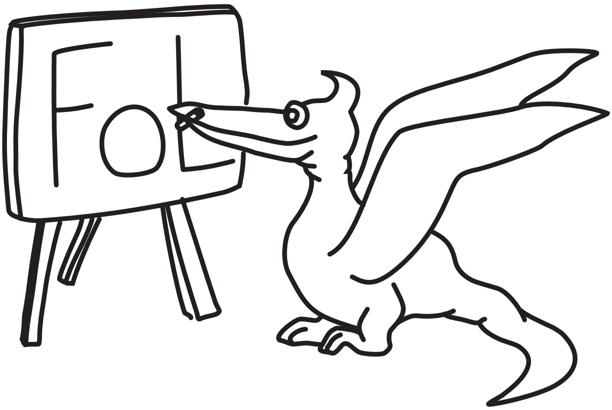
Solution of 2nd Online Physics Brawl



Problem FoL.1 ... curtain

Imagine a thin rigid homogeneous rod with mass $m = 1\,\mathrm{kg}$ and length $l = 2\,\mathrm{m}$, which is attached to a horizontal rail by a small massless ring at its end, so that it can slide without friction. As the ring accelerates at a constant rate $a = 5\,\mathrm{m\cdot s^{-2}}$, the rod makes a constant angle φ to the vertical. Find this angle given that the whole situation takes place on the Earth's surface, thus in the presence of the vertical acceleration due to gravity $g = 10\,\mathrm{m\cdot s^{-2}}$, neglecting the effects of air resistance. State your answer in radians. *Náry, looking at the mechanics of curtain.*

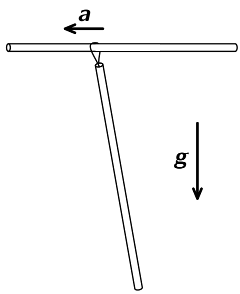


Fig. 1: To the problem 1

Let us describe the system in the frame of reference comoving with the ring. In this frame, the rod has only one degree of freedom, namely the angle φ, which was said to be constant. Hence the rod can be considered at rest with respect to this frame and we are left with determining the conditions for equilibrium. Apart from the force due to gravity, there is a ficticious force acting on every element of the rod, which is due to the accelerating frame of reference. Resolving the forces acting on the rod in the direction perpendicular to it (the parallel components are compensated for by the tension in the rod) and equating the corresponding moments, we find the following condition for the rod to be at equilibrium

$$\mathrm{tg}\,\varphi = \frac{a}{g}\,.$$

Substituting the numerical values for a and g, we obtain $\varphi \doteq 0.46\,\mathrm{rad}$.

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Problem FoL.2 ... hangmen

There are two balls of mass m and electric charge q (with the same sign) hanging from two slings of length l fixed in the same point. These balls are placed in air with density $\varrho_\mathrm{a} = 1.2\,\mathrm{kg\cdot m^{-3}}$. Due to the electric repulsion of the balls, the slings are forming an angle α. If we put the same aparate in the olive oil with relative permitivity $\varepsilon_\mathrm{r} = 3$ and density $\varrho_o = 900\,\mathrm{kg\cdot m^{3}}$, the angle will remain the same. Consider the permitivity of the air to be same as permitivity od vacuum. What is the density of the balls?
f(Aleš) was reading about little balls and decided to recompute the problem for nicer situation.

There are two forces affecting the balls, one electric and the other one gravitational. Electric force can be computed from Coulomb's law and the gravitational which can be computed from law of gravity, hence

$$F_\mathrm{e} = \frac{1}{4\pi\varepsilon_0\varepsilon_\mathrm{r}} \frac{q^2}{4\sin^2\left(\frac{\alpha}{2}\right) l^2}\,,$$

$$F_g = V\left(\varrho_\mathrm{b} - \varrho_\mathrm{e}\right) g\,,$$

where ϱ_e is the density of the environment.

Those two forces are perpendicular to each other. Gravitational force is vertical while the electric force is horizontal.

From geometry

$$\frac{F_\mathrm{e}}{F_g} = \mathrm{tg}\left(\frac{\alpha}{2}\right).$$

Let's mark the forces after inserting the aparate to the olive oil with primes. Condition for the constant angle gives us

$$\frac{F_\mathrm{e}}{F_g} = \frac{F'_\mathrm{e}}{F'_g}\,,$$

from here

$$\frac{1}{\varrho_\mathrm{b} - \varrho_\mathrm{a}} = \frac{1}{\left(\varrho_\mathrm{b} - \varrho_o\right)\varepsilon_\mathrm{r}}\,,$$

where we assumed that the permitivity of the air is the same as of vacuum.

Then

$$\varrho_\mathrm{b} = \frac{\varrho_\mathrm{o}\varepsilon_\mathrm{r} - \varrho_\mathrm{a}}{\varepsilon_\mathrm{r} - 1}\,.$$

Plugging in numbers, $\varrho_\mathrm{b} \doteq 1349,4\,\mathrm{kg{\cdot}m^{-3}}$.

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Problem FoL.3 ... blue rock

Center of mass of a mountaineer climbing on a rock is $h = 24\,\mathrm{m}$ above ground. The last belay (place where the climber's rope runs through a metallic circle attached to the rock) is at a height $h_0 = 20\,\mathrm{m}$. The climber slips and falls. How closest to the ground does he get during his fall? Young's modulus of the rope is $E = 100\,\mathrm{MPa}$, its radius $r = 0.5\,\mathrm{cm}$ and mass of the climber $m = 70\,\mathrm{kg}$. Neglect mass of the rope and all friction. Assume that the rope is attached to the climber in his center of mass. All distances are given with respect to a securing device which is attached to the ground and does not move during the fall. The local gravitational acceleration is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Honza thought carefully before climbing.*

Let's denote l the increase in the length of the rope during the fall. From the conservation of energy we have

$$mg\left(2\left(h - h_0\right) + l\right) = \frac{1}{2}\frac{E\pi r^2}{h} l^2\,.$$

Solving the quadratic equation we find l. The height h_f in which the climber stops falling is then $h_\mathrm{f} = h - 2(h - h_0) - l \doteq 7.7\,\mathrm{m}$.

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Problem FoL.4 ... gas block

There is a movable divider in a closed cylindrical vessel, separating it into two chambers. One of the chambers contains $25\,\mathrm{mg}$ *of* $\mathrm{N_2}$ *while the second one contains* $40\,\mathrm{mg}$ *of* He. *Assume that the equilibrium state was attained. What is the ratio of the lenghts of the chambers in the equilibrium state? Assume ideal-gas behaviour. Your answer should be less than 1.*

Janapka was going through her old exercise books.

The pressure in both chambers is the same, since we are in equilibrium state. Let us compute the amount of substance in both chambers. We need to look up molar masses M in the tables. For nitrogen, it is $28\,\mathrm{g{\cdot}mol^{-1}}$ while for helium, we have $4\,\mathrm{g{\cdot}mol^{-1}}$. We will plug in the numbers into the formula for the amount of substance, where $n = m/M$ and m is the mass of the gas in the chamber. Let us denote the area of the base of the bottle by S and the lenghts by L. Then, taking into account the ideal gas law, we can write

$$p = \frac{n_1 RT}{SL_1} = \frac{n_2 RT}{SL_2}\,,$$

whence $L_1/L_2 = n_1/n_2 \doteq 0.089$.

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Problem FoL.5 ... marble play

We want to galvanise (in copper sulphate solution) an iron sphere of mass $m_{\mathrm{Fe}} = 8\,\mathrm{kg}$ *and density* $\varrho_{\mathrm{Fe}} = 8\,\mathrm{g{\cdot}cm^{-3}}$. *We aim for a layer of copper* $\Delta r = 2\,\mathrm{mm}$ *thick. The density of copper is* $\varrho_{\mathrm{Cu}} = 9\,\mathrm{g{\cdot}cm^{-3}}$, *molar mass is* $M_{\mathrm{Cu}} = 63.5\,\mathrm{kg{\cdot}mol^{-1}}$ *and we apply constant current* $I = 0.5\,\mathrm{A}$. *How long does the process take? State your answer in days, ceiled (i.e. 3.238 days are 4 days).*

Kiki was remembering the chemistry olympiad.

Essential for the solution is finding out how much copper m_{Cu} is deposited onto the surface of the sphere during the electrolysis. We know the density of the sphere and its mass, hence we can compute its volume, which is $V_{\mathrm{Fe}} = m_{\mathrm{Fe}}/\varrho_{\mathrm{Fe}}$. The volume can also be expressed as $V = 4\pi r^3/3$, so we can infer the radius of the sphere r, which will grow to $R = r + \Delta r$, where $\Delta r = 0.002\,\mathrm{m}$. The volume of the resulting sphere will be $V = 4\pi R^3/3$. Copper and iron do not mix in the process, so we can deduce the volume of the copper layer, which is $V_{\mathrm{Cu}} = V - V_{\mathrm{Fe}}$. The mass of the layer can be computed as $m_{\mathrm{Cu}} = V_{\mathrm{Cu}}\varrho_{\mathrm{Cu}}$. We can input the known quantities to obtain

$$\begin{aligned}
m_{\mathrm{Cu}} &= V_{\mathrm{Cu}}\varrho_{\mathrm{Cu}}\,,\\
m_{\mathrm{Cu}} &= \left(V - V_{\mathrm{Fe}}\right)\varrho_{\mathrm{Cu}}\,,\\
m_{\mathrm{Cu}} &= \left(\frac{4}{3}\pi R^3 - \frac{m_{\mathrm{Fe}}}{\varrho_{\mathrm{Fe}}}\right)\varrho_{\mathrm{Cu}}\,,\\
m_{\mathrm{Cu}} &= \left(\frac{4}{3}\pi\left(\sqrt[3]{\frac{3m_{\mathrm{Fe}}}{4\pi\varrho_{\mathrm{Fe}}}} + \Delta r\right)^3 - \frac{m_{\mathrm{Fe}}}{\varrho_{\mathrm{Fe}}}\right)\varrho_{\mathrm{Cu}}\,.
\end{aligned}$$

We will use Faraday's laws of electrolysis to infer the time needed to produce such amount of copper. We write $m = AIt$, where A is the corresponding electrochemical equivalent $A = M_{\mathrm{Cu}}/(Fz)$, $M_{\mathrm{Cu}} = 0.0635\,\mathrm{kg{\cdot}mol^{-1}}$ is the molar mass of copper, $F = 96\,485\,\mathrm{C{\cdot}mol^{-1}}$ is the

Faraday's constant and $z = 2$ is the number of electrons released during the reaction when a copper cation with oxidation number II changes into a copper atom with oxidation number 0. The time elapsed can be computed as

$$t = \frac{m_\mathrm{C} u}{AI}\,,$$
$$t = \frac{m_\mathrm{C} u F z}{I M_\mathrm{C} u}\,.$$

Plugging in the numbers, we get $t \doteq 5\,462\,\mathrm{ks}$, which is equivalent to 62 days after ceiling.

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Problem FoL.6 ... springy one

We happen to have forgotten a spring with natural length $l = 0.5\,\mathrm{m}$ and a spring constant of $k = 5\,\mathrm{N{\cdot}m^{-1}}$ in the outer space. There is a mass $m_1 = 1\,\mathrm{kg}$ attached to one end of the spring while at the other end, there is a mass $m_2 = 3\,\mathrm{kg}$. Compute the period of small oscillations of the spring. *Paťo was going through his old exercise book.*

The centre of mass of an isolated system considered with respect to an inertial frame of reference cannot accelerate. Hence, in the centre of mass reference frame, the periods of oscillations of both masses must be the same. The centre of mass is at the distance

$$x = l\frac{m_2}{m_1 + m_2}$$

from m_1. Let us regard the spring as consisting of two springs connected in series at the centre of mass. The spring constants of these springs are inversely proportional to their respective lengths (the shorter spring is more rigid than the longer one), hence for the spring connected to m_1 we have

$$k_1 = k\frac{l}{x} = k\frac{m_1 + m_2}{m_2}\,.$$

The period of oscillations of the whole system is equal to the period of oscillations of this part of it, so

$$T = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}}\,.$$

Plugging in the numbers, we get $T \doteq 2.43\,\mathrm{s}$.

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Problem FoL.7 ... expandable balloon

A closed helium balloon is taking off from the Earth's surface, where the temperature and pressure are $300\,\mathrm{K}$ and $101\,\mathrm{kPa}$ respectively. Eventually, it will reach the point where the temperature and pressure are $258\,\mathrm{K}$ and $78\,\mathrm{kPa}$ respectively. Assuming that the balloon is of spherical shape initially with radius $10\,\mathrm{m}$, find the factor by which its radius will increase. The balloon is in thermodynamic equilibrium with its environment. Do not take into account the surface tension of the balloon. *Tomáš was dreaming about flying in a hot air balloon.*

We can use the ideal gas law to write

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}\,.$$

Hence the ratio V_2/V_1 can be expressed as

$$\frac{p_1 T_2}{p_2 T_1} \doteq 1.114\,.$$

The initial volume of the balloon is $V_1 = 4\pi r_1^3/3$. We will express the new volume as

$$V_2 = V_1 \frac{p_1 T_2}{p_2 T_1} = \frac{4\pi}{3} r_2^3 = \frac{4\pi}{3} r_1^3 \cdot \frac{p_1 T_2}{p_2 T_1}\,.$$

Hence the resulting ratio is

$$\sqrt[3]{\frac{p_1 T_2}{p_2 T_1}} \doteq 1.037\,.$$

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Problem FoL.8 ... resistant prism

The edges of a square pyramid are made out of wires which are conductively connected at all vertices. Compute the resistance across the opposite vertices on a diagonal of the base square, given that the resistance of one metre of the wire is $1\,\Omega$*, the height of the pyramid is* $\sqrt{7}\,\mathrm{m}$ *and the base length is* $2\,\mathrm{m}$*.* *Pikoš, while marking solutions.*

We want to find the resistance across the opposite vertices of the base of a square pyramid. By symmetry of the problem, we note that all the remaining vertices are at the same potential, including the top of the pyramid. Hence there is no current through the wires mutually connecting these vertices and we can discard them. We are left with three pairs of resistors connected in parallel, each pair containing identical resistors. As for the two pairs defining the base, the resistors have resistance $2\,\Omega$ while in the case of the third pair, the resistance of the constituent resistors is (by Pythagoras' theorem)

$$\sqrt{\left(\sqrt{7}\right)^2 + \left(\frac{\sqrt{2^2+2^2}}{2}\right)^2}\,\Omega = 3\,\Omega\,.$$

Thus the sought-after resistance is $1.5\,\Omega$.

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Problem FoL.9 ... satellite

Consider a planet and its natural satellite orbiting about their common centre of mass, where the motion takes place in a plane. The magnitude of the tangent velocity of the satellite, defined with respect to the centre of mass of the system, is $2.5\,\mathrm{km\cdot s^{-1}}$*. Find the ratio of the mass of the planet to the mass of the satellite in order for the centre of mass of the system to be located at the planet's surface, given that the orbits are circular. The mass of the planet is* $M_\mathrm{p} = 7.6\cdot 10^{24}\,\mathrm{kg}$*, it has a radius of* $R_\mathrm{p} = 7\,436\,\mathrm{km}$ *and the radius of the satellite is* $R_\mathrm{m} = 1\,943\,\mathrm{km}$

Nicola was thinking about the two-body problem.

