ω_1 and ω_2 collide head-on. Under what conditions can an electron-positron pair be created?

Solution

The naive answer is to say the energy present must exceed the rest energy,

$$\hbar\omega_1 + \hbar\omega_2 \geq 2m_e.$$

However, this is incorrect because the electron and positron will inevitably have kinetic energy, since the photons initially have a net momentum. The lowest total kinetic energy is achieved when the electron and positron come out with the same velocity, which is the velocity of the center of mass frame of the photons.

The total four-momentum of the photons is

$$(\hbar(\omega_1 + \omega_2), \hbar(\omega_1 - \omega_2))$$

in the lab frame, and $(E_{\text{cm}}, 0)$ in the center of mass frame. Therefore,

$$E_{\text{cm}}^2 = \hbar^2((\omega_1 + \omega_2)^2 - (\omega_1 - \omega_2)^2) = 4\hbar^2\omega_1\omega_2.$$

In the center of mass frame, the electron and positron can be produced at rest, so the condition is $E_{\text{cm}} \geq 2m_e$, which means

$$\hbar\sqrt{\omega_1\omega_2} \geq m_e.$$

[3] **Problem 9.** In a particle collider, a proton of mass m is given *kinetic* energy E and collided with an initially stationary proton.

(a) What is the minimum E required to produce a proton-antiproton pair, $p + p \to p + p + p + \overline{p}$?

(b) How about N proton-antiproton pairs, where $N = 1$ in part (a)?

The scaling behavior of the answer you found in part (b) is the reason many particle colliders use two beams going in opposite directions, even though managing two beams precisely enough to collide them at the desired points is technically challenging.

[3] **Problem 10** (MPPP 196)**.** Two ultrarelativistic particles with negligible rest mass collide with oppositely directed momenta p_1 and p_2 elastically. Find the minimum possible angle between their velocities after the collision.

[3] **Problem 11.** IPhO 2003, problem 3A.

[4] **Problem 12.** APhO 2007, problem 3B. A comprehensive relativistic dynamics problem.

3 Relativistic Systems

Idea 4

The truly nonintuitive part of the result $E = mc^2$ is that changes in internal energy cause changes in mass. As a simple example, if you take a box of gas and heat it up, it'll have more mass than before, in every sense: the system will have more inertia, it'll have more momentum and kinetic energy when moving, it'll be heavier, and it'll exert more gravitational force on other objects. Some of the questions below illustrate how this can occur.

[3] **Problem 13.** The facts that $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$ are conserved are fundamentally new results of relativity, so the logically cleanest way to set up the theory is to simply make these postulates, without any further justification. But this certainly isn't the most *convincing* way, if you don't already believe that relativity is true.

The most striking new result is the huge rest energy $E = mc^2$. Throughout his life, Einstein came up with many derivations of this result, starting from more familiar postulates. In this problem, we'll cover Baierlein's simplified version of Einstein's 1946 derivation of $E = mc^2$. Specifically, we will prove that when the energy content of a body at rest decreases by ΔE, its mass decreases by $\Delta E/c^2$. The result then follows if one assumes that a zero-mass object has no rest energy.

Consider an object of mass M at rest, and suppose it emits photons with equal and opposite momenta p_γ upward and downward simultaneously. Let m be the final mass of the object.

(a) Now consider the same process in a frame moving with speed $v \ll c$ to the left. By using conservation of momentum in the x direction, show that

$$M = m + \frac{2p_\gamma}{c}.$$

Don't use the relativistic momentum formula here, because we're trying to imagine we don't already know relativity. Just use the fact that at $v \ll c$ the Galilean formula works.

(b) Using energy conservation, conclude the desired result.

(c) The derivation also works if one considers a frame moving upward with speed $v \ll c$. Carry out this analysis.

(d) The physicist Hans Ohanian has claimed that all of Einstein's derivations of $E = mc^2$, including this one, were inadequate. What do you think?

Example 6: USAPhO 2023 B2

A spaceship of mass m is propelled by light produced by lasers on Earth, with total power P. The light evenly impacts a sail on the spaceship, and reflects directly backwards. If the spaceship starts near Earth at rest, how long will it take, in the Earth's frame, to accelerate the spaceship to a speed v_f?

Solution

The spaceship is accelerated by the light, because light carries momentum. Consider a piece of the beam with total momentum dp_x in the Earth's frame, which impacts the spaceship when it has speed v. Lorentz transforming to the ship's frame, this momentum is $dp'_x = \gamma(1-v)\, dp_x$, and it is flipped in sign upon reflection to $-dp'_x$. Lorentz transforming that final momentum back to the Earth's frame gives a final momentum $-\gamma^2(1-v)^2\, dp_x$. Thus, the change in the spaceship's momentum is

$$dP_x = \left(1 + \gamma^2(1-v)^2\right) dp_x = \frac{2}{1+v}\, dp_x.$$

Considering the rate at which the beam impacts the spaceship gives $dp_x = P(1-v)\, dt$, so

$$\frac{dP_x}{dt} = \frac{1-v}{1+v}\,(2P).$$

On the other hand, using the definition of relativistic momentum gives

$$\frac{dP_x}{dt} = \frac{m\, dv/dt}{(1-v^2)^{3/2}}.$$

Combining these results and separating and integrating yields

$$\frac{2Pt}{m} = \int_0^{v_f} \frac{dv}{(1-v)^2\sqrt{1-v^2}}.$$

Note that we implicitly assumed m was a constant, which is valid because the mirror is perfectly reflective: the spaceship doesn't absorb any energy, so its rest mass doesn't change.

[3] **Problem 14.** Consider a completely black cube of density ρ and side length L sitting in free space. In some particular frame, plane electromagnetic waves of intensity I (in units of $\mathrm{W/m^2}$) approach the cube from the left and right, striking two faces of it head on. Neglect any radiation from the cube. If the cube has an initial velocity $v \ll c$ in this frame, find its displacement after a long time.

(Hint: solving the problem exactly will be very messy; it's better to approximate early, since we only want an answer correct in the limit $v/c \to 0$.)

[4] **Problem 15.** A rocket of initial mass M_0 starts from rest and propels itself forward along the x axis by emitting photons backward.

(a) Show that the final velocity of the rocket relative to the initial frame is

$$\frac{v}{c} = \frac{x^2 - 1}{x^2 + 1} = \tanh(\log x), \quad x = \frac{M_0}{M_f}$$

where M_f is the final rest mass of the rocket. (Hint: for this part, no integration is needed.)

(b) More generally, show that if the rocket fuel comes out at a speed u relative to the rocket,

$$\frac{v}{c} = \frac{x^{2u/c} - 1}{x^{2u/c} + 1} = \tanh((u/c)\log x)$$

where x is defined as above. (Hint: to avoid nasty differential equations, relate dm and dv.)

(c) Show that this reduces to the nonrelativistic rocket equation in the limit $u/c \to 0$.

(d) Show that in the limit $v/c \to 0$, the result of part (a) *also* reduces to the nonrelativistic rocket equation with exhaust speed c. Why does this work, given that photons are the most relativistic possible things?

[3] **Problem 16** (Cahn)**.** An empty box of total mass M and perfectly reflecting walls is at rest in the lab frame. Then N photons are introduced into the box, each with angular frequency ω_0 in a standing wave configuration; one can think of these photons as continually bouncing back and forth with velocity $\pm c\,\hat{\mathbf{x}}$, with zero total momentum.

(a) State what the rest mass M_{tot} of the system will be when the photons are present.

(b) Consider the momentum of the system in an inertial frame moving along the x axis with speed $v \ll c$. Using the first order Doppler shift and assuming that at any moment, half the photons are moving left and half the photons are moving right, show that $p = M_{\text{tot}}v$. This provides a dynamical explanation of exactly how photons contribute to the inertia of an object.

(c) Unfortunately, it is *not* true that half the photons are moving right at any given time. Show that the fraction of photons moving to the right is modified by an amount of order v/c, and find the total momentum accounting for this effect.

(d) **[A]** The analysis of part (b) is nice and neat, and you can sometimes find it in textbooks. But part (c) shows that this simple analysis is wrong! What's going on? (This requires considering the stress-energy tensor, which is beyond the scope of Olympiad physics.)

Remark

In Newtonian mechanics, we know that for an isolated system, $\mathbf{p}_{\text{tot}} = M_{\text{tot}}\mathbf{v}_{\text{CM}}$. In relativity, however, the idea of a "center of mass" no longer makes any sense. For example, suppose a particle with mass m decays into two photons. Each of the photons has no mass, so the center of mass is no longer defined! You can always define the mass of an overall system as

$\sqrt{E_{\text{tot}}^2 - p_{\text{tot}}^2}$, and this quantity remains equal to m, but it's no longer the sum of the masses of the individual parts. Since you can't break the mass of the system into parts, you can't sum over the parts to define a center of mass.

However, you can still define a "center of energy",

$$\mathbf{x}_{\text{CE}} = \frac{\sum_i \mathbf{x}_i E_i}{\sum_i E_i}$$

where E_i is the energy of particle i. It turns out that in relativity, we always have

$$\mathbf{p}_{\text{tot}} = \frac{E_{\text{tot}}}{c^2} \mathbf{v}_{\text{CE}}$$

which is called the "center of energy theorem". (Specifically, it comes from applying Noether's theorem to the symmetry of Lorentz boosts.) Of course, this reduces to $\mathbf{p}_{\text{tot}} = M_{\text{tot}} \mathbf{v}_{\text{CM}}$ in the nonrelativistic limit, since in that case almost all the energy is rest energy, $E = mc^2$.

4 Relativistic Dynamics

The previous questions could be solved by just using momentum and energy conservation. In this section we'll consider some deeper problems, which require considering the detailed dynamics.

Idea 5

In relativity, the force four-vector is defined as

$$f^\mu = \frac{dp^\mu}{d\tau} = ma^\mu.$$

There's a bit of a subtlety here. In relativity, the invariant mass of a system can change when it absorbs energy, even if it doesn't exchange any particles with its environment. For example, putting a system on the stove gives it energy but not momentum, thereby changing $m = \sqrt{E^2 - p^2}$. That's a perfectly valid four-force, but it feels strange to call it a "force". Therefore, we often restrict to four-forces that don't change the invariant mass, and since

$$\frac{dm^2}{d\tau} = \frac{d}{d\tau}(p \cdot p) = 2mu \cdot f$$

that corresponds to demanding $f \cdot u = 0$. These are sometimes called "pure" forces.

Idea 6

There's also a second way to define force in special relativity, with three-vectors. The first subtlety here is that you could define it as $d\mathbf{p}/dt$ or $m\mathbf{a}$, but the two differ in relativity. Since accelerations transform in a rather nasty way, as we saw in **R1**, the usual choice is to define

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

The second subtlety is that, whenever we define forces as three-forces, we usually implicitly assume that they fix the invariant mass m, i.e. we automatically rule out "put it on a stove" forces. Otherwise, there wouldn't be any way to tell how the energy changes over time.

[4] **Problem 17.** In this problem, we'll derive some properties of the three-force and four-force. For reference, see section 12.5 of Morin.

(a) Show that for a particle traveling along the $\hat{\mathbf{x}}$ direction,

$$\mathbf{F} = m(\gamma^3 a_x, \gamma a_y, \gamma a_z).$$

This is the relativistic three-vector analogue of $\mathbf{F} = m\mathbf{a}$, but it implies that force is no longer parallel to acceleration, which will be important in the problems below.

(b) Now let S' be the momentary rest frame of that particle. In this frame, since the particle is at rest, the nonrelativistic expression $\mathbf{F}' = m\mathbf{a}'$ holds. By using the transformation of acceleration derived in **R1**, show that

$$\mathbf{F} = (F'_x, F'_y/\gamma, F'_z/\gamma).$$

That is, transverse forces are redshifted in relativity, while longitudinal forces are unchanged.

(c) Show that the components of the four-force are

$$f^\mu = \left(\gamma \frac{dE}{dt}, \gamma \mathbf{F}\right).$$

Use the relativistic transformation of the four-force to rederive the result of part (b).

(d) The four-impulse is defined as

$$\Delta p^\mu = \int f^\mu \, d\tau.$$

But you can also consider the Lorentz scalar

$$\int f^\mu \, dx_\mu.$$

This ought to be something nice and simple that you already know about. What is it?

Remark

In popular science books and some older textbooks, relativistic dynamics is introduced using the idea of relativistic mass, $m_r = \gamma m$. This definition implies the simple results $E = m_r c^2$ and $\mathbf{p} = m_r \mathbf{v}$, so these books often say that relativistic dynamics is just like ordinary dynamics, except that moving objects have more mass. This picture is misleading because it breaks down once you go beyond one dimension: in problem 17, you showed that $\mathbf{F}$ is not even parallel to $\mathbf{a}$, so there's no definition of mass that recovers Newtonian mechanics. You instead need separate "transverse" and "longitudinal" relativistic masses,

$$\mathbf{F} = m_\perp \mathbf{a}_\perp + m_\parallel \mathbf{a}_\parallel, \quad m_\perp = \gamma m, \quad m_\parallel = \gamma^3 m.$$

I think this picture is honestly more confusing than helpful, though. It's better to avoid talking about mass and acceleration too much, and focus more on momentum and energy.

You'll also see arguments that relativistic mass is useful when thinking about gravity. In general relativity, all energy produces gravity equally. If you have a box with n particles bouncing around, which all have Lorentz factor γ and rest mass m, then the energy of the box is the same as that of n particles at rest, with mass m_r. So it looks like the gravity sourced by the particles is described by their relativistic mass. Unfortunately, this argument is *also* wrong, because in general relativity pressure also produces gravity. In the limit $\gamma \to \infty$, describing a gas of ultrarelativistic particles, the pressure contribution means we get twice as much gravitational attraction as would be predicted from the energy alone.

Example 7

A circular pendulum consists of a mass m attached to a string of length ℓ, with the other end fixed. Suppose the mass rotates in a small circle of radius $r \ll \ell$, with a nonrelativistic velocity in the lab frame. Find the angular frequency of the oscillations in the lab frame, and in a frame where the entire setup moves vertically with a relativistic speed v.

Solution

In the lab frame, this is a standard rotational mechanics problem. By the small angle approximation, the horizontal component of the three-force is $F_\perp = mgr/L$. This is equal to

$$F_\perp = ma_\perp = m\omega^2 r$$

from which we immediately conclude $\omega = \sqrt{g/L}$. We can use the results of problem 17 to find the answer in the other frame. The two effects are that the transverse force is redshifted, and the force's relation with acceleration is different,

$$F_\perp = \frac{mgr}{\gamma L}, \quad F_\perp = \gamma m a_\perp = \gamma m \omega^2 r.$$

Combining these results, we find

$$\omega = \frac{1}{\gamma}\sqrt{\frac{g}{L}}.$$

Of course, γ is just the usual time dilation factor. We knew this had to be the answer, because time dilation follows directly from the postulates of relativity, but now we can explicitly show this is the right answer in this specific example. (With similar reasoning, you can show that a mass-spring system oscillates slower, too.)

Remark

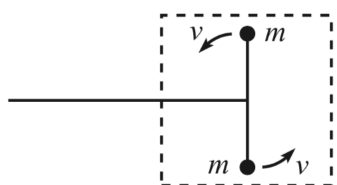
It's important not to misunderstand the meaning of the above example. Like many old physicists, Oleg Jefimenko decided one day that relativity had to be completely wrong. His argument was along the lines of the previous example: he showed that length contraction and time dilation could be derived dynamically in some simple cases, without the need to switch frames. Therefore, they can't be "real".

This argument doesn't make sense. It's like saying that energy can't be real because you can solve many mechanics problems with just $F = ma$, without needing to invoke energy conservation. (Though amazingly, some people actually do spend years arguing whether force or energy is "more real", in a debate that resembles rival high school cheerleading squads, when it's better to realize that they're both wonderful tools with complementary uses.)

Furthermore, it actually turns out to be *extremely* difficult to derive the core results of relativistic dynamics (such as the "transverse" and "longitudinal" masses, already measured by the turn of the 20^{th} century) without using relativistic assumptions. In the early 1900s, many physicists tried to explain the dynamics of the electron solely in terms of its electromagnetic fields. Since the field energy and field momentum of a moving point charge are infinite, it was necessary to take a model of the electron with finite size, but there were many possibilities, leading to many different expressions for the transverse mass, as well as persistent issues like the 4/3 problem mentioned in **E7**.

Relativity circumvents all of these issues. If you accept the postulates of relativity, you don't need to care whether the electron is shaped like a sphere, an ellipsoid, a torus, or a dumbbell: as long as its dynamics obey Lorentz symmetry, its four-momentum is a four-vector, and the usual results follow. And that's just as well, because with the advent of quantum mechanics, we learned that the electron is not like *any* of these classical models. But the relativistic result still holds, because our quantum theories obey the postulates of relativity too. This flexibility comes about because, like thermodynamics, relativity isn't so much a physical theory, as it is a framework within which many theories can be formulated.

[3] **Problem 18** (Morin 12.8)**.** Consider a dumbbell made of two equal masses, m. The dumbbell spins around, with its center pivoted at the end of a stick.



If the speed of the masses is v, then the energy of the system is $2\gamma m$. Treated as a whole, the system is at rest. Therefore, the mass of the system must be $2\gamma m$. (Imagine enclosing it in a box, so that you can't see what's going on inside.) Convince yourself that the system does indeed behave like a mass of $M = 2\gamma m$, by pushing on the stick (when the dumbbell is in the "transverse" position shown in the figure) and showing that $F = dp/dt = Ma$.

Idea 7

The Lorentz force is a three-force as defined in problem 17. That is, we have

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \frac{d\mathbf{p}}{dt}$$

and the force keeps the invariant mass fixed.

Example 8

A point charge q of mass m is initially at rest, and experiences a uniform electric field E. What time t does it take the object to move a distance x?

Solution

In **R1**, we found $x(t)$ for a uniformly accelerated rocket, which assumed a constant three-force in the momentarily comoving frame. By contrast, here we have a constant three-force $F = qE$ in the *lab* frame. However, we showed in problem 17 that forces along the direction of motion are the same in both frames, so these two problems are actually identical!

So we already know the answer to the problem, but it turns out that in the lab frame perspective, there's a slick alternative derivation that yields the result in one step. The trick is to consider the energy and momentum. Recall from problem 1 that the three-force F obeys $F = dp/dt$ and $F = dE/dx$. Therefore, when the object reaches its destination,

$$E = m + Fx, \quad p = Ft.$$

But we also know that $E^2 = p^2 + m^2$, so plugging the results in and solving for t gives

$$t = \sqrt{x^2 + \frac{2mx}{F}}$$

which is compatible with our earlier expression for $x(t)$. The reason this was so easy is that momentum and energy behave simply in relativity, while position and velocity don't.

Example 9

The LHC accelerates protons to an energy of $E = 7\,\text{TeV}$, and is a tunnel of radius $R = 4.3\,\text{km}$. If the protons are kept in a circular orbit in the tunnel by a magnetic field of magnitude B, find the required value of B. If the value of B is kept constant, what would be the radius of a future collider which accelerates protons to an energy of $20\,\text{TeV}$?

Solution

The centripetal force required is

$$F = \left|\frac{d\mathbf{p}}{dt}\right| = \omega p$$

where ω is the angular velocity. The speed of the protons is very close to c, so the angular velocity is $\omega \approx c/R$, and the momentum is $p \approx E/c$. The deflecting force is $qvB \approx qcB$, so

$$qcB \approx \omega p \approx \frac{E}{R}.$$

Therefore, we have

$$B = \frac{E}{qcR} = \frac{7 \times 10^{12}}{(3 \times 10^8)(4.3 \times 10^3)}\,\text{T} = 5.4\,\text{T}.$$

This is slightly lower than what is actually used, because magnets don't take up the entire tunnel. Since $R \propto E$, the future collider would need a radius of

$$R' = \frac{20\,\text{TeV}}{7\,\text{TeV}} R = 12\,\text{km}.$$

Remark

You might be wondering how to write the Lorentz force as a four-force. It certainly should be possible, since we know electromagnetism is compatible with relativity (indeed, it led us to relativity in the first place), but it seems challenging because electromagnetism is so naturally written in terms of three-vectors. It turns out that the proper way to express the electromagnetic field in relativity is to join the electric and magnetic fields together, making them the components of an antisymmetric rank 2 tensor,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

called the field strength tensor. Then the four-force is

$$f^\mu = q u_\nu F^{\mu\nu}$$

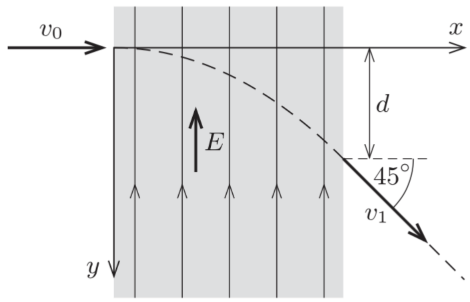
where u_ν is the four-velocity. Note that this ensures the rest mass of the particle is fixed, as

$$f \cdot u = q u_\mu u_\nu F^{\mu\nu} = -q u_\mu u_\nu F^{\nu\mu} = -f \cdot u$$

using the antisymmetric property, so $f \cdot u = 0$. (In fact, the requirement to keep the rest mass fixed is quite restrictive, so this is one of the simplest possible relativistic force laws.)

[2] **Problem 19.** USAPhO 2013, problem A3. A warmup question using the above facts.

[3] **Problem 20** (MPPP 192)**.** An electron moving with speed $v_0 = 0.6c$ enters a homogeneous electric field that is perpendicular to its velocity.



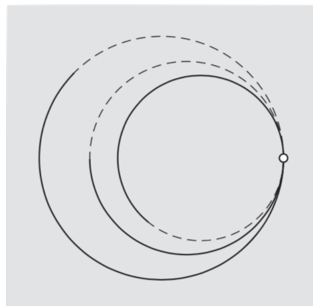
When the electron leaves the field, its velocity makes an angle 45° with its initial direction.

(a) Find the speed v_1 of the electron after it has crossed the electric field.

(b) Find the distance d shown above, if the strength of the electric field is $E = 510\,\text{kV/m}$.

Note that the rest energy of an electron is 510 keV.

[3] **Problem 21** (MPPP 194). The trajectories of charged particles, moving in a homogeneous magnetic field, can be seen by observing the tracks they leave in cloud chambers. Because the particles are moving quickly, it is impossible to see the tracks being formed; instead, one must infer what happened from the shapes of the tracks. Is it possible that, when a charged particle decays into two other charged particles, the trail segments close to the decay point (before the particles have started to slow down significantly) are arcs of circles that touch each other, as shown?



If so, identify which track belongs to the original particle. If not, explain why not.

[3] **Problem 22.** USAPhO 2006, problem A4.

[3] **Problem 23.** USAPhO 2022, problem B2. A nice problem on deriving the time dilation formula for an electrostatic "clock".

[3] **Problem 24.** Consider a particle at the origin at time $t = 0$, with initial x-momentum p_0 and total energy E_0. A constant three-force F acts on the particle in the $-y$ direction.

(a) Calculate $y(t)$. (Hint: don't write down any equations containing γ, because it depends on $v_x(t)$, which we don't know yet.)

(b) Calculate $x(t)$.

(c) Combine these results to get $y(x)$. This is the path of a relativistic projectile.

[5] **Problem 25.** IPhO 1994, problem 1. Print out the custom answer sheets before starting.

Remark

The setup of problem 25 is a nice model for mesons, particles composed of two quarks. And it's not just something made up for an Olympiad; it is a simple version of the MIT "bag model", which was one of the most important advances in the field in the 1970s. In fact, if you look at the original paper, which has thousands of citations, you'll find the answer to the IPhO question in figure 3!

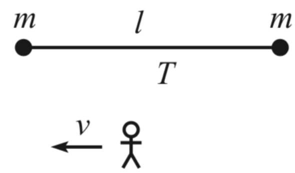
Idea 8

In string theory, strings carry a constant tension T, in the sense that the force $\mathbf{F} = d\mathbf{p}/dt$ exerted on one piece of string by its neighbors is T in the momentary rest frame of that piece. The strings may stretch or shrink freely, and have zero mass when they have zero length.

[3] **Problem 26** (Morin 12.16)**.** A simple exercise involving relativistic string.

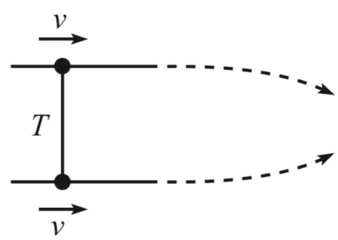
(a) Two masses m are connected by a string of length ℓ and constant tension T. The masses are released simultaneously, and they collide and stick together. What is the mass, M, of the resulting blob?

(b) Consider this scenario from the point of view of a frame moving to the left at speed v.



The energy of the resulting blob must be $\gamma M c^2$. Show that you obtain the same result by computing the work done on the two masses.