Since we know the speed of the moon in its orbit, we can derive the distance h of the moon's center from the planet's surface. We know that $F_G = F_\mathrm{d}$, where F_G is the magnitude of the force due to gravity on the moon and F_d is the magnitude of the centripetal force making the moon follow the circular orbit. Hence we have

$$\kappa \frac{M_\mathrm{p} M_\mathrm{m}}{(R_\mathrm{p}+h)^2} = \frac{M_\mathrm{m} v^2}{h}\,,$$

where κ is the gravitational constant. Solving the above equation for h, we have

$$0 = h^2 + \left(2R_\mathrm{p} - \frac{\kappa M_\mathrm{p}}{v^2}\right) h + R_\mathrm{p}^2\,. \tag{1}$$

Denoting $x = h/R_\mathrm{p}$, we obtain

$$0 = x^2 + \left(2 - \frac{\kappa M_\mathrm{p}}{v^2 R_\mathrm{p}}\right) x + 1\,.$$

We need the centre of mass at the surface of the planet. By definition of the centre of mass, we write

$$R_\mathrm{p} = \frac{(h+R_\mathrm{p})M_\mathrm{m}}{M_\mathrm{m}+M_\mathrm{p}}\,.$$

Thus, multiplying through by $M_\mathrm{m} + M_\mathrm{p}$ and dividing by $M_\mathrm{m} R_\mathrm{p}$, we get

$$\frac{M_\mathrm{p}}{M_\mathrm{m}} + 1 = x + 1 \quad \Rightarrow \quad x = \frac{M_\mathrm{p}}{M_\mathrm{m}}\,.$$

By solving the equation (1), we get $x = M_\mathrm{p}/M_\mathrm{m} \doteq 8.79$.

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Problem FoL.10 ... a loop

Consider an electron-emitting source with an emitting speed of $v = 1.5\cdot 10^7\,\mathrm{m\cdot s^{-1}}$*. At a point* P*, the electrons enter a homogeneous magnetic field with a magnitude of* $B = 1\cdot 10^{-3}\,\mathrm{T}$*. The vector of electrons' velocity at* P *makes an angle* $\varphi = 15^\circ$ *to the magnetic field vector. Find the distance of* P *from the point where the electrons again (for the first time) cross the field line going through* P*.*

Zdeněk has teleported inside a monitor.

We need to resolve the velocity vector v into two components. One component is perpendicular to the field lines, $v_x = v\sin\varphi$, while the other one is parallel to them, $v_y = v\cos\varphi$. The Lorentz force is acting as a centripetal force, so

$$Qv_xB = \frac{mv_x^2}{r}\,,$$

whence we can express r as

$$r = \frac{mv_x}{QB}\,.$$

Then the period of the orbital motion can be obtained as

$$T = \frac{2\pi r}{v_x} = \frac{2\pi m}{QB}\,.$$

This is basically the time the electron needs to cross the field line once more. In the meantime, it will travel through a distance s in the direction parallel to the field lines

$$s = v_y\frac{2\pi m}{QB}\,.$$

Substituting for the charge and the mass of the electron, we get $s \doteq 0.52\,\mathrm{m}$.

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Problem FoL.11 ... ecologically passive

A passionate tree hugger (weighing only $m = 50\,\mathrm{kg}$) learned that the city council decided to chop down his favourite tree. He climbed onto the top of his homogeneous green friend believing that he would keep the tree killers away. However, the lumberjacks came and cut down the $h = 10\,\mathrm{m}$ tall tree weighing $M = 1\,\mathrm{t}$. What was the speed with which the tree hugger, initially resting at the tree top, hit the ground? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Terka, while climbing a tree.

The crucial one is the law of conservation of energy here. The potential energy of the tree hugger and the tree is

$$E_\mathrm{p} = g\left(mh + \frac{h}{2}M\right)\,.$$

The kinetic energy is given by the moment of inertia I of the system as

$$I = \frac{1}{2}I\omega^2\,,$$

where ω is the angular speed of the tree hitting the ground. The moment of inertia can be computed with aid of the parallel axis theorem as

$$E_\mathrm{k} = \frac{1}{12}Mh^2 + M\left(\frac{h}{2}\right)^2 + mh^2\,.$$

Conserving the total energy, we have

$$hg\left(2m + M\right) = \left(\frac{1}{3}M + m\right)h^2\omega^2\,.$$

Whence we obtain ω as well as $v = \omega h$:

$$v = \sqrt{\frac{3M + 6m}{M + 3m} gh}\,.$$

Plugging in the numbers, we get $v \doteq 16.8\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.12 ... nucleus on a diet

Imagine we have a nucleus with nucleon number $A = 36$, proton number $Z = 17$ and the mass of $m_\mathrm{k} = 5.99965 \cdot 10^{-26}\,\mathrm{kg}$ What would be the sum of bonding energies in nucleus in one mole of nuclei of this element? Provide your answer in TJ.

Kiki was bored during the lecture on anorganic chemistry.

The element contains $Z = 17$ protons and $N = A - Z = 19$ neutrons. Mass of proton as a standalone particle is $m_\mathrm{p} = 1.6725 \cdot 10^{-27}\,\mathrm{kg}$, mass of neutron is $m_\mathrm{n} = 1.6749 \cdot 10^{-27}\,\mathrm{kg}$. Just adding these two, the nucleus should weigth $m_\mathrm{t} = 17m_\mathrm{p} + 19m_\mathrm{n}$. The real mass of the nucleus is smaller. The difference is directly proportional to bonding energy

$$\delta m = m_\mathrm{t} - m_\mathrm{k}$$

according to $E = \delta m\, c^2$. We are interested in energy of one mole of the nuclei, $E_\mathrm{m} = N_A \cdot E$, where $N_A = 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}$ is Avogadro number. Plugging in the numbers, we get $E_\mathrm{m} \doteq 1.40 \cdot 10^{13}\,\mathrm{J} = 14.0\,\mathrm{TJ}$.

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Problem FoL.13 ... music of the locomotives

Two locomotives A and B are moving with velocities $v_A = 15\,\mathrm{m{\cdot}s^{-1}}$ in the right direction and $v_B = 30\,\mathrm{m{\cdot}s^{-1}}$ in the left direction, facing each other on the paralel railways. Locomotive A whistles on a frequence $200\,\mathrm{Hz}$. The speed of the sound is $340\,\mathrm{m{\cdot}s^{-1}}$. Let's assume that some of the sonic waves will be reflected from the locomotive B back to locomotive A. Which frequency will be heard by the engineer in the locomotive A? Assume that rightwards direction is positive and the environment is not moving. *Janapka was playing with trains.*

The problem makes use of Doppler effect. First, let's compute the frequency which will be heard by engineer in locomotive B. Locomotive A is moving to the right, in the positive direction, so locomotive B will have velocity $-|v_B|$. Environment is not moving. The velocity of the source is equal to the velocity of locomotive A, moving in the positive direction. The speed of sound is denoted as v. Frequency which is heard by engineer in locomotive B is

$$f_B = f_0 \frac{c + |v_B|}{c - |v_A|} \doteq 228\,\mathrm{Hz}\,.$$

This is the frequency sent from locomotive B to locomotive A. So let's apply Doppler's law once more to get the final frequency

$$f_A = f_B \frac{c + |v_A|}{c - |v_B|} \doteq 261\,\mathrm{Hz}\,.$$

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Problem FoL.14 ... capacitors in the circuit

Voltage in the circuit on the attached image is $10\,\mathrm{V}$ *and each capacitor has capacity* $10\,\mathrm{\mu F}$. *Compute the charge (in* $\mathrm{\mu C}$*) on the capacitor* C_1. *Dominika on a branch.*

We use the numbering of capacitors as in the image. Let's recall the rules for computing the final capacity if the capacitors are in series (C_s) or parallel (C_p)

$$\frac{1}{C_\mathrm{s}} = \frac{1}{C_a} + \frac{1}{C_b}\,, \quad C_\mathrm{p} = C_a + C_b\,.$$

Using this formula, we replace the capacitors $1, 2, 3$ with equivalent C_{123} and capacitors $1, 2, 3, 4$ with equivalent C_{1234}

$$C_{123} = \frac{3}{2}C, \qquad C_{1234} = \frac{3}{5}C.$$

Charge in the right branch of circuit (with capacitors $C_{1\ldots4}$) is $Q_{1234} = U \cdot C_{1234}$, which is the same as the charge of capacitor C_4 and also $C_{1\ldots3}$: $Q_{1234} = Q_3 = Q_{123}$. Voltage on $C_{1\ldots3}$ is $U_{123} = Q_{123}/C_{123}$. Capacitors C_1 and C_2 will have the same charge, $Q_1 = Q_2 = U_{123} \cdot C_{12} = U_{123} \cdot \frac{C}{2}$. Generally

$$Q_1 = \frac{1}{2}U_{123}C = \frac{1}{2}CU\frac{C_{1234}}{C_{123}} = \frac{1}{5}CU\,.$$

Plugging the numbers in, we get $Q_1 = 20\,\mathrm{\mu C}$.

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Problem FoL.15 ... glasses or contact lenses

Pepa needs glasses with focal length $f_\mathrm{b} = 50\,\mathrm{cm}$ *to clearly see his favourite crossword at a distance* $D = 25\,\mathrm{cm}$ *from his eye lens. The glasses are at* $d = 2\,\mathrm{cm}$ *from his eye lens which itself has focal length* $f_\mathrm{o} = 2\,\mathrm{cm}$. *What focal distance (in cetimetres) must Pepa's contact lenses – which are in direct contact with the eye – have so that Pepa still could see the crossword clearly without any change in the focal length of the eye lens? Treat all lenses as thin.*

Lukáš stared into crosswords.

First we calculate distance from the eye lens in which the image of the crossword is produced. Crossword is at a distance $d_\mathrm{b} = D - d$ from the glasses. This is less than f_b which means the image will be on the same side of the lens as the crossword. Image position with respect to the glasses is obtained from the lensmaker's equation

$$-s_\mathrm{b} = \frac{1}{\frac{1}{f_\mathrm{b}} - \frac{1}{D-d}}\,.$$

The image is then at a distance $s_\mathrm{o} = s_\mathrm{b} + d$ from the eye lens. From this directly follows final position of the image x:

$$x = \frac{1}{\frac{1}{f_\mathrm{o}} - \frac{1}{s_\mathrm{o}}} = \frac{1}{\frac{1}{f_\mathrm{o}} - \frac{f_\mathrm{b}-D+d}{d(f_\mathrm{b}-D+d)+f_\mathrm{b}(D-d)}} = \frac{1204}{575}\mathrm{cm}\,.$$

Focal distance f_c of the contact lens must fulfill

$$\frac{1}{f_\mathrm{c}}+\frac{1}{f_\mathrm{o}}=\frac{1}{x}+\frac{1}{D}\,,$$

which leads to

$$f_\mathrm{c}=\frac{1}{\frac{1}{D}-\frac{f_\mathrm{b}-D+d}{d(f_\mathrm{b}-D+d)+f_\mathrm{b}(D-d)}}\doteq 56.9\,\mathrm{cm}$$

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Problem FoL.16 ... superluminal electron

Assume the Bohr model of an ionized atom with fixed nucleus (with proton number Z) and only one electron. Which would be the lowest possible proton number of the atom in order to obtain superluminal velocity of the orbiting electron? Assume that the electron is in a ground state. Speed of light is $c = 299.8\cdot 10^6\,\mathrm{m\cdot s^{-1}}$, charge of the electron is $e = -1.6022\cdot 10^{-19}\,\mathrm{C}$, Coulomb's constant is $k_\mathrm{e} = 8.987\cdot 10^9\,\mathrm{N\cdot m^2\cdot C^{-2}}$ and the reduced Planck constant is $\hbar = 1.0546\cdot 10^{-34}\,\mathrm{J\cdot s}$. Use classical, not relativistic, physics. *Jakub wanted to destroy the world.*

Velocity of the electron in the ground state can be obtained directly from the Bohr model

$$v_\mathrm{e}=\frac{Ze^2k_\mathrm{e}}{\hbar}\,.$$

We want this velocity to be superluminal, hence

$$v_\mathrm{e}=\frac{Ze^2k_\mathrm{e}}{\hbar}>c\,.$$

Playing around with this algebracic expression, we get condition for Z

$$Z>\frac{c\hbar}{e^2k_\mathrm{e}}\doteq 137.05\,.$$

Number of protons is an integer, so the velocity of the electron crossed the superluminal barrier for $Z = 138$.

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Problem FoL.17 ... ecocar

Find the mass of a compressed gas which is equivalent to one litre of petrol with a heating value of $L = 30\,\mathrm{MJ\cdot l^{-1}}$, given that we use air as our working gas with an initial pressure of $p_0 = 200p_\mathrm{a}$, where $p_\mathrm{atm} = 10^5\,\mathrm{hPa}$ is the atmospheric pressure. Further assume that the density of air under atmospheric pressure is $\varrho_\mathrm{atm} = 1.3\,\mathrm{kg\cdot m^{-3}}$, that it exhibits ideal-gas behaviour and that the expansion process is carried out isothermally without any loss of energy.
Lukáš came up with this on the trip in the mountains.

Let us write the ideal gas law for the original, intermediate and atmospheric pressure

$$p_0V_0=nRT\,,\quad pV=nRT\quad p_\mathrm{atm}V_\mathrm{atm}=nRT\,.$$

The temperature and the amount of substance are constant. The work done by gas during the expansion process is

$$W = \int_{V_0}^{V_{\mathrm{atm}}} p\mathrm{d}V = \int_{V_0}^{V_{\mathrm{atm}}} p_0 V_0/V \mathrm{d}V = p_0 V_0 \ln \frac{V_{\mathrm{atm}}}{V_0} = p_0 V_0 \ln \frac{p_0}{p_{\mathrm{atm}}} \,.$$

The density of air is $\varrho = m/V_{\mathrm{atm}}$, for which we can write

$$p_{\mathrm{atm}} = \frac{\varrho_{\mathrm{atm}}}{M_{\mathrm{m}}} RT \,,$$

where M_{m} is molecular mass of the air. By the ideal gas law for the initial state, we can express $p_0 V_0$ as.

$$p_0 V_0 = \frac{m}{M_{\mathrm{m}}} RT = m \frac{p_{\mathrm{atm}}}{\varrho_{\mathrm{atm}}} \,.$$

The work has already been computed before, so let us assume that it is equal to the heating value of one litre of petrol

$$L V_{\mathrm{B}} = W = p_0 V_0 \ln \frac{p_0}{p_{\mathrm{atm}}} = m \frac{p_{\mathrm{atm}}}{\varrho_{\mathrm{atm}}} \ln \frac{p_0}{p_{\mathrm{atm}}} \,,$$

whence

$$m = \frac{L V_{\mathrm{B}} \varrho_{\mathrm{atm}}}{p_{\mathrm{atm}} \ln \frac{p_0}{p_{\mathrm{atm}}}} \doteq 73.6\,\mathrm{kg} \,.$$

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Problem FoL.18 ... swinging barrel

A cylindrical object with a radius of $r = 0.5\,\mathrm{m}$, height $l = 3\,\mathrm{m}$ and mass of two tonnes is floating on the water of density ϱ so that the axis of the cylinder remains vertical. Let us displace the cylinder from its equilibrium position vertically by $\Delta x = 1\,\mathrm{mm}$. Find the period of oscillations of the cylinder (in seconds). The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Pikoš's problem from the school days.

There is a force due to gravity $F_G = mg$ acting on the cylinder, where m is the mass and g is the acceleration due to gravity. The other force acting on the cylinder is the buoyant force

$$F_{\mathrm{v}} = V \varrho g \,,$$

where V is the volume of the submerged part of the cylinder and ϱ is the density of the liquid (water). At the equilibrium position, the resultant force acting on the cylinder is zero. Let us denote the height of the submerged part as x_0 (at the equilibrium position). Then $mg = \pi r^2 x_0 \varrho g$. If we displace the cylinder from the equilibrium position by x upwards, then the height of the submerged part will be $x_0 - x$ and the magnitude of the resultant force will be equal to

$$F = mg - \pi r^2 (x - x_0) \varrho g \,.$$

Substituting for x from the previous equation we obtain

$$F = -\pi r^2 x \varrho g \,,$$

thus the force acting on the cylinder is proportional to the displacement, hence we can see that it is a simple harmonic oscilator with the effective spring constant $\pi r^2 \varrho g$. Therefore, the sought-after period of oscillations is

$$T = 2\pi\sqrt{m/(\pi r^2 \varrho g)} \doteq 3.2\,\mathrm{s}\,.$$

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Problem FoL.19 ... capacitor duo

Find the charge in Coulombs on the capacitor C_2 if you know the following: At the beginning, the switch S_0 was off and the switch S_1 was on, as displayed on the picture. There was zero voltage on both capacitors. Then we switched S_0 on and waited until the current stopped flowing. Subsequently, we switched S_1 off and again waited for the circuit to come into a steady state. At the end, we measured the charge on the capacitor C_2. The voltage across the ideal voltage source (DC) is $U = 17\,\mathrm{V}$ and both capacitors have the same capacitance $C = 1\,\mu\mathrm{F}$.

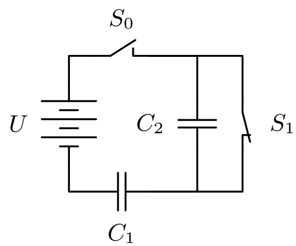

Fig. 2: To the problem 19

Náry felt like doing some electrotechnics.

This experiment represents a process in which we add an extra capacitor into a capacitor circuit in a steady state. This new element acts as a simple conductor because it is added at a moment when all differences in the electric potential are balanced. There is a zero voltage on the added capacitor and thus also a zero accumulated charge.

Validity of this statement can be seen from the second Kirchhoff's circuit law and the conservation of the electric charge.

$$\frac{Q+q}{C} + \frac{q}{C} = U\,, \tag{2}$$

where Q is the original charge on the positively charged plate of the capacitor C_1 and q is an extra charge on this plate after switching off S_1. This is then the charge lost by the plate of C_2 which is connected with the positively charged plate of C_1.

From the first phase of the process we know

$$\frac{Q}{C} = U\,, \tag{3}$$

which, combined with (2) gives $q = 0\,\mathrm{C}$.

Charge on the capacitor C_2 will be $0\,\mathrm{C}$.

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Problem FoL.20 ... heureka

Consider a cube with edge length $a = 1\,\mathrm{m}$ and density $\varrho_0 = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$ and a container with a liquid of density (at its level) ϱ_0. The density of the liquid increases linearly with depth, so $\varrho(h) = \varrho_0 + \alpha h$, where $\alpha = 25\,\mathrm{kg{\cdot}m^{-4}}$. How deep does the cube sink, given that the height of the level of the liquid does not change having immersed the cube in the liquid? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. State your answer in centimetres.

Terka was bathing in strange liquids.

Let us use the Archimedes' principle: a thin slab of the cube with a height of $\mathrm{d}x$ submerged at a depth of x is acted on by a force corresponding to the weight $ga^2(\varrho_0 + \alpha x)\mathrm{d}x$ of the liquid of the same volume as the slab. The total mass of the water replaced by the cube's body must be equal to the total mass of the cube, so

$$\int_0^H (\varrho_0 + \alpha x)\mathrm{d}x = a^3 \varrho_0\,,$$

which leads to a quadratic equation. This can be solved as

$$0 = \alpha H^2 + 2\varrho_0 H - 2a^3\varrho_0\,,$$

$$H = \frac{\sqrt{\varrho_0^2 + 2\alpha a^3 \varrho_0} - \varrho_0}{\alpha}\,.$$

Plugging in the numbers, we get $H \doteq 98.8\,\mathrm{cm}$.

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Problem FoL.21 ... ecology above all

Find the efficiency (energy returned on energy invested - ERoEI) of storing the energy in the form of compressed air. If there is surplus of electrical energy produced in solar power plants, we start to compress the air adiabatically so that we eventually achieve a compression ratio of $k = 10$. If we need the energy back, we let the air expand adiabatically to the original pressure. However, before we commenced the expansion process the air cooled down to the temperature before the compression took place. Assume that the air is a diatomic gas exhibiting ideal-gas properties. State your answer in terms of percentage.

Lukáš was listening to a programme about the cars running on compressed air.

Quantities indexed by 1 describe the initial state while the ones indexed by 2 describe the state after the adiabatic compression, 3 stands for the state before the adiabatic expansion and 4 is for the state after the expansion.

Let us derive the energy supplied to the gas during the adiabatic compression first (energy invested). We can write

$$T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\kappa - 1} = T_1 k^{\kappa - 1}\,.$$

Then the energy invested is

$$E_{\mathrm{in}} = RnT_1 \left(k^{\kappa-1} - 1\right) .$$

For determining the parameters describing the state 3, the ideal gas law needs to be employed. We know that $T_3 = T_1$, hence

$$p_3 = kp_1 .$$

We should note that $p_4 = p_1$. Then we can write for the adiabatic expansion

$$T_4^{\kappa} = T_3^{\kappa} \left(\frac{p_3}{p_1}\right)^{1-\kappa} = T_1^{\kappa} k^{1-\kappa} \quad \Rightarrow \quad T_4 = T_1 k^{\frac{1-\kappa}{\kappa}} .$$

The energy returned in this process is

$$E_{\mathrm{out}} = RnT_1 \left(1 - k^{\frac{1-\kappa}{\kappa}}\right) .$$

At this point, we can compute the efficiency as

$$\eta = \frac{E_{\mathrm{out}}}{E_{\mathrm{in}}} = \frac{1 - k^{\frac{1-\kappa}{\kappa}}}{k^{\kappa-1} - 1} .$$

Let us recall that for a diatomic gas we have $\kappa = 1.4$, so the answer is $\eta \doteq 31.9\,\%$.

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Problem FoL.22 ... resistor blackbox

Consider a box containing three resistors with unknown resistances. These resistors are connected through ideal conductors. There are four output terminals leading outside the box. We measure the resistance across different pairs of terminals for all possible combinations of terminals. Five such measurements yield $9\,\Omega$, $12\,\Omega$, $8\,\Omega$, $3\,\Omega$ and $14\,\Omega$. Let us now connect the ohmmeter across the last unmeasured pair of terminals. What resistance do we find?

Náry is tragic.

Unless we want to measure zero resistance, there should not be any loop in the wiring, hence the resistors must be connected either in a star configuration or in series. Equipped with this condition we can uniquely determine both the precise way of wiring as well as the individual values of resistances of the resistors and hence the resistance across the last choice of terminals.

Let us denote the resistances by R_1, R_2 and R_3. Based on how these resistors may be connected we infer that the resistances measured across the terminals can either be a combination of sums of R_1, R_2 and R_3 or directly R_1, R_2 or R_3. There are exactly 7 such cases. Since we obtained our measurements for 5 possible combinations of terminals out of 6, we must have necessarily measured at least two values which correspond directly to some of R_1, R_2 and R_3. This implies that the least resistance measured across a pair of terminals must be one of R_1, R_2 or R_3, since assuming the opposite we come to a quick contradiction. Denoting the least resistance R_1, we observe that $R_1 = 3\,\Omega$.

In the next step we subtract R_1 from all of the remaining measured resistances. There exist exactly 3 pairs of resistors, which we potentially could have measured and which satisfy the following: when we subtract R_1 from the resistance of the first one we get the resistance of the second one. Since we conducted 5 measurements, there must be at least one such a pair in the list of measured resistances. Indeed, the values $12\,\Omega$ and $9\,\Omega$ differ exactly by $3\,\Omega$. Since this is the only such pair and simultaneously $12\,\Omega$ is not the highest value measured, we must have that $9\,\Omega$ is the resistance of the second resistor, denoted by R_2 (think this carefully through!).

Since $8\,\Omega$ is less than R_2, it must be equal either to R_3 or $R_1 + R_3$. However, it would be impossible to place the terminals in a star or serial wiring in such a way so as to obtain a resistance of $14\,\Omega$, if we had $R_3 = 8\,\Omega$. Hence $R_3 = R_2 - R_1 = 5\,\Omega$.

Based on the above derived results we easily conclude that the resistors are connected in a star and that the resistance across the last choice of terminals is $R_3 = 5\,\Omega$.

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Problem FoL.23 ... orbiters

Determine the magnitude of a magnetic field which is present at the centre of mass of the system of two planets charged with equal charges of $Q = 100\,\mathrm{TC}$ *and with equal masses of* $M = 5\cdot 10^{24}\,\mathrm{kg}$ *orbiting each other at a distance of* $d = 500\,000\,\mathrm{km}$ *apart. In your calculations, regard the two planets as point charges and assume that their orbits are circular. State your answer in terms of* nT. *Lukáš wanted to create an unconventional problem no matter what.*

The planets exert an attractive force on each other, which is of magnitude

$$F = \frac{1}{d^2}\left(GM^2 - \frac{Q^2}{4\pi\varepsilon_0}\right).$$

In order to balance the centrifugal with the attractive force (from the point of view of the frame of reference rotating with the planets), we have to satisfy

$$v = \sqrt{\frac{dF}{2M}}\,.$$

This means that the current flowing around the centre of mass will be

$$I = \frac{2Qv}{\pi d}$$

which allows us to compute the magnitude of the magnetic field at the centre of mass as

$$B = \mu_0\frac{I}{\pi d} = \mu_0\frac{2Qv}{(\pi d)^2} = \mu_0\frac{2Q\sqrt{\frac{dF}{2M}}}{(\pi d)^2} \doteq 57.23\,\mathrm{nT}\,.$$

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Problem FoL.24 ... mercujet

Consider a flask filled with a potion of density $\varrho = 13\,600\,\mathrm{kg{\cdot}m^{-3}}$. The flask is sealed at the top, with a small capillary inserted in the seal. The potion is heated to the temperature corresponding to its boiling point under the atmospheric pressure $p_\mathrm{a} = 101\,325\,\mathrm{Pa}$. By how much will the level of the potion in the capillary rise? Neglect both the change in the surface tension with the temperature as well as the thermal expansion of both the potion and the flask. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. The flask is placed in vacuum. Assume that at the beginning of the process, the partial pressure of the potion vapor is zero.

Lukáš was playing with a PET bottle.

If we heat the potion up to temperature corresponding to its boiling point under the atmospheric pressure, the partial pressure of its vapor is the same as the atmospheric pressure, hence the potion will rise to a height of

$$h = \frac{p_\mathrm{a}}{\varrho g} \doteq 759\,\mathrm{mm}\,,$$

because the pressure above the level of the potion in the flask is p_a and there is zero pressure above the level of the potion in the capillary.

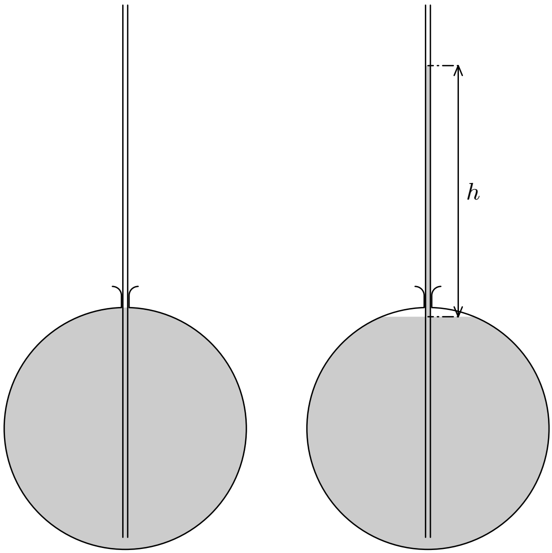

Fig. 3: english label 24

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Problem FoL.25 ... dimensionless hydrogen

Consider the ground state energy of the hydrogen atom (a system consisting of a proton and an electron) in the non-relativistic model with the proton fixed at a point. If the electron is *in the infinite distance from the proton, the energy of the configuration is defined to be zero. Find the ground state energy in terms of dimensionless units defined by putting the mass of the electron $m_\mathrm{e} = 1$, reduced Planck constant $\hbar = 1$ and $k_\mathrm{e}e^2 = 1$, where k_e is the Coulomb's constant and e denotes the electron's charge.* *Jakub was forced to think*

Considering the Bohr's model of atom, the energy of the ground state of hydrogen is

$$E_0 = -\frac{m_\mathrm{e} e^4 k_\mathrm{e}^2}{2\hbar^2}\,.$$

We simply substitute 1 for the quantities mentioned in the task to obtain the energy in dimensionless units

$$E_0 = -\frac{1}{2}\frac{(m_\mathrm{e})\left(k_\mathrm{e}e^2\right)^2}{(\hbar)^2} = -\frac{1}{2}\,.$$

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Problem FoL.26 ... jumping dog

Imagine a marble placed in a height of $h = 1\,\mathrm{m}$ above the ground. At some point, we release the marble and simultaneously start to push the ground towards the marble with a speed of $V = 1.2\,\mathrm{m{\cdot}s^{-1}}$ with respect to the initial rest frame of the system. Given that the coefficient of restitution between the marble and the ground is $e = 0.3$, find the time needed for the marble to steady on the moving ground. We define the coefficient of restitution as the ratio of speeds of the marble with respect to the ground before and after each collision.
Kuba played with a marble.

Let us look at the situation from the inertial frame of reference connected with the moving ground. The initial speed of the marble in this frame is V. The marble will hit the surface at the time

$$t_1 = \sqrt{2\frac{h}{g} + \left(\frac{V}{g}\right)^2} - V/g$$

having a speed of

$$v_1 = V + gt_1 = \sqrt{2gh + V^2}\,.$$

Let us take into account the coefficient of restitution e. After the collision the marble will have a speed of $v_2 = ev_1$ and with this one, it will hit the surface again at the time

$$t_2 = 2\frac{v_2}{g} = 2e\frac{v_1}{g}\,,$$

subsequently hitting the surface again with a speed of $v_3 = ev_2$ etc. By induction, write

$$\forall n \in \mathbb{N}\backslash\{1\} : t_n = 2e^{(n-1)}\frac{v_1}{g}\,,$$

where t_1 is given above. If $e < 1$ the series $\sum t_n$ converges. Let T be the time we are looking for. Then

$$T = \sum_{n=1}^{\infty} t_n = t_1 + 2\frac{v_1}{g}\sum_{n=2}^{\infty} e^{(n-1)} = t_1 + 2\frac{v_1}{g}\sum_{n=1}^{\infty} e^n\,,$$

giving

$$T = t_1 + 2\frac{v_1}{g}\frac{e}{1-e} = \frac{\sqrt{2gh + V^2}}{g}\frac{1+e}{1-e} - V/g\,.$$

Plugging in the numbers, we get $T \doteq 0.74\,\mathrm{s}$.

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Problem FoL.27 ... P5

What is the average number of photons arriving from Pluto's spherical moon P5 onto the mirror of the Hubble Space Telescope every second? Assume the following: the moon has a diameter of $D = 20\,\mathrm{km}$, its distance from the Sun is $L = 32\,\mathrm{AU}$ and it has an albedo of $a = 0.3$. The Hubble Space Telescope is equipped with a mirror of diameter $d = 2\,\mathrm{m}$. You can regard the Sun as being a monochromatic light source, emitting at a wavelength of $\lambda = 550\,\mathrm{nm}$ with the solar constant being $P_\mathrm{S} = 1\,400\,\mathrm{W{\cdot}m^{-2}}$. Further assume that the Hubble Space Telescope is also located at a distance of L from P5. Do not consider the absorption in the interplanetary medium, assume the isotropic scattering of photons by P5 and do not take into account the photons absorbed and subsequently radiated back by P5.

Lukáš read some stuff about exoplanets.

Let us compute the power that the moon receives from the Sun

$$P_\mathrm{m} = P_\mathrm{S}\frac{1\,\mathrm{AU}^2}{L^2}\cdot\frac{\pi}{4}D^2\,.$$

This power (reduced by the albedo) is uniformly scattered by the moon onto a sphere with a radius of L and an area of $4\pi L^2$. However, we detect only that part corresponding to the area of the telescope's mirror being $S_\mathrm{d} = \pi d^2/4$. Hence the power detected is

$$P_\mathrm{o} = aP_\mathrm{m}\frac{\pi d^2/4}{4\pi L^2} = P_\mathrm{S}a\frac{1\,\mathrm{AU}^2}{L^2}\cdot\frac{\pi}{4}D^2\frac{\pi d^2/4}{4\pi L^2} = P_\mathrm{S}a\frac{\pi}{64}\frac{1\,\mathrm{AU}^2D^2d^2}{L^4} \doteq 1.398\cdot 10^{-18}\,\mathrm{W}\,.$$

Further we have to determine the energy of one photon of given wavelength. It is true that $E = hc/\lambda = 3.638\cdot 10^{-19}\,\mathrm{J}$. Hence, for the number of photons detected every second we have $P_\mathrm{o}/E = 3.842\,\mathrm{Bq}$, where Bq is a unit with the same dimension as Hz. Though this one is used for random processes, while the latter is for periodic ones. The telescope will receive 3.8 photons per second on average.