[3] **Problem 27** (Morin 12.37)**.** Two equal masses are connected by a relativistic string with tension T. The masses are constrained to move with speed v along parallel lines, as shown.



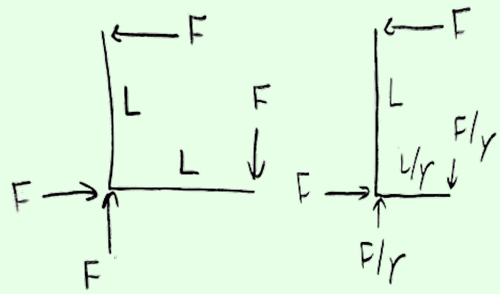
The constraints are then removed, and the masses are drawn together. They collide and make one blob which continues to move to the right. Is the following reasoning correct?

> The forces on the masses point in the y direction. Therefore, there is no change in the momentum of the masses in the x direction. But the mass of the resulting blob is greater than the sum of the initial masses (because they collide with some relative speed). Therefore, the speed of the resulting blob must be less than v (to keep p_x constant), so the whole apparatus slows down in the x direction.

If your answer is “no,” exactly what’s wrong about the reasoning above?

Example 10: Right Angle Lever Paradox

In 1909, Lewis and Tolman found one of the first relativistic paradoxes. Consider a rigid lever in static equilibrium, with both arms of length L, experiencing the forces shown at left.



In a frame where the lever moves to the right with speed v, one of the lever arms will be contracted to L/γ, as shown at right. In addition, by the results of problem 17, the vertical external forces will be redshifted to F/γ. This implies a net torque of

$$\tau = FL - \frac{F}{\gamma}\frac{L}{\gamma} = FLv^2.$$

The paradox is, given that $\boldsymbol{\tau} = d\mathbf{L}/dt$, why doesn't the lever rotate?

Solution

The resolution is that, in the frame shown at right, the angular momentum of the lever is constantly increasing. The horizontal forces are continually doing equal and opposite work on the lever, resulting in a upward flow of energy of rate Fv in the vertical arm. As explained in **E7** and earlier in this problem set, in relativity, energy flow is equal to momentum density, so the total upward momentum in the vertical arm is FLv. Therefore,

$$\frac{dL}{dt} = \frac{dx}{dt}(FLv) = FLv^2$$

exactly as expected.

Remark

The resolution of the right angle lever paradox is very controversial, with dozens of papers written on the subject, so we should discuss what it even means to "resolve" a paradox. As long as we believe relativity is self-consistent, we already know what's going to happen: the lever won't rotate. Everything the lever does is determined by $\mathbf{F} = d\mathbf{p}/dt$ alone, so if it looks like angular momentum considerations give a different answer, that just means we haven't formulated the latter correctly. The reason there are so many different resolutions out there is just that people choose different ways to define torque and angular momentum.

The solution above is the standard one, and its implicit definition of angular momentum can be motivated by Noether's theorem. That's a reasonable choice, since it's a specific output of a useful and general theorem, and we thereby know for sure that it's conserved for isolated systems. Unfortunately, explaining the definition takes some advanced math.

We define the angular momentum density tensor

$$M^{\mu\nu\rho}(x) = x^\mu T^{\nu\rho}(x) - x^\nu T^{\mu\rho}(x)$$

where the right-hand side contains the stress-energy tensor, from the solution to problem 16. The total angular momentum is an antisymmetric rank 2 tensor,

$$J^{\mu\nu}(t) = \int d\mathbf{x}\, M^{\mu\nu\rho}(x).$$

Noether's theorem states that it is this quantity that is conserved for an isolated system, due to symmetry under rotations and boosts. More specifically, the three spatial components J^{xy}, J^{yz}, and J^{zx} just make up ordinary angular momentum, e.g. for a single point particle they would assemble into the vector $\mathbf{r} \times \mathbf{p} = \mathbf{r} \times (\gamma m \mathbf{v})$. And the other components J^{0x}, J^{0y} and J^{0z} have to do with the center of mass motion.

If there is an external four-force per unit proper volume $f^\mu(x)$, which in terms of the stress-energy tensor implies $\partial_\mu T^{\mu\nu} = f^\nu$, the rate of change of angular momentum is

$$\frac{dJ^{\mu\nu}}{dt} = \tau^{\mu\nu}, \quad \tau^{\mu\nu} = \int d\mathbf{x}\, x^\mu f^\nu(x) - x^\nu f^\alpha(x)$$

which looks quite similar to the Newtonian expression. The component of this equation relevant to this paradox is $dJ^{xy}/dt = \tau^{xy}$, where

$$J^{xy} = \int d\mathbf{x}\, x T^{y0} - y T^{x0}, \quad \tau^{xy} = \sum_k x^{(k)} F_y^{(k)} - y^{(k)} F_x^{(k)}$$

where the index k sums over the four forces, and the T^{i0} stand for the density of momentum in the i direction. From this point on, the solution proceeds as above.

There is something a bit strange here, though. In the lever's rest frame, the angular momentum is zero, so if $J^{\mu\nu}$ were a tensor, it would have to be zero in all frames, but instead it rises to arbitrarily high values in the other frame. The reason is that when there are external torques, $J^{\mu\nu}$ isn't a tensor at all, just like how the four-momentum wasn't a four-vector in the solution to problem 16. That's one of the reasons there's a controversy: there just doesn't exist any definition that has all the nice properties one might want.

Relativity II: Dynamics

Chapter 12 of Morin covers relativistic dynamics, as does chapter 13 of Kleppner, or chapter 12 of Wang and Ricardo, volume 2. For four-vectors in relativistic dynamics, finish chapter 13 of Morin, or chapter 14 of Kleppner. For a deeper explanation of four-vectors, see chapter 2 of Schutz. There is a total of **83** points.

1 Energy and Momentum

Idea 1

The relativistic generalizations of energy and momentum are

$$E = \gamma mc^2, \quad \mathbf{p} = \gamma m\mathbf{v}.$$

These quantities are conserved, and m is defined as the rest mass. Note that m is *not* conserved in inelastic processes, while E *is* conserved; this is precisely the opposite of what happens nonrelativistically. The relativistic energy E automatically counts all contributions to the energy, including internal energy and rest energy mc^2.

[5] **Problem 1.** A few useful facts about energy and momentum, for future reference.

(a) Recalling the definition of the four-velocity from **R1**, show that

$$(E/c, \mathbf{p}) = mu^\mu$$

where u^μ is the four-velocity. Setting $c = 1$, this establishes $p^\mu = (E, \mathbf{p})$ is a four-vector.

(b) Suppose a particle is at rest in frame S'. Confirm explicitly that the components of the four-momentum p^μ transform as expected when going to frame S.

(c) Setting $c = 1$ for all future parts, show that the norm of the four-momentum is

$$p^\mu p_\mu = E^2 - |\mathbf{p}|^2 = m^2.$$

This is a very useful result that can simplify the solutions to many problems below, especially ones that simply ask for a final mass m. In this case one can often compute a single four-momentum and find its norm to get the answer.

(d) The expressions in idea 1 for E and $\mathbf{p}$ don't work for photons, since γ is infinite and m is zero. Instead, show that for a photon we have $p^\mu = \hbar k^\mu$.

(e) A system's center of mass frame is the one where its momentum is zero. For a system with total energy E and momentum $\mathbf{p}$, show that the center of mass has velocity $\mathbf{v} = \mathbf{p}/E$.

(f) In Newtonian mechanics, the kinetic energy K of an object with fixed mass m satisfies $dK = \mathbf{v} \cdot d\mathbf{p}$. Show that this also holds in relativity, assuming the rest mass m is fixed.

(g) As we'll discuss in more detail below, the force three-vector is defined as $\mathbf{F} = d\mathbf{p}/dt$ in relativistic mechanics. Show that $dK = \mathbf{F} \cdot d\mathbf{x}$, continuing to assume that m is fixed.

Solution. (a) We saw in **R1** that $u^\mu = (\gamma c, \gamma \mathbf{v})$. Multiplying by m gives

$$mu^\mu = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p})$$

as desired.

(b) Suppose we boost by velocity u. Then, the new speed is $\frac{u+v}{1+uv}$, so the new value of γ is

$$\gamma' = \left(1 - \frac{(u+v)^2}{(1+uv)^2}\right)^{-1/2} = (1+uv)\gamma_u\gamma.$$

Thus, the boosted values of E and p are

$$E' = \gamma' m = \gamma_u(E + up), \quad p' = \gamma' mv = \gamma_u(p + uE).$$

These are exactly the expected Lorentz transformation properties.

(c) The norm is

$$E^2 - p^2 = \gamma^2 m^2 - \gamma^2 m^2 v^2 = \gamma^2 m^2 (1-v)^2 = m^2$$

as desired.

(d) This follows directly from the de Broglie relations $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$.

(e) In this frame, $p' = 0$. Then using the result of part (b), we have $p - vE = 0$ where v is the velocity of the center of mass in the original frame. Therefore, $\mathbf{v} = \mathbf{p}/E$.

(f) Starting with $E^2 = p^2 + m^2$ and taking the differential of both sides,

$$2E\,dE = 2\mathbf{p} \cdot d\mathbf{p}.$$

Solving for dE, we have

$$dE = \frac{\mathbf{p}}{E} \cdot d\mathbf{p} = \mathbf{v} \cdot d\mathbf{p}$$

where we used the result of problem 1. Since K and E are the same up to a constant anyway, we conclude $dK = \mathbf{v} \cdot d\mathbf{p}$ as desired.

(g) We have $\mathbf{F} \cdot d\mathbf{x} = (\mathbf{F}\,dt) \cdot (d\mathbf{x}/dt) = \mathbf{v} \cdot d\mathbf{p} = dK$ using the result of part (f), as desired.

Idea 2

In relativistic dynamics problems, it is almost always better to work with energy and momentum than velocity; one typically shouldn't even mention velocities unless the problem asks for or gives them.

We'll start with some very simple problems to warm up.

Example 1: KK 13.5

A particle of mass m and speed v collides and sticks to a stationary particle of mass M. Find the final speed of the composite particle.

Solution

The total four momentum is $(\gamma m + M, \gamma m v)$, so the speed is

$$v = \frac{p}{E} = \frac{\gamma m v}{\gamma m + M}.$$

Example 2: Morin 12.2

Two photons of energy E collide at an angle θ and create a particle of mass M. What is M?

Solution

The total four-momentum is

$$p^\mu = (2E, E(1+\cos\theta), E\sin\theta).$$

The mass is just the norm of the four-momentum, so

$$M = \sqrt{4E^2 - E^2(1+\cos\theta)^2 - E^2\sin^2\theta} = E\sqrt{2 - 2\cos\theta} = 2E\sin(\theta/2)/c^2$$

where we restored the factors of c at the end.

[1] **Problem 2** (Morin 12.4)**.** A stationary mass M_A decays into masses M_B and M_C. What are the energies of these two masses?

Solution. In the lab frame, the momenta of the masses B and C adds to zero, so $p_B^2 = p_C^2$, so

$$E_B^2 - M_B^2 = E_C^2 - M_C^2.$$

We also know that $E_B + E_C = M_A$, so simplifying gives

$$E_B - E_C = \frac{M_B^2 - M_C^2}{M_A}.$$

Therefore, we conclude

$$E_B = \frac{M_A^2 + M_B^2 - M_C^2}{2M_A}, \quad E_C = \frac{M_A^2 - M_B^2 + M_C^2}{2M_A}.$$

[2] **Problem 3.** USAPhO 2012, problem A1.

[2] **Problem 4.** A particle with mass M and energy E moves towards a detector when it suddenly decays and emits a photon in its direction of motion. The detector measures a photon angular frequency of ω. What was the photon's angular frequency in the rest frame of the decaying particle?

Solution. It's not hard to solve this using four-momentum conservation, but a nice alternative is to use the Doppler shift formula from **R1**. Letting p be the particle's momentum in the lab frame,

$$\omega = \omega' \sqrt{\frac{1+v}{1-v}} = \omega' \sqrt{\frac{E+p}{E-p}} = \omega' \frac{E+p}{\sqrt{E^2 - p^2}} = \omega' \frac{E+p}{M}.$$

Thus, the answer is

$$\omega' = \frac{M\omega}{E + \sqrt{E^2 - M^2}}.$$

[3] **Problem 5.** USAPhO 2002, problem A2.

Now let's try some more involved problems.

Example 3: Woodhouse 7.5

A particle of rest mass m moves with velocity $\mathbf{u}$ and collides elastically with a second particle, also of rest mass m, which is initially at rest. After the collision, the particles have velocities $\mathbf{v}$ and $\mathbf{w}$. Show that if θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\cos\theta = \frac{(1-\sqrt{1-v^2})(1-\sqrt{1-w^2})}{vw}.$$

Solution

First, a remark: in Newtonian mechanics, you learn that in an inelastic collision, the kinetic energy is dissipated into microscopic thermal motion. This often leads students to ask: if we keep track of the motion of all particles in detail, then are all collisions actually perfectly elastic? According to particle physics, the answer is no. You really can lose kinetic energy by converting it to mass-energy, in collisions which change the identity of the particles or produce new particles. Therefore, at particle colliders, we say a collision is elastic if the particles that come out are precisely the same as the ones that came in. For this example, that means the final particles still have rest mass m.

Conservation of energy and momentum imply

$$1 + \gamma_u = \gamma_v + \gamma_w, \quad \gamma_u \mathbf{u} = \gamma_v \mathbf{v} + \gamma_w \mathbf{w}.$$

To get an expression with $\cos\theta$, we take the norm squared of the momentum equation,

$$\gamma_u^2 u^2 = \gamma_v^2 v^2 + \gamma_w^2 w^2 + 2\gamma_v\gamma_w vw\cos\theta.$$

This can be substantially simplified by noting that $\gamma_u^2 u^2 = \gamma_u^2 - 1$, giving

$$2vw\gamma_v\gamma_w\cos\theta = \gamma_u^2 - \gamma_v^2 - \gamma_w^2 + 1.$$

The appearance of so many squares motivates us to square both sides of the energy equation,

$$1 + 2\gamma_u + \gamma_u^2 = \gamma_v^2 + \gamma_w^2 + 2\gamma_v\gamma_w.$$

Using this to simplify the right-hand side of the previous equation,

$$2vw\gamma_v\gamma_w\cos\theta = 2\gamma_v\gamma_w - 2\gamma_u = 2(\gamma_v\gamma_w - \gamma_v - \gamma_w + 1) = 2(\gamma_v - 1)(\gamma_w - 1)$$

where in the second step we used conservation of energy. After solving for $\cos\theta$, we get the desired result. This was a bit of a slog, but it's representative of the hardest calculations you'll ever have to do for special relativity problems.

As a check on that result, note that in the nonrelativistic limit we get $\cos\theta = 0$, indicating a 90° angle, which you saw in **M3**. At relativistic speeds, the opening angle gets smaller, which is a manifestation of the "beaming" effect you saw in **R1**. This is a familiar effect, commonly observed in particle physics experiments.

[3] **Problem 6** (Morin 12.6)**.** A ball of mass M and energy E collides head-on elastically with a stationary ball of mass m. Show that the final energy of mass M is

$$E' = \frac{2mM^2 + E(m^2 + M^2)}{2Em + m^2 + M^2}.$$

This problem is a little messy, but you can save yourself some trouble by noting that $E' = E$ must be a root of the equation you get for E'.

Solution. Let the answer be x. The final momentum is $(E + m, p)$, split between $P_M = (x, p_M)$ and P_m. Now, $P_m = (E + m, p) - (x, p_M)$, so taking the norm squared, we see that

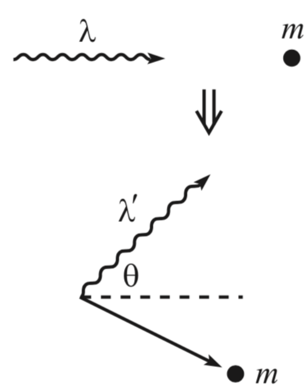
$$\begin{aligned}
&m^2 = (E + m - x)^2 - (\sqrt{E^2 - M^2} - \sqrt{x^2 - M^2})^2 \\
&\implies m^2 = (E^2 + m^2 + x^2 + 2Em - 2Ex - 2mx) - E^2 + M^2 - x^2 + M^2 + 2\sqrt{(E^2 - M^2)(x^2 - M^2)} \\
&\implies 0 = 2Em - 2Ex - 2mx + 2M^2 + 2\sqrt{(E^2 - M^2)(x^2 - M^2)} \\
&\implies (E^2 - M^2)(x^2 - M^2) = (mx + Ex - Em - M^2)^2.
\end{aligned}$$

This is manifestly a quadratic in x, and we know that one root is $x = E$, so applying Vieta's formulas and some tedious algebra reveals that

$$x = \frac{2mM^2 + E(m^2 + M^2)}{2Em + m^2 + M^2}$$

as desired.

[3] **Problem 7** (Morin 12.7)**.** In Compton scattering, a photon collides with a stationary electron.



(a) If the photon scatters at an angle θ, show that the resulting wavelength λ' is given in terms of the original wavelength λ by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

where m is the mass of the electron.

(b) While Compton scattering can occur for photons of any frequency, it is usually used in reference to X-rays, which have very high frequencies. Why?

Solution. (a) The original momentum of the system is $(E+m, E, 0)$ where E is the original energy of the photon. Let x be the new energy of the photon. Then $P_\gamma = (x, x\cos\theta, x\sin\theta)$, and $P_m = (E+m, E, 0) - x(1, \cos\theta, \sin\theta)$. Taking the norm squared, we see that

$$\begin{aligned} m^2 &= (E+m-x)^2 - (E - x\cos\theta)^2 - x^2\sin^2\theta \\ \implies 0 &= 2Em - 2Ex - 2mx + 2Ex\cos\theta \\ \implies x &= \frac{Em}{m + E(1-\cos\theta)} = c^2\left(c^2/E + \frac{1}{m}(1-\cos\theta)\right)^{-1}. \end{aligned}$$

Now, $\lambda = hc/x = \lambda + \lambda_C(1-\cos\theta)$ where $\lambda_C = h/mc$.

(b) The wavelength shift is independent of frequency, and since $c = f\lambda$ the frequency shift (which is what we measure directly) is larger if the frequency begins large. The energy loss for visible photons is hardly noticeable, while it is very large for X-rays.

Indeed, for such photons we usually talk about Thomson scattering (as in **E7**) which does not change the frequency of the photon at all. At the level of relativistic dynamics, Thomson scattering is nothing more than the low-frequency limit of Compton scattering. Incidentally, at even higher frequencies, the result has more subtle corrections due to quantum field theory effects, and the cross section is given by the Klein–Nishina formula.

[3] **Problem 8.** USAPhO 2017, problem A4. However, to make it a little harder, solve part (a) without assuming E_b is small.

2 Optimal Collisions

These collision problems are conceptually simple, but somewhat more mathematically challenging.

Idea 3

The minimum energy configuration of a system of particles with fixed total momentum is the one where they all move with the same velocity. This is easiest to show by boosting to the center of mass frame (i.e. the frame with zero total momentum) and then boosting back.

Example 4: KK 14.3

A high energy photon (γ ray) collides with a proton at rest. A neutral pi meson is produced according to the reaction

$$\gamma + p \to p + \pi^0.$$

What is the minimum energy the γ ray must have for this reaction to occur? The rest mass of a proton is 938 MeV and the rest mass of a neutral pion is 135 MeV.

Solution

The total four-momentum is $(E + m_p, E)$ where E is the energy of the γ ray in the lab frame. This four-momentum has norm $2Em_p + m_p^2$. Crucially, the norms of four-momenta don't change upon changing frames, so the total four-momentum in the center of mass frame is

$$\left(\sqrt{2Em_p + m_p^2}\,, 0\right)$$

because the total spatial momentum vanishes by definition. On the other hand, we also know that the reaction can just barely happen when both the proton and pion are produced at rest in the center of mass frame, with a final four-momentum of $(m_p + m_\pi, 0)$. Hence we have

$$\sqrt{2Em_p + m_p^2} = m_p + m_\pi$$

and plugging in the numbers gives $E = 145\,\text{MeV}$. As expected, this is a little bit more than the mass-energy of the pion, because the final system inevitably has some kinetic energy too.

Example 5

Two photons of angular frequencies ω_1 and ω_2 collide head-on. Under what conditions can an electron-positron pair be created?

Solution

The naive answer is to say the energy present must exceed the rest energy,

$$\hbar\omega_1 + \hbar\omega_2 \geq 2m_e.$$

However, this is incorrect because the electron and positron will inevitably have kinetic energy, since the photons initially have a net momentum. The lowest total kinetic energy is achieved when the electron and positron come out with the same velocity, which is the velocity of the center of mass frame of the photons.

The total four-momentum of the photons is

$$(\hbar(\omega_1 + \omega_2), \hbar(\omega_1 - \omega_2))$$

in the lab frame, and $(E_{\text{cm}}, 0)$ in the center of mass frame. Therefore,

$$E_{\text{cm}}^2 = \hbar^2((\omega_1 + \omega_2)^2 - (\omega_1 - \omega_2)^2) = 4\hbar^2\omega_1\omega_2.$$

In the center of mass frame, the electron and positron can be produced at rest, so the condition is $E_{\text{cm}} \geq 2m_e$, which means

$$\hbar\sqrt{\omega_1\omega_2} \geq m_e.$$

[3] **Problem 9.** In a particle collider, a proton of mass m is given *kinetic* energy E and collided with an initially stationary proton.

(a) What is the minimum E required to produce a proton-antiproton pair, $p + p \to p + p + p + \overline{p}$?

(b) How about N proton-antiproton pairs, where $N = 1$ in part (a)?

The scaling behavior of the answer you found in part (b) is the reason many particle colliders use two beams going in opposite directions, even though managing two beams precisely enough to collide them at the desired points is technically challenging.

Solution. (a) Let p be the momentum of the proton. The total four momentum is then

$$p^\mu = (E + 2m, p).$$

We end up with four particles of mass m. From the idea above, the threshold energy is minimized when all of these particles have the same velocity, so they each have $p_i = p/4$. Then the final four-momentum is

$$p^\mu = 4(\sqrt{m^2 + p^2/16}, p/4).$$

Setting the energies equal, we have

$$\sqrt{16m^2 + p^2} = E + 2m$$

and using $E^2 = p^2 + m^2$ and simplifying gives

$$2Em = 12m^2, \quad E = 6m.$$

With this calculation in mind, the Bevatron at Berkeley was designed to accelerate protons to a kinetic energy of about $6.2m$. It discovered the antiproton in 1955, winning the 1959 Nobel prize.

(b) Now we have $2N + 2$ particles of mass m at the end, which have $p_i = p/(2N + 2)$. Now we instead have

$$p^\mu = (2N + 2)(\sqrt{m^2 + (p/(2N + 2))^2},\, p/(2N + 2))$$

and setting the energies equal again gives

$$\sqrt{(2N + 2)^2 m^2 + E^2 + 2Em} = E + 2m$$

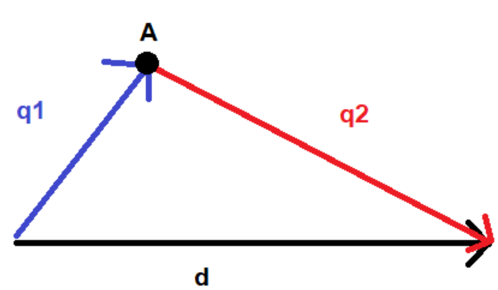
and solving gives

$$E = (2N^2 + 4N)m.$$

In other words, the energy required scales up quadratically in the mass-energy of the stuff you want to create!

[3] **Problem 10** (MPPP 196). Two ultrarelativistic particles with negligible rest mass collide with oppositely directed momenta p_1 and p_2 elastically. Find the minimum possible angle between their velocities after the collision.

Solution. Let $\mathbf{q}_1, \mathbf{q}_2$ be the two new momenta of the new (still ultra-relativistic) particles. We see that $\mathbf{q}_1 + \mathbf{q}_2 = (p_1 - p_2)\hat{\mathbf{x}} \equiv \mathbf{d}$ and $q_1 + q_2 = p_1 + p_2 \equiv 2a$ (energy).



We see that the point A lies on an ellipse with foci at the endpoints of $\mathbf{d}$. The obtuse angle in the picture is actually the supplement of what we want, so we want to maximize the obtuse angle. This happens when A is on the perpendicular bisector of $\mathbf{d}$.

Let 2θ be the obtuse angle. We see that $\sin\theta = (d/2)/a = (p_1 - p_2)/(p_1 + p_2)$, giving a maximum possible angle of

$$\theta_{\text{max}} = \pi - 2\sin^{-1}\left(\frac{p_1 - p_2}{p_1 + p_2}\right) = 2\cos^{-1}\left(\frac{p_1 - p_2}{p_1 + p_2}\right).$$

[3] **Problem 11.** IPhO 2003, problem 3A.

[4] **Problem 12.** APhO 2007, problem 3B. A comprehensive relativistic dynamics problem.

3 Relativistic Systems

Idea 4

The truly nonintuitive part of the result $E = mc^2$ is that changes in internal energy cause changes in mass. As a simple example, if you take a box of gas and heat it up, it'll have more mass than before, in every sense: the system will have more inertia, it'll have more momentum and kinetic energy when moving, it'll be heavier, and it'll exert more gravitational force on other objects. Some of the questions below illustrate how this can occur.

[3] **Problem 13.** The facts that $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$ are conserved are fundamentally new results of relativity, so the logically cleanest way to set up the theory is to simply make these postulates, without any further justification. But this certainly isn't the most *convincing* way, if you don't already believe that relativity is true.

The most striking new result is the huge rest energy $E = mc^2$. Throughout his life, Einstein came up with many derivations of this result, starting from more familiar postulates. In this problem, we'll cover Baierlein's simplified version of Einstein's 1946 derivation of $E = mc^2$. Specifically, we will prove that when the energy content of a body at rest decreases by ΔE, its mass decreases by $\Delta E/c^2$. The result then follows if one assumes that a zero-mass object has no rest energy.

Consider an object of mass M at rest, and suppose it emits photons with equal and opposite momenta p_γ upward and downward simultaneously. Let m be the final mass of the object.

(a) Now consider the same process in a frame moving with speed $v \ll c$ to the left. By using conservation of momentum in the x direction, show that

$$M = m + \frac{2p_\gamma}{c}.$$

Don't use the relativistic momentum formula here, because we're trying to imagine we don't already know relativity. Just use the fact that at $v \ll c$ the Galilean formula works.

(b) Using energy conservation, conclude the desired result.

(c) The derivation also works if one considers a frame moving upward with speed $v \ll c$. Carry out this analysis.

(d) The physicist Hans Ohanian has claimed that all of Einstein's derivations of $E = mc^2$, including this one, were inadequate. What do you think?

Solution. (a) The initial momentum is Mv. After emitting the photons, the body still has the same speed, so its final momentum is mv. Using Galilean velocity addition, the photons are emitted at a slight angle in this frame, contributing momentum $2p_\gamma/c$.

(b) Since the speeds are low, the $mv^2/2$ and $Mv^2/2$ contributions to the energy are second order and hence negligible. Energy $2p_\gamma c$ goes into photons, so an equal amount must have come out of rest energy. But the change in mass is $2p_\gamma/c$, so $\Delta E = \Delta M\, c^2$.

Finally, assuming that the rest energy of a particle goes to zero as its mass does to zero, which seems reasonable, gives $E = Mc^2$.

(c) Initially, the mass M has momentum downwards of Mv, and after the photons are emitted, the mass m has momentum mv which is made up for by the photons of different momenta due to Doppler shifting. Since energy and momenta are proportional to frequency, which is proportional to $1 \pm v/c$, the difference in the momenta of the photons is $p_\gamma(2v/c)$ so we get $M = m + 2p_\gamma/c$. For energy, we have $\frac{1}{2}Mv^2 + \Delta E = \frac{1}{2}mv^2 + p_\gamma c(1 + v/c + 1 - v/c)$, and with second order v terms we have $\Delta E = 2p_\gamma c = \Delta M c^2$. The rest will be the same as above.

(d) This is a very subjective question, so opinions will vary. Here's my personal opinion.

Special relativity contains nonrelativistic mechanics as a special case. Therefore, there is no need to motivate any of the results of special relativity using arguments from nonrelativistic physics – relativity stands on its own. Instead one can derive the results of nonrelativistic physics by taking limits of the results of special relativity. (It's just like quantum mechanics: you don't derive Schrodinger's equation from $F = ma$, you derive $F = ma$ as a limiting behavior of Schrodinger's equation.) Because of this, there is absolutely nothing illogical about simply *defining* $E = \gamma mc^2$. We then believe it because it reduces to results we already know about ($E = mv^2/2$ in the nonrelativistic limit) and also produces new verified predictions (nuclear power works).

(It's also worth noting that in nonrelativistic physics, the definition of energy simply follows from it being the conserved quantity associated with time translations. If we continue to define energy that way in special relativity, we automatically get $E = \gamma mc^2$. So it's not like $E = \gamma mc^2$ is some ad hoc, independent assumption on top of what we assumed in **R1**.)

Given the above, what is the point of trying to derive the rest energy expression at all? It's just to make people more comfortable with the new ideas of relativity. In physics you can often derive the same result in multiple ways. The rest energy follows automatically from the full framework of relativity, but it also follows by using *part* of the framework of relativity and part of the framework of nonrelativistic physics. This is useful if you're trying to explain why rest energy makes sense, to people who don't already believe in it: you get to the result using fewer unfamiliar assumptions, and possibly only ones that have already been tested experimentally. That's why arguments like these were important historically, when scientists were first grappling with relativity, and pedagogically, when students first encounter relativity.

A derivation using this kind of "hybrid" framework is necessarily weaker. For example, we had to make the somewhat random assumption above that a zero-mass object has no rest energy. You could argue that the only way to deduce *that* is to start with $E = mc^2$, making the argument "circular". But that doesn't really matter. The point of such a derivation is just to provide motivation, by explaining something new and unfamiliar in terms of things that are more believable. If you find the result that a zero-mass object has no rest energy believable, then the derivation works for you.

Example 6: USAPhO 2023 B2

A spaceship of mass m is propelled by light produced by lasers on Earth, with total power P. The light evenly impacts a sail on the spaceship, and reflects directly backwards. If the spaceship starts near Earth at rest, how long will it take, in the Earth's frame, to accelerate the spaceship to a speed v_f?

Solution

The spaceship is accelerated by the light, because light carries momentum. Consider a piece of the beam with total momentum dp_x in the Earth's frame, which impacts the spaceship when it has speed v. Lorentz transforming to the ship's frame, this momentum is $dp'_x = \gamma(1-v)\, dp_x$, and it is flipped in sign upon reflection to $-dp'_x$. Lorentz transforming that final momentum back to the Earth's frame gives a final momentum $-\gamma^2(1-v)^2\, dp_x$. Thus, the change in the spaceship's momentum is

$$dP_x = \left(1 + \gamma^2(1-v)^2\right) dp_x = \frac{2}{1+v}\, dp_x.$$

Considering the rate at which the beam impacts the spaceship gives $dp_x = P(1-v)\, dt$, so

$$\frac{dP_x}{dt} = \frac{1-v}{1+v}\,(2P).$$

On the other hand, using the definition of relativistic momentum gives

$$\frac{dP_x}{dt} = \frac{m\, dv/dt}{(1-v^2)^{3/2}}.$$

Combining these results and separating and integrating yields

$$\frac{2Pt}{m} = \int_0^{v_f} \frac{dv}{(1-v)^2\sqrt{1-v^2}}.$$

Note that we implicitly assumed m was a constant, which is valid because the mirror is perfectly reflective: the spaceship doesn't absorb any energy, so its rest mass doesn't change.

[3] **Problem 14.** Consider a completely black cube of density ρ and side length L sitting in free space. In some particular frame, plane electromagnetic waves of intensity I (in units of W/m^2) approach the cube from the left and right, striking two faces of it head on. Neglect any radiation from the cube. If the cube has an initial velocity $v \ll c$ in this frame, find its displacement after a long time. (Hint: solving the problem exactly will be very messy; it's better to approximate early, since we only want an answer correct in the limit $v/c \to 0$.)

Solution. We'll set $c = 1$ for convenience, and expand everything to lowest order in v. The cube sees the frequency of the light beams redshifted and blueshifted, by a factor of $1 - v$ and $1 + v$

respectively. This affects the intensities by the same factor, so the net force on the box is

$$\frac{dp}{dt} = -2Pv, \quad P = IL^2, \quad m = \rho L^3.$$

In the nonrelativistic limit $p = mv$, so it's tempting to conclude that

$$\frac{dp}{dt} = m\frac{dv}{dt} = -2Pv.$$

However, this is incorrect, even in the limit of small v! The reason is that as v gets smaller, the drag force gets smaller, but the cube still is absorbing energy at the same rate, and this energy can be sizable. So even in the nonrelativistic limit, we have to account for the change of the rest mass of the cube. Specifically, we have

$$\frac{dm}{dt} \approx \frac{dE}{dt} = 2P$$

where we used $E \approx m$, valid in the nonrelativistic limit. Then

$$\frac{dp}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = m\frac{dv}{dt} + 2Pv.$$

We thus arrive at the equation

$$m\frac{dv}{dt} = -4Pv.$$

Multiplying both sides by dt and integrating both sides, we find

$$\Delta x = \frac{mv_0}{4P} = \frac{\rho L v_0}{4I}.$$

Thus, even the naively negligible term changes the answer by a factor of 2.

[4] **Problem 15.** A rocket of initial mass M_0 starts from rest and propels itself forward along the x axis by emitting photons backward.

(a) Show that the final velocity of the rocket relative to the initial frame is

$$\frac{v}{c} = \frac{x^2 - 1}{x^2 + 1} = \tanh(\log x), \quad x = \frac{M_0}{M_f}$$

where M_f is the final rest mass of the rocket. (Hint: for this part, no integration is needed.)

(b) More generally, show that if the rocket fuel comes out at a speed u relative to the rocket,

$$\frac{v}{c} = \frac{x^{2u/c} - 1}{x^{2u/c} + 1} = \tanh((u/c)\log x)$$

where x is defined as above. (Hint: to avoid nasty differential equations, relate dm and dv.)

(c) Show that this reduces to the nonrelativistic rocket equation in the limit $u/c \to 0$.

(d) Show that in the limit $v/c \to 0$, the result of part (a) *also* reduces to the nonrelativistic rocket equation with exhaust speed c. Why does this work, given that photons are the most relativistic possible things?

Solution. (a) We see that the four momentum goes from $(M_0, 0)$ to $(\gamma M_f, \gamma M_f v)$. Since the difference is given by photons, we must have

$$-\gamma M_f v = \gamma M_f - M_0 \implies \gamma M_f (1+v) = M_0 \implies \frac{1+v}{1-v} = x^2.$$

Solving for v and restoring c, we have

$$\frac{v}{c} = \frac{x^2-1}{x^2+1}$$

as desired.

(b) This part does require integration. The reason that part (a) didn't require integration is that all the emitted photons have the same speed in the original frame, because light always travels at c. But in this case, the emitted fuel will have varying speed in the original frame, depending on when it was emitted, so we have to actually do the calculation.

Since our variable x is in terms of mass, it's useful to relate the decrease in mass dm of the rocket with its increase in speed dv. Let's consider the very first instant the rocket is on. The decrease in the rocket's energy is dm (the kinetic energy is picks up is proportional to dv^2, which is negligible). All of this energy must be in the fuel, which is traveling with speed u, which means the mass of the fuel obeys

$$dm = \gamma_u \, dm_f.$$

The momentum carried by this bit of fuel is

$$dp = \gamma_u u \, dm_f = u \, dm.$$

This is equal to the momentum change of the rocket, $dp = m\, dv$. So combining everything,

$$-\frac{dm}{m} = \frac{dv}{u}.$$

In fact, this is exactly the same as the first half of the derivation of the ordinary rocket equation.

Now, in general this equation works as long as we're working in the momentarily comoving frame of the rocket. The place the relativity comes in is that the dv in this frame is not the same as the dv in the original frame. If the rocket has speed v in the original frame, then after accelerating by dv in its momentarily comoving frame, it ends up with speed

$$v' = \frac{v + dv}{1 + v\, dv} \approx v + dv - v^2\, dv = v + (1-v^2)\, dv$$

in the original frame. Therefore, we actually have in general

$$-\frac{dm}{m} = \frac{1}{u}\frac{dv}{1-v^2}$$

and integrating both sides gives

$$\log x = \frac{1}{u}\int_0^v \frac{dv}{1-v^2} = \frac{1}{2u}\int_0^v \frac{dv}{1-v} + \frac{dv}{1+v} = \frac{1}{2u}\log\frac{1+v}{1-v}.$$

Solving for v gives the result.

(c) We can use the approximation

$$x^{2u/c} = e^{(2u/c)\log x} \approx 1 + \frac{2u}{c}\log x$$

to arrive at

$$\frac{v}{c} \approx \frac{(2u/c)\log x}{2} \approx \frac{u}{c}\log x.$$

In other words, $v = u\log x$ which is precisely the nonrelativistic rocket equation. (Here we have implicitly assumed that $(u/c)\log x$ is small, which is equivalent to assuming that the rocket doesn't get to relativistic speeds. If u/c is nonrelativistic, this should be true for any reasonable value of x.)

(d) At first glance, this shouldn't make any sense. When $u/c \to 1$, the rocket fuel is always moving extremely relativistically, so how can we take the nonrelativistic limit? But pressing on, let's consider the limit $v/c \to 0$. This corresponds to $x \to 0$, so

$$v \approx \frac{(x-1)(x+1)}{2}\,c \approx (x-1)\,c = \frac{M_0 - M_f}{M_f}\,c.$$

On the other hand, the nonrelativistic rocket equation gives

$$v = u\log\frac{M_0}{M_f} = c\log\frac{M_0}{M_f} = c\log\left(1 + \frac{M_0 - M_f}{M_f}\right) \approx \frac{M_0 - M_f}{M_f}\,c$$

which matches.

Why does this work? Notice that the first half of the derivation in part (b) gives precisely the same result as the ordinary rocket equation; the only thing that matters from the standpoint of propelling the rocket is how much momentum you get from the fuel per energy spent. In the nonrelativistic limit, this ratio is $p/E \approx p/mc^2 = u/c^2$. When we apply the nonrelativistic rocket equation to relativistic fuel, we're implicitly using the "dumb" extrapolation $p/E = u/c^2$ for all speeds u. But this is in fact exactly true in relativity, because the factors of γ cancel out! For example, for photons we indeed have $p/E = 1/c$.

Thus, the only step where we actually need relativity is the velocity addition performed in the second half of part (b), but this effect is negligible as long as v/c is small, no matter how big u/c is.

[3] **Problem 16** (Cahn)**.** An empty box of total mass M and perfectly reflecting walls is at rest in the lab frame. Then N photons are introduced into the box, each with angular frequency ω_0 in a standing wave configuration; one can think of these photons as continually bouncing back and forth with velocity $\pm c\,\hat{\mathbf{x}}$, with zero total momentum.

(a) State what the rest mass M_{tot} of the system will be when the photons are present.

(b) Consider the momentum of the system in an inertial frame moving along the x axis with speed $v \ll c$. Using the first order Doppler shift and assuming that at any moment, half the photons are moving left and half the photons are moving right, show that $p = M_{\text{tot}}v$. This provides a dynamical explanation of exactly how photons contribute to the inertia of an object.

(c) Unfortunately, it is *not* true that half the photons are moving right at any given time. Show that the fraction of photons moving to the right is modified by an amount of order v/c, and find the total momentum accounting for this effect.

(d) [A] The analysis of part (b) is nice and neat, and you can sometimes find it in textbooks. But part (c) shows that this simple analysis is wrong! What's going on? (This requires considering the stress-energy tensor, which is beyond the scope of Olympiad physics.)

Solution. (a) Since $E = mc^2$, the rest mass is

$$M_{\text{tot}} = M + \frac{N\hbar\omega_0}{c^2}.$$

(b) Since $v \ll c$, we will use the equation $p = M_{\text{tot}}v$. We clearly have momentum Mv from the box itself. Meanwhile, the photons are Doppler shifted, so their total momentum is

$$p_\gamma = \frac{N}{2}\frac{\hbar\omega_0}{c}(1 + v/c) - \frac{N}{2}\frac{\hbar\omega_0}{c}(1 - v/c) = \frac{Nv\hbar\omega_0}{c^2}.$$

Dividing the momentum by v, we find the same result as in part (a).

(c) The fraction of photons moving to the right/left is $(1 \pm v/c)(N/2)$, which implies that

$$p_\gamma = \frac{N}{2}\frac{\hbar\omega_0}{c}(1 + v/c)^2 - \frac{N}{2}\frac{\hbar\omega_0}{c}(1 - v/c)^2 = \frac{2Nv\hbar\omega_0}{c^2}.$$

This appears to ruin the conclusion of part (b), and there is no other first-order effect to fix it.

Now we resolve the paradox. For simplicity, we'll analyze the system only at first order in v/c. There are numerous other effects at second order, such as the relativistic corrections to the Doppler shift and momentum, but these will complicate the analysis without adding much insight.

The resolution is very subtle, so to warm up, let's consider a simpler situation. In **R3**, you will learn that the charge density and current density can be combined into a four-vector $J^\mu = (\rho, \mathbf{J})$. If you integrate J^0 over all of space, you get the total electric charge Q. And it can be shown that whenever you integrate the zeroth component of a four-vector over all space, you get a Lorentz scalar. That is, the total charge is the same in all frames.

However, this *isn't* always true if you don't integrate over all of space. For example, suppose we had a segment of wire with a perfectly steady current flowing through it. In the wire's frame, it's neutral, and each new charge enters the left end as another charge exits the right end. But in a frame with a velocity along the wire, the loss of simultaneity effect implies that the wire has a net charge! That is, "the amount of charge on the wire" is *not* a Lorentz scalar. (This insight is essential to solving many of the problems in **R3**.) The amount of charge in a system is only necessarily a Lorentz scalar when there's no current flowing through it.

The same subtlety applies to energy and momentum. The total four-momentum of an isolated system (i.e. through which no external energy or momentum enters or leaves) is indeed a four-vector. That's why, for all the collision problems in this problem set, we could treat the four-momenta of particles long before or after the collision as four-vectors. But the photons in the box are not a closed system, because they are constantly interacting with the box, and as a result their four-momentum is not a four-vector. That's why the total momentum of the photons, in a frame where the box is moving, is not what we expect. However, the total momentum of the photons and box together is exactly what we expect, i.e. it is precisely $M_{\text{tot}}v$ in the nonrelativistic limit. The rest of the solution will show this explicitly.

To do this properly, we must introduce the stress-energy tensor $T^{\mu\nu}$, which is analogous to p^μ in the same way that J^μ is analogous to Q. Concretely, in a one-dimensional universe with only x and t directions, it is

$$T^{\mu\nu} = \begin{pmatrix} u & S \\ S & \sigma \end{pmatrix}$$

where the components have the following meanings.

- $T^{00} = u$ is the energy density.
- $T^{01} = S$ is the momentum density, i.e. what we must integrate over space to get momentum. We call this S because it coincides with the Poynting vector for a light wave.
- T^{10} is the current of energy in the x direction. For example, a particle of mass m and velocity v would have $T^{10} = mc^2 v$. It turns out that in general $T^{10} = T^{01}$.
- T^{11} is the current of x-momentum in the x direction, i.e. it has units of momentum per time. Physically, a flow of momentum is equivalent to a pressure.

Upon a Lorentz transformation, the stress energy tensor transforms differently from a four-vector. For a four-vector we would have

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

but for the stress-energy tensor we have

$$\begin{pmatrix} u' & S' \\ S' & \sigma' \end{pmatrix} = \gamma^2 \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} u & S \\ S & \sigma \end{pmatrix} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}.$$

Expanding to first order in v, we have

$$S' = (u + \sigma)v + O(v^2).$$

The momentum of the photons is found by integrating S', giving

$$p_\gamma = \int_0^{L/\gamma} S'\, dx = L(u + \sigma)v + O(v^2).$$

The first term, Luv, is just what we would naively expect by transforming the four-momentum of the photons as a four-vector, and it's the answer we find in part (b). The pressure exerted by the walls yields the additional contribution $vL\sigma$. The energy density in the rest frame is simply $u = N\hbar\omega_0/L$, while the pressure exerted by the walls is $\sigma = N\hbar\omega_0/L$. Summing the terms gives

$$p_\gamma = 2Nv\hbar\omega_0$$

just as we found more directly in part (c).

Now we're in a position to see where the extra momentum is. The walls of the box cause a constant current of x-momentum to flow rightward through the photons. Hence the internal forces of the box must have an equal and opposite current of x-momentum leftward. Thus, by the same argument as above, in the primed frame p_{box} contains a contribution $-L\sigma v$ which precisely cancels the unwanted $L\sigma v$ contribution in the photons. Hence the total momentum is indeed

$$p_{\text{tot}} = Mv + N\hbar\omega_0 v$$

as it must be. For a similar setup, see this paper, which considers a capacitor containing an electromagnetic field, modeled classically instead of in terms of photons.

Remark

In Newtonian mechanics, we know that for an isolated system, $\mathbf{p}_{\text{tot}} = M_{\text{tot}}\mathbf{v}_{\text{CM}}$. In relativity, however, the idea of a "center of mass" no longer makes any sense. For example, suppose a particle with mass m decays into two photons. Each of the photons has no mass, so the center of mass is no longer defined! You can always define the mass of an overall system as $\sqrt{E_{\text{tot}}^2 - p_{\text{tot}}^2}$, and this quantity remains equal to m, but it's no longer the sum of the masses of the individual parts. Since you can't break the mass of the system into parts, you can't sum over the parts to define a center of mass.

However, you can still define a "center of energy",

$$\mathbf{x}_{\text{CE}} = \frac{\sum_i \mathbf{x}_i E_i}{\sum_i E_i}$$

where E_i is the energy of particle i. It turns out that in relativity, we always have

$$\mathbf{p}_{\text{tot}} = \frac{E_{\text{tot}}}{c^2}\,\mathbf{v}_{\text{CE}}$$

which is called the "center of energy theorem". (Specifically, it comes from applying Noether's theorem to the symmetry of Lorentz boosts.) Of course, this reduces to $\mathbf{p}_{\text{tot}} = M_{\text{tot}}\mathbf{v}_{\text{CM}}$ in the nonrelativistic limit, since in that case almost all the energy is rest energy, $E = mc^2$.

4 Relativistic Dynamics

The previous questions could be solved by just using momentum and energy conservation. In this section we'll consider some deeper problems, which require considering the detailed dynamics.

Idea 5

In relativity, the force four-vector is defined as

$$f^\mu = \frac{dp^\mu}{d\tau} = ma^\mu.$$

There's a bit of a subtlety here. In relativity, the invariant mass of a system can change when it absorbs energy, even if it doesn't exchange any particles with its environment. For example, putting a system on the stove gives it energy but not momentum, thereby changing $m = \sqrt{E^2 - p^2}$. That's a perfectly valid four-force, but it feels strange to call it a "force". Therefore, we often restrict to four-forces that don't change the invariant mass, and since

$$\frac{dm^2}{d\tau} = \frac{d}{d\tau}(p \cdot p) = 2mu \cdot f$$

that corresponds to demanding $f \cdot u = 0$. These are sometimes called "pure" forces.

Idea 6

There's also a second way to define force in special relativity, with three-vectors. The first subtlety here is that you could define it as $d\mathbf{p}/dt$ or $m\mathbf{a}$, but the two differ in relativity. Since accelerations transform in a rather nasty way, as we saw in **R1**, the usual choice is to define

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

The second subtlety is that, whenever we define forces as three-forces, we usually implicitly assume that they fix the invariant mass m, i.e. we automatically rule out "put it on a stove" forces. Otherwise, there wouldn't be any way to tell how the energy changes over time.

[4] **Problem 17.** In this problem, we'll derive some properties of the three-force and four-force. For reference, see section 12.5 of Morin.

(a) Show that for a particle traveling along the $\hat{\mathbf{x}}$ direction,

$$\mathbf{F} = m(\gamma^3 a_x, \gamma a_y, \gamma a_z).$$

This is the relativistic three-vector analogue of $\mathbf{F} = m\mathbf{a}$, but it implies that force is no longer parallel to acceleration, which will be important in the problems below.

(b) Now let S' be the momentary rest frame of that particle. In this frame, since the particle is at rest, the nonrelativistic expression $\mathbf{F}' = m\mathbf{a}'$ holds. By using the transformation of acceleration derived in **R1**, show that

$$\mathbf{F} = (F'_x, F'_y/\gamma, F'_z/\gamma).$$

That is, transverse forces are redshifted in relativity, while longitudinal forces are unchanged.

(c) Show that the components of the four-force are

$$f^\mu = \left(\gamma \frac{dE}{dt}, \gamma \mathbf{F}\right).$$

Use the relativistic transformation of the four-force to rederive the result of part (b).

(d) The four-impulse is defined as

$$\Delta p^\mu = \int f^\mu \, d\tau.$$

But you can also consider the Lorentz scalar

$$\int f^\mu \, dx_\mu.$$

This ought to be something nice and simple that you already know about. What is it?