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Problem FoL.28 ... weird gravity

Imagine that you are standing on the inner side of the wall of a huge open-ended cylinder with a base radius of $R = 1000\,\mathrm{m}$, The cylinder is rotating about its axis with a constant angular speed so that the centrifugal acceleration you can feel is the same as the acceleration due to gravity on the Earth's surface $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. The cylinder itself is virtually massless and it is placed in vacuum, outside the influence of gravitating bodies. Imagine you throw a ball straight up from the surface you are standing on, giving it an initial speed of $v = 10\,\mathrm{m{\cdot}s^{-1}}$ in *a direction perpendicular to the surface. Compute how far from you the ball lands. The distance is measured along the surface, in the frame of reference connected with the rotating cylinder. State your answer in metres.* *Kuba whirling around.*

If there is to be a normal acceleration of g on the inner side of the wall, the angular speed must be $\omega = \sqrt{g/R}$. We need to remember that there are no forces acting on the flying ball in the non-rotating reference frame which is thus inertial, so in this frame the ball is either at rest or it moves in a straight line with constant speed. In fact, in such frame the ball's trajectory will be a straight line which makes an angle α with the direction normal to the surface, where

$$\operatorname{tg}\alpha = \frac{\sqrt{gR}}{v}.$$

The speed of the ball in this frame is

$$v' = \sqrt{v^2 + gR},$$

so the straight trajectory of the ball hits the wall of the cylinder at time

$$t = \frac{2R\cos\alpha}{\sqrt{v^2 + gR}} = \frac{2Rv}{v^2 + gR}.$$

But during this time, the cylinder rotates through an angle of ωt, so the distance to the point of landing measured along the surface is

$$d = R(\pi - 2\alpha - \omega t) = R\left(\pi - 2\operatorname{arctg}\frac{\sqrt{gR}}{v} - \frac{2Rv}{v^2 + gR}\sqrt{\frac{g}{R}}\right).$$

Plugging in the numbers, $d \doteq 1.36\,\mathrm{m}$.

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Problem FoL.29 ... gramme of hexagram

Consider a fractal which resulted from infinite plunging of hexagrams into each other (see the figure). Only the coloured parts have mass and their total mass is exactly one gramme. The massive parts are made of homogeneous material and the radius of the fractal is one centimetre (the radius is measured from the centre to the furthermost vertex of the fractal). What is the moment of inertia with respect to the axis perpendicular to the plane of the fractal and going through the centre of the fractal? The result should be stated in terms of $\mathrm{g{\cdot}cm^2}$. *The moment of inertia of an equilateral triangle with a side lenght of* a_t *and mass* m_t *with respect to the axis perpendicular to the plane of the triangle and going through its centre of mass is*

$$I_\mathrm{t} = \frac{1}{12} m a_\mathrm{t}^2\,.$$

Karel was thinking about moments of inertia.

Let the radius of the fractal be r. By looking at the geometry of the problem, we note that the ratio of the dimensions of every inner and outer triangle is $q = 1/\sqrt{3}$. Let us denote the area of the coloured part depicted on the first diagram by S_1, the area of the coloured part which

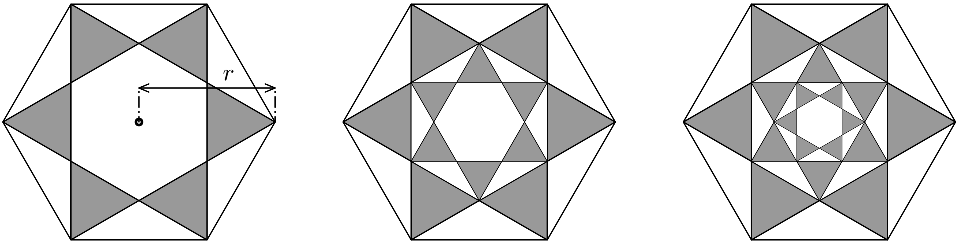

Fig. 4: To the problem 29

was added in the the second diagram will be denoted by S_2 etc. The area is proportional to the square of the linear dimensions of a tringle, so

$$S_{n+1} = q^2 S_n\,.$$

The total area is then

$$S = \sum_{n=1}^{\infty} S_n = S_1 \frac{1}{1-q^2} = \frac{3}{2} S_1\,.$$

The lengths of the edges of the equilateral triangles in the first diagram are $a_1 = r/\sqrt{3}$, so the area of the whole fractal is

$$S = \frac{3}{2} S_1 = \frac{3}{2} \cdot 6 a_1^2 \frac{\sqrt{3}}{4} = r^2 \frac{3\sqrt{3}}{4}\,.$$

Hence the surface density is

$$\varrho = \frac{m}{S} = \frac{4}{3\sqrt{3}} \frac{m}{r^2}\,.$$

The moment of inertia of the shape depicted on the first diagram, I_1, is six times the moment of inertia of an equilateral triangle with respect to the axis going through the centre of the fractal. To compute this moment of inertia, we need to employ the parallel axis theorem

$$\frac{I_1}{6} = I_{\mathrm{t}1} + m_{\mathrm{t}1} \cdot l_{\mathrm{t}1}^2,$$

where $I_{\mathrm{t}1}$ is the moment of inertia of the triangle with respect to the axis going through its centre of mass, $m_{\mathrm{t}1}$ is its mass and $l_{\mathrm{t}1}$ is the distance between the centre of mass of the triangle and the centre of the fractal. We know the dimensions of the triangles as well as their surface density and $l_{\mathrm{t}1} = \frac{2}{3} r$. From the geometry of the problem, we can write

$$I_1 = 6\left(I_{\mathrm{t}1} + m_{\mathrm{t}1} \cdot l_{\mathrm{t}1}^2\right) = \frac{17}{54} m r^2\,.$$

If the surface density is constant, the moment of inertia is proportional to the fourth power of the dimension ($I \propto m r^2$ and $m \propto r^2$), so for the plunged stars

$$I_{n+1} = q^4 I_n\,.$$

Summing the resulting geometrical progression, we get a finite moment of inertia

$$I = \sum_{n=1}^{\infty} I_n = I_1 \frac{1}{1-q^4} = \frac{9}{8} I_1 = \frac{17}{48} mr^2 \,.$$

Substituting the numerical values, we get $I = 0.354\,\mathrm{g\cdot cm^2}$.

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Problem FoL.30 ... crystal mirror

Consider an aquarium suspended in the air. We can assume that it is infinitely big, so no matter from which direction the rays are coming, they always hit a wall (see the figure). The walls are made of glass with a refractive index of $n_1 > 1$. The aquarium is filled with an unknown transparent liquid with a refractive index of $n_2 > 1$. The refractive index of air is $n = 1$. There is a sheet of paper placed under the aquarium with the solution to this problem written down on it. Find the smallest refractive index of the liquid so that we would not be able to see the paper from aside of the aquarium (see the figure).

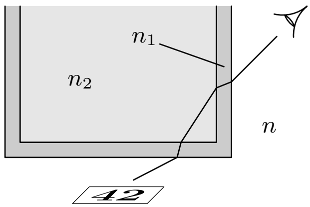

Fig. 5: To the problem 30

f(Aleš) brought up a problem during the brainstorming.

The light is passing through several layers with different refractive indices. For such a situation, we write

$$n_1 \sin\varphi_1 = n_2 \sin\varphi_2 = \cdots = n_N \sin\varphi_N \,. \tag{4}$$

Hence when passing through the aquarium wall for the first time, the rays refract independently on the refractive index of the glass and the direction in which they are deflected is affected by the refractive index of the liquid only. Since we have $n_1 > n$, the rays always pass into the aquarium.

The rays can hit the bottom of the aquarium under a range of angles of $(0, \pi/2)$. We see from the equation (4) that should the total internal reflection occur, it will certainly happen no later than at the last two interfaces. In other words, if there is to be a total internal reflection on the interfaces liquid–glass or glass-air, it would occur on the interface liquid–air as well. Hence, it is true that

$$n \sin\frac{\pi}{2} = n_2 \sin\varphi_2 \,,$$

whence

$$n_2 = \frac{1}{\sin\varphi_2} \,,$$

where φ_2 is the angle of incidence of the rays passing through the liquid onto the glass. From the symmetry of the problem, by considering the rays travelling in the opposite direction, we can write $\varphi_2 = \pi/4$. Hence we end up with a condition

$$n_2 \geq \sqrt{2}\,,$$

thus $n_2 = \sqrt{2}$, which is approximately 1.41.

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Problem FoL.31 ... running mirror

Consider a system containing a converging lens with a focal lenght of $f = 20\,\mathrm{cm}$ and a movable convex mirror with a radius of curvature of $3f$ (see the figure). At $t = 0$, when the mirror and the lens are in contact, we start to move the mirror with a speed of $v = 1\,\mathrm{m{\cdot}s^{-1}}$ away from the lens. What should be the position of the object as a function of time, in order for its image to stay at a distance of $2f$ leftwards from the lens? Assume that you can write $x(t) = f\cdot(v^2t^2 + vft - 3f^2)/(v^2t^2 - A)$ for the position of the object. Determine the constant A. Assume that the speed of light is infinite and that you can use the paraxial approximation.

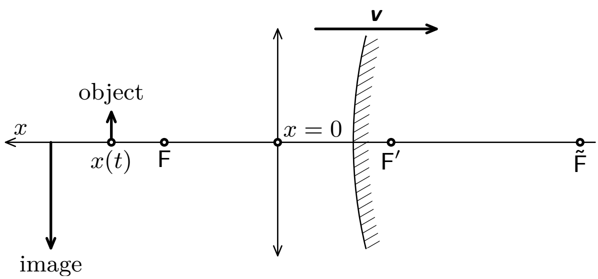

Fig. 6: To the problem 31

Lukáš, sitting on an optical bench.

Let us denote the position of the object by x, the position of the image produced by the lens by x_c, the position of the image after imaging by the mirror by x_cz and the position after the final imaging by the lens by x_czc. Further denote by d the distance of the mirror from the lens. Then we can write

$$\frac{1}{x} + \frac{1}{x_\mathrm{c}} = \frac{1}{f}\,, \tag{5}$$

$$\frac{1}{d - x_\mathrm{c}} + \frac{1}{d - x_\mathrm{cz}} = \frac{2}{r}\,, \tag{6}$$

$$\frac{1}{x_\mathrm{cz}} + \frac{1}{x_\mathrm{czc}} = \frac{1}{f}\,, \tag{7}$$

Further we have to substitute $x_\mathrm{czc} = 2f$ and $r = -3f$ due to the standard sign convention. Without these two assumptions, we can write the position of the object as a function of the

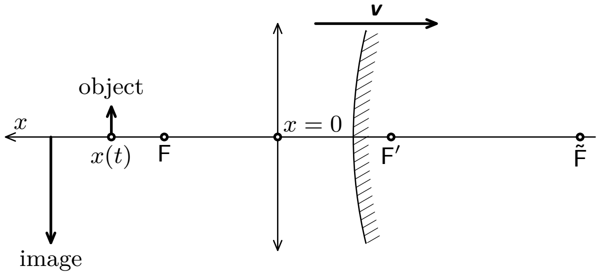

Fig. 7: To the problem 31

position of its final image and the distance of the mirror from the lense, which yields somewhat complicated expression. We then have

$$x = \frac{d^2 f + df^2 - 3f^3}{d^2 - \frac{5}{2} f^2}\,,$$

whence, substituting $d = vt$

$$x(t) = f \cdot \frac{v^2 t^2 + vft - 3f^2}{v^2 t^2 - \frac{5}{2} f^2}\,.$$

We observe that $A = 5f^2/2 = 0.1\,\mathrm{m}^2$.

This approach to the problem is somewhat technical and we need not have followed this path on our way to the final solution, since it is sufficient to determine the position of the mirror for one particular position of the object.

For this purpose we choose $x = f$. Then, by (5), we have $x_\mathrm{c} = \infty$. Further we know that $x_\mathrm{czc} = 2f$ and by (7) we get $x_\mathrm{cz} = 2f$, hence for d by (6) we have

$$\frac{1}{d - \infty} + \frac{1}{d + f} = -\frac{2}{3f} \quad \Rightarrow \quad \frac{1}{d - 2f} = -\frac{2}{3f} \quad \Rightarrow \quad d = \frac{1}{2} f\,.$$

This situation occurs at $t = d/v = f/(2v)$. We substituted in to the ansatz given in the task and having divided by f, we obtain

$$1 = \frac{v^2 t^2 + vft - 3f^2}{v^2 t^2 - A} \quad \Rightarrow \quad v^2 t^2 - A = v^2 t^2 + vft - 3f^2 \quad \Rightarrow \quad A = \frac{5}{2} f^2 = 0.1\,\mathrm{m}^2\,.$$

The numerical value of the sought-after constant A je $0.1\,\mathrm{m}^2$.

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Problem FoL.32 ... charged robber

A robber weighing $m = 50\,\mathrm{kg}$ was running away from the policemen. He decided to save himself by jumping from a cliff. Luckily enough, he stole a charge of $q = 10\,\mathrm{C}$. There is a homogeneous magnetic field below the cliff, reaching the height $a = 10\,\mathrm{m}$. The field is perpendicular to the vertical and its magnitude is $B = 10\,\mathrm{T}$. What is the maximum height of the cliff h (in metres) *from which the robber can jump without reaching the surface? Do not take into account the air resistance. The acceleration due to gravity is* $9.81\,\mathrm{m\cdot s^{-2}}$ *and the velocity of the robber at the moment of entering the magnetic field has a vertical component only. Assume that the magnetic field is pointing in the right direction so the robber does not hit the cliff.*

From the head of Tomáš B.

Let us assume that the cliff is on the right hand side, so we chose a right-handed cartesian coordinates system so that the x axis points to the left and y axis is points down. Let the $y = 0$ plane be the boundary of the region with the magnetic field pointing in the direction of z. Employing the Lorentz force, the equations of motion can be written as

$$m\ddot{x} = qB\dot{y}\,,$$
$$m\ddot{y} = -qB\dot{x} + mg\,.$$

Let us start to measure the time at the moment when the robber enters the magnetic field. At this moment we have $\dot{x}(0) = 0$ and $y(0) = 0$. After integrating the first equation and substituting into the second one, we get

$$\ddot{y} = -\left(\frac{qB}{m}\right)^2 y + g\,.$$

The vertical motion of the robber can apparently be described by the equation of simple harmonic oscilator, so

$$\frac{1}{2}m\dot{y}^2(0) + \frac{1}{2}\frac{q^2B^2}{m}\delta^2 = \frac{1}{2}\frac{q^2B^2}{m}(a-\delta)^2$$

whence we can obtain

$$h = \frac{q^2B^2a^2}{2gm^2} \doteq 20.4\,\mathrm{m}\,.$$

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Problem M.1 ... drive safely!

Two cars with different types of tyres are decelerating in summer on a straight dry road from a velocity $v_0 = 100\,\mathrm{km{\cdot}h^{-1}}$. Initially, they are riding next to each other and start braking at the same time. When the car with summer tyres comes to the rest the other one, with winter tyres, is still moving with a velocity $v_1 = 37\,\mathrm{km{\cdot}h^{-1}}$ and stops after 6 more meters. What is the fraction of the horizontal deceleration of the car with winter tyres to the horizontal deceleration of the other car? *Michal heard this on radio.*

Let us denote decelerations of the cars with winter and summer tyres a_w and a_s respectively. It obviously holds

$$v_0 - a_w(v_0/a_s) = v_1\,.$$

Thus

$$(1 - a_w/a_s) = v_1/v_0\,,$$

which allows us to obtain the desired fraction $a_w/a_s \doteq 0.63$.