Solution. (a) Using the chain rule and the definition of $\mathbf{p}$,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \gamma m\mathbf{a} + m\mathbf{v}\frac{d\gamma}{dt}.$$

Thus, the y and z components in the desired expression are correct, while the x component (i.e. the part parallel to $\mathbf{v}$ itself) has an extra contribution due to the second term. We have

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt} = \gamma^3 v a_x$$

using a result from **R1**, so

$$F_x = \gamma m a_x (1 + \gamma^2 v^2) = m\gamma^3 a_x$$

as desired.

(b) We see that

$$\mathbf{F} = m(\gamma^3 a_x, \gamma a_y, \gamma a_z) = m(\gamma^3 a'_x/\gamma^3, \gamma a'_y/\gamma^2, \gamma a_z/\gamma^2) = (F'_x, F'_y/\gamma, F'_z/\gamma)$$

where we used $\mathbf{F}' = m\mathbf{a}'$ in the last step.

(c) We just note that

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma\frac{d}{dt}$$

which gives

$$f^\mu = \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt} = \left(\gamma\frac{dE}{dt}, \gamma\frac{d\mathbf{p}}{dt}\right) = \left(\gamma\frac{dE}{dt}, \gamma\mathbf{F}\right).$$

In the primed frame of part (b), the components are

$$f^{\mu'} = \left(0, \mathbf{F}'\right).$$

Applying a Lorentz transformation to the original frame, we have

$$f^x = \gamma F'_x, \quad f^y = F'_y, \quad f^z = F'_z.$$

Since we know that $f^i = \gamma F_i$, we find

$$F_x = F'_x, \quad F_y = F'_y/\gamma, \quad F_z = F'_z/\gamma$$

as desired.

(d) Using the chain rule, we have

$$I = \int f^\mu \frac{dx_\mu}{d\tau}\, d\tau = \int f \cdot u \, d\tau = \int \frac{1}{2m}\frac{dm^2}{d\tau}\, d\tau = \Delta m$$

so I gives the change in rest mass, which is of course a scalar, and just zero in most cases.

Remark

In popular science books and some older textbooks, relativistic dynamics is introduced using the idea of relativistic mass, $m_r = \gamma m$. This definition implies the simple results $E = m_r c^2$ and $\mathbf{p} = m_r \mathbf{v}$, so these books often say that relativistic dynamics is just like ordinary dynamics, except that moving objects have more mass. This picture is misleading because it breaks down once you go beyond one dimension: in problem 17, you showed that $\mathbf{F}$ is not even parallel to $\mathbf{a}$, so there's no definition of mass that recovers Newtonian mechanics. You instead need separate "transverse" and "longitudinal" relativistic masses,

$$\mathbf{F} = m_\perp \mathbf{a}_\perp + m_\parallel \mathbf{a}_\parallel, \quad m_\perp = \gamma m, \quad m_\parallel = \gamma^3 m.$$

I think this picture is honestly more confusing than helpful, though. It's better to avoid talking about mass and acceleration too much, and focus more on momentum and energy.

You'll also see arguments that relativistic mass is useful when thinking about gravity. In general relativity, all energy produces gravity equally. If you have a box with n particles bouncing around, which all have Lorentz factor γ and rest mass m, then the energy of the box is the same as that of n particles at rest, with mass m_r. So it looks like the gravity sourced by the particles is described by their relativistic mass. Unfortunately, this argument is *also* wrong, because in general relativity pressure also produces gravity. In the limit $\gamma \to \infty$, describing a gas of ultrarelativistic particles, the pressure contribution means we get twice as much gravitational attraction as would be predicted from the energy alone.

Example 7

A circular pendulum consists of a mass m attached to a string of length ℓ, with the other end fixed. Suppose the mass rotates in a small circle of radius $r \ll \ell$, with a nonrelativistic velocity in the lab frame. Find the angular frequency of the oscillations in the lab frame, and in a frame where the entire setup moves vertically with a relativistic speed v.

Solution

In the lab frame, this is a standard rotational mechanics problem. By the small angle approximation, the horizontal component of the three-force is $F_\perp = mgr/L$. This is equal to

$$F_\perp = ma_\perp = m\omega^2 r$$

from which we immediately conclude $\omega = \sqrt{g/L}$. We can use the results of problem 17 to find the answer in the other frame. The two effects are that the transverse force is redshifted, and the force's relation with acceleration is different,

$$F_\perp = \frac{mgr}{\gamma L}, \quad F_\perp = \gamma m a_\perp = \gamma m \omega^2 r.$$

Combining these results, we find

$$\omega = \frac{1}{\gamma}\sqrt{\frac{g}{L}}.$$

Of course, γ is just the usual time dilation factor. We knew this had to be the answer, because time dilation follows directly from the postulates of relativity, but now we can explicitly show this is the right answer in this specific example. (With similar reasoning, you can show that a mass-spring system oscillates slower, too.)

Remark

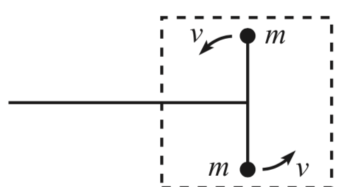
It's important not to misunderstand the meaning of the above example. Like many old physicists, Oleg Jefimenko decided one day that relativity had to be completely wrong. His argument was along the lines of the previous example: he showed that length contraction and time dilation could be derived dynamically in some simple cases, without the need to switch frames. Therefore, they can't be "real".

This argument doesn't make sense. It's like saying that energy can't be real because you can solve many mechanics problems with just $F = ma$, without needing to invoke energy conservation. (Though amazingly, some people actually do spend years arguing whether force or energy is "more real", in a debate that resembles rival high school cheerleading squads, when it's better to realize that they're both wonderful tools with complementary uses.)

Furthermore, it actually turns out to be *extremely* difficult to derive the core results of relativistic dynamics (such as the "transverse" and "longitudinal" masses, already measured by the turn of the 20^{th} century) without using relativistic assumptions. In the early 1900s, many physicists tried to explain the dynamics of the electron solely in terms of its electromagnetic fields. Since the field energy and field momentum of a moving point charge are infinite, it was necessary to take a model of the electron with finite size, but there were many possibilities, leading to many different expressions for the transverse mass, as well as persistent issues like the 4/3 problem mentioned in **E7**.

Relativity circumvents all of these issues. If you accept the postulates of relativity, you don't need to care whether the electron is shaped like a sphere, an ellipsoid, a torus, or a dumbbell: as long as its dynamics obey Lorentz symmetry, its four-momentum is a four-vector, and the usual results follow. And that's just as well, because with the advent of quantum mechanics, we learned that the electron is not like *any* of these classical models. But the relativistic result still holds, because our quantum theories obey the postulates of relativity too. This flexibility comes about because, like thermodynamics, relativity isn't so much a physical theory, as it is a framework within which many theories can be formulated.

[3] **Problem 18** (Morin 12.8)**.** Consider a dumbbell made of two equal masses, m. The dumbbell spins around, with its center pivoted at the end of a stick.



If the speed of the masses is v, then the energy of the system is $2\gamma m$. Treated as a whole, the system is at rest. Therefore, the mass of the system must be $2\gamma m$. (Imagine enclosing it in a box, so that you can't see what's going on inside.) Convince yourself that the system does indeed behave like a mass of $M = 2\gamma m$, by pushing on the stick (when the dumbbell is in the "transverse" position shown in the figure) and showing that $F = dp/dt = Ma$.

Solution. Consider speeding up the system by dv to the left. The relativistic velocity addition formula for u plus dv becomes

$$\frac{u+dv}{1+\frac{u\,dv}{c^2}} = (u+dv)(1-udv/c^2) = u + dv(1-u^2/c^2).$$

Let γ_u be $1/\sqrt{1-u^2}$. Let γ'_u be the gamma factor for $u + dv(1-u^2)$. One can easily check that $\gamma'_u = \gamma(1+u\,dv)$. Thus, the change in momentum due to the extra dv is

$$\gamma m(1+u\,dv)(u+dv(1-u^2)) - \gamma m u = \gamma m\,dv,$$

which is surprisingly what one would naively expect. Thus, the total change in momentum of the system is simply $dp = 2\gamma m\,dv$, so $dp/dt = M\,dv/dt$, as desired.

Idea 7

The Lorentz force is a three-force as defined in problem 17. That is, we have

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \frac{d\mathbf{p}}{dt}$$

and the force keeps the invariant mass fixed.

Example 8

A point charge q of mass m is initially at rest, and experiences a uniform electric field E. What time t does it take the object to move a distance x?

Solution

In **R1**, we found $x(t)$ for a uniformly accelerated rocket, which assumed a constant three-force in the momentarily comoving frame. By contrast, here we have a constant three-force $F = qE$ in the *lab* frame. However, we showed in problem 17 that forces along the direction of motion are the same in both frames, so these two problems are actually identical!

So we already know the answer to the problem, but it turns out that in the lab frame perspective, there's a slick alternative derivation that yields the result in one step. The trick is to consider the energy and momentum. Recall from problem 1 that the three-force F obeys $F = dp/dt$ and $F = dE/dx$. Therefore, when the object reaches its destination,

$$E = m + Fx, \quad p = Ft.$$

But we also know that $E^2 = p^2 + m^2$, so plugging the results in and solving for t gives

$$t = \sqrt{x^2 + \frac{2mx}{F}}$$

which is compatible with our earlier expression for $x(t)$. The reason this was so easy is that momentum and energy behave simply in relativity, while position and velocity don't.

Example 9

The LHC accelerates protons to an energy of $E = 7\,\text{TeV}$, and is a tunnel of radius $R = 4.3\,\text{km}$. If the protons are kept in a circular orbit in the tunnel by a magnetic field of magnitude B, find the required value of B. If the value of B is kept constant, what would be the radius of a future collider which accelerates protons to an energy of $20\,\text{TeV}$?

Solution

The centripetal force required is

$$F = \left|\frac{d\mathbf{p}}{dt}\right| = \omega p$$

where ω is the angular velocity. The speed of the protons is very close to c, so the angular velocity is $\omega \approx c/R$, and the momentum is $p \approx E/c$. The deflecting force is $qvB \approx qcB$, so

$$qcB \approx \omega p \approx \frac{E}{R}.$$

Therefore, we have

$$B = \frac{E}{qcR} = \frac{7 \times 10^{12}}{(3 \times 10^8)(4.3 \times 10^3)}\,\text{T} = 5.4\,\text{T}.$$

This is slightly lower than what is actually used, because magnets don't take up the entire tunnel. Since $R \propto E$, the future collider would need a radius of

$$R' = \frac{20\,\text{TeV}}{7\,\text{TeV}} R = 12\,\text{km}.$$

Remark

You might be wondering how to write the Lorentz force as a four-force. It certainly should be possible, since we know electromagnetism is compatible with relativity (indeed, it led us to relativity in the first place), but it seems challenging because electromagnetism is so naturally written in terms of three-vectors. It turns out that the proper way to express the electromagnetic field in relativity is to join the electric and magnetic fields together, making them the components of an antisymmetric rank 2 tensor,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

called the field strength tensor. Then the four-force is

$$f^\mu = q u_\nu F^{\mu\nu}$$

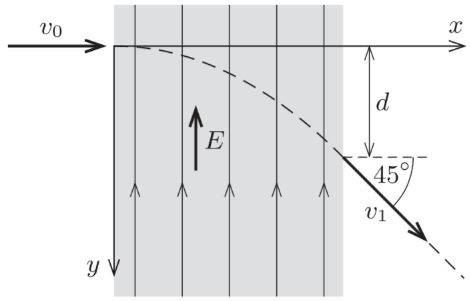
where u_ν is the four-velocity. Note that this ensures the rest mass of the particle is fixed, as

$$f \cdot u = q u_\mu u_\nu F^{\mu\nu} = -q u_\mu u_\nu F^{\nu\mu} = -f \cdot u$$

using the antisymmetric property, so $f \cdot u = 0$. (In fact, the requirement to keep the rest mass fixed is quite restrictive, so this is one of the simplest possible relativistic force laws.)

[2] **Problem 19.** USAPhO 2013, problem A3. A warmup question using the above facts.

[3] **Problem 20** (MPPP 192)**.** An electron moving with speed $v_0 = 0.6c$ enters a homogeneous electric field that is perpendicular to its velocity.



When the electron leaves the field, its velocity makes an angle 45° with its initial direction.

(a) Find the speed v_1 of the electron after it has crossed the electric field.

(b) Find the distance d shown above, if the strength of the electric field is $E = 510\,\text{kV/m}$.

Note that the rest energy of an electron is 510 keV.

Solution. (a) Since we are working with three-forces here, we use $\mathbf{F} = d\mathbf{p}/dt$. This tells us that the component of momentum p_x is unchanged. Since the velocity is at a 45° angle, so is the momentum, so $p_y = p_x$. Thus, the momentum increases by a factor of $\sqrt{2}$. The momentum per mass started at $0.6/0.8 = 3/4$, so its now $\frac{3}{4}\sqrt{2}$. Thus,

$$\frac{v_1}{\sqrt{1-v_1^2}} = \frac{3\sqrt{2}}{4} \implies \frac{v_1^2}{(1-v_1^2)} = \frac{9}{8} \implies v_1 = \frac{3c}{\sqrt{17}}.$$

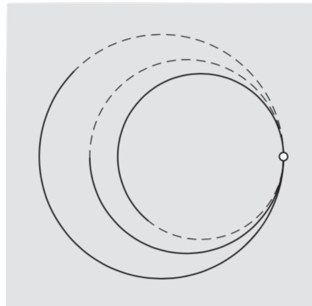
Note that this implies that v_x has *decreased*, even though the electric 3-force had no x-component. As we warned above, this is a manifestation of the fact that $\mathbf{F}$ is no longer parallel to $\mathbf{a}$ in relativity.

(b) As we showed in problem 1, the basics of work still work the same in relativity. The amount of work done on the electron is eEd, while the energy change is $m\Delta\gamma$, where

$$\Delta\gamma = \frac{1}{\sqrt{1-9/17}} - \frac{1}{\sqrt{1-9/25}} = \frac{\sqrt{17}}{\sqrt{8}} - \frac{5}{3}.$$

Plugging in the numbers gives $d = 20.8\,\text{cm}$.

[3] **Problem 21** (MPPP 194)**.** The trajectories of charged particles, moving in a homogeneous magnetic field, can be seen by observing the tracks they leave in cloud chambers. Because the particles are moving quickly, it is impossible to see the tracks being formed; instead, one must infer what happened from the shapes of the tracks. Is it possible that, when a charged particle decays into two other charged particles, the trail segments close to the decay point (before the particles have started to slow down significantly) are arcs of circles that touch each other, as shown?



If so, identify which track belongs to the original particle. If not, explain why not.

Solution. Number the three tracks as 1, 2, and 3 starting from the inside, and let their radii be $r_1 < r_2 < r_3$. We know that even for relativistic motion, the momentum of a particle is $p = qBr$. We can then use conservation of momentum and conservation of charge to investigate each case.

Case 1: Particle 1 decays, implying that a particle comes in along track 1, and particles leave along tracks 2 and 3. The curvatures of the tracks imply

$$q_1 > 0, \quad q_2 > 0, \quad q_3 > 0.$$

Conservation of charge and momentum imply

$$q_1 = q_2 + q_3, \quad q_1 r_1 = q_2 r_2 + q_3 r_3.$$

By combining these equations, we may solve for r_1 to find

$$r_1 = \frac{q_2 r_2 + q_3 r_3}{q_2 + q_3}.$$

However, this is impossible because we know r_1 is smaller than both r_2 and r_3.

Case 2: Particle 2 decays, which implies

$$q_1 < 0, \quad q_2 < 0, \quad q_3 > 0.$$

Conservation of charge and momentum imply

$$q_2 = q_1 + q_3, \quad |q_2 r_2| = |q_1 r_1| - |q_3 r_3|.$$

Being careful with minus signs, momentum conservation implies

$$-q_2 r_2 = -q_1 r_1 - q_3 r_3.$$

Again solving for r_1, we find

$$r_1 = \frac{q_3 r_3 + (-q_2) r_2}{q_3 + (-q_2)}$$

which is a contradiction for the same reason as in case 1.

Case 3: Particle 3 decays, which implies

$$q_1 < 0, \quad q_2 > 0, \quad q_3 < 0.$$

Conservation of charge and momentum imply

$$q_3 = q_1 + q_2, \quad |q_3 r_3| = |q_1 r_1| - |q_2 r_2|.$$

Again being careful with minus signs, momentum conservation implies

$$-q_3 r_3 = -q_1 r_1 - q_2 r_2.$$

Again solving for r_1, we find

$$r_1 = \frac{q_2 r_2 + (-q_3) r_3}{q_2 + (-q_3)}$$

which is again a contradiction. Thus, the series of tracks shown is impossible.

[3] **Problem 22.** USAPhO 2006, problem A4.

[3] **Problem 23.** USAPhO 2022, problem B2. A nice problem on deriving the time dilation formula for an electrostatic "clock".

[3] **Problem 24.** Consider a particle at the origin at time $t = 0$, with initial x-momentum p_0 and total energy E_0. A constant three-force F acts on the particle in the $-y$ direction.

(a) Calculate $y(t)$. (Hint: don't write down any equations containing γ, because it depends on $v_x(t)$, which we don't know yet.)

(b) Calculate $x(t)$.

(c) Combine these results to get $y(x)$. This is the path of a relativistic projectile.

Solution. We use the technique of example 8, setting $c = 1$ throughout.

(a) By the definition of three-force and the work-energy theorem,

$$p_x = p_0, \quad p_y = -Ft, \quad E = E_0 - Fy.$$

To find $y(t)$, we use the fact that $v_y = p_y/E$, so

$$\frac{dy}{dt} = -\frac{Ft}{E_0 - Fy}.$$

Separating and integrating, then using the initial condition gives

$$y^2 - \frac{2E_0}{F} y = t^2.$$

Solving the quadratic in y gives

$$y(t) = \frac{E_0}{F} - \sqrt{\frac{E_0^2}{F^2} + t^2}.$$

(b) Similarly, we have

$$\frac{dx}{dt} = \frac{p_x}{E} = \frac{p_0}{E_0 - Fy} = \frac{p_0}{\sqrt{E_0^2 + F^2 t^2}}$$

where we used the result of part (a). Separating and integrating,

$$x = \int_0^t \frac{p_0 \, dt}{\sqrt{E_0^2 + F^2 t^2}}.$$

Nondimensionalizing the integral, it can be performed with the hyperbolic trigonometric substitution $t = (E_0/F) \sinh\theta$, giving

$$x(t) = \frac{p_0}{F} \sinh^{-1} \frac{Ft}{E_0}.$$

(c) To get $y(x)$, we invert the above to get $t(x)$ and plug it into our expression for $y(t)$. We have

$$\frac{Ft}{E_0} = \sinh \frac{Fx}{p_0}$$

and plugging this in gives

$$y(x) = \frac{E_0}{F}(1 - \cosh(Fx/p_0c))$$

where we restored c in the last step. In other words, relativistic projectile motion follows an inverted catenary! To check the nonrelativistic limit, we just note that

$$\cosh u = 1 + \frac{u^2}{2} + \dots$$

which tells us that

$$y(x) \approx -\frac{1}{2}\frac{E_0}{F}\left(\frac{Fx}{p_0c}\right)^2 \approx -\frac{1}{2}\frac{mF}{p_0^2}x^2 \approx -\frac{1}{2}\frac{F}{mv_0^2}x^2$$

which is indeed the usual parabola.

[5] **Problem 25.** IPhO 1994, problem 1. Print out the custom answer sheets before starting.

Remark

The setup of problem 25 is a nice model for mesons, particles composed of two quarks. And it's not just something made up for an Olympiad; it is a simple version of the MIT "bag model", which was one of the most important advances in the field in the 1970s. In fact, if you look at the original paper, which has thousands of citations, you'll find the answer to the IPhO question in figure 3!

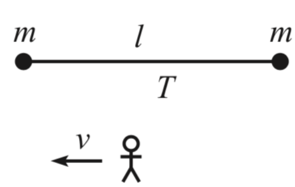
Idea 8

In string theory, strings carry a constant tension T, in the sense that the force $\mathbf{F} = d\mathbf{p}/dt$ exerted on one piece of string by its neighbors is T in the momentary rest frame of that piece. The strings may stretch or shrink freely, and have zero mass when they have zero length.

[3] **Problem 26** (Morin 12.16)**.** A simple exercise involving relativistic string.

(a) Two masses m are connected by a string of length ℓ and constant tension T. The masses are released simultaneously, and they collide and stick together. What is the mass, M, of the resulting blob?

(b) Consider this scenario from the point of view of a frame moving to the left at speed v.



The energy of the resulting blob must be γMc^2. Show that you obtain the same result by computing the work done on the two masses.

Solution. (a) The total work done on the masses is ℓT, so by energy conservation this must manifest as rest energy in the final blob, $M = 2m + \ell T/c^2$.

(b) Let $c = 1$. The initial energy is $2\gamma m$, so we need to show that the work done is $\gamma \ell T$.

At first glance, this is puzzling, because the initial distance between the masses in this frame is ℓ/γ. Therefore, naively applying $W = \int F\,dx$, we have

$$W = \int T\,dx_1 - \int T\,dx_2 = T \int dx_1 - dx_2 = T\ell/\gamma$$

which is wrong. The resolution is that we have assumed the masses are released simultaneously in the original frame, which means they aren't released simultaneously in this frame.

The mass on the left will start accelerating first, and after some time, the mass on the right will accelerate. In the original frame, these two events have $\Delta x = \ell$ and $\Delta t = 0$. Thus, applying the Lorentz transformation,

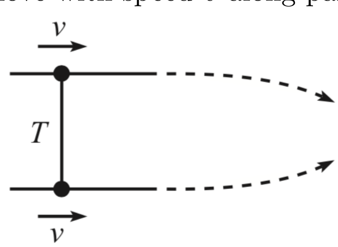
$$\Delta x' = \gamma \Delta x = \gamma \ell.$$

Suppose that after it starts experiencing the tension, the left mass moves a distance x_0 before it collides with the right mass. Then the above calculation shows that after the right mass starts experiencing the tension, it moves a distance $x_0 - \Delta x'$ until collision. Thus,

$$W = T(x_0 - (x_0 - \Delta x')) = \gamma \ell T$$

as desired.

[3] **Problem 27** (Morin 12.37). Two equal masses are connected by a relativistic string with tension T. The masses are constrained to move with speed v along parallel lines, as shown.



The constraints are then removed, and the masses are drawn together. They collide and make one blob which continues to move to the right. Is the following reasoning correct?

> The forces on the masses point in the y direction. Therefore, there is no change in the momentum of the masses in the x direction. But the mass of the resulting blob is greater than the sum of the initial masses (because they collide with some relative speed). Therefore, the speed of the resulting blob must be less than v (to keep p_x constant), so the whole apparatus slows down in the x direction.

If your answer is "no," exactly what's wrong about the reasoning above?

Solution. The reasoning is incorrect. To see this, we can consider working in the initial rest frame of the system. In this frame, the masses just approach each other and collide, ending up at rest. So in the original frame, the whole apparatus must keep going at the same speed as before.

There are two ways to see what's going on. First, consider just the top mass, and work throughout in the original frame. Then the incorrect statement is the very first sentence: the three-force on the top mass is not always in the y direction. Recall the relativistic transformation of the three-force derived in problem 17. This tells us that if we align the x' axis with the *instantaneous* motion of the particle, then

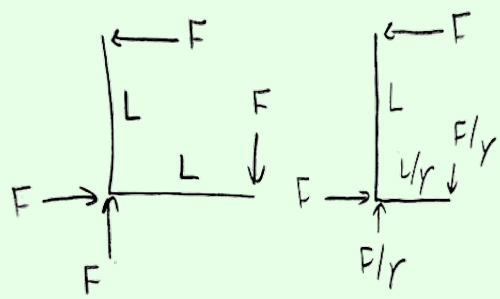
$$\mathbf{F} = (F'_{x'}, F'_{y'}/\gamma, F'_{z'}/\gamma).$$

Once the top mass gets moving, it has velocity components along both x and y, so the x' axis must be tilted accordingly. Upon applying this formula (i.e. redshifting the y' component of the force), we end up with a nonzero x component of the force, so the logic above fails.

Alternatively, we can consider the entire system, of the masses and string. In this case, the statement that fails is the second parenthetical, "to keep p_x constant". The issue here is that the string itself has a linear mass density of T/c^2, due to the energy stored in it in the stretching process, and hence also carries momentum. This needs to be accounted for in the momentum conservation equation, and gives the "missing" momentum we need. Note that this is totally compatible with the previous paragraph; the force discussed there is precisely how this string momentum ends up transferred to the masses.

Example 10: Right Angle Lever Paradox

In 1909, Lewis and Tolman found one of the first relativistic paradoxes. Consider a rigid lever in static equilibrium, with both arms of length L, experiencing the forces shown at left.



In a frame where the lever moves to the right with speed v, one of the lever arms will be contracted to L/γ, as shown at right. In addition, by the results of problem 17, the vertical external forces will be redshifted to F/γ. This implies a net torque of

$$\tau = FL - \frac{F}{\gamma}\frac{L}{\gamma} = FLv^2.$$

The paradox is, given that $\boldsymbol{\tau} = d\mathbf{L}/dt$, why doesn't the lever rotate?

Solution

The resolution is that, in the frame shown at right, the angular momentum of the lever is constantly increasing. The horizontal forces are continually doing equal and opposite work on the lever, resulting in a upward flow of energy of rate Fv in the vertical arm. As explained in **E7** and earlier in this problem set, in relativity, energy flow is equal to momentum density, so the total upward momentum in the vertical arm is FLv. Therefore,

$$\frac{dL}{dt} = \frac{dx}{dt}\,(FLv) = FLv^2$$

exactly as expected.

Remark

The resolution of the right angle lever paradox is very controversial, with dozens of papers written on the subject, so we should discuss what it even means to "resolve" a paradox. As long as we believe relativity is self-consistent, we already know what's going to happen: the lever won't rotate. Everything the lever does is determined by $\mathbf{F} = d\mathbf{p}/dt$ alone, so if it looks like angular momentum considerations give a different answer, that just means we haven't formulated the latter correctly. The reason there are so many different resolutions out there is just that people choose different ways to define torque and angular momentum.

The solution above is the standard one, and its implicit definition of angular momentum can be motivated by Noether's theorem. That's a reasonable choice, since it's a specific output of a useful and general theorem, and we thereby know for sure that it's conserved for isolated systems. Unfortunately, explaining the definition takes some advanced math.

We define the angular momentum density tensor

$$M^{\mu\nu\rho}(x) = x^\mu T^{\nu\rho}(x) - x^\nu T^{\mu\rho}(x)$$

where the right-hand side contains the stress-energy tensor, from the solution to problem 16. The total angular momentum is an antisymmetric rank 2 tensor,

$$J^{\mu\nu}(t) = \int d\mathbf{x}\, M^{\mu\nu\rho}(x).$$

Noether's theorem states that it is this quantity that is conserved for an isolated system, due to symmetry under rotations and boosts. More specifically, the three spatial components J^{xy}, J^{yz}, and J^{zx} just make up ordinary angular momentum, e.g. for a single point particle they would assemble into the vector $\mathbf{r} \times \mathbf{p} = \mathbf{r} \times (\gamma m \mathbf{v})$. And the other components J^{0x}, J^{0y} and J^{0z} have to do with the center of mass motion.

If there is an external four-force per unit proper volume $f^\mu(x)$, which in terms of the stress-energy tensor implies $\partial_\mu T^{\mu\nu} = f^\nu$, the rate of change of angular momentum is

$$\frac{dJ^{\mu\nu}}{dt} = \tau^{\mu\nu}, \quad \tau^{\mu\nu} = \int d\mathbf{x}\, x^\mu f^\nu(x) - x^\nu f^\alpha(x)$$

which looks quite similar to the Newtonian expression. The component of this equation relevant to this paradox is $dJ^{xy}/dt = \tau^{xy}$, where

$$J^{xy} = \int d\mathbf{x}\, xT^{y0} - yT^{x0}, \quad \tau^{xy} = \sum_k x^{(k)} F_y^{(k)} - y^{(k)} F_x^{(k)}$$

where the index k sums over the four forces, and the T^{i0} stand for the density of momentum in the i direction. From this point on, the solution proceeds as above.

There is something a bit strange here, though. In the lever's rest frame, the angular momentum is zero, so if $J^{\mu\nu}$ were a tensor, it would have to be zero in all frames, but instead it rises to arbitrarily high values in the other frame. The reason is that when there are external torques, $J^{\mu\nu}$ isn't a tensor at all, just like how the four-momentum wasn't a four-vector in the solution to problem 16. That's one of the reasons there's a controversy: there just doesn't exist any definition that has all the nice properties one might want.

Relativity III: Fields

Relativity in electromagnetism is covered in chapter 5 of Purcell and then sprinkled in throughout the rest of the book, notably in sections 6.7 and 9.7, and appendix H. For a more advanced discussion, see chapter 3 of Schutz for tensors, and chapter 12 of Griffiths and chapters I-34, II-13, and II-25 through II-28 of the Feynman lectures for relativistic electromagnetism. For a brief taste of general relativity, see chapter 14 of Morin, and chapter II-42 of the Feynman lectures. For a great, accessible introduction to tests of general relativity, see *Was Einstein Right?* by Will. There is a total of **90** points.

1 Electromagnetic Field Transformations

Idea 1: Field Transformations

If the electromagnetic field is $(\mathbf{E}, \mathbf{B})$ in one reference frame, then in a reference frame moving with velocity $\mathbf{v}$ with respect to this frame, the components of the field parallel to $\mathbf{v}$ are

$$E'_\parallel = E_\parallel, \quad B'_\parallel = B_\parallel$$

while the components perpendicular are

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp), \quad \mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_\perp\right).$$

As alluded to in **R2**, this is the transformation rule for the components of a rank 2 antisymmetric tensor.

Under these transformations, Maxwell's equations remain true in all inertial frames, and the Lorentz force transforms properly as well. Furthermore, a Lorentz transformation does not change the total amount of charge in a system, where total charge is defined by Gauss's law via the electric flux through a surface containing the system.

Remark

There are many ways of deriving the field transformations. The tensor method alluded to above is the mathematically cleanest, but the conceptually clearest is to think about how some simple setups must Lorentz transform, if Maxwell's equations are to remain true. For example, boosting a capacitor increases the charge density on the plates because of length contraction, which is why $\mathbf{E}'_\perp$ contains $\gamma\mathbf{E}_\perp$. (Further examples are given in chapter 5 of Purcell, which is essential reading for this section.) Another method is to demand that the Lorentz force obeys the transformation of three-force derived in **R2**.

[4] **Problem 1.** Basic facts about the electric and magnetic fields of a moving charge.

(a) Show that the field of a point charge q at the origin moving with constant velocity v is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \frac{1 - v^2}{(1 - v^2 \sin^2\theta)^{3/2}} \hat{\mathbf{r}}$$

in units where $c = 1$, and θ is the angle from $\mathbf{v}$. In particular, the field is still radial.

(b) Verify that the charge of this moving charge is still q. It may be useful to consult the integral table in appendix K of Purcell.

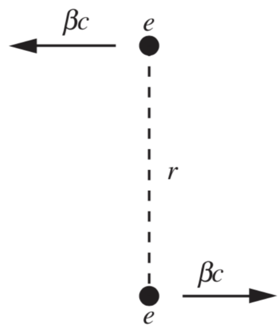
(c) Argue that the magnetic field of this point charge must be exactly

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}.$$

(d) Verify that the previous result is correct in nonrelativistic electromagnetism (i.e. using Coulomb's law and the Biot–Savart law).

The result of part (a), first found by Heaviside in 1888, implies that the field lines and equipotential surfaces of a moving charge contract by a factor of γ in the direction of motion. In fact, this was what inspired Lorentz and Fitzgerald to propose length contraction in the first place! Since it's very hard to measure the Coulomb field of a relativistic electron, this prediction was first directly verified in 2022.

[3] **Problem 2** (Purcell 5.29). Two protons are moving antiparallel to each other, along lines separated by a distance r, with the same speed v in the lab frame, as shown.



Consider the moment the protons are a distance r apart.

(a) Show that the three-force experienced by each proton due to the electric field of the other is

$$F = \frac{\gamma e^2}{4\pi\epsilon_0 r^2}.$$

(b) Compute the three-force experienced by one of the protons by transforming to its rest frame, computing the force there, then transforming back to the lab frame. In particular, show that this is not equal to the result of part (a).

(c) Show that the discrepancy is resolved if the magnetic three-force is also included.

Recall from **R2** that the Lorentz three-force is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. You will also have to use the three-force transformation laws you derived there.

[3] **Problem 3** (Purcell 5.24). In the rest frame of a particle with charge q, another particle with charge q is approaching with relativistic velocity $\mathbf{v}$. Assume that both particles are extremely massive, and hence their velocities are nearly constant. The second particle passes a minimum distance b from the first.

(a) Show that the impulse acquired by each particle is perpendicular to $\mathbf{v}$ with magnitude $q^2/2\pi\epsilon_0 vb$. (Hint: you can avoid doing a nasty integral by using Gauss's law.)

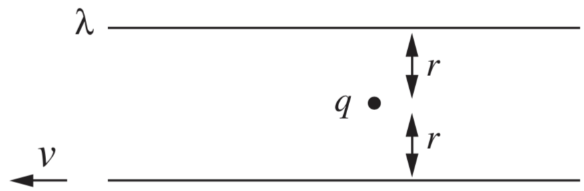
(b) If the particles have mass m, roughly how large does m have to be for the above result to be a good approximation?

[3] **Problem 4.** USAPhO 2014, problem B2. This isn't the clearest of problems, but it introduces and justifies the Galilean field transformations we first saw in **E4**.

[3] **Problem 5** (Purcell 5.30)**.** Consider an infinite wire oriented along $\hat{\mathbf{x}}$ with linear charge density λ and current I. Show that under a Lorentz boost along $\hat{\mathbf{x}}$, (λ, I) transforms like (ct, x).

[2] **Problem 6** (Purcell 6.22)**.** A neutral wire carries current I. A stationary charge q is nearby; the Lorentz force on this charge is zero. Verify this remains true in a frame moving parallel to the wire with velocity $\mathbf{v}$, by using the Lorentz transformations of the fields.

[3] **Problem 7** (Purcell 6.69)**.** Two very long sticks each have uniform linear proper charge density λ. One stick is stationary in the lab frame, while the other moves to the left with speed v, as shown.



They are $2r$ apart, and a stationary point charge q lies midway between them. Find the Lorentz three-force on the charge in the lab frame, and also in the frame of the bottom stick, and verify the forces relate properly.

Remark: Are Wires Neutral?

The classic example in problem 6 starts by assuming the wire is neutral is the lab frame. But in problem 5, you showed that if a current-carrying wire is neutral is one reference frame, then it's *not* neutral in other reference frames. So how do we know which frame a *real* wire is neutral in? Is it the lab frame, the frame where the current vanishes, or something else?

It actually depends on the details, so for concreteness, let's consider two very long, parallel wires, connected at one end by a battery and at the other end by a resistor. Before current starts flowing, the whole system is neutral. So if the wires picked up a net charge density, the battery would have to have a large compensating charge, which would make it blow up. Or, to say it another way, if a net charge appears in the rest of the system, it pulls a compensating charge out of the battery, so the battery keeps the wires net neutral. You can then show that in a boosted frame, the wires stay net neutral, as you'd expected.

But this argument only shows that the wires have *opposite* charge densities $\pm\lambda$ in the lab frame. Can we show that $\lambda = 0$? Actually, we can't, because it's not true! As briefly discussed in a problem in **E2**, wires in circuits do carry charges in the lab frame, even if everything is ideal. One simple way to see this here is to note that the wires are at different electrical potentials. That's only possible if there's an electric field between them, which is created by the charge densities carried by the wires.

[3] **Problem 8.** The vectors $\mathbf{E}$ and $\mathbf{B}$ cannot go into four-vectors, as they transform among each other, but rather fit together into an antisymmetric rank two tensor. As a result, there is a different set of associated invariant quantities.

(a) Show that under the relativistic field transformations, the quantities $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ are both invariant. (Hint: this can be done using vector notation, using $\mathbf{E}_\perp \cdot \mathbf{E}_\parallel = \mathbf{B}_\perp \cdot \mathbf{B}_\parallel = 0$.) These are the two basic invariants, out of which all other invariants can be constructed.

(b) Suppose that in an inertial frame, $\mathbf{E}$ is zero at a given point and $\mathbf{B}$ is nonzero. Is it possible to find an inertial frame where $\mathbf{B}$ is zero at that point?

(c) Recall from **E7** that, in units where $\epsilon_0 = \mu_0 = 1$, the energy density of the electromagnetic field is $\mathcal{E} = E^2/2 + B^2/2$, and the Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{B}$. Show that $\mathcal{E}^2 - |\mathbf{S}|^2$ is invariant. (Hint: don't use the field transformations for this part.)

Remark: Is Magnetism Real?

Purcell's electromagnetism textbook is exceptional because it shows that a force like magnetism must exist, if one believes Coulomb's law and relativity. The idea is simple. We know how forces transform between frames, and given some reasonable assumptions, can also deduce how electric fields transform between frames. If electric fields were all there were, then electric forces would have to transform just like three-forces, but they don't. So there must be some other force to make up the difference, and it turns out to be precisely the magnetic force. We saw an example of this in problem 2.

It is important not to misunderstand this beautiful idea. Many people, upon reading such arguments, believe that magnetism "doesn't exist" because it's "all just electric fields". Sometimes people even say that magnetic forces are a "mistake" caused by "forgetting about" relativistic corrections. This is all totally backwards. Sometimes time dilation in one frame can be explained in terms of length contraction in another, but that doesn't mean that length contraction doesn't exist, or is a mistake – it's perfectly real in that particular frame. (Furthermore, while you can always get rid of the magnetic *force* on one particle at one moment by going to that particle's rest frame, there are plenty of situations where you can't remove the magnetic field, as we saw in problem 8!)

The real lesson of relativity isn't that magnetic fields are a mistake, it's that electric and magnetic fields are as intertwined as space and time, as you can see from their transformation properties. Just as space and time combine into a four-vector, electric and magnetic fields combine, in an equal footing, into the electromagnetic field tensor.

Remark: Electromagnetism in Covariant Form

Problem 5 is a first step to showing that $J^\mu = (\rho, \mathbf{J})$ is a four-vector, where ρ is the charge density and $\mathbf{J}$ is the current density. Note that the continuity equation for charge, as mentioned in **T2**, can be simply written in four-vector notation as

$$\partial_\mu J^\mu = 0.$$

As another example, you can show that the four-current of a single charged particle q is $J^\mu = q u^\mu$. We can go even further and write the whole of electromagnetism in terms of four-vectors and tensors. Maxwell's equations can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

The invariant quantities found in problem 8 can be written in terms of the field strength tensor as $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi–Civita symbol. These are the only two ways to "contract all the indices" to get a scalar.

Remark: Elegant Notation

Sometimes people dislike the index notation above because of all the little Greek letters floating around. If you *only* want to deal with vectors, vector notation is often better. It hides all the indices, at the cost of requiring you to introduce special symbols like $\cdot$ and $\times$ to specify the vector operations you want to do. The reason we don't use a vector-like notation for tensors is because there are too many operations you can do with them (e.g. "contract the 3rd index of a rank 4 tensor with the 1st index of a rank 2 tensor") to define separate symbols for each one; indices are just more efficient. On the other hand, if you only work with totally antisymmetric tensors, then there are only a few possible operations, and one can use the elegant, index-free "differential form" notation. In this notation, Maxwell's equations are

$$d \star F = J$$

where d is called the exterior derivative, $\star$ is the Hodge dual, and the fact that the electromagnetic fields are derivatives of potentials is expressed as

$$F = dA.$$

So is this the *best*, *most true* formulation of Maxwell's equations? Well, as Feynman once pointed out, you can easily do better. For example, you can define the "unworldliness"

$$U = |\mathbf{F} - m\mathbf{a}|^2 + (\nabla \cdot \mathbf{E} - \rho/\epsilon_0)^2 + \ldots.$$

Then *all* physical laws can be expressed in terms of the amazingly simple equation

$$\boxed{\boxed{\boxed{U = 0.}}}$$

But this doesn't actually help, because to use the equation for anything, you need to plug in the definition of U, and then you're back to where you were before. In general, more elegant notation is often more brittle: it only works well in a smaller set of situations. (For example, with differential form notation, you just can't write down the stress-energy tensor of the electromagnetic field, because that's symmetric rather than antisymmetric.) Index notation is great because it works as long as indices are contracted in pairs, which holds as long as you're dealing with laws that are independent of coordinate system. In general, there's no need to be ideological about notation; it's just a tool, and we should use the best tool for each job. If anyone tells you that their preferred notation for vectors or tensors will revolutionize physics, keep your hand on your wallet.

[4] **Problem 9.** Consider an electromagnetic wave of the form

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = B_0 \cos(kz - \omega t)\hat{\mathbf{y}}.$$

As usual, you may work in units where $c = 1$.

(a) What do Maxwell's equations imply about the relation between E_0 and B_0, and k and ω?

(b) Now consider a frame moving with velocity v along the $\hat{\mathbf{z}}$ direction. Show that the electromagnetic wave continues to have the same basic functional form for $\mathbf{E}'(z', t')$ and $\mathbf{B}'(z', t')$, but with new parameters E_0', B_0', k', and ω'. Using these results, show that the energy density of the wave is smaller by a factor of $(1 - v)/(1 + v)$.

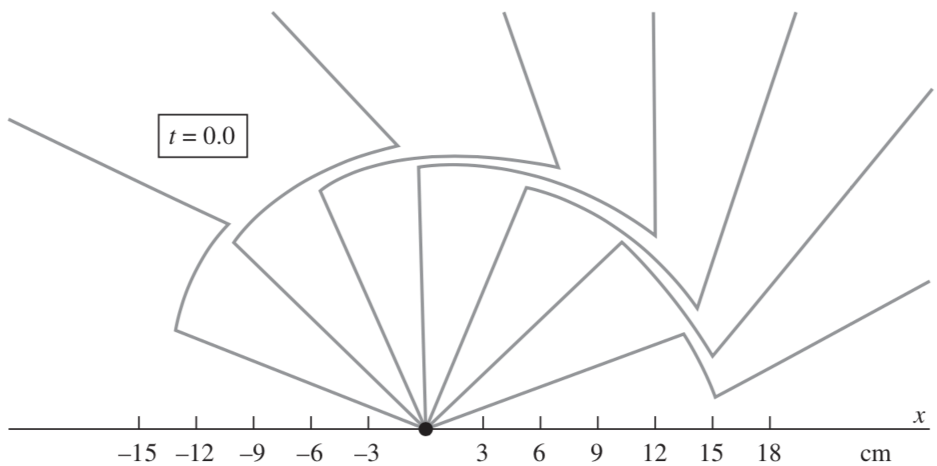
(c) The energy of a photon in an electromagnetic wave of angular frequency ω is $E = \hbar\omega$. Show that for a finite-sized electromagnetic wave, the initial and boosted frames agree on the number of photons. This was one of the hints Einstein used to conclude light was made of photons.

(d) Now consider another question Einstein pondered: what does the light wave look like if we try to "catch up" with it, taking $v \to c$? Is this consistent with the invariants of problem 8?

Idea 2

If a uniformly moving point charge suddenly stops moving, then the field outside a spherical shell, centered at the charge when it stopped moving, expanding at speed c, is precisely that calculated in problem 1. The same occurs if the point charge suddenly changes its velocity; information about the change only propagates at c.

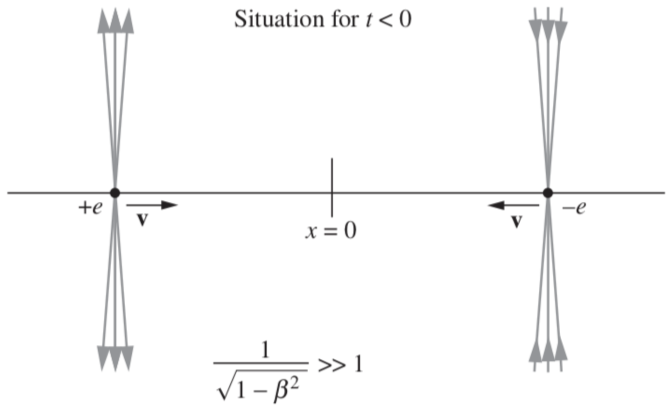
[1] **Problem 10** (Purcell 5.18). In the figure below, you see an electron at time $t = 0$ and the associated electric field at that instant.



(a) Describe what has been going on, as quantitatively as you can.

(b) Where was the electron at the time $t = -0.75\,\text{ns}$?

[2] **Problem 11** (Purcell 5.19). The figure below shows two highly relativistic particles with opposite charge approaching the origin.



They collide at the origin at time $t = 0$ and remain there as a neutral entity. Sketch the field lines at some time $t > 0$.

[3] **Problem 12.** Work through the derivation of the Larmor formula in Appendix H of Purcell.

[3] **Problem 13** (Purcell H.4)**.** The Larmor formula only applies to particles moving nonrelativistically. To get a result valid for faster particles, we can simply transform into an inertial frame F' where the particle is nonrelativistic, apply the Larmor formula, then transformed back to the lab frame.

(a) Consider an relativistic electron moving perpendicularly to a magnetic field $\mathbf{B}$. Defining the radiation power as $P_{\text{rad}} = dE/dt$, find P'_{rad}, the power in a frame instantaneously comoving with the electron.

(b) Argue that in this context, $P_{\text{rad}} = P'_{\text{rad}}$, and conclude that

$$P_{\text{rad}} = \frac{\gamma^2 v^2 e^4 B^2}{6\pi\epsilon_0 m^2 c^3}.$$

Thus, the power increases rapidly as $v \to c$. Incidentally, a "relativistic" way to write the general result is

$$P_{\text{rad}} = \frac{q^2}{6\epsilon_0 c^3}\left(\frac{1}{m}\frac{dp^\mu}{d\tau}\right)^2$$

which clearly reduces to the Larmor formula in the nonrelativistic limit.

(c) This radiation is also called synchrotron radiation. Qualitatively, how does its angular distribution differ from radiation from an accelerating nonrelativistic charge?

Remark: Gravitoelectromagnetism

As mentioned in **E1**, there's a close analogy between electrostatic fields, which are sourced by charge density ρ_e, and gravitational fields, which are sourced by energy density ρ. Therefore, if you apply the analogy and run the same arguments as in Purcell, you would expect there to be a "gravitomagnetic" field, which is sourced by momentum density $\mathbf{J} = \rho\mathbf{v}$. That's indeed correct! In the theory of gravitoelectromagnetism, the force on a point mass is

$$\mathbf{F} = m(\mathbf{E}_g + 4\mathbf{v} \times \mathbf{B}_g)$$

where the gravitoelectric and gravitomagnetic fields $\mathbf{E}_g$ and $\mathbf{B}_g$ satisfy

$$\nabla \cdot \mathbf{E}_g = 4\pi G\rho, \quad \nabla \cdot \mathbf{B}_g = 0, \quad \nabla \times \mathbf{E}_g = -\dot{\mathbf{B}}_g, \quad \nabla \times \mathbf{B}_g = 4\pi G\mathbf{J} + \dot{\mathbf{E}}_g.$$

From this you can draw some interesting conclusions. For example:

- Two masses moving parallel to each other will have an extra attraction due to the gravitomagnetic force.
- A rotating object will produce a gravitomagnetic field which can cause gyroscopes to precess; this is called the Lense–Thirring, or frame dragging effect, which has been measured by satellites such as Gravity Probe B. (There is also a significantly larger "geodetic" effect caused by the curvature of spacetime around the Earth, but this isn't captured within gravitoelectromagnetism.)
- A mass at rest, inside a cylinder which suddenly starts to rotate, will pick up a small angular velocity in the same direction due to the induction of a gravitoelectric field.
- Gravitational waves are generated by accelerating masses and carry energy, just like electromagnetic radiation.

Now you might be puzzled by two things: first, how does gravitoelectromagnetism relate to general relativity, and second, why is there an extra 4 in one of the equations above? Well, the truth is that Purcell's arguments don't really work for gravity. These arguments crucially depend on electric charge $Q = \int \rho_e \, d\mathbf{x}$ being Lorentz invariant, which in our more sophisticated language was necessary to ensure $j^\mu = (\rho_e, \rho_e \mathbf{v})$ is a four-vector. However, the total energy $E = \int \rho \, d\mathbf{x}$ is not Lorentz invariant – instead it's itself a component of a four-vector. Thus, $(\rho, \rho\mathbf{v})$ isn't a four-vector, so none of the arguments really work: the theory of gravitoelectromagnetism is just not Lorentz invariant at all.

Instead, gravitoelectromagnetism is properly derived as a limiting case of general relativity, valid when all the masses involved are moving slowly, $v \ll c$. The fact that general relativity is a theory of a rank 2 tensor field, the metric $g_{\mu\nu}$, is responsible for the extra factors of 2 above. Even though it's only approximately true, gravitoelectromagnetism is a very useful tool for analyzing precision tests of general relativity, since it's much easier to calculate with.

On the other hand, there's also a lot of nonsense written about it by people who don't understand it. For example, a lot of internet luminaries are certain that it can be used to replace dark matter, even though, using just the basic equations above, you can see that the gravitomagnetic force is $(v/c)^2$ times smaller than the usual gravitational force. That makes it about 10^6 times too small to explain the anomalous rotation of galaxies.