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Problem M.2 ... wheelspokes

Consider an eight-spoked wheel with radius $r = 30\,\mathrm{cm}$ rotating with angular speed $\omega = 2.5\pi\,\mathrm{rad{\cdot}s^{-1}}$ about a fixed axle through its centre. There are some boys shooting a bow in the direction of the wheel, trying to make the arrows pass freely through the gaps between the spokes. The length of one arrow is $l = 23\,\mathrm{cm}$. Assuming that the spokes and the arrows are negligibly thin, find the minimal speed of the arrows so that the boys would succeed in their objective. State your answer in metres per second. *Zdeněk and his head spinning all around.*

Let us write the frequency in terms of the angular angular speed

$$f = \frac{\omega}{2\pi}\,.$$

For the arrow not to be hit by a moving spoke, it must pass through the wheel in less than one eigth of the period T, where

$$\frac{T}{8} = \frac{1}{8f}\,.$$

This must be equal to the time which it takes the arrow to travel through the distance equal to its length, thus

$$v = 8fl \doteq 2.3\,\mathrm{m{\cdot}s^{-1}}\,.$$

Note that the information about the radius is completely redundant, unless we consider the spokes and arrows to be of finite thickness.

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Problem M.3 ... spleen on a bridge

Bored person is standing on a bridge with the height $h = 15\,\mathrm{m}$ and is dropping pebbles on cars passing along a straight road beneath him. In a distance he spots an approaching motorcyclist and decides to hit him. He estimates the motorcyclists's instantaneous velocity to be $v = 72\,\mathrm{km{\cdot}h^{-1}}$ and his horizontal distance $d = 500\,\mathrm{m}$. He calculates when he should drop the pebble and he indeed drops it at the calculated moment. However, when the pebble hits the ground the motorcyclist is already $x = 50\,\mathrm{m}$ behind the intended point of collision. What is the difference, in kilometers per hour, between the estimate of the motorcyclist's velocity and his actual velocity if we assume that the initial horizontal distance is guessed precisely and the velocity was constant throughout the motion? *Kiki, during a stroll in Brno.*

Legthy instructions are compensated by an easy and quick solution. The time t when the pebble hits the ground is given by $t = d/v$. If the motorcyclist already drove a distance $s = d + x$ at that time, then his actual velocity was $v_\mathrm{a} = s/t$. Therefore it holds

$$v_\mathrm{a} = \frac{(d + x)}{d/v}\,.$$

After plugging in numbers in appropriate units we substract the estimated velocity and evaluate their difference $\Delta v = v_\mathrm{a} - v$ which is numerically $7.2\,\mathrm{km{\cdot}h^{-1}}$.

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Problem M.4 ... broken altimeter

A curious sky observer notices an airplane which is approaching in a way that it eventually passes exactly above his head. At one point, when the airplane is still approaching, the observer measures that it is $\alpha_1 = 1.3\,\mathrm{rad}$ above the horizon. However, a noise from its engines is coming from a direction $\alpha_2 = 0.5\,\mathrm{rad}$ above the horizon. The observer measures an angular velocity of the airplane at the moment when it is passing above his head $\omega = 0.125\,\mathrm{rad{\cdot}s^{-1}}$. Based on these inputs calculate the height h of the airplane, assuming it is constant throughout the motion. The speed of sound is $c = 340\,\mathrm{m{\cdot}s^{-1}}$ and unvarying with the altitude. Neglect the finite speed of light. *Kuba was crossing half of the globe.*

We assume that the airplane is travelling parallelly with the surface of Earth. At the moment when the airplane is passing above the observer we can write

$$v = \omega \cdot h,$$

where v is a velocity of the airplane.

Sound which the observer hears under the angle α_2 is coming from the distance

$$d = \frac{h}{\sin\alpha_2}\,.$$

And the distance between point which the observer sees under the angle α_1 and the point from which he hears the sound is

$$s = h\left(\frac{1}{\operatorname{tg}\alpha_2} - \frac{1}{\operatorname{tg}\alpha_1}\right)\,.$$

The airplane travelled the distance s in the same time as the sound covered the distance d. Therefore it holds

$$\frac{d}{c} = \frac{s}{v} = \frac{1}{\omega}\left(\frac{1}{\operatorname{tg}\alpha_2} - \frac{1}{\operatorname{tg}\alpha_1}\right) .$$

Substituion for d from the equation above finally yields

$$h = \frac{c\sin\alpha_2}{\omega}\left(\frac{1}{\operatorname{tg}\alpha_2} - \frac{1}{\operatorname{tg}\alpha_1}\right) .$$

And after numerical evaluation we get $h \doteq 2\,025\,\mathrm{m}$.

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Problem E.1 ... a different one

Filip, who is colour blind and equipped with a red He–Ne *laser (providing light with wavelength* $\lambda_1 = 633\,\mathrm{nm}$*), decided to measure the refractive index of his little glass made of borosilicate glass (designated as BSC7) for the wavelength of his laser (corresponding to red colour). The method used was to measure the critical angle of refraction of the laser beam incident on glass-vacuum interface and to infer the refractive index thereof. However, because of well-sustained mess in the container where he stores his lasers, he used a green laser (wavelength* $\lambda_2 = 555\,\mathrm{nm}$*) instead of the red one, by mistake. Find the fractional error of his result in terms of permille (parts per thousand) assuming that the measurement was not subject to other errors of any kind. For the BSC7 glass we have the respective refractive indices for the wavelengths* $\lambda_1 = 633\,\mathrm{nm}$ *and* $\lambda_2 = 555\,\mathrm{nm}$ *equal to* $n_1 = 1.51508$ *and* $n_2 = 1.51827$*.*

Honza stumbled upon while in optics lab.

It does not take one too long to find out that based on the measurement strategy described in the task, it is possible to find the refractive index for given wavelength directly, since $\sin\alpha_\mathrm{c} = \frac{1}{n}$, where α_c is the critical angle and n is the sought-after refractive index. Hence, in order to find the answer we only need to know the refractive indices for given wavelengths in given material. The answer then reads

$$p = \frac{n_2 - n_1}{n_1} \doteq 2.1\,‰\,.$$

Jan Česal

Problem E.2 ... firefly

A neon lamp is connected through a resistor to a rigid source of alternating voltage of a root mean squre voltage $230\,\mathrm{V}$ *and frequency* $50\,\mathrm{Hz}$*. Its ignition voltage (the striking voltage) is* $120\,\mathrm{V}$ *and the maintaining voltage is* $80\,\mathrm{V}$*. How long it will stay lit during one half-period? Assume that all resistors in the circuit are such that you do not have to take drop in the current into consideration. Please provide the result in* ms*.*

f(Aleš) wanted to read in the evening but he didn't have any lamp.

Time dependence of a voltage is given by $u(t) = U\sin(\omega t)$. The root mean square voltage is defined by $U_\mathrm{rms} = U\sqrt{2}$. Therefore we can write for the ignition voltage

$$U_\mathrm{I} = \sqrt{2}U_\mathrm{rms}\sin(\omega t_1)\,,$$

where t_1 is the time when the lamp lits, if we start measuring time when the voltage is zero. If we express the time t_1

$$t_1 = \frac{\arcsin\left(\frac{U_\mathrm{I}}{\sqrt{2}U_\mathrm{rms}}\right)}{\omega}\,.$$

Following the same procedure we get time t_2 when the neon lamp goes out

$$t_2 = \frac{\arcsin\left(\frac{U_\mathrm{M}}{\sqrt{2}U_\mathrm{rms}}\right)}{\omega}\,,$$

where U_M is the maintaining voltage. We also put into use a familiar expression $\omega = 2\pi f$.

For both times we get two results. In the case of the time t_1 we are interested in the lesser one and in the case of the time t_2 in the bigger one in order to determine whole time when the neon lamp stays lit. Desired answer is then $t = t_2 - t_1$. Numerically we get

$$t_1 \doteq 1.20\,\mathrm{ms}\,,$$
$$t_2 \doteq 9.21\,\mathrm{ms}\,,$$

and thus

$$t \doteq 8.01\,\mathrm{ms}\,.$$

Finally we round the answer to 8 ms.

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Problem E.3 ... shut it down!

A square elevator with mass $m = 1000\,\mathrm{kg}$ and a side length of $a = 3\,\mathrm{m}$ is moving in a very long square shaft of a side length $b = 4\,\mathrm{m}$ with a speed of $v = 2\,\mathrm{m{\cdot}s^{-1}}$. Since this is the experimental physics building the elevator has a point charge $q = 1\,\mathrm{C}$ embedded in the middle of its floor. The most problematic lab just created a strong homogenous electric field with potential difference between the walls of the shaft $U = 1000\,\mathrm{V}$. The electric field is perpendicular to the motion of the elevator and also perpendicular to the walls of the shaft. The suspension of the elevator is so long that you can assumme that it moves along a straight line. What is the maximal time for which the electric field can last so the elevator still does not hit the wall of the shaft during the field's action?
f(Aleš) had an afternoon filled with thoughts about elevators and electricity kept meddling into it.

The elevator is subject to the electric force $F_\mathrm{e} = qE$ with electric intensity given by $E = U/b$. The acceleration in the perpendicular direction is

$$a = \frac{qU}{mb}\,.$$

The elevator cannot move in the perpendicular direction further than $s = (b - a)/2$ which will take time

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2\left(\frac{b-a}{2}\right)mb}{qU}} = \sqrt{\frac{(b-a)\,mb}{qU}}\,.$$

Numerically, we get $t = \sqrt{4}\,\mathrm{s} = 2\,\mathrm{s}$.

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Problem E.4 ... anullator

Consider two long parallel conductors $\{a\}^{\{a\}}$ metres apart, where $\{a\}$ denotes the numerical value of a physical quantity a, so $a = \{a\}[a]$, where we write $[a]$ for the unit of a. Both conductors are placed in vacuum and the currents through them flow in the directions opposite to one another. Assume that a current of $I_1 = 1\,\mathrm{A}$ flows through the conductor nr. 1. Find the current *flowing through the conductor nr. 2 given that the magnetic field is zero at a perpendicular distance of* $\{b\}^{\{b\}}$ *from the conductor nr. 1 (that one which is further from the conductor nr. 2). Further assume that the distance* a *is* $30\,277\,604\,100\,\mathrm{m}$ *longer than one astronomical unit, that the light travels through the distance* b *in 10 minutes and that the astronomical unit is precisely* $149\,597\,870\,700\,\mathrm{m}$. *f(Aleš) wrote a sweet dot.*

The magnitude of a magnetic field $\boldsymbol{B}$ at a distance of r from a conductor can be written as

$$B = \frac{\mu}{2\pi}\frac{I}{r}\,.$$

We need to have $\boldsymbol{B}_1 + \boldsymbol{B}_2 = 0$, so

$$\frac{\mu}{2\pi}\left(\frac{I_1}{r_1} - \frac{I_2}{r_2}\right) = 0\,,$$

whence

$$|I_2| = \left|I_1\frac{r_2}{r_1}\right|\,.$$

Now, let us remember that $r_2 = \{a\}^{\{a\}}\,\mathrm{m} + \{b\}^{\{b\}}\,\mathrm{m}$ and $r_1 = \{b\}^{\{b\}}\,\mathrm{m}$. Further, the speed of light is precisely $299\,792\,458\,\mathrm{m{\cdot}s^{-1}}$.

Being realistic, you cannot really do power of those huge numbers, so let us rather notice that both the same, hence

$$I_2 = 1\,\mathrm{A}\cdot\frac{k+k}{k} = 2\,\mathrm{A}\,.$$

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Problem X.1 ... in the subway

There are two escalators, one is leading out of the subway station, the other one into it. They move in the opposite dirrections with velocities $0.5\,c$, where c is speed of light in vacuum. Bob is in a hurry to get to the platform and he's running with velocity $0.6\,c$ with respect to the escalator. Bobek, on the other hand, is trying to get out of the subway with velocity $0.4\,c$, related to escalator. We are the observers standing on the platform. What is the velocity of their mutual movement according to our measurement?

Dominika took a subway for the first time in her life.

The velocities are high enough so we have to think in relativistic terms (since we don't want to violate the laws of physics by moving faster than light). We use a following formula

$$\frac{u+v_i}{1+uv_i/c^2}\,.$$

u is the velocity of the escalator while v_i is velocity of Bob (Bobek) with respect to the escalator. We obtain two velocities

$$v_{\mathrm{Bob}} \doteq 0.85\,c\,,$$
$$v_{\mathrm{Bobek}} \doteq 0.75\,c\,.$$

Adding these two will give us velocity we are looking for

$$v = |v_{\mathrm{Bob}} + v_{\mathrm{Bobek}}| = \frac{u+v_1}{1+uv_1/c^2} + \frac{u+v_2}{1+uv_2/c^2} \doteq 1.596\,c\,.$$

Rounding this number we get $1.60\,c$.

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Problem X.2 ... crash

A proton with invariant mass of $938\,272.0\,\mathrm{keV}$ and kinetic enegry of $1\,\mathrm{MeV}$ hits a nucleus of the isotope ${}^{7}_{3}\mathrm{Li}$ with mass of $7.016003\,m_u$ and induces a decay to two non-excited α particles with invariant mass of $3.727379\,\mathrm{GeV}\cdot c^{-2}$. What will be the total kinetic energy in MeV of these two particles? Consider $m_u = 931.2720\,\mathrm{MeV}\cdot c^{-2}$ and $c = 299\,792\,458\,\mathrm{m{\cdot}s^{-1}}$.

Even f(Aleš) used to play marbles.

We use the energy conservation law which states that

$$T = T_{\mathrm{p}} + \left(m_{\mathrm{Li}} + m_{\mathrm{p}} - 2m_\alpha\right) c^2\,,$$

where T is the wanted kinetic energy, T_{p} is the kinetic energy of the proton, m_{Li} is the mass of the lithium nucleus, m_{p} is the proton mass and m_α is the mass of the α particle.

We convert everything to electronvolts and after evaluating the equation we get $T \doteq 18\,\mathrm{MeV}$.

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Problem X.3 ... blip

A particle has a mass increased sevenfold from its invariant mass due to its movement. You can follow the particle on a path $l = 1\,\mathrm{m}$ long with a measuring device. How fast must the device be to register the particle, i. e., what is the shortest time interval it has to distinguish in order to register the particle? The speed of light is $c = 299\,792\,458\,\mathrm{m{\cdot}s^{-1}}$. State the result in ns. *f(Aleš) writing when healthy, writing when sick.*

The energy of the particle is

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = n m_0 c^2 \,.$$

From that follows that

$$\frac{1}{1 - \frac{v^2}{c^2}} = n^2 \,,$$

so that

$$v = \sqrt{c^2 \left(1 - \frac{1}{n^2}\right)} = \frac{c}{n} \sqrt{n^2 - 1} \,.$$

The particle flies through the distance l in time τ given by

$$\tau = \frac{l}{v}$$

$$\tau = \frac{l}{\frac{c}{n}\sqrt{n^2 - 1}} \,.$$

Evaluation gives $v \doteq 3.37021 \cdot 10^{-9}\,\mathrm{s}$ which is $v \doteq 3.4\,\mathrm{ns}$.