In fact, now is a good time to issue a warning. There's a concept called Lizardman's constant, which is the fact that in any survey, no matter how it's designed, about 3% of the answers will be complete nonsense. 3% of people will enthusiastically tell you that they were born on Mars, that the Moon landing was faked, or that the Earth is run by lizardmen. That's because there's an irreducible fraction of people that are mistaken, crazy, or just plain trolling.

The internet is a wonderful place to learn introductory physics, because it's relatively straightforward, so the sincere and competent outnumber the crazy. But as you go to more advanced topics, the fraction of people who know what's going on, and who have the time and energy to tell you, rapidly drops, while the 3% stays just as large. Now that you're at the end of this curriculum, you're also at the point where the *majority* of internet commentators on the topics you're learning are completely wrong. Fortunately, you're also learning what sources are good, and developing the knowledge needed to check things for yourself. As you continue learning tougher subjects, these skills will keep you on the right track.

2 Charges in Fields

[5] **Problem 14.** IPhO 1991, problem 2. A problem on the subtle relativistic "hidden momentum".

Idea 3: Scalar and Vector Potentials

In **E1**, we learned about the electric (or "scalar") potential $\phi(\mathbf{x})$, which obeys $\mathbf{E} = -\nabla\phi$. More generally, the scalar potential can depend on both space and time, as can the vector potential $\mathbf{A}(\mathbf{x}, t)$, and these two quantities yield the electric and magnetic fields by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Olympiad problems rarely require thinking about the vector potential, but it's essential to formulate the following idea.

Idea 4: Canonical Momentum

Noether's theorem yields a conserved quantity for every symmetry. If a system is symmetric under time translations, then its energy is conserved. Spatial translational symmetry yields momentum conservation, and rotational symmetry yields angular momentum conservation.

We won't prove Noether's theorem, but we'll illustrate it for a nonrelativistic particle of mass m and charge q. First, if ϕ and $\mathbf{A}$ are both time-independent, then the conserved energy is

$$E = \frac{1}{2}mv^2 + q\phi.$$

This is quite familiar. Note that $\mathbf{A}$ doesn't appear because in this case, the only role of $\mathbf{A}$ is to determine the magnetic field, which does no work.

As for space-translational symmetry, if ϕ and $\mathbf{A}$ are both space-independent, then the conserved momentum, called the "canonical" momentum, is

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}.$$

This is less familiar, so let's check it explicitly. Since we have assumed ϕ and $\mathbf{A}$ are both space-independent, we simply have $\mathbf{B} = 0$ and $\mathbf{E} = -\partial\mathbf{A}/\partial t$, so

$$\frac{d\mathbf{p}}{dt} = m\mathbf{a} - q\mathbf{E} = 0$$

as desired. This tells us that $q\mathbf{A}$ is like a "potential momentum", similar to how $q\phi$ is a potential energy. (Since the canonical momentum is such an important property, it is usually denoted by $\mathbf{p}$ whenever it's in play, while the Newtonian "mechanical"/"kinetic" momentum is demoted to $\boldsymbol{\pi} = m\mathbf{v}$.) However, in this case, the tool of canonical momentum doesn't tell us much we didn't already know.

Canonical momentum becomes useful in situations with only *partial* translational symmetry. For example, suppose that ϕ and $\mathbf{A}$ are both independent of x, but not y and z. Then the fields can be quite complicated, as can the particle's motion, but p_x will still be conserved!

In addition, the canonical momentum is the building block used for more complex situations. For example, if ϕ and $\mathbf{A}$ are both invariant under rotations about the z-axis, then

$$J_z = (\mathbf{r} \times \mathbf{p}) \cdot \hat{\mathbf{z}}$$

is conserved. Moreover, the adiabatic invariant of **M4** must be written in terms of the canonical momentum. For example, for periodic motion along the x-axis, it is

$$I = \oint p_x \, dx$$

while for periodic circular motion in the xy plane, it is

$$I = \oint J_z \, d\theta.$$

Finally, though all the following problems will assume the particles are nonrelativistic, the results above go through unchanged in relativistic mechanics provided that $\boldsymbol{\pi} = m\mathbf{v}$ is replaced with the relativistic momentum $\gamma m\mathbf{v}$.

[4] **Problem 15.** Let's check some of the statements made above in a simple case. Consider a situation with zero electric field and a constant uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) Show that this situation can be described with

$$\phi = 0, \quad \mathbf{A} = \frac{B}{2}(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}) = \frac{Br}{2}\hat{\boldsymbol{\theta}}$$

which is symmetric under translations along the z-axis and rotations about the z-axis.

(b) The symmetries of this problem tell us that p_z and J_z are conserved. Conservation of $p_z = \pi_z$ is obvious, because there are no forces in the z-direction. As for J_z, find an explicit expression for it, in terms of the ordinary angular momentum L_z, B, and r.

Back in **E4**, we encountered IPhO 1996, problem 2, which instructs the reader to solve a tricky problem by pulling out a magical conserved quantity; it's just J_z. Next, let's suppose the magnetic field's magnitude varies in time, corresponding to

$$\phi = 0, \quad \mathbf{A} = \frac{B(t)r}{2}\hat{\boldsymbol{\theta}}.$$

Of course, the changing magnetic field is accompanied by an induced electric field, as $\mathbf{E} = -\partial\mathbf{A}/\partial t$.

(c) Suppose that B is initially equal to B_0, and the particle is orbiting in a circle of radius r_0 about the z-axis. The field is *slowly* changed to B_1. What is the new radius r_1 of the orbit?

(d) What if instead the field is very quickly changed to B_1?

(e) Even more generally, let's suppose the field is $\mathbf{B} = B(r,t)\,\hat{\mathbf{z}}$, which can depend on both time and the distance r to the z-axis. Assuming the field changes slowly, find a compact expression for J_z in terms of r, the value of B at the particle's radius, and B_{av}, the average value of B within the circle formed by the particle. This will immediately yield the answer to the "betatron" example in **E4**.

[2] **Problem 16.** Here's another quick application of the conserved J_z identified in problem 15. As discussed in **E8**, electron orbits can be modified in a magnetic field, leading to diamagnetism. Many textbooks try to motivate this by considering forces on classical electrons, but such arguments don't actually work: it can be shown that for classical systems in thermal equilibrium, diamagnetic effects always cancel out. A legitimate derivation requires some quantum mechanics.

In **X1**, we discussed how the electron orbits in a hydrogen atom have $L_z = n\hbar$. Let's suppose the states $n = \pm 1$ are occupied, corresponding to electrons of charge q and mass m performing a circular orbit of radius r in opposite directions. The magnetic moments associated with these orbits cancel. But in the presence of a uniform magnetic field B, perpendicular to the plane of the orbit, the quantization condition becomes $J_z = n\hbar$, and the magnetic moments no longer cancel.

Find an approximate expression for the net magnetic moment, in terms of q, m, r, and B. Assume B is small, so that the magnetic force is small compared to the electrostatic force.

[2] **Problem 17.** Let's consider one more simple application of canonical momentum. Suppose a point charge of mass m and charge q experiences the uniform constant fields $\mathbf{E} = E\hat{\mathbf{x}}$ and $\mathbf{B} = B\hat{\mathbf{y}}$.

(a) Write a corresponding ϕ and $\mathbf{A}$ which are independent of y and z.

(b) What are the associated conserved quantities?

By now, we've covered most of the applications of canonical momentum for point charges. When it shows up on modern Olympiad problems, there's generally a twist. For instance, we can replace the point charge with an electric dipole, as illustrated in the following three tough questions.

[5] **Problem 18.** 🕓 APhO 2001, problem 2.

[5] **Problem 19.** 🕓 EuPhO 2022, problem 3.

[5] **Problem 20.** Physics Cup 2021, problem 1. This one requires more electromagnetism background.

[5] **Problem 21.** 🕓 GPhO 2017, problem 3. A problem on a "shock wave" hitting an electron. Don't be intimidated by the language; you don't need to know anything about shock wave physics to do this question.

3 Gravitational Fields

Idea 5

In classical mechanics, you've seen that a uniform gravitational field behaves a lot like the fictitious force due to a uniform acceleration. The equivalence principle states that the two behave exactly identically, in all possible contexts; it was one of the key ideas that led to the development of general relativity.

[4] **Problem 22.** In this problem, we given one of the classic justifications for gravitational redshift, the fact that photons redshift when moving against a gravitational field. Suppose that point B is a height h above point A, in a gravitational field g. A set of electrons and positrons with total rest mass M are converted into photons of frequency f at point A. The photons fly upward to point B, where they are converted back into electrons and positrons. Assume throughout that g is small.

(a) Find the total mass M' at point B.

(b) Find the frequency f' of the photons measured at point B.

(c) Since the frequencies of photons can be used as a clock, the result of part (b) shows that gravitational fields cause time dilation, which applies to everything, not just photons. Show that your result in part (b) is equivalent to the statement that times are dilated by a factor of $1 + \phi/c^2$, where ϕ is the gravitational potential and $\phi/c^2 \ll 1$.

We should also be able to understand part (b) using the equivalence principle. To confirm this, suppose that two observers C and D begin at rest, with D a distance h to the right of C. At a certain moment, both observers begin accelerating to the right with a small acceleration a.

(d) If C emits light of frequency f (in C's rest frame), show that D observes light of frequency f', where f' matches your answer to part (b).

(e) The predicted frequency shift was observed in the 1959 Pound–Rebka experiment, where gamma rays were transmitted from the top to the bottom of a tower of height $h = 22.5\,\text{m}$. What is the fractional change in energy of the photons?

(f) Gamma ray photons of energy $14\,\text{keV}$ were used in the Pound–Rebka experiment. According to the energy-time uncertainty principle, what is the minimum time needed to detect the effect?

Remark

You might be a little worried that the result of part (c) above does not seem to be invariant under a large, constant shift of ϕ, even though in Newtonian mechanics we can always do this. In fact, in that case the same analysis is essentially valid, but the "extra" gravitational time dilation is canceled out by other effects, which unfortunately can't be explained without full general relativity. In other words, the analysis above is only valid when ϕ is small.

If you find this confusing, you're not alone. In 2018, there was some excitement as researchers claimed to explain a long-standing anomaly in particle physics, making a mistake precisely along these lines. (A rebuttal is given here.)

[3] **Problem 23.** In this problem we consider the effects of relativity on a clock on the surface of the Earth, which has mass M and radius R. It rotates about its axis in time T, as measured by an observer at infinity who is at rest relative to the center of the planet

(a) Consider a clock C that lies on the surface of the planet at a point on the equator. Compute the time measured by the clock C after a single rotation of the planet, incorporating both special relativity and gravitational time dilation. Which effect is bigger?

(b) Repeat part (a) for a clock C' on a satellite orbiting the planet, in a circular orbit a height h above the equator.

(c) Using the numbers $M = 5.97 \times 10^{24}\,\text{kg}$, $R = 6.4 \times 10^{6}\,\text{m}$, and $h = 2 \times 10^{7}\,\text{m}$, estimate the difference in time elapsed per day for the two clocks, counting only the effect of special relativity, or only the effect of gravitational time dilation.

This paper explains how the Global Positioning System accounts for both of these effects to work.

[5] **Problem 24.** APhO 2014, problem 3. Gravitational fields bend light; this problem is about the geometry of gravitational lensing. Print out the official answer sheets and record your answers on them.

[5] **Problem 25.** IPhO 1995, problem 1. This problem is about the applications of gravitational redshift, and also serves as a nice review of **R2**.

[3] **Problem 26.** IPhO 2023, problem 2, parts C.1 through C.4. A neat problem on how the Shapiro delay, a classic test of general relativity, can be used to measure the masses of neutron stars.

Remark: Visualizing Relativity

You've probably heard that in general relativity, gravity is explained by the curvature of spacetime. In other words, freely falling objects always move in straight lines through spacetime; they only look like they're accelerating downward because we are constantly being accelerated upward. This is nicely illustrated here and explained in greater detail in this paper.

There is a common analogy for this involving picturing space as a distorted rubber sheet. It's a very bad analogy, because things will only accelerate towards the valleys in the sheets if you have gravity pointing down the sheet. In other words, the analogy tries to explain gravity by assuming you have spatial curvature *and* gravity. This misses the beautiful key point of relativity, which is that the gravity can be explained by *spacetime* curvature alone.

The fact that freely falling objects move in straight lines means that an object sitting on the surface of the Earth is actually being constantly accelerated. But this leads to a common followup question: in this picture, the surfaces of America and India are constantly accelerated in opposite directions, so why doesn't the Earth tear itself apart? Indeed, in special relativity this would make no sense. It's only possible because of spacetime curvature.

This can be explained with a spatial curvature analogy. Consider two people walking east, side by side, with one just north of the equator and the other south. In order to stay a *constant* distance apart, the person walking on the north will constantly have to bear to the right, while the person walking on the south will have to bear to the left, because the Earth's surface is spatially curved. Similarly, in a situation with spacetime curvature, America and India need constant opposite accelerations to maintain the same distance.

There's a neat way to visualize this situation called the "river model", which was rediscovered and animated here. The basic idea is that we think of space as a river that is constantly flowing towards the center of the Earth. Observers in America and India constantly need to paddle in opposite directions against the river to stay in place. This is also a good way to think about the event horizon of a black hole, which is where the river starts to flow faster than light.

In this remark I've given three analogies about spacetime, so which of them is "correct"? None, really. The analogies don't tell us what spacetime is. They're just different ways of verbally describing what the equations of general relativity say. They each imperfectly describe some aspects of the equations, and fail to capture others. (Any simple analogy *must* fail to capture the content of a theory, because if it really were simpler and just as valid, then that analogy would be the theory instead!) There is no actual spacetime rubber or river; those are just stories we tell ourselves to make the mathematics more appealing to our animal-descended minds. Of course, philosophers debate over whether the attitude I've expressed in this paragraph is right. It's called "anti-realism", and I wrote about it here.

Relativity III: Fields

Relativity in electromagnetism is covered in chapter 5 of Purcell and then sprinkled in throughout the rest of the book, notably in sections 6.7 and 9.7, and appendix H. For a more advanced discussion, see chapter 3 of Schutz for tensors, and chapter 12 of Griffiths and chapters I-34, II-13, and II-25 through II-28 of the Feynman lectures for relativistic electromagnetism. For a brief taste of general relativity, see chapter 14 of Morin, and chapter II-42 of the Feynman lectures. For a great, accessible introduction to tests of general relativity, see *Was Einstein Right?* by Will. There is a total of **90** points.

1 Electromagnetic Field Transformations

Idea 1: Field Transformations

If the electromagnetic field is $(\mathbf{E}, \mathbf{B})$ in one reference frame, then in a reference frame moving with velocity $\mathbf{v}$ with respect to this frame, the components of the field parallel to $\mathbf{v}$ are

$$E'_{\parallel} = E_{\parallel}, \quad B'_{\parallel} = B_{\parallel}$$

while the components perpendicular are

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}), \quad \mathbf{B}'_{\perp} = \gamma \left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_{\perp} \right).$$

As alluded to in **R2**, this is the transformation rule for the components of a rank 2 antisymmetric tensor.

Under these transformations, Maxwell's equations remain true in all inertial frames, and the Lorentz force transforms properly as well. Furthermore, a Lorentz transformation does not change the total amount of charge in a system, where total charge is defined by Gauss's law via the electric flux through a surface containing the system.

Remark

There are many ways of deriving the field transformations. The tensor method alluded to above is the mathematically cleanest, but the conceptually clearest is to think about how some simple setups must Lorentz transform, if Maxwell's equations are to remain true. For example, boosting a capacitor increases the charge density on the plates because of length contraction, which is why $\mathbf{E}'_{\perp}$ contains $\gamma \mathbf{E}_{\perp}$. (Further examples are given in chapter 5 of Purcell, which is essential reading for this section.) Another method is to demand that the Lorentz force obeys the transformation of three-force derived in **R2**.

[4] **Problem 1.** Basic facts about the electric and magnetic fields of a moving charge.

(a) Show that the field of a point charge q at the origin moving with constant velocity v is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \frac{1 - v^2}{(1 - v^2 \sin^2 \theta)^{3/2}} \hat{\mathbf{r}}$$

in units where $c = 1$, and θ is the angle from $\mathbf{v}$. In particular, the field is still radial.

(b) Verify that the charge of this moving charge is still q. It may be useful to consult the integral table in appendix K of Purcell.

(c) Argue that the magnetic field of this point charge must be exactly

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}.$$

(d) Verify that the previous result is correct in nonrelativistic electromagnetism (i.e. using Coulomb's law and the Biot–Savart law).

The result of part (a), first found by Heaviside in 1888, implies that the field lines and equipotential surfaces of a moving charge contract by a factor of γ in the direction of motion. In fact, this was what inspired Lorentz and Fitzgerald to propose length contraction in the first place! Since it's very hard to measure the Coulomb field of a relativistic electron, this prediction was first directly verified in 2022.

Solution. For concreteness, let the charge be moving along the z direction, and let the primed frame be the rest frame of the charge.

(a) By rotational symmetry, it suffices to show this holds in the xz plane. Consider the point

$$(x, z) = (r \sin\theta, r\cos\theta)$$

at time $t = 0$. Lorentz transforming to the primed frame, this point corresponds to

$$(x', z') = (r\sin\theta, \gamma r\cos\theta).$$

By Coulomb's law, the electric field at that point is

$$E'_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{((x')^2 + (z')^2)^{3/2}}, \quad E'_z = \frac{q}{4\pi\epsilon_0} \frac{z'}{((x')^2 + (z')^2)^{3/2}}.$$

Applying the field transformations, the field in the original frame is

$$E_x = \gamma E_{x'} = \frac{q}{4\pi\epsilon_0} \frac{\gamma r \sin\theta}{r^3(\sin^2\theta + \gamma^2\cos^2\theta)^{3/2}}, \quad E_z = E'_z = \frac{q}{4\pi\epsilon_0} \frac{\gamma r\cos\theta}{r^3(\sin^2\theta + \gamma^2\cos^2\theta)^{3/2}}.$$

This is a radial field, as desired, and swapping the γ's for v's gives the desired result.

There's a simple explanation for why the field is radial. The fact that the field is radial in the charge's rest frame just means $z/x = E_z/E_x$. When we Lorentz transform, z is Lorentz contracted by a factor of γ, while E_x is enhanced by a factor of γ, so we still have the equality of ratios $z'/x' = E'_z/E'_x$. This should make it clear that the fact that the field stays radial here is a fortuitous coincidence, since x^μ and $(\mathbf{E}, \mathbf{B})$ transform totally differently in general.

(b) The electric flux through a unit sphere surrounding the charge must still be q/ϵ_0, so we need to show that

$$1 = \frac{1}{4\pi} \int \frac{1 - v^2}{(1 - v^2\sin^2\theta)^{3/2}}\, d\Omega.$$

Setting up spherical coordinates as usual, the $d\phi$ integral gives 2π, leaving

$$\frac{1 - v^2}{2} \int_0^\pi \frac{\sin\theta\, d\theta}{(1 - v^2\sin^2\theta)^{3/2}} = \frac{1 - v^2}{2} \left. \frac{-\cos\theta}{(1 - v^2)\sqrt{1 - v^2\sin^2\theta}} \right|_0^\pi$$

where we used equation (K.15) from Purcell. Plugging in the limits gives 1 as desired.

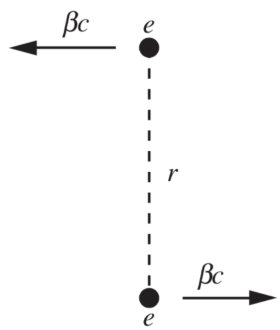
(c) We know $\mathbf{B}' = 0$, and using the field transformations immediately gives the result.

(d) Applying Coulomb's law, we have

$$\frac{1}{c^2}\mathbf{v} \times \mathbf{E} = (\mu_0\epsilon_0)\frac{qv}{4\pi\epsilon_0 r^2}\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{qv}{r^2}\hat{\boldsymbol{\theta}}$$

which is precisely what the Biot–Savart law predicts.

[3] **Problem 2** (Purcell 5.29)**.** Two protons are moving antiparallel to each other, along lines separated by a distance r, with the same speed v in the lab frame, as shown.



Consider the moment the protons are a distance r apart.

(a) Show that the three-force experienced by each proton due to the electric field of the other is

$$F = \frac{\gamma e^2}{4\pi\epsilon_0 r^2}.$$

(b) Compute the three-force experienced by one of the protons by transforming to its rest frame, computing the force there, then transforming back to the lab frame. In particular, show that this is not equal to the result of part (a).

(c) Show that the discrepancy is resolved if the magnetic three-force is also included.

Recall from **R2** that the Lorentz three-force is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. You will also have to use the three-force transformation laws you derived there.

Solution. (a) Using part (a) of problem 1 gives

$$eE = \frac{e^2}{4\pi\epsilon_0 r^2}\frac{1-v^2}{(1-v^2)^{3/2}} = \frac{\gamma e^2}{4\pi\epsilon_0 r^2}$$

as desired.

(b) In the rest frame of one proton, we have

$$F' = eE' = \frac{e^2}{4\pi\epsilon_0 r^2}\frac{1-v'^2}{(1-v'^2)^{3/2}}, \quad v' = \frac{2v}{1+v^2}.$$

Simplifying a bit yields

$$F' = \frac{e^2}{4\pi\epsilon_0 r^2}\frac{1+v^2}{1-v^2}.$$

Using our result from **R2** that transverse forces are redshifted,

$$F = \frac{F'}{\gamma} = \frac{\gamma e^2}{4\pi\epsilon_0 r^2}(1 + v^2).$$

This is more than the result we found in part (a).

(c) The extra three-force due to the magnetic field adds to the electric force,

$$evB = ev^2 E = \frac{\gamma e^2}{4\pi\epsilon_0 r^2}\, v^2$$

where we used part (c) of problem 1. This is exactly the missing piece.

[3] **Problem 3** (Purcell 5.24)**.** In the rest frame of a particle with charge q, another particle with charge q is approaching with relativistic velocity $\mathbf{v}$. Assume that both particles are extremely massive, and hence their velocities are nearly constant. The second particle passes a minimum distance b from the first.

(a) Show that the impulse acquired by each particle is perpendicular to $\mathbf{v}$ with magnitude $q^2/2\pi\epsilon_0 vb$. (Hint: you can avoid doing a nasty integral by using Gauss's law.)

(b) If the particles have mass m, roughly how large does m have to be for the above result to be a good approximation?

Solution. (a) Let the stationary particle be at the origin, and let the velocity of the moving particle be along the x-axis. Then the impulse experienced by the moving particle is

$$J = \int F\, dt = \frac{q}{v}\int_{-\infty}^{\infty} E_z(x, b)\, dx.$$

On the other hand, suppose we consider the electric flux through an infinite cylinder of radius b, oriented along the x-axis. Then

$$\Phi = \int \mathbf{E} \cdot d\mathbf{S} = 2\pi b \int_{-\infty}^{\infty} E_z(x, b)\, dx.$$

Therefore, we conclude

$$J = \frac{q}{v}\frac{\Phi}{2\pi b} = \frac{q^2}{2\pi\epsilon_0 vb}.$$

The great thing about this derivation is that exactly the same reasoning applies to the impulse experienced by the stationary charge. It can be written as a similar integral, except that $E_z(x, b)$ is now the electric field of a moving charge. But the impulse is the exactly the same in magnitude because the Gauss's law argument still works. That's good to know, as it ensures momentum is conserved.

(b) First off, we've ignored magnetic forces, even though the particles will pick up transverse velocity and hence begin to feel them. Since magnetic forces are small compared to electric forces when the (tranverse) speeds of the charges are nonrelativistic, we should require

$$J \ll mc$$

to be safe. Now, given this assumption, we can focus on the electric forces. Here we have assumed the charges don't move transversely during the whole process. The final transverse velocity is of order J/m, and the total time the interaction takes is of order b/v, so we need

$$\frac{J}{m}\frac{b}{v} \ll b$$

which simplifies to

$$J \ll mv.$$

Since this is strictly stronger than the other condition, this is the only one we really need. In other words, this kind of calculation only works if the transverse speed J/m the charges pick up is small compared to the original speed, i.e. if the angular deflection is small.

[3] **Problem 4.** USAPhO 2014, problem B2. This isn't the clearest of problems, but it introduces and justifies the Galilean field transformations we first saw in **E4**.

[3] **Problem 5** (Purcell 5.30)**.** Consider an infinite wire oriented along $\hat{\mathbf{x}}$ with linear charge density λ and current I. Show that under a Lorentz boost along $\hat{\mathbf{x}}$, (λ, I) transforms like (ct, x).