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Problem X.4 ... cracking right now

An unknown sample of rock contains $2\,\%$ of thorium, which contains $0.05\,\%$ of the ${}^{232}_{90}\mathrm{Th}$ radionuclide. The weight of the sample is $100\,\mathrm{g}$. What is the activity of this sample if the half-life of thorium is $1.4 \cdot 10^{10}$ years? Assume that nothing else decays except the mentioned thorium. *f(Aleš) recalled the loading of equipment for a camp.*

Let $a = 0.2\,\%$, $b = 0.05\,\%$ and $m = 0.1\,\mathrm{kg}$. The atomic mass of thorium is $A_\mathrm{r} \doteq 232$. The activity is given by

$$A(t) = \lambda N(t) \,,$$

where λ is the decay constant defined as $\lambda = \ln 2 / T$ and N is the number of decaying nuclei. These are the nuclei of the radioactive thorium ${}^{232}_{90}\mathrm{Th}$, whose number can be find from the ratio of the weight of the radionuclide in the sample to the mass of one atom of ${}^{232}_{90}\mathrm{Th}$.

$$N = \frac{abm}{A_\mathrm{r} m_u} \,,$$

where A_r is the atomic mass of thorium and m_u is the atomic mass unit, explicitly

$$N = \frac{0.020.05 \cdot 10^{-2} \cdot 0.1}{232 \cdot 1.66 \cdot 10^{-27}} \doteq 2.60 \cdot 10^{18} \,.$$

If we consider that a year has $3.16 \cdot 10^7$ s, we get

$$A = \frac{\ln 2}{1.4 \cdot 10^{10} \cdot 3.16 \cdot 10^7\,\mathrm{s}} \cdot \frac{0.02 \cdot 5 \cdot 10^{-4} \cdot 0.1}{232 \cdot 1.66 \cdot 10^{-27}} \doteq 4.07\,\mathrm{Bq}$$

So the activity is about $4.1\,\mathrm{Bq}$.

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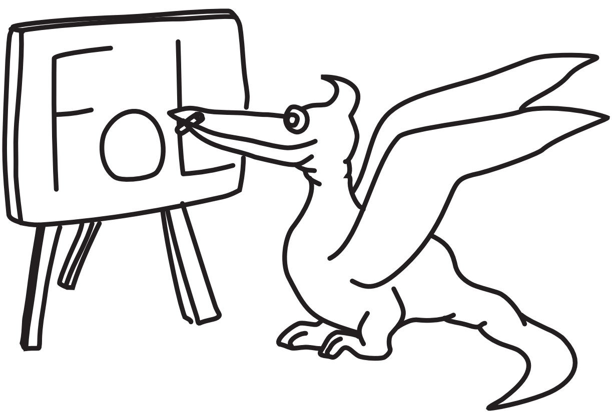
Solution of 3rd Online Physics Brawl



Problem FoL.1 ... jumping dog

An escaping prisoner needed to jump from one rooftop to another one because it's just the thing escaping prisoners do. The first building, from which he jumped, is $H = 16\,\mathrm{m}$ tall and the second one is $h = 11.6\,\mathrm{m}$, the buildings are $d = 4\,\mathrm{m}$ apart. The velocity of the prisoner at the moment of jump is $v = 3.8\,\mathrm{m{\cdot}s^{-1}}$ and he is jumping parallel to the sufrace. Determine the missing/redundant distance (using the -/+ signs) after the jump relative to the rooftop of the second building. Air resistance is negligible.Assume gravitational acceleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Kiki has found and remade this HRW problem while drinking tea.

First, we estimate the time needed for the jump, $t = \sqrt{2\Delta h/g}$ where Δh is the height difference between the two buildings. Knowing the time, we can compute the distance jumped on the x-axis $x = vt\cos\alpha$, where $\alpha = O$.

$$x = vt\cos\alpha$$

$$x = v\sqrt{\frac{2\Delta h}{g}}$$

Filling in the numbers, we get $x \doteq 3.60\,\mathrm{m}$, meaning that the prisoner would need $0.40\,\mathrm{m}$ more to reach the second rooftop, hence we write the result as $\Delta \doteq -0.40\,\mathrm{m}$.

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Problem FoL.2 ... source of radiation

Compute the wavelength of radiation passing through an optical grid. The distance of the optical grid from the projection screen is $2.0\,\mathrm{m}$, the distance between 0th and 2nd maximum is $6.0\,\mathrm{cm}$. Period of the optical grid is $5.0\cdot 10^{-6}\,\mathrm{m}$. The result should be in nanometers.

Monika couldn't believe her eyes.

Diffraction on the optical grid is described by the following relation

$$\sin\alpha = \frac{k\lambda}{a}\,,$$

where α is an incidence angle of the ray measured from the perpendicular, k is an order of the maximum (here $k = 2$), λ is the wavelength of the radiation and a is the period of the grid. The distance from the projection screen to the optical grid is l and the distance between 0^{th} and 2^{nd} maximum is b. For $b \ll l$, we can write $\sin\alpha \approx b/l$. Knowing this, we can use the relation for the diffraction and calculate the wavelength:

$$\lambda \approx \frac{ab}{kl} = 75\,\mathrm{nm}\,.$$

The radioation has $\lambda \doteq 75\,\mathrm{nm}$, hence it falls into the ultraviolet part of electromagnetic spectrum.

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Problem FoL.3 ... fluctuational

Quantum electrodynamics which connects quantum theory with the relativity brought revolution into our understanding of the vacuum. Vacuum isn't emptiness and nothingness anymore, it's a state with the lowest energy. The lowest possible energy state cannot be zero since we have to take into account the uncertainty principle – which leads us to the fluctuations of the vacuum. The strongest mechanical demonstration of quantum fluctuations is the attraction force between two mirrors (two parallel metallic plates without any charge) separated by a narrow gap. Around the system, there are waves with all frequencies but inside the gap, there are only such waves which correspond to the resonance frequency of the gap. This results in a small, yet measurable force which pushes the mirrors closer to each other. It's magnitude depends on the area of the mirrors S (it's natural to assume linear dependence here) and the width of the gap d. There are two fundamental constants in the relation, c and h:

$$F = K c^{\alpha} h^{\beta} d^{\gamma} S\,.$$

Using dimensional analysis, determine the coefficients α, β and γ and the force for the values $S = 1\,\mathrm{cm}^2$ and $d = 1\,\mathrm{\mu m}$. K is a dimensionless constant, its value is $K = \pi/480$ as can be derived with precise calculations.

Zdeněk went through his old excercise book and was surprised to no end.

Dimensional analysis results in following equation

$$\mathrm{kg}^1\mathrm{m}^1\mathrm{s}^{-2} = \mathrm{m}^{\alpha+2\beta+\gamma+2}\mathrm{s}^{-\alpha-\beta}\mathrm{kg}^{\beta}\,.$$

Comparing the coefficients results in $\alpha = 1$, $\beta = 1$ and $\gamma = -4$. Substituing these numbers into the given relation for F, we obtain

$$F = \frac{\pi h c S}{480 d^4}\,.$$

With the given values, the force is $F \doteq 1.3 \cdot 10^{-7}\,\mathrm{N}$. This effect is known as Casimir effect.

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Problem FoL.4 ... czech energetics

Imagine you transform a 500 Euro banknote into energy. How many times more valuable does it become? The banknote weights 1.1 g, 1 kWh of electric energy costs 20 cents. Round the results to tens.

Try to discover the real value of money.

The energy gained from the banknote van be computed using the relation $E = mc^2$. For the 500 euro banknote, it's 27.5 GWh. Multiplying by the price for one kWh and dividing by the nominal value of the banknote, the banknote becomes 11 000 times more valuable than it's nominal value.

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Problem FoL.5 ... fall one

What is the stabilized velocity of a falling leaf, if the leaf has enough time to reach an equilibrium. Consider the grammage of the leaf similar to be to that of office paper ($80\,\mathrm{g{\cdot}m^{-2}}$) and that it has the shape is of a hemisphere. Consider Newton's relation for air resistance and take air density to be $\varrho = 1.29\,\mathrm{kg{\cdot}m^{-3}}$ and the drag coefficient $C = 0.33$. Assume gratitational acceleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. Those who are afraid of falling leaves have a guilty conscience.

Putting the air resistance force and gravity to equality, $C\varrho S v^2/2 = mg$ and considering that from the knowledge of grammage, we can compute the mass $m = \sigma S$, we express the stabilized velocity as

$$v = \sqrt{\frac{2\sigma g}{C\varrho}} = 1.9\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem FoL.6 ... choo choo train

A little train named Denis decided to go through one turn repetitively, speeding up till he derails. The radius of the turn is $R = 190\,\mathrm{m}$, tilt of the rail is $\alpha = 5^\circ$, the rail width is $d = 1.4\,\mathrm{m}$ and the center of the mass of the train is $h = 1.6\,\mathrm{m}$ from the rails. What's the difference between minimal and maximal speed with which the train could go through the turn without derailing. The result should be in kilometres per hour. The gravitational acceleration is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. The train moves horizontally. Jakub wants a driving licence for a train.

First of all, let's find out the minimum velocity allowed, using the angle between the perpendicular of the train and the abscissa between the center of the mass of the train and the rail. From the geometrical point of view, φ is

$$\varphi = \operatorname{arctg}\frac{d}{2h} \doteq 23^\circ\,,$$

which is more than the slope of the rail, $\alpha = 5^\circ$. Hence the train can stand on the rail and won't fall off. So the minimum velocity is zero. The maximal velocity $v_{\max}$ will occur in the moment when the total force acting on the train will point from the center of the mass of the train towards the outer rail. The angle between total acting force and the gravitational force is $\alpha + \varphi$. Hence for the ratio of centrifugal and gravitational force, we can write

$$\frac{F_\mathrm{o}}{G} = \frac{\frac{mv^2}{R}}{mg} = \operatorname{tg}(\alpha + \varphi)\,,$$

from which we can derive the maximal velocity

$$v_{\max} = \sqrt{Rg\operatorname{tg}(\alpha + \operatorname{arctg}\frac{d}{2h})} \doteq 115\,\mathrm{km{\cdot}h^{-1}}\,,$$

which is also the difference between the minimal and maximal velocities, since the minimal one is zero.

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Problem FoL.7 ... collision!!!

A positron and a helium nucleus are approaching each other, both are moving in a straight line, with the same speed $v = 2\,000\,\mathrm{km\cdot s^{-1}}$. What will the distance be between them in the moment when the positron stops (in the reference frame of the lab)? Express in pm. *Solve it clasically.*

Tomáš Bárta wanted to assign a particle physics task.

At the beginning, the mass of the positron is m_e and its velocity is $v_{\mathrm{e},0} = v$, for the helium nucleus, we have mass m_H and velocity $v_{\mathrm{H},0} = -v$ (it is aimed in the opposite direction). The moment we are interested in is when the velocity of positron decreases to zero and when it occurs the velocity of the positron is $v_{\mathrm{e},1} = 0$ and the velocity of the helium nucleus $v_{\mathrm{H},1}$. From the conservation of momentum, we can write

$$m_\mathrm{e} v_{\mathrm{e},0} + m_\mathrm{H} v_{\mathrm{H},0} = m_\mathrm{e} v_{\mathrm{e},1} + m_\mathrm{H} v_{\mathrm{H},1}\,.$$

Plugging in the numbers, we arrive at

$$v_{\mathrm{H},1} = v\frac{m_\mathrm{e} - m_\mathrm{H}}{m_\mathrm{H}}\,.$$

Then, using the conservation of energy, we can derive the distance of the particles. Assume zero potential energy at the beginning, since the distance between the particles was infinite. Hence

$$\frac{1}{2} m_\mathrm{e} v_{\mathrm{e,O}}^2 + \frac{1}{2} m_\mathrm{H} v_{\mathrm{H,O}}^2 = \frac{1}{2} m_\mathrm{e} v_{\mathrm{e},1}^2 + \frac{1}{2} m_\mathrm{H} v_{\mathrm{H},1}^2 + k\,\frac{Q_1 Q_2 e^2}{d}\,,$$

where k is Coulomb constant, Q_1 and Q_2 are the charges of positron and of helium nucleus respectively in the multiples of an elementary charge (($Q_1 = 1$, $Q_2 = 2$), e is the elementary charge and d is the distance between the particles. Plugging in the known velocities,

$$\frac{1}{2}(m_\mathrm{e} + m_\mathrm{H}) v^2 = \frac{1}{2} v^2 \frac{(m_\mathrm{H} - m_\mathrm{e})^2}{m_\mathrm{H}} + k\,\frac{Q_1 Q_2 e^2}{d}\,.$$

In the case of these equations, we know all the variables' values, so we can plug in the numbers and

$$d = \frac{2k\; Q_1 Q_2 e^2}{v^2 (3m_\mathrm{H} - m_\mathrm{e})}\frac{m_\mathrm{H}}{m_\mathrm{e}} \doteq 84.4\,\mathrm{pm}\,.$$

As we can see, the result and the radius of the helium nucleus differ by at least four orders of magnitude, so the particles won't collide.

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Problem FoL.8 ... semicircular analyzer

There is a specific instrument to analyze the distribution of the electrons flying out of the experimental device. It consists of two semicircles of $R = 20\,\mathrm{cm}$ radius, with a magnetic field between them. A half of the cut edge of the semicircle is the detector, while in the second half, there is, in the distance $d = 15\,\mathrm{cm}$ from the center, a crevice through which the electrons are ejected into the space with magnetic field. Assume the electrons are moving at nonrelativistic speed. Derive the ratio η of maximal and minimal velocity of the electrons, measurable with this device.

Aleš wrote down the first thing which came onto his mind.

Lorenz force will act on every particle which passes through the crevice into the analyzing device. The particle falls onto the detector in such manner that the point of the impact will be $2r$ from the crevice where r is a so called Larmor radius. Larmor radius can be derived from the knowledge of the forces acting on the particle. There is actually only one force in this case, the centrifugal force, which is represented by the Lorenz force. Lorenz force depends on the initial velocity of the particle v, its charge q and mass m and also on the magnetic induction B. Lorenz force ($\boldsymbol{F}_\mathrm{L} = q\boldsymbol{v} \times \boldsymbol{B}$) in the smartly chosen coordinate system (in the directions $\boldsymbol{B}$ and $\boldsymbol{v}$) can be written as

$$\frac{mv^2}{r} = qvB\,,$$
$$r = \frac{mv}{qB}\,.$$

If this radius corresponds to half of the distance between the closest point of the crevice

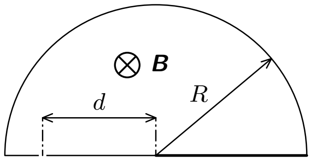

Fig. 1: Analyzator

and the detector, the minimal speed must be $v_\mathrm{min} = dqB/2m$, while in the case of the furthest point, the speed will be maximal $v_\mathrm{max} = (d+R)qB/2m$. Since we are interested in the ratio, we can write:

$$\eta = \frac{v_\mathrm{max}}{v_\mathrm{min}} = \frac{\dfrac{(d+R)qB}{2m}}{\dfrac{dqB}{2m}} = \frac{d+R}{d}\,.$$

Plugging in the numbers, we arrive at $\eta \doteq 2.33$.

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Problem FoL.9 ... upgrade

Lately, CERN upgraded their super collider. They increased the energy of the accelerated protons from $3.5\,\mathrm{TeV}$ to $7\,\mathrm{TeV}$. We would like to know how it affected the velocity of the protons. Invariant mass of the proton is $938\,\mathrm{MeV}/c^2$.

Jakub was a bit interested by this question.

We will use the equation $E = \gamma mc^2$, which will help us derive Lorentz factor in both cases (which is approximately 7463 for $7\,\mathrm{TeV}$ and 3731 for $3.5\,\mathrm{TeV}$). Then we can rewrite the equations

$$\gamma = \sqrt{\frac{1}{1-\dfrac{v^2}{c^2}}}$$

as

$$v = \frac{\sqrt{\gamma^2 - 1}}{\gamma} c\,.$$

Since we are interested in the difference between the velocities, we simply deduct one value from the other

$$\Delta v = \left(\frac{\sqrt{\gamma_1^2 - 1}}{\gamma_1} - \frac{\sqrt{\gamma_2^2 - 1}}{\gamma_2} \right) c\,.$$

Hence the result is approximately $8.1\,\mathrm{m \cdot s^{-1}}$, hence it's very small.