Solution. This can get kind of complicated if you think about a completely general set of charges in the wire. On the other hand, we already know that the way electromagnetic fields transform doesn't depend on what makes the fields, and we know that Maxwell's equations relate the fields to the charge density and current. Thus, the transformation of (λ, I) shouldn't depend on precisely what's responsible for the λ and I, so we can take a concrete choice that's easy to work with.

Specifically, let's suppose the wire is built entirely out of point charges of linear number density n and charge q moving with velocity $u\hat{\mathbf{x}}$. Then we have

$$\lambda = nq, \quad I = nqu.$$

Now we move to a frame moving to the right with speed v. In this frame, the charges are moving with speed

$$u' = \frac{u - v}{1 - uv}$$

and, by applying length contraction, the linear number density is

$$n' = \frac{\gamma_{u'}}{\gamma_u} n = \sqrt{\frac{1 - u^2}{1 - u'^2}}\, n = (1 - uv)\gamma_v n.$$

Putting things together, we have

$$\lambda' = n'q = (1 - uv)\gamma_v nq, \quad I' = n'qu' = (u - v)\gamma_v nq.$$

Writing this in terms of the unprimed quantities, we conclude

$$\lambda' = \gamma_v(\lambda - vI), \quad I' = \gamma_v(I - v\lambda)$$

which is precisely the desired result.

[2] **Problem 6** (Purcell 6.22)**.** A neutral wire carries current I. A stationary charge q is nearby; the Lorentz force on this charge is zero. Verify this remains true in a frame moving parallel to the wire with velocity $\mathbf{v}$, by using the Lorentz transformations of the fields.

Solution. Applying the transformations of problem 5, in that frame we have

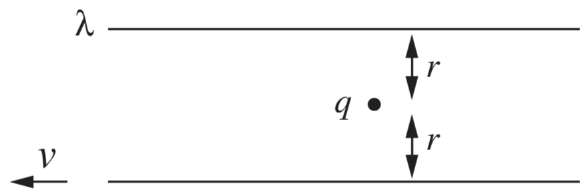
$$\lambda' = -\gamma_v v I, \quad I' = \gamma_v I.$$

In other words, the wire picks up a net charge. The electric and magnetic forces are

$$qE = \frac{\gamma_v v I q}{2\pi r \epsilon_0}, \quad qvB = \frac{\mu_0 \gamma_v v I q}{2\pi r}$$

which balance because $\mu_0 \epsilon_0 = 1$, in our units where $c = 1$.

[3] **Problem 7** (Purcell 6.69)**.** Two very long sticks each have uniform linear proper charge density λ. One stick is stationary in the lab frame, while the other moves to the left with speed v, as shown.



They are $2r$ apart, and a stationary point charge q lies midway between them. Find the Lorentz three-force on the charge in the lab frame, and also in the frame of the bottom stick, and verify the forces relate properly.

Solution. Let upward-pointing forces be denoted with a positive sign. In the lab frame, we only have an electric force. The charge on the bottom stick is length contracted, so

$$F = q\left(\frac{\gamma\lambda}{2\pi\epsilon_0 r} - \frac{\lambda}{2\pi\epsilon_0 r}\right) = \frac{q\lambda}{2\pi\epsilon_0 r}(\gamma - 1).$$

In the frame of the bottom stick, the bottom stick has charge density λ, while the top stick has

$$\lambda' = \gamma\lambda, \quad I' = \gamma v \lambda$$

where I' is directed to the right. Now the charge experiences both an electric and a magnetic force. The electric force is

$$F'_E = -\frac{q\lambda}{2\pi\epsilon_0 r}(\gamma - 1)$$

by the same logic as in the lab frame. The magnetic force is

$$F'_B = qvB = \frac{\mu_0 \gamma v^2 q\lambda}{2\pi r} = \frac{q\lambda}{2\pi\epsilon_0 r}(\gamma v^2).$$

The sum of the two is

$$F' = \frac{q\lambda}{2\pi\epsilon_0 r}(\gamma v^2 - \gamma + 1) = \frac{q\lambda}{2\pi\epsilon_0 r}\left(1 - \frac{1}{\gamma}\right) = \frac{F}{\gamma}$$

exactly as expected.

Remark: Are Wires Neutral?

The classic example in problem 6 starts by assuming the wire is neutral is the lab frame. But in problem 5, you showed that if a current-carrying wire is neutral is one reference frame, then it's *not* neutral in other reference frames. So how do we know which frame a *real* wire is neutral in? Is it the lab frame, the frame where the current vanishes, or something else?

It actually depends on the details, so for concreteness, let's consider two very long, parallel wires, connected at one end by a battery and at the other end by a resistor. Before current starts flowing, the whole system is neutral. So if the wires picked up a net charge density, the battery would have to have a large compensating charge, which would make it blow up. Or, to say it another way, if a net charge appears in the rest of the system, it pulls a compensating charge out of the battery, so the battery keeps the wires net neutral. You can then show that in a boosted frame, the wires stay net neutral, as you'd expected.

But this argument only shows that the wires have *opposite* charge densities $\pm\lambda$ in the lab frame. Can we show that $\lambda = 0$? Actually, we can't, because it's not true! As briefly discussed in a problem in **E2**, wires in circuits do carry charges in the lab frame, even if everything is ideal. One simple way to see this here is to note that the wires are at different electrical potentials. That's only possible if there's an electric field between them, which is created by the charge densities carried by the wires.

[3] **Problem 8.** The vectors $\mathbf{E}$ and $\mathbf{B}$ cannot go into four-vectors, as they transform among each other, but rather fit together into an antisymmetric rank two tensor. As a result, there is a different set of associated invariant quantities.

(a) Show that under the relativistic field transformations, the quantities $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ are both invariant. (Hint: this can be done using vector notation, using $\mathbf{E}_\perp \cdot \mathbf{E}_\parallel = \mathbf{B}_\perp \cdot \mathbf{B}_\parallel = 0$.) These are the two basic invariants, out of which all other invariants can be constructed.

(b) Suppose that in an inertial frame, $\mathbf{E}$ is zero at a given point and $\mathbf{B}$ is nonzero. Is it possible to find an inertial frame where $\mathbf{B}$ is zero at that point?

(c) Recall from **E7** that, in units where $\epsilon_0 = \mu_0 = 1$, the energy density of the electromagnetic field is $\mathcal{E} = E^2/2 + B^2/2$, and the Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{B}$. Show that $\mathcal{E}^2 - |\mathbf{S}|^2$ is invariant. (Hint: don't use the field transformations for this part.)

Solution. (a) Setting $c = 1$ for convenience as usual, we have

$$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E}'_\perp \cdot \mathbf{B}'_\perp + E'_\parallel B'_\parallel = \gamma^2(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp) \cdot (\mathbf{B}_\perp - \mathbf{v} \times \mathbf{E}_\perp) + E_\parallel B_\parallel.$$

The cross terms vanish (try an explicit example if you don't see why), which leaves

$$\mathbf{E}' \cdot \mathbf{B}' = \gamma^2(1 - v^2)\mathbf{E}_\perp \cdot \mathbf{E}_\perp + E_\parallel B_\parallel = \mathbf{E} \cdot \mathbf{B}$$

as desired. As for the other quantity, we have

$$\begin{aligned}
E'^2 - B'^2 &= E'^2_\parallel + \mathbf{E}'_\perp \cdot \mathbf{E}'_\perp - B'^2_\parallel - \mathbf{B}'_\perp \cdot \mathbf{B}'_\perp \\
&= E^2_\parallel + \gamma^2(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp)^2 - B^2_\parallel - \gamma^2(\mathbf{B}_\perp - \mathbf{v} \times \mathbf{E}_\perp)^2 \\
&= E^2_\parallel + \gamma^2(\mathbf{E}_\perp \cdot \mathbf{E}_\perp + v^2 \mathbf{B}_\perp \cdot \mathbf{B}_\perp) - B^2_\parallel - \gamma^2(\mathbf{B}_\perp \cdot \mathbf{B}_\perp + v^2 \mathbf{E}_\perp \cdot \mathbf{E}_\perp) \\
&= E^2_\parallel + \mathbf{E}_\perp \cdot \mathbf{E}_\perp - B^2_\parallel - \mathbf{B}_\perp \cdot \mathbf{B}_\perp \\
&= E^2 - B^2
\end{aligned}$$

as desired, where in the third equality the cross-terms canceled.

(b) This is impossible, because then $E^2 - B^2$ would have different signs in the two frames.

(c) If we plugged in the field transformations, the algebra would get extremely messy. Instead, we use the hint that this invariant can be constructed out of the ones found in part (a), so

$$\begin{aligned}
\mathcal{E}^2 - |\mathbf{S}|^2 &= \left(\frac{1}{2}E^2 + \frac{1}{2}B^2\right)^2 - |\mathbf{E} \times \mathbf{B}|^2 \\
&= \left(\frac{1}{2}E^2 + \frac{1}{2}B^2\right)^2 + (\mathbf{E} \cdot \mathbf{B})^2 - E^2 B^2 \\
&= \left(\frac{1}{2}E^2 - \frac{1}{2}B^2\right)^2 + (\mathbf{E} \cdot \mathbf{B})^2.
\end{aligned}$$

This is constructed out of the invariants in part (a), so it is invariant as well.

Remark: Is Magnetism Real?

Purcell's electromagnetism textbook is exceptional because it shows that a force like magnetism must exist, if one believes Coulomb's law and relativity. The idea is simple. We know how forces transform between frames, and given some reasonable assumptions, can also deduce how electric fields transform between frames. If electric fields were all there were, then electric forces would have to transform just like three-forces, but they don't. So there must be some other force to make up the difference, and it turns out to be precisely the magnetic force. We saw an example of this in problem 2.

It is important not to misunderstand this beautiful idea. Many people, upon reading such arguments, believe that magnetism "doesn't exist" because it's "all just electric fields". Sometimes people even say that magnetic forces are a "mistake" caused by "forgetting about" relativistic corrections. This is all totally backwards. Sometimes time dilation in one frame can be explained in terms of length contraction in another, but that doesn't mean that length contraction doesn't exist, or is a mistake – it's perfectly real in that particular frame. (Furthermore, while you can always get rid of the magnetic *force* on one particle at one moment by going to that particle's rest frame, there are plenty of situations where you can't remove the magnetic field, as we saw in problem 8!)

The real lesson of relativity isn't that magnetic fields are a mistake, it's that electric and magnetic fields are as intertwined as space and time, as you can see from their transformation properties. Just as space and time combine into a four-vector, electric and magnetic fields combine, in an equal footing, into the electromagnetic field tensor.

Remark: Electromagnetism in Covariant Form

Problem 5 is a first step to showing that $J^\mu = (\rho, \mathbf{J})$ is a four-vector, where ρ is the charge density and $\mathbf{J}$ is the current density. Note that the continuity equation for charge, as mentioned in **T2**, can be simply written in four-vector notation as

$$\partial_\mu J^\mu = 0.$$

As another example, you can show that the four-current of a single charged particle q is $J^\mu = qu^\mu$. We can go even further and write the whole of electromagnetism in terms of four-vectors and tensors. Maxwell's equations can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

The invariant quantities found in problem 8 can be written in terms of the field strength tensor as $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi–Civita symbol. These are the only two ways to "contract all the indices" to get a scalar.

Remark: Elegant Notation

Sometimes people dislike the index notation above because of all the little Greek letters floating around. If you *only* want to deal with vectors, vector notation is often better. It hides all the indices, at the cost of requiring you to introduce special symbols like $\cdot$ and $\times$ to specify the vector operations you want to do. The reason we don't use a vector-like notation for tensors is because there are too many operations you can do with them (e.g. "contract the 3rd index of a rank 4 tensor with the 1st index of a rank 2 tensor") to define separate symbols for each one; indices are just more efficient. On the other hand, if you only work with totally antisymmetric tensors, then there are only a few possible operations, and one can use the elegant, index-free "differential form" notation. In this notation, Maxwell's equations are

$$d \star F = J$$

where d is called the exterior derivative, $\star$ is the Hodge dual, and the fact that the electromagnetic fields are derivatives of potentials is expressed as

$$F = dA.$$

So is this the *best*, *most true* formulation of Maxwell's equations? Well, as Feynman once pointed out, you can easily do better. For example, you can define the "unworldliness"

$$U = |\mathbf{F} - m\mathbf{a}|^2 + (\nabla \cdot \mathbf{E} - \rho/\epsilon_0)^2 + \ldots.$$

Then *all* physical laws can be expressed in terms of the amazingly simple equation

$$\boxed{\boxed{\boxed{U = 0.}}}$$

But this doesn't actually help, because to use the equation for anything, you need to plug in the definition of U, and then you're back to where you were before. In general, more elegant notation is often more brittle: it only works well in a smaller set of situations. (For example, with differential form notation, you just can't write down the stress-energy tensor of the electromagnetic field, because that's symmetric rather than antisymmetric.) Index notation is great because it works as long as indices are contracted in pairs, which holds as long as you're dealing with laws that are independent of coordinate system. In general, there's no need to be ideological about notation; it's just a tool, and we should use the best tool for each job. If anyone tells you that their preferred notation for vectors or tensors will revolutionize physics, keep your hand on your wallet.

[4] **Problem 9.** Consider an electromagnetic wave of the form

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = B_0 \cos(kz - \omega t)\hat{\mathbf{y}}.$$

As usual, you may work in units where $c = 1$.

(a) What do Maxwell's equations imply about the relation between E_0 and B_0, and k and ω?

(b) Now consider a frame moving with velocity v along the $\hat{\mathbf{z}}$ direction. Show that the electromagnetic wave continues to have the same basic functional form for $\mathbf{E}'(z',t')$ and $\mathbf{B}'(z',t')$, but with new parameters E_0', B_0', k', and ω'. Using these results, show that the energy density of the wave is smaller by a factor of $(1-v)/(1+v)$.

(c) The energy of a photon in an electromagnetic wave of angular frequency ω is $E = \hbar\omega$. Show that for a finite-sized electromagnetic wave, the initial and boosted frames agree on the number of photons. This was one of the hints Einstein used to conclude light was made of photons.

(d) Now consider another question Einstein pondered: what does the light wave look like if we try to "catch up" with it, taking $v \to c$? Is this consistent with the invariants of problem 8?

Solution. (a) From **E7**, we know that $E_0 = B_0$ and $k = \omega$.

(b) The electromagnetic field only has perpendicular components. Using the field transformations,

$$\mathbf{E}'(z',t') = \gamma(E_0 \cos(kz - \omega t) - vB_0 \cos(kz - \omega t))\hat{\mathbf{x}}.$$

In units where $c = 1$, we have $E_0 = B_0$ for an electromagnetic wave, so this simplifies to

$$\mathbf{E}'(z',t') = \gamma(1-v)E_0 \cos(kz - \omega t)\hat{\mathbf{x}}.$$

Repeating the reasoning for the magnetic field, we conclude

$$E_0' = \gamma(1-v)E_0, \quad B_0' = \gamma(1-v)B_0$$

which still obeys $E_0' = B_0'$ as expected. Thus, the energy density is reduced by a factor of

$$\gamma^2(1-v)^2 = \frac{1-v}{1+v}$$

as stated. To find k' and ω', we can simply apply the Lorentz transformations,

$$kz - \omega t = k(\gamma(z' + vt')) - \omega(\gamma(t' + vz')) = \gamma(k - \omega v)z' - \gamma(\omega - kv)t'.$$

This indicates that $\mathbf{E}'(z', t')$ is still a plane wave proportional to $\cos(k'z' - \omega' t')$, where

$$k' = \gamma(k - \omega v), \quad \omega' = \gamma(\omega - kv)$$

which is of course just the statement that $(\omega, \mathbf{k})$ is a four-vector, derived in **R1**. Using the fact that $\omega = k$, we conclude

$$k' = \omega' = \gamma(1 - v)\omega = \sqrt{\frac{1-v}{1+v}}\,\omega$$

which is of course just the usual Doppler shift.

(c) The number of photons is the ratio of the total energy in the wave to the energy of each photon. Since the energy of each photon is reduced by a factor of $\sqrt{(1-v)/(1+v)}$ in the boosted frame, we need to show that the total energy of the wave is reduced by the same factor. This results from the combination of two effects.

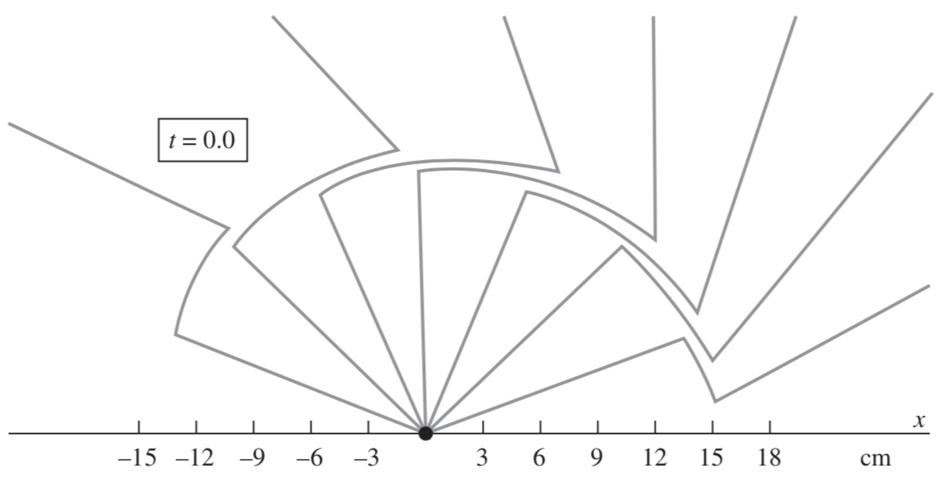
First, we know the energy density is reduced by the factor $(1-v)/(1+v)$. Second, the wavenumber is reduced by a factor of $\sqrt{(1-v)/(1+v)}$, which means the wavelength is increased by $\sqrt{(1+v)/(1-v)}$. Since the number of wavelengths contained in the wave is the same in every reference frame, this means the volume of the wave is increased by $\sqrt{(1+v)/(1-v)}$. Multiplying these factors gives the desired result.

(d) In this case we have $E_0', B_0', \omega', k' \to 0$, so the light wave disappears! That is, you can never "catch up" to a light wave. This result is completely compatible with the invariants from part (a) of problem 8, which both vanish for a plane electromagnetic wave. The invariant in part (c) vanishes as well, since $\mathcal{E} = |\mathbf{S}|$ for a plane wave.

Idea 2

If a uniformly moving point charge suddenly stops moving, then the field outside a spherical shell, centered at the charge when it stopped moving, expanding at speed c, is precisely that calculated in problem 1. The same occurs if the point charge suddenly changes its velocity; information about the change only propagates at c.

[1] **Problem 10** (Purcell 5.18)**.** In the figure below, you see an electron at time $t = 0$ and the associated electric field at that instant.



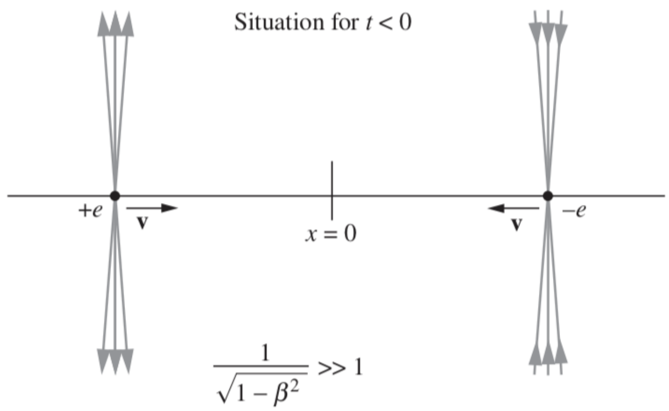
(a) Describe what has been going on, as quantitatively as you can.

(b) Where was the electron at the time $t = -0.75\,\text{ns}$?

Solution. (a) Clearly, the particle isn't moving at $t = 0$. Since there's a kink in the field lines at $r = 15\,\text{cm}$, it must have quickly stopped at $t = -r/c = -0.5\,\text{ns}$, since the speed of light is $c = 30\,\text{cm/ns}$. We also see that the field lines outside this shell are straight, and point towards the location $x = 12\,\text{cm}$. This implies that shortly before the charge stopped, it was moving with constant velocity $v = |x/t| = 24\,\text{cm/ns} = 0.8c$.

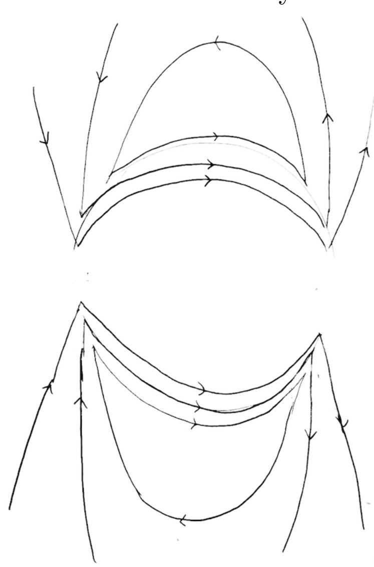
(b) By combining the results from (a), it must have been at $x = -(24\,\text{cm/ns})(0.25\,\text{ns}) = -6\,\text{cm}$.

[2] **Problem 11** (Purcell 5.19)**.** The figure below shows two highly relativistic particles with opposite charge approaching the origin.



They collide at the origin at time $t = 0$ and remain there as a neutral entity. Sketch the field lines at some time $t > 0$.

Solution. Before the charges collide, we have a kind of distorted dipole field. After they collide, we still have a dipole field at $r > ct$, with the positive charge on the right and the negative charge on the left. For $r < ct$ the field is zero, and at the shell $r \approx ct$ there is a thin shell that connects up the field lines. This transverse pulse of radiation steadily moves outward over time.



Some crackpots claim that particle annihilation is impossible because that would imply the electric field has to "instantly vanish", contradicting relativity. As you can see, that's not the case. The resulting pulse of radiation travels outward at the speed of light. The electric field only vanishes inside the expanding shell.

[3] **Problem 12.** Work through the derivation of the Larmor formula in Appendix H of Purcell.

[3] **Problem 13** (Purcell H.4)**.** The Larmor formula only applies to particles moving nonrelativistically. To get a result valid for faster particles, we can simply transform into an inertial frame F' where the particle is nonrelativistic, apply the Larmor formula, then transformed back to the lab frame.

(a) Consider an relativistic electron moving perpendicularly to a magnetic field $\mathbf{B}$. Defining the radiation power as $P_{\text{rad}} = dE/dt$, find P'_{rad}, the power in a frame instantaneously comoving with the electron.

(b) Argue that in this context, $P_{\text{rad}} = P'_{\text{rad}}$, and conclude that

$$P_{\text{rad}} = \frac{\gamma^2 v^2 e^4 B^2}{6\pi\epsilon_0 m^2 c^3}.$$

Thus, the power increases rapidly as $v \to c$. Incidentally, a "relativistic" way to write the general result is

$$P_{\text{rad}} = \frac{q^2}{6\epsilon_0 c^3}\left(\frac{1}{m}\frac{dp^\mu}{d\tau}\right)^2$$

which clearly reduces to the Larmor formula in the nonrelativistic limit.

(c) This radiation is also called synchrotron radiation. Qualitatively, how does its angular distribution differ from radiation from an accelerating nonrelativistic charge?

Solution. (a) In the frame comoving with the electron, it's not relativistic, so we can just apply the Larmor formula,

$$P'_{\text{rad}} = \frac{e^2 a'^2}{6\pi\epsilon_0 c^3}.$$

In this frame, the only force is the electric force, so

$$a' = \frac{eE'}{m} = \frac{e\gamma v B}{m}.$$

Putting it together, we conclude

$$P'_{\text{rad}} = \frac{e^4\gamma^2 v^2 B^2}{6\pi\epsilon_0 m^2 c^3}.$$

(b) We have $P'_{\text{rad}} = dE'/dt'$. Now, in the primed frame, the electron is just accelerating transversely, with no component along the unprimed frame's $\mathbf{v}$. Thus, when we Lorentz transform back to the unprimed frame, we simply get $dE = \gamma dE'$ and $dt = \gamma dt'$. The γ factors cancel out, giving the desired result.

(c) In the primed frame, the radiation power comes out with a wide angular distribution, but none of it comes out along the direction of motion of the charge, and most of it comes out roughly transverse to the motion. But when we boost back to the original frame, where the charge is moving very quickly, the radiation's direction gets a big component along the charge's direction of motion. Thus, almost all the radiation is "beamed" in a narrow cone along the charge's motion (as we saw in **R1**), though there still is zero radiation intensity *exactly* along the charge's direction.