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Problem FoL.10 ... let there be equality

In the container, there is a mixture of $m_1 = 50\,\mathrm{g}$ of iodide ^{131}I and $m_2 = 20\,\mathrm{g}$ of strontium ^{90}Sr. How long will it take to have the same number of atoms of each element in that container?

Taken from Chemistry Olympics.

Half-life of iodide is $T_1 = 8.02\,\mathrm{d}$ and its amount of substance is $M_1 = 131\,\mathrm{g \cdot mol^{-1}}$, while the values for strontium are $T_2 = 28.8\,\mathrm{y}$ and $M_2 = 90.0\,\mathrm{g \cdot mol^{-1}}$. We want an equal number of atoms of both elements, hence

$$\frac{m_1}{M_1} 2^{-\frac{t}{T_1}} = \frac{m_2}{M_2} 2^{-\frac{t}{T_2}}\,,$$

in other words

$$\frac{m_1 M_2}{m_2 M_1} = 2^{\frac{t}{T_1} - \frac{t}{T_2}}\,,$$

where

$$t = \frac{T_1 T_2}{(T_2 - T_1)\ln 2} \ln \frac{m_1 M_2}{m_2 M_1} \doteq 6.26\,\mathrm{d}\,.$$

Notice that strontium's half-life is much longer than iodide's and the amount of substance of iodide is almost twice as big as of strontium. We can use this information to estimate that the time will be somewhat close to T_1. This can be a quick check just before we hand in the problem.

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Problem FoL.11 ... apple cider

Consider a homogeneous cylinder with both pedestals of mass $M = 100\,\mathrm{g}$ and height $H = 20\,\mathrm{cm}$. The cylinder is hanging in the air so its axis of symmetry is identical with the direction of the gravitational acceleration. At first, the cylinder is full of apple cider of mass $m_0 = 500\,\mathrm{g}$. Let's make a small hole in the bottom of the cylinder, so the cider runs out of it. Compute the height of the surface of the cider for the lowest position of the center of mass of the system. Give the result in cm.

Domča likes fall fruits and their derivatives.

Let's write the distance between the bottom of the cylinder and the center of the mass as y (the coordinate system is positive in the upwards direction). The center of mass of the cylinder is in a constant height $H/2$, the cider's center of mass is in $h/2$, while h is the level of the liquid, which depends on the current mass of the cider, m. Hence if we want to know the center of mass of the whole system, we need to compute an average of the centers of mass, which is

$$y = \frac{MH + mh}{2(m + M)}\,.$$

The mass of the cider can be computed as the product of the volume and the density ϱ, where the volume can be computed from the knowledge of the area of the cylinder's bottom S and the current level of the cider $m = Sh\varrho$. Using this, we can rewrite the first equation as

$$y = \frac{MH + h^2 S\varrho}{2(hS\varrho + M)}\,.$$

Since we are interested in the minimum, we have to compute the derivative of the relation with respect to h. The derivative should be zero (we are looking for a stationary point), hence we will obtain a quadratic equation

$$h^2 S\ \varrho + 2Mh - MH = 0\,.$$

Solving the equation and taking into account that only the positive root makes sense, we arrive to

$$y_{\min} = \frac{MH}{m_0}\left(\sqrt{1 + \frac{m_0}{M}} - 1\right)\,.$$

Plugging in the numbers, we get the level of cider to be 5.8 cm.

Solution without derivatives also exists and we leave it to readers.

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Problem FoL.12 ... Wien's filter

If you ever need to filter particles of a certain velocity from others, you can simply do that using perpendicular magnetic and electric fields. Let's assume both of them are homogeneous and oriented so that the particle flying through them flies in a perpendicular direction to both of them. The forces with which the fields are acting on the particles are anti parallel. What is the velocity the particle must have in order to reach the detector or in other words to pass the velocity filter which is aligned with the original direction of motion of the particle? Electric field intensity is $E = 9 \cdot 10^3\,\mathrm{V{\cdot}m^{-1}}$ and the magnetic induction is $B = 3 \cdot 10^{-2}\,\mathrm{T}$.

From the experimental physicist's life.

We want the particle to travel straight, hence the Lorentz force $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ has to be zero. Considering the correct orientation of both fields and the fact that the particle is traveling perpendicular to the magnetic induction, we can express the velocity as $v = E/B$. Plugging in the numbers, we get $v \doteq 3 \cdot 10^5\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.13 ... pseudo-ice tea

Kiki felt like having some ice tea, but she didn't know how to prepare it. So she tried to do it in the following way: she put $m_{\mathrm{w}} = 250\,\mathrm{g}$ of water at temperature $t_{\mathrm{w}} = 20\,^{\circ}\mathrm{C}$ into a kettle along with $m_i = 350\,\mathrm{g}$ of ice at temperature $t_i = 0\,^{\circ}\mathrm{C}$. After some time (precisely when the contents of the kettle reached the thermal equilibrium), she figured out that she might want to turn on the kettle. How long will it take to boil the water in the kettle, whose power input and efficiency are known to be $P = 1.8\,\mathrm{kW}$ and $80\,\%$ respectively?

Kiki was drinking tea while thinking of problems for the competition.

Before turning on the electric kettle, we have to compute the change in temperature of the water from t_{w} to t_1 at which the mixture of ice and water reaches equilibrium. Let us assume that not all of the ice will be melted. The mass of the remaining ice is then

$$m_{\mathrm{i}}' = m_{\mathrm{i}} - \frac{m_{\mathrm{w}} c (t_{\mathrm{w}} - t_1)}{l_{\mathrm{i}}}\,,$$

where $t_1 = 0^{\circ}\mathrm{C}$. This gives $m_{\mathrm{i}}' \doteq 287\,\mathrm{g} > 0\,\mathrm{g}$, hence our assumption was right. After turning on the kettle, the rest of the ice will melt and become liquid water. This water added to the original water is the total amount of water we want to heat up from temperature $t_1 = 0^{\circ}\mathrm{C}$ to $t_2 = 100\,^{\circ}\mathrm{C}$. The specific latent heat of fusion for ice is $l_{\mathrm{f}} = 334\,\mathrm{kJ \cdot kg^{-1}}$ and the specific heat capacity of water is $c = 4.18\,\mathrm{kJ \cdot kg^{-1}}$. We can also write $Pt\eta = Q$, where t is the time and Q is the heat energy necessary for the water to start boiling. Hence the time needed can be expressed as

$$\begin{aligned} t &= \frac{Q}{P\eta}\,,\\ t &= \frac{m_{\mathrm{i}}' + (m_{\mathrm{i}} + m_{\mathrm{w}}) c (t_2 - t_1)}{P\eta} = \frac{m_{\mathrm{i}} l_{\mathrm{f}} - m_{\mathrm{w}} c (t_{\mathrm{w}} - t_1) + (m_{\mathrm{i}} + m_{\mathrm{w}}) c (t_2 - t_1)}{P\eta}\,. \end{aligned}$$

Plugging in the numbers, we get $t \doteq 241\,\mathrm{s}$ ($t \doteq 4\,\mathrm{min}\,1\,\mathrm{s}$).

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Problem FoL.14 ... climbing acetone

Find the elevation of acetone at temperature $20\,^{\circ}\mathrm{C}$ *in a capillary with diameter* $0.6\,\mathrm{mm}$. *Surface tension of acetone is* $0.023\,4\,\mathrm{N \cdot m^{-1}}$. *Your result should be stated in centimetres.*

Kiki remembered a set of problems from physical chemistry.

Force due to surface tension causing elevation is $F = \sigma l$, where σ is surface tension and l is the length of the surface rim. Elevation force is in equilibrium with gravitational force $F_{\mathrm{g}} = mg$, where m is the mass of the column of liquid. The length of the rim can be written as $l = 2\pi r$, where r is the radius of the capillary. The mass m can be expressed as $m = \varrho V = \varrho \pi 2h$, where ϱ is the density of the acetone and h is the height of the column of liquid we are trying to find. Equating the forces, we have

$$h = \frac{2\sigma}{r\varrho g}\,.$$

Plugging in the numbers yields $h = 2.0\,\mathrm{cm}$. The density of acetone can be found on the Internet, it is approximately $\varrho = 790\,\mathrm{kg{\cdot}m^{-3}}$.

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Problem FoL.15 ... the round one

Imagine a small glass ball with the property that it focuses parallel rays of light incident perpendicularly on its surface onto its back side. What is the refractive index of the glass the ball is made of? Work in paraxial approximation. The result should be expressed as a multiple of the refractive index of ball's surroundings.

Zdeněk found a very suspicious ball in his drawer.

Let us draw the ray diagram of light passing through the glass ball and denote the distances and angles as in the figure. From the isosceles triangle with two equal sides R we can write

$$\alpha = 180^\circ - (180^\circ - 2\beta) = 2\beta\,.$$

Using paraxial approximation $b \ll R$, we can write Snell's law as $n_1\alpha = n_2\beta$, where n_1 is the refractive index of the ball's surroundings and n_2 is the refractive index of the ball itself. Substituting for α yields in $n_2 = 2n_1$, hence the refractive index of the sphere must be twice as big as the refractive index of the environment.

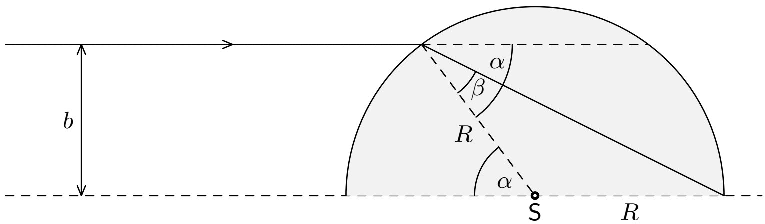

Fig. 2: Light deflection in sphere

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Problem FoL.16 ... the bulb and the capacitor

When Oliver was in New York, he bought a GE 31546 60A1 P VRS ES 110 120V BE type light bulb. When he came back to Czech republic, he didn't really want to throw it away, so in order for the bulb to work in Czech republic he had to connect it in series with an ideal capacitor. What should be the capacitance of this capacitor so that the correct nominal value of the voltage across the light bulb is obtained? The alternating current in European distribution network has frequency $50\,\mathrm{Hz}$ *and the household outlet voltage in Czech Republic is* $230\,\mathrm{V}$. *Do not take into account the internal resistance of the source. The answer should be stated in* μF.

Jimmy was inspired by a friend of his, who bought a wrong bulb.

Since the capacitor and the light bulb are connected in series, we can assume that per unit time, there is same charge passing through both of the them, so the current has to be the same through both the bulb and the capacitor. This current can be obtained from the characteristics of the bulb as $I = P/U = 0.5\,\mathrm{A}$. The capacitor can be described by

$$X_\mathrm{C} = \frac{1}{\omega C} \Rightarrow C = \frac{1}{2\pi\nu X_\mathrm{C}} = \frac{I}{2\pi\nu U_\mathrm{C}}\,.$$

It remains to determine the voltage across the capacitor. Current leads the voltage so we can write $U^2 = U_C^2 + U_\mathrm{B}^2$, since the bulb acts as a resistance and so there is no phase difference between the current and the voltage across the bulb. From this we get our final expression

$$C = \frac{P}{2\pi\nu U_\mathrm{Ž}\sqrt{U^2 - U_\mathrm{Ž}^2}} \doteq 8.11\,\mathrm{\mu F}\,.$$

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Problem FoL.17 ... the rod on the wires

Consider a rod of mass $m = 97\,\mathrm{kg}$ suspended on two steel wires Q and W with equal radii $r = 1.3\,\mathrm{mm}$ and elastic moduli $E = 210 \cdot 10^9\,\mathrm{Pa}$ in such a way that the rod is parallel to the horizontal, as we can see in the figure. Wire Q was originally (before we used it to suspend the rod) $l_0 = 2.7\,\mathrm{m}$ long while wire W was by $\Delta l = 2\,\mathrm{mm}$ longer. Let us denote the horizontal distances of the wires from rod's center of mass by d_Q, d_W, as we did in the figure. What is the ratio d_Q/d_W? The acceleration due to gravity is $g = 10\,\mathrm{m\cdot s^{-2}}$. The result should be stated up to two significant figures. Assume that the radii of the wires stay unchanged.

Domča was playing with ropes made of construction steel.

The wire Q behaves according to the Hook's law, so the force acting on it must satisfy

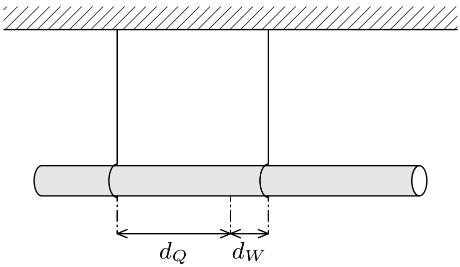

Fig. 3: Rod on wires

$F_Q = SE\Delta l/l_0 = \pi r^2 E\Delta l/l_0$. This force, along with the force F_W with which the rod acts on the wire W, must compensate the gravitational force mg, so we have $F_W = mg - F_Q$. Since the rod is at rest and there is no torsion, the torques acting on it must be in equilibrium – relative to the centre of mass, we can write

$$F_\mathrm{Q} \cdot d_\mathrm{Q} = F_\mathrm{W} \cdot d_\mathrm{W}\,.$$

Now, the ratio can be written easily as

$$\frac{d_\mathrm{Q}}{d_\mathrm{W}} = \frac{F_\mathrm{W}}{F_\mathrm{Q}} = \frac{mg - \pi r^2 E \Delta l / l_0}{\pi r^2 E \Delta l / l_0}\,.$$

Plugging in the numbers, the ratio of the lengths is 0.17.

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Problem FoL.18 ... totem

In one of the Plitvička lakes, there is a place where the lake is 2 m *deep and a vertical totem sticks out of the water reaching to the height of one meter above the water level. The rays coming from the setting Sun located at the altitude of* 30° *above the horizon are incident on the totem. How long is the shadow (in meters) cast by the totem onto the bottom of the lake?*

Dominika was watching Karl May's classics.

The length of the shadow cast on the water level is s_h, while its length cast on the bottom of the lake is s_d. Refractive indices of water and air are $n = 1.33$ and $n' = 1$ respectively. We know that the angle of incidence of the light is 60°, so the light casts a shadow on the water level with a length of $s_\mathrm{h} = (1\,\mathrm{m})/(\mathrm{tg}\,30^\circ)$. The light is then refracted according to Snell's law

$$n' \sin 60^\circ = n \sin \alpha\,,$$

where *alpha* is the angle of refraction of the rays entering the water. The shadow on the bottom of the lake will be longer due to these refracted rays, so

$$s_\mathrm{d} = s_\mathrm{h} + 2\,\mathrm{m} \cdot \mathrm{tg}\left(\arcsin\left(\frac{1}{n}\sin 60^\circ\right)\right) = 3.45\,\mathrm{m}\,.$$

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Problem FoL.19 ... solar mania

The Sun shines onto the Discworld at an angle of 39° *measured relative to its horizon. In such a situation the illuminance cast on its surface is* $E_1 = 80 \cdot 10^3\,\mathrm{lx}$. *Find the illuminance cast on the surface of the Discworld when the Sun is only* 30° *above the horizon.*

f(Aleš) was lacking light.

For the illumination cast on a surface we have

$$E = \frac{I}{h^2}\cos\alpha = \frac{I}{h^2}\cos\left(\frac{\pi}{2} - \beta\right)\,,$$

where α is the angle of incidence of the incoming rays and β is the angle which the rays make with the horizon. For the ratio of illuminances for both angles of incidence we can write

$$\frac{E_2}{E_1} = \frac{\frac{I}{h^2}\cos\left(\frac{\pi}{2} - \beta_2\right)}{\frac{I}{h^2}\cos\left(\frac{\pi}{2} - \beta_1\right)}\,.$$

From this equation, we can write

$$E_2 = E_1 \frac{\cos\left(\frac{\pi}{2} - \beta_2\right)}{\cos\left(\frac{\pi}{2} - \beta_1\right)}\,.$$

Plugging in the numbers, we get $E_2 \doteq 63\,600\,\mathrm{lx}$.