Remark: Gravitoelectromagnetism

As mentioned in **E1**, there's a close analogy between electrostatic fields, which are sourced by charge density ρ_e, and gravitational fields, which are sourced by energy density ρ. Therefore, if you apply the analogy and run the same arguments as in Purcell, you would expect there to be a "gravitomagnetic" field, which is sourced by momentum density $\mathbf{J} = \rho\mathbf{v}$. That's indeed correct! In the theory of gravitoelectromagnetism, the force on a point mass is

$$\mathbf{F} = m(\mathbf{E}_g + 4\mathbf{v} \times \mathbf{B}_g)$$

where the gravitoelectric and gravitomagnetic fields $\mathbf{E}_g$ and $\mathbf{B}_g$ satisfy

$$\nabla \cdot \mathbf{E}_g = 4\pi G\rho, \quad \nabla \cdot \mathbf{B}_g = 0, \quad \nabla \times \mathbf{E}_g = -\dot{\mathbf{B}}_g, \quad \nabla \times \mathbf{B}_g = 4\pi G\mathbf{J} + \dot{\mathbf{E}}_g.$$

From this you can draw some interesting conclusions. For example:

- Two masses moving parallel to each other will have an extra attraction due to the gravitomagnetic force.
- A rotating object will produce a gravitomagnetic field which can cause gyroscopes to precess; this is called the Lense–Thirring, or frame dragging effect, which has been measured by satellites such as Gravity Probe B. (There is also a significantly larger "geodetic" effect caused by the curvature of spacetime around the Earth, but this isn't captured within gravitoelectromagnetism.)
- A mass at rest, inside a cylinder which suddenly starts to rotate, will pick up a small angular velocity in the same direction due to the induction of a gravitoelectric field.
- Gravitational waves are generated by accelerating masses and carry energy, just like electromagnetic radiation.

Now you might be puzzled by two things: first, how does gravitoelectromagnetism relate to general relativity, and second, why is there an extra 4 in one of the equations above? Well, the truth is that Purcell's arguments don't really work for gravity. These arguments crucially depend on electric charge $Q = \int \rho_e \, d\mathbf{x}$ being Lorentz invariant, which in our more sophisticated language was necessary to ensure $j^\mu = (\rho_e, \rho_e\mathbf{v})$ is a four-vector. However, the total energy $E = \int \rho \, d\mathbf{x}$ is not Lorentz invariant – instead it's itself a component of a four-vector. Thus, $(\rho, \rho\mathbf{v})$ isn't a four-vector, so none of the arguments really work: the theory of gravitoelectromagnetism is just not Lorentz invariant at all.

Instead, gravitoelectromagnetism is properly derived as a limiting case of general relativity, valid when all the masses involved are moving slowly, $v \ll c$. The fact that general relativity is a theory of a rank 2 tensor field, the metric $g_{\mu\nu}$, is responsible for the extra factors of 2 above. Even though it's only approximately true, gravitoelectromagnetism is a very useful tool for analyzing precision tests of general relativity, since it's much easier to calculate with.

On the other hand, there's also a lot of nonsense written about it by people who don't understand it. For example, a lot of internet luminaries are certain that it can be used to replace dark matter, even though, using just the basic equations above, you can see that the gravitomagnetic force is $(v/c)^2$ times smaller than the usual gravitational force. That makes it about 10^6 times too small to explain the anomalous rotation of galaxies.

In fact, now is a good time to issue a warning. There's a concept called Lizardman's constant, which is the fact that in any survey, no matter how it's designed, about 3% of the answers will be complete nonsense. 3% of people will enthusiastically tell you that they were born on Mars, that the Moon landing was faked, or that the Earth is run by lizardmen. That's because there's an irreducible fraction of people that are mistaken, crazy, or just plain trolling.

The internet is a wonderful place to learn introductory physics, because it's relatively straightforward, so the sincere and competent outnumber the crazy. But as you go to more advanced topics, the fraction of people who know what's going on, and who have the time and energy to tell you, rapidly drops, while the 3% stays just as large. Now that you're at the end of this curriculum, you're also at the point where the *majority* of internet commentators on the topics you're learning are completely wrong. Fortunately, you're also learning what sources are good, and developing the knowledge needed to check things for yourself. As you continue learning tougher subjects, these skills will keep you on the right track.

2 Charges in Fields

[5] **Problem 14.** IPhO 1991, problem 2. A problem on the subtle relativistic "hidden momentum".

Idea 3: Scalar and Vector Potentials

In **E1**, we learned about the electric (or "scalar") potential $\phi(\mathbf{x})$, which obeys $\mathbf{E} = -\nabla\phi$. More generally, the scalar potential can depend on both space and time, as can the vector potential $\mathbf{A}(\mathbf{x}, t)$, and these two quantities yield the electric and magnetic fields by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Olympiad problems rarely require thinking about the vector potential, but it's essential to formulate the following idea.

Idea 4: Canonical Momentum

Noether's theorem yields a conserved quantity for every symmetry. If a system is symmetric under time translations, then its energy is conserved. Spatial translational symmetry yields momentum conservation, and rotational symmetry yields angular momentum conservation.

We won't prove Noether's theorem, but we'll illustrate it for a nonrelativistic particle of mass m and charge q. First, if ϕ and $\mathbf{A}$ are both time-independent, then the conserved energy is

$$E = \frac{1}{2}mv^2 + q\phi.$$

This is quite familiar. Note that $\mathbf{A}$ doesn't appear because in this case, the only role of $\mathbf{A}$ is to determine the magnetic field, which does no work.

As for space-translational symmetry, if ϕ and $\mathbf{A}$ are both space-independent, then the conserved momentum, called the "canonical" momentum, is

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}.$$

This is less familiar, so let's check it explicitly. Since we have assumed ϕ and $\mathbf{A}$ are both space-independent, we simply have $\mathbf{B} = 0$ and $\mathbf{E} = -\partial \mathbf{A}/\partial t$, so

$$\frac{d\mathbf{p}}{dt} = m\mathbf{a} - q\mathbf{E} = 0$$

as desired. This tells us that $q\mathbf{A}$ is like a "potential momentum", similar to how $q\phi$ is a potential energy. (Since the canonical momentum is such an important property, it is usually denoted by $\mathbf{p}$ whenever it's in play, while the Newtonian "mechanical"/"kinetic" momentum is demoted to $\boldsymbol{\pi} = m\mathbf{v}$.) However, in this case, the tool of canonical momentum doesn't tell us much we didn't already know.

Canonical momentum becomes useful in situations with only *partial* translational symmetry. For example, suppose that ϕ and $\mathbf{A}$ are both independent of x, but not y and z. Then the fields can be quite complicated, as can the particle's motion, but p_x will still be conserved!

In addition, the canonical momentum is the building block used for more complex situations. For example, if ϕ and $\mathbf{A}$ are both invariant under rotations about the z-axis, then

$$J_z = (\mathbf{r} \times \mathbf{p}) \cdot \hat{\mathbf{z}}$$

is conserved. Moreover, the adiabatic invariant of **M4** must be written in terms of the canonical momentum. For example, for periodic motion along the x-axis, it is

$$I = \oint p_x \, dx$$

while for periodic circular motion in the xy plane, it is

$$I = \oint J_z \, d\theta.$$

Finally, though all the following problems will assume the particles are nonrelativistic, the results above go through unchanged in relativistic mechanics provided that $\boldsymbol{\pi} = m\mathbf{v}$ is replaced with the relativistic momentum $\gamma m\mathbf{v}$.

[4] **Problem 15.** Let's check some of the statements made above in a simple case. Consider a situation with zero electric field and a constant uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) Show that this situation can be described with

$$\phi = 0, \quad \mathbf{A} = \frac{B}{2}(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}) = \frac{Br}{2}\hat{\boldsymbol{\theta}}$$

which is symmetric under translations along the z-axis and rotations about the z-axis.

(b) The symmetries of this problem tell us that p_z and J_z are conserved. Conservation of $p_z = \pi_z$ is obvious, because there are no forces in the z-direction. As for J_z, find an explicit expression for it, in terms of the ordinary angular momentum L_z, B, and r.

Back in **E4**, we encountered IPhO 1996, problem 2, which instructs the reader to solve a tricky problem by pulling out a magical conserved quantity; it's just J_z. Next, let's suppose the magnetic field's magnitude varies in time, corresponding to

$$\phi = 0, \quad \mathbf{A} = \frac{B(t)r}{2}\,\hat{\boldsymbol{\theta}}.$$

Of course, the changing magnetic field is accompanied by an induced electric field, as $\mathbf{E} = -\partial\mathbf{A}/\partial t$.

(c) Suppose that B is initially equal to B_0, and the particle is orbiting in a circle of radius r_0 about the z-axis. The field is *slowly* changed to B_1. What is the new radius r_1 of the orbit?

(d) What if instead the field is very quickly changed to B_1?

(e) Even more generally, let's suppose the field is $\mathbf{B} = B(r,t)\,\hat{\mathbf{z}}$, which can depend on both time and the distance r to the z-axis. Assuming the field changes slowly, find a compact expression for J_z in terms of r, the value of B at the particle's radius, and B_{av}, the average value of B within the circle formed by the particle. This will immediately yield the answer to the "betatron" example in **E4**.

Solution. (a) This follows immediately from evaluating the curl of $\mathbf{A}$, either in Cartesian or cylindrical coordinates.

(b) The z-component of the canonical angular momentum is

$$J_z = (\mathbf{r} \times (m\mathbf{v} + q\mathbf{A})) \cdot \hat{\mathbf{z}} = L_z + \frac{qBr^2}{2}.$$

(c) Because the field is slowly changed, the particle remains in a circular orbit centered on the z-axis, so we can simply use the conservation of J_z to find the answer. It is

$$J_z = -mvr + \frac{qBr^2}{2}$$

where the minus sign is because the two contributions to J_z have opposite signs. On the other hand, for circular motion we have $mv^2/r = qvB$, which tells us that the two terms are simply proportional to each other. So in this case, we just have

$$J_z = -\frac{qBr^2}{2}$$

and the answer to the question is

$$r_1 = r_0\sqrt{B_0/B_1}.$$

We could also have gotten to this conclusion using the adiabatic theorem, but it's not any different, since here $I = 2\pi J_z$. And of course, you can also derive it directly using Newton's laws, in a manner similar to the betatron example in **E4**.

(d) When the field is quickly changed, the particle simply receives a sharp impulse. The final orbit is still a circle, but it won't be centered on the z-axis. Conservation of J_z during the impulse gives

$$m\,\Delta v = \frac{qr_0\,\Delta B}{2} = \frac{qr_0}{2}(B_1 - B_0)$$

which of course can also be deduced by Newton's laws. The final speed is

$$v' = \frac{qB_0r_0}{m} + \Delta v = \frac{qr_0}{m}\frac{B_0 + B_1}{2}$$

which is related to the final radius by $v' = qB_1r_1/m$. We thus conclude

$$r_1 = r_0\frac{B_0 + B_1}{2B_1}.$$

For a more challenging problem which uses similar ideas, see Physics Cup 2017, problem 3.

(e) In this case, we have

$$J_z = -mvr + qrA_\theta$$

where A_θ is independent of θ. To evaluate this, we note that

$$A_\theta = \frac{1}{2\pi}\int_0^{2\pi} A_\theta\,d\theta = \frac{1}{2\pi r}\oint_C \mathbf{A}\cdot d\mathbf{r} = \frac{\Phi_B(r)}{2\pi r} = \frac{r}{2}B_{\text{av}}(r).$$

where C is the particle's momentary circular orbit, and we used Stokes' theorem and then the definition of B_{av}. We still have $mvr = qB(r)r^2$, so

$$J_z = qr^2\left(-B + \frac{B_{\text{av}}}{2}\right).$$

If the quantity in parentheses doesn't vanish, then conservation of J_z implies that r has to change as the field is changed. As we found in **E4** using Newton's laws, r can stay the same if $B = B_{\text{av}}/2$, in which case J_z simply vanishes.

[2] **Problem 16.** Here's another quick application of the conserved J_z identified in problem 15. As discussed in **E8**, electron orbits can be modified in a magnetic field, leading to diamagnetism. Many textbooks try to motivate this by considering forces on classical electrons, but such arguments don't actually work: it can be shown that for classical systems in thermal equilibrium, diamagnetic effects always cancel out. A legitimate derivation requires some quantum mechanics.

In **X1**, we discussed how the electron orbits in a hydrogen atom have $L_z = n\hbar$. Let's suppose the states $n = \pm 1$ are occupied, corresponding to electrons of charge q and mass m performing a circular orbit of radius r in opposite directions. The magnetic moments associated with these orbits cancel. But in the presence of a uniform magnetic field B, perpendicular to the plane of the orbit, the quantization condition becomes $J_z = n\hbar$, and the magnetic moments no longer cancel.

Find an approximate expression for the net magnetic moment, in terms of q, m, r, and B. Assume B is small, so that the magnetic force is small compared to the electrostatic force.

Solution. This problem was inspired by this paper. Following problem 15, we note that for the $n = 1$ orbit,

$$J_z = L_z + \frac{1}{2}qBr'^2 = \hbar$$

where r' is the orbit radius when the magnetic field is on. In the absence of a magnetic field, we have $L_z = \hbar$, so the change in L_z is

$$\Delta L_z = -\frac{1}{2} q B r'^2.$$

Since the magnetic field is weak, $r' \approx r$, and since ΔL_z is proportional to B, which is already small, we can neglect the difference between r and r'. Finally, using a result from **E5**, we have

$$\Delta \mu_z = \frac{q}{2m} \Delta L_z = -\frac{q^2 B r^2}{4m}.$$

For the $n = -1$ orbit, the change in magnetic moment has the same sign, giving a total of

$$\mu_z = -\frac{q^2 B_z r^2}{2m}.$$

[2] **Problem 17.** Let's consider one more simple application of canonical momentum. Suppose a point charge of mass m and charge q experiences the uniform constant fields $\mathbf{E} = E\hat{\mathbf{x}}$ and $\mathbf{B} = B\hat{\mathbf{y}}$.

(a) Write a corresponding ϕ and $\mathbf{A}$ which are independent of y and z.

(b) What are the associated conserved quantities?

Solution. (a) The simplest possible answer is $\phi = -Ex$ and $\mathbf{A} = -Bx\hat{\mathbf{z}}$.

(b) The conserved quantities are $p_y = mv_y$ and $p_z = mv_z - qBx$. Conservation of p_y is trivial, as there are no forces in the y-direction. As for p_z, it's conserved because

$$\frac{dp_z}{dt} = ma_z - qBv_x$$

which is simply the z-component of the Lorentz force law, as we saw in the corresponding problem in **E4**. So in this case the canonical momentum doesn't tell you much new, though in a more subtle situation, such as when the particle is relativistic, it can be useful.

By now, we've covered most of the applications of canonical momentum for point charges. When it shows up on modern Olympiad problems, there's generally a twist. For instance, we can replace the point charge with an electric dipole, as illustrated in the following three tough questions.

[5] **Problem 18.** APhO 2001, problem 2.

[5] **Problem 19.** EuPhO 2022, problem 3.

Solution. See the official solutions here.

[5] **Problem 20.** Physics Cup 2021, problem 1. This one requires more electromagnetism background.

Solution. See the official solutions here.

[5] **Problem 21.** GPhO 2017, problem 3. A problem on a "shock wave" hitting an electron. Don't be intimidated by the language; you don't need to know anything about shock wave physics to do this question.

Solution. See the official solutions here.

3 Gravitational Fields

Idea 5

In classical mechanics, you've seen that a uniform gravitational field behaves a lot like the fictitious force due to a uniform acceleration. The equivalence principle states that the two behave exactly identically, in all possible contexts; it was one of the key ideas that led to the development of general relativity.

[4] **Problem 22.** In this problem, we given one of the classic justifications for gravitational redshift, the fact that photons redshift when moving against a gravitational field. Suppose that point B is a height h above point A, in a gravitational field g. A set of electrons and positrons with total rest mass M are converted into photons of frequency f at point A. The photons fly upward to point B, where they are converted back into electrons and positrons. Assume throughout that g is small.

(a) Find the total mass M' at point B.

(b) Find the frequency f' of the photons measured at point B.

(c) Since the frequencies of photons can be used as a clock, the result of part (b) shows that gravitational fields cause time dilation, which applies to everything, not just photons. Show that your result in part (b) is equivalent to the statement that times are dilated by a factor of $1 + \phi/c^2$, where ϕ is the gravitational potential and $\phi/c^2 \ll 1$.

We should also be able to understand part (b) using the equivalence principle. To confirm this, suppose that two observers C and D begin at rest, with D a distance h to the right of C. At a certain moment, both observers begin accelerating to the right with a small acceleration a.

(d) If C emits light of frequency f (in C's rest frame), show that D observes light of frequency f', where f' matches your answer to part (b).

(e) The predicted frequency shift was observed in the 1959 Pound–Rebka experiment, where gamma rays were transmitted from the top to the bottom of a tower of height $h = 22.5\,\text{m}$. What is the fractional change in energy of the photons?

(f) Gamma ray photons of energy $14\,\text{keV}$ were used in the Pound–Rebka experiment. According to the energy-time uncertainty principle, what is the minimum time needed to detect the effect?

Solution. (a) By conservation of energy,

$$Mc^2 = M'c^2 + M'gh$$

from which we conclude, using the fact that g is small, that

$$M' = M\left(1 - \frac{gh}{c^2}\right).$$

(b) Each photon has its energy reduced by a factor of $1 - gh/c^2$, and since $E = hf$,

$$f' = f\left(1 - \frac{gh}{c^2}\right).$$

(c) If we measure time through the frequency of light, then

$$\Delta t' = \Delta t \left(1 + \frac{gh}{c^2}\right)$$

again to lowest order in g. This is the desired result, since $\phi = gh$. It implies that higher clocks tick faster.

(d) Since the acceleration is small, it takes about a time h/c for the light to arrive at D. By this time, D has picked up a velocity ah/c, so by the Doppler shift,

$$f' = f\sqrt{\frac{1 - ah/c^2}{1 + ah/c^2}} = f'\left(1 - \frac{ah}{c^2}\right)$$

where we again work to lowest order in a.

(e) Plugging in the numbers, $gh/c^2 = 2.5 \times 10^{-15}$.

(f) The change in energy is $\Delta E = h\Delta f$, and the uncertainty principle says we need time

$$\Delta t \gtrsim \frac{h}{\Delta E} \sim \frac{h}{(2.5 \times 10^{-15})(14\,\text{keV})} = 10^{-4}\,\text{s}.$$

In the real experiment, Pound and Rebka used two identical samples of iron as the emitted and receiver, and vibrated one of them vertically at a few tens of Hz. Whenever the relative velocity was just enough to cancel out the gravitational redshift effect, absorption occurred. Since the Δt required was substantially lower than the period of the vibration, the vibration didn't mess up the experiment.

Remark

You might be a little worried that the result of part (c) above does not seem to be invariant under a large, constant shift of ϕ, even though in Newtonian mechanics we can always do this. In fact, in that case the same analysis is essentially valid, but the "extra" gravitational time dilation is canceled out by other effects, which unfortunately can't be explained without full general relativity. In other words, the analysis above is only valid when ϕ is small.

If you find this confusing, you're not alone. In 2018, there was some excitement as researchers claimed to explain a long-standing anomaly in particle physics, making a mistake precisely along these lines. (A rebuttal is given here.)

[3] **Problem 23.** In this problem we consider the effects of relativity on a clock on the surface of the Earth, which has mass M and radius R. It rotates about its axis in time T, as measured by an observer at infinity who is at rest relative to the center of the planet

(a) Consider a clock C that lies on the surface of the planet at a point on the equator. Compute the time measured by the clock C after a single rotation of the planet, incorporating both special relativity and gravitational time dilation. Which effect is bigger?

(b) Repeat part (a) for a clock C' on a satellite orbiting the planet, in a circular orbit a height h above the equator.

(c) Using the numbers $M = 5.97 \times 10^{24}\,\text{kg}$, $R = 6.4 \times 10^6\,\text{m}$, and $h = 2 \times 10^7\,\text{m}$, estimate the difference in time elapsed per day for the two clocks, counting only the effect of special relativity, or only the effect of gravitational time dilation.

This paper explains how the Global Positioning System accounts for both of these effects to work.

Solution. (a) The clock C always has speed $v = 2\pi R/T$, so if we only counted time dilaton,

$$T_C = T\sqrt{1-(2\pi R/cT)^2} \approx T\left(1 - \frac{1}{2}\left(\frac{2\pi R}{cT}\right)^2\right).$$

It is also at a lower gravitational potential than a clock at infinity, so counting only gravitational time dilation,

$$T_C = T\left(1 + \Delta\phi/c^2\right) = T\left(1 - GM/Rc^2\right).$$

Of course in reality both effects occur, and at leading order they just add, giving

$$T_C = T\sqrt{1-(2\pi R/cT)^2}\left(1 + \Delta\phi/c^2\right) \approx T\left(1 - \frac{1}{2}\left(\frac{2\pi R}{cT}\right)^2 - \frac{GM}{Rc^2}\right).$$

The two effects are equal when $v^2 = 2GM/R$, which describes escape velocity. Since the Earth is rotating a lot slower than that, the gravitational time dilation effect is much larger.

(b) We can just repeat the exercise, the only difference being that the clock C' has speed $v = \sqrt{GM/(R+h)}$, which implies

$$T_C = T\sqrt{1-(2\pi R/cT)^2} \approx T\left(1 - \frac{1}{2}\frac{GM}{(R+h)c^2} - \frac{GM}{(R+h)c^2}\right).$$

The gravitational time dilation effect is still larger, but only by a factor of 2.

(c) Plugging in the numbers, we have

$$\left(\frac{2\pi R}{cT}\right)^2 = 2.4 \times 10^{-12}, \quad \frac{GM}{Rc^2} = 7.0 \times 10^{-10}, \quad \frac{GM}{(R+h)c^2} = 1.7 \times 10^{-10}.$$

If we just consider the time dilation effect, the time difference per day is

$$\frac{T}{2}\left(\frac{GM}{(R+h)c^2} - \left(\frac{2\pi R}{cT}\right)^2\right) = 7\,\mu\text{s}.$$

If we consider just gravitational time dilation, the time difference per day is

$$T\left(\frac{GM}{(R+h)c^2} - \frac{GM}{Rc^2}\right) = -46\,\mu\text{s}.$$

Not only is the gravitational effect important, it's more important than the time dilation effect!

[5] **Problem 24.** APhO 2014, problem 3. Gravitational fields bend light; this problem is about the geometry of gravitational lensing. Print out the official answer sheets and record your answers on them.

[5] **Problem 25.** IPhO 1995, problem 1. This problem is about the applications of gravitational redshift, and also serves as a nice review of **R2**.

[3] **Problem 26.** IPhO 2023, problem 2, parts C.1 through C.4. A neat problem on how the Shapiro delay, a classic test of general relativity, can be used to measure the masses of neutron stars.

Remark: Visualizing Relativity

You've probably heard that in general relativity, gravity is explained by the curvature of spacetime. In other words, freely falling objects always move in straight lines through spacetime; they only look like they're accelerating downward because we are constantly being accelerated upward. This is nicely illustrated here and explained in greater detail in this paper.

There is a common analogy for this involving picturing space as a distorted rubber sheet. It's a very bad analogy, because things will only accelerate towards the valleys in the sheets if you have gravity pointing down the sheet. In other words, the analogy tries to explain gravity by assuming you have spatial curvature *and* gravity. This misses the beautiful key point of relativity, which is that the gravity can be explained by *spacetime* curvature alone.

The fact that freely falling objects move in straight lines means that an object sitting on the surface of the Earth is actually being constantly accelerated. But this leads to a common followup question: in this picture, the surfaces of America and India are constantly accelerated in opposite directions, so why doesn't the Earth tear itself apart? Indeed, in special relativity this would make no sense. It's only possible because of spacetime curvature.

This can be explained with a spatial curvature analogy. Consider two people walking east, side by side, with one just north of the equator and the other south. In order to stay a *constant* distance apart, the person walking on the north will constantly have to bear to the right, while the person walking on the south will have to bear to the left, because the Earth's surface is spatially curved. Similarly, in a situation with spacetime curvature, America and India need constant opposite accelerations to maintain the same distance.

There's a neat way to visualize this situation called the "river model", which was rediscovered and animated here. The basic idea is that we think of space as a river that is constantly flowing towards the center of the Earth. Observers in America and India constantly need to paddle in opposite directions against the river to stay in place. This is also a good way to think about the event horizon of a black hole, which is where the river starts to flow faster than light.

In this remark I've given three analogies about spacetime, so which of them is "correct"? None, really. The analogies don't tell us what spacetime is. They're just different ways of verbally describing what the equations of general relativity say. They each imperfectly describe some aspects of the equations, and fail to capture others. (Any simple analogy *must* fail to capture the content of a theory, because if it really were simpler and just as valid, then that analogy would be the theory instead!) There is no actual spacetime rubber or river; those are just stories we tell ourselves to make the mathematics more appealing to our animal-descended minds. Of course, philosophers debate over whether the attitude I've expressed in this paragraph is right. It's called "anti-realism", and I wrote about it here.