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Problem FoL.20 ... a decaying one

Imagine we have a $20.0\,\mathrm{g}$ sample of an unknown radioactive element whose nuclei are known to contain 232 nucleons. In the sample, $2.12 \cdot 10^{11}$ decay events per minute are observed to occur. Find the half-life of the element. The product of the decay is assumed to be stable.
f(Aleš) having fun during a lecture on nuclear physics.

For the number of particles we can write

$$N_0 = \frac{m}{Am_\mathrm{u}}\,,$$

where A is the mass number, m_u atomic mass unit and m is the mass of the radioactive material considered. The number of decay events is given by

$$N' = N_0 \lambda t\,,$$

where t is time and λ is the decay constant, which can be expressed in terms of the above defined quantities as

$$\lambda = \frac{N' A m_\mathrm{u}}{mt}\,.$$

For the half-life T we can derive

$$T = \frac{\ln 2}{\lambda}\,.$$

Substituting for λ, we get

$$T = \frac{mt \ln 2}{N' A m_u}\,.$$

Plugging in the given numerical values and taking $m_\mathrm{u} \doteq 1.66 \cdot 10^{-27}\,\mathrm{kg}$, we obtain the half-life of $T \doteq 1.02 \cdot 10^{13}\,\mathrm{s}$.

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Problem FoL.21 ... thermo-sphere

It is widely known that ordinary light bulbs emit a substantially larger portion of their radiation in infrared than they do in visible. Imagine our heating went on strike and we wish, using our light bulb, to warm up our hands from the temperature $T_1 = 15\,^\circ\mathrm{C}$ to $T_2 = 35\,^\circ\mathrm{C}$. We go on and cover the bulb entirely by our hands, thus exploiting all the thermal power emitted by the incandescent tungsten filament. Find the time needed to bring our frozen hands to the required temperature given the temperature of the tungsten filament is known to be $T_W = 3\,000\,\mathrm{K}$, its *length $l = 10^{-1}$ m and its diameter $d = 10^{-4}$ m. We estimate the mass and the heat capacity of our hands as $m = 1\,\mathrm{kg}$ and $c = 3\,000\,\mathrm{J{\cdot}K^{-1}{\cdot}kg^{-1}}$ respectively.*

Mirek and his striking heating.

The total amount of heat necessary to warm our hands is given by $Q = mc(T_2 - T_1)$. The total radiative power of the light bulb can be inferred from the Stefan-Boltzmann law as $P = \sigma \pi d l T_{\mathrm{W}}^4$. The time needed is then given by

$$t = \frac{Q}{P} = \frac{mc(T_2 - T_1)}{\sigma \pi d l T_{\mathrm{W}}^4}\,.$$

Substituting the numerical values, we obtain $416\,\mathrm{s} = 6\,\mathrm{min}\,56\,\mathrm{s}$. Hence we observe that the light bulb can be used as a small heater.

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Problem FoL.22 ... analogue hydrometer

Due to its higher density, cold water stays close to the bottom of a rectangular vessel which is filled up to the height of $h = 30\,\mathrm{cm}$. We assume that the density of water in the vessel grows linearly with increasing depth – at the water level, the density is equal to $\varrho_l = 996\,\mathrm{kg{\cdot}m^{-3}}$, while the density ϱ_{b} at the bottom of the vessel is unknown. Determine this density using the fact, that a homogeneous rod with density $\varrho_{\mathrm{r}} = 997\,\mathrm{kg{\cdot}m^{-3}}$ and length h immersed in the water and fixed by one of its ends at the water level makes an angle of $\varphi = 60^\circ$ with the vertical.

Mirek was thinking about alternative measuring instruments.

At depth x, the density is determined by

$$\varrho(x) = \varrho_{\mathrm{l}} + \frac{x}{h}\left(\varrho_{\mathrm{b}} - \varrho_{\mathrm{l}}\right).$$

For the rod to be at equilibrium, it is required that the total torque acting on the rod is zero, i.e.

$$\int \mathrm{d}M = 0\,.$$

We can express the elementary torque acting on an infinitely small section of the rod as $\mathrm{d}M = x(\mathrm{d}F_{\mathrm{vz}} - \mathrm{d}F_{\mathrm{g}})$ where the elementary forces $\mathrm{d}F_{\mathrm{vz}}$ and $\mathrm{d}F_{\mathrm{g}}$ are given by

$$\mathrm{d}F_{\mathrm{vz}} = \frac{mg\varrho(x)}{\varrho_{\mathrm{r}} h}\mathrm{d}x\,,$$
$$\mathrm{d}F_{\mathrm{g}} = \frac{mg}{h}\mathrm{d}x\,,$$

where m is the mass of the rod. Integrating from 0 to $h\cos\varphi$ (where our x-axis is directed vertically downwards and $h\cos\varphi$ is the x-coordinate of the lower end of the rod), we have

$$\int_0^{h\cos\varphi} x\left(\frac{mg}{\varrho_{\mathrm{r}} l}\left(\varrho_{\mathrm{l}} + \frac{x}{h}(\varrho_{\mathrm{b}} - \varrho_{\mathrm{l}})\right) - \frac{mg}{l}\right)\mathrm{d}x = 0\,,$$
$$\frac{\varrho_{\mathrm{l}}}{2\varrho_{\mathrm{r}} l} + \frac{h\cos\varphi}{3\varrho_{\mathrm{r}} h l}\left(\varrho_{\mathrm{b}} - \varrho_{\mathrm{l}}\right) - \frac{1}{2l} = 0\,.$$

It remains to express ϱ_b and substitute the numerical values. Eventually, we obtain

$$\varrho_\mathrm{b} = \frac{3}{2\cos\varphi}\left(\varrho_\mathrm{r} - \varrho_\mathrm{l}\right) + \varrho_\mathrm{l} \doteq 999\,\mathrm{kg{\cdot}m^{-3}}\,.$$

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Problem FoL.23 ... pressurized box

Assume we have $n = 1\,\mathrm{mol}$ of carbon dioxide (CO_2) in a closed vessel with volume $V = 1\,\mathrm{l}$. The vessel is in thermal equilibrium with its surroundings at temperature $T = 297\,\mathrm{K}$. How does the estimate of pressure in the vessel based on the ideal gas law (denote p_id) differ from the the estimate based on the van der Waals equation (1) of state for non-ideal fluid (denote p_Waals)?

$$\left(p_\mathrm{Waals} + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \tag{1}$$

Determine $\left(p_{id} - p_\mathrm{Waals}\right)/p_\mathrm{id}$. Use the following values of a, b for CO_2:

$$a = 0.3653\,\mathrm{Pa{\cdot}m^6{\cdot}mol^{-2}}\,,$$
$$b = 4.280\cdot 10^{-5}\,\mathrm{m^3{\cdot}mol^{-1}}\,.$$

The molar gas constant is $R = 8.31\,\mathrm{J{\cdot}K^{-1}{\cdot}mol^{-1}}$.

Karel wanted to mention van der Waals gas.

Let us express the estimates of pressure from both equations

$$p_\mathrm{id} = \frac{nRT}{V}\,,$$
$$p_\mathrm{Waals} = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}\,.$$

It remains to calculate the ratio

$$\frac{p_\mathrm{id} - p_\mathrm{Waals}}{p_\mathrm{id}} = 1 - \frac{V}{V - nb} + \frac{na}{RTV} \doteq 10.3\,\%\,.$$

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Problem FoL.24 ... world champion in high jump

Brian Griffin (a dog) has one spring mounted to each of his hind legs. Each of these springs has an unstretched length of $l = 0.5\,\mathrm{m}$ and spring constant $k = 3\cdot 10^5\,\mathrm{kg{\cdot}s^{-2}}$. The dog then jumps to a vertical height of $h = 10\,\mathrm{m}$ and when he falls down, the springs become hooked into the ground so the dog starts oscillating. What is the amplitude of the undamped oscillations for a dog weighing $m = 500\,\mathrm{kg}$? Assume gravitational accleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek envied Brian his funny means of transport.

At the top of his trajectory, the dog has potential energy $E_1 = mgh$. After the impact, the springs absorb energy $E_\mathrm{p} = \frac{1}{2}(k+k)y^2$ and the dog retains energy $E_2 = mg(l-y)$ where y is the maximum compression of each spring. Using the law of conservation of mechanical energy, we obtain a quadratic equation for y

$$ky^2 - mgy - mg(h-l) = 0\,.$$

Using the given numerical values, its positive root is found to be $y \doteq 0.402\,\mathrm{m}$. However, this is not the amplitude. To obtain the amplitude, we have to subtract the compression of springs at equilibrium $y_0 = mg/2k \doteq 0.008\,\mathrm{m}$. Hence the amplitude is $y_\mathrm{a} \doteq 0.394\,\mathrm{m}$.

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Problem FoL.25 ... pulleys

Calculate the vertical acceleration of the weight with mass m_1 in the pulley system depicted in the figure, assuming that the system is initially at rest. The masses of individual weights are $m_1 = 400\,\mathrm{g}$, $m_2 = 200\,\mathrm{g}$, $m_3 = 100\,\mathrm{g}$. Use $g = 10\,\mathrm{m\cdot s^{-2}}$ as the value for the acceleration due to gravity. Pulleys and strings are weightless and friction can be neglected.

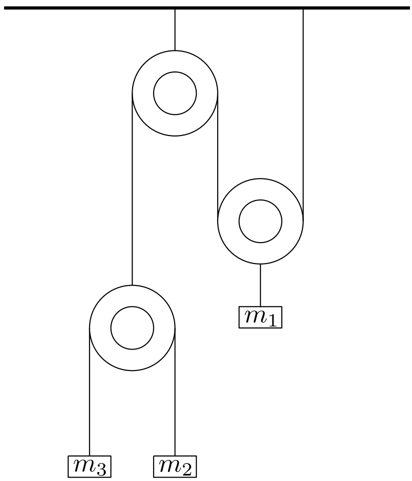

Fig. 4: Pulleys machine

Mirek was amazed at simple machines.

Let us denote by T_2 the tension in the string joining weights 2 and 3. Similarly, let T_1 be the tension in the string attached to the leftmost free pulley, wound around the fixed pulley and the rightmost free pulley. Accelerations of the individual weights are denoted by a_1, a_2 and a_3,

a_4 is the acceleration of the leftmost pulley. Assume that all the above defined accelerations are directed downwards. Then we can describe the system with following equations

$$\begin{aligned} m_1 a_1 &= m_1 g - 2T_1 \,, \\ m_2 a_2 &= m_2 g - T_2 \,, \\ m_3 a_3 &= m_3 g - T_2 \,, \\ T_1 &= 2T_2 \,, \\ 2a_1 &= -a_4 \,. \end{aligned}$$

We define the weights' accelerations relative to the leftmost pulley as a_2' and a_3' respectively. For these we can write $a_2' = -a_3'$, $a_2 = a_2' + a_4$ and $a_3 = -a_2' + a_4$, which we use to derive the sixth equation

$$a_1 = -\frac{1}{4}(a_2 + a_3)\,.$$

We use the last three equations to substitute into the first three, so we obtain

$$\begin{aligned} -\frac{1}{4} m_1 (a_2 + a_3) &= m_1 g - 4T_2 \,, \\ m_2 a_2 &= m_2 g - T_2 \,, \\ m_3 a_3 &= m_3 g - T_2 . \end{aligned}$$

Adding the second equation to the third and comparing the result with the first, we get

$$\frac{16T_2}{m_1} - 4g = 2g - T_2 \left(\frac{1}{m_2} + \frac{1}{m_3} \right) .$$

Solving this for T_2 yields

$$T_2 = 6g \frac{m_1 m_2 m_3}{m_1 m_2 + m_1 m_3 + 16 m_2 m_3} .$$

Now we substitute for T_2 into the equations relating accelerations $a_2 = g - T_2/m_2$, $a_3 = g - T_2/m_3$ and using fifth and sixth relation, we get

$$a_1 = g \frac{m_1 m_2 + m_1 m_3 - 8 m_2 m_3}{m_1 m_2 + m_1 m_3 + 4 m_2 m_3} = -2\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem FoL.26 ... boxed light

How many photons of blue light does it take to achieve pressure $p = 1\,\mathrm{bar} = 10^5\,\mathrm{Pa}$ *in an empty cube-shaped box? The wavelength of the photons is* $\lambda = 450\,\mathrm{nm}$ *and the edge length of the cube is* $a = 10\,\mathrm{cm}$*. The interior of the cube is perfectly reflective.*

Thermodynamics class acted upon Karel.

We will use the same reasoning as when deriving the pressure of an ordinary gas. A wall of area S is being hit by $Nct/(6V)$ photons over a time period t, where N is the number of particles, V is the volume of the box, c is the speed of light and $1/6$ is the particles' effective direction factor – only one sixth of them is moving in the direction of positive x-axis. When a photon hits the wall, its momentum is changed by $2h/\lambda$. The change in total momentum of all photons reflected at the area S during time period t is

$$\Delta p_{\mathrm{m}} = \frac{hcNtS}{3\lambda V}\,.$$

Force is defined as the rate of change of momentum while pressure is a force per unit area acting perpendicularly to a surface. Thus

$$p = \frac{hcN}{3\lambda V}\,.$$

We need to find the number of particles

$$N = \frac{3p\lambda V}{hc} \doteq 6.79 \cdot 10^{20}\,.$$

With this number of particles, the radiation energy density in the box is about $3 \cdot 10^5\,\mathrm{J{\cdot}m^{-3}}$, which is roughly six orders of magnitude greater than the radiation energy density at the surface of the Sun.

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Problem FoL.27 ... enlightened carriage

A carriage of mass $m = 100\,\mathrm{g}$ can move along rails without friction. Assume there is a vertical mirror attached to one side of the carriage. We then focus all the light emitted by a light bulb with power $P = 60\,\mathrm{W}$ into a ray which is incident perpendicularly onto the mirror. Assuming that the carriage was initially at rest, how long will it take the carriage to travel a distance $l = 1\,\mathrm{m}$ (in seconds)? You can assume that the mirror is perfectly reflective and that the whole power of the light bulb transferred into radiation.

Jakub wanted a contactless turbo-propulsion.

The light which is reflected from the mirror possesses certain momentum. After the reflection, the sign of the momentum will change. Since the momentum is conserved, the carriage must have gained some momentum in the process. Momentum of a photon with wavelength λ is $p = h/\lambda$, where h is the Planck's constant. Force acting upon the carriage can be computed by dividing the change in momentum by the time period over which the change took place, i.e.

$$F = \frac{\Delta p}{\Delta t}\,.$$

The change in momentum of a photon is is equal to twice its momentum itself since, upon reflection, the photon changes the direction in which it moves. The relation between wavelength λ and the frequency f of a photon is $\lambda = c/f$. Substituting for λ from this, we get

$$F = \frac{2hf}{c\Delta t}\,.$$

Energy E of a photon is $E = hf$. Substituting for E from this, we arrive at

$$F = \frac{2E}{c\Delta t}\,.$$

This energy is the energy of a photon (photons) reflected over a time period of Δt and it also corresponds to the energy output of the light bulb over the same period. Dividing the energy by the time period we obtain the power of the bulb P

$$F = \frac{2P}{c}\,.$$

This constant force causes constant acceleration

$$a = \frac{F}{m} = \frac{2P}{cm}$$

of the carriage. Thus the time t needed for the carriage to cover distance l satisfies $l = at^2/2$. Expressing t from this and plugging in the numbers, we get

$$t = \sqrt{\frac{lcm}{P}} \doteq 707\,\mathrm{s}\,.$$

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Problem FoL.28 ... Earth-cylinder

Assume that the Earth turned into an infinite cylinder whose radius and density are the same as those of our real Earth with the distance to the Moon also remaining unchanged. What will be the speed of the Moon (which remains spherical) in its orbit around the cylinder? Assume that the Earth's radius and density are $6\,378\,\mathrm{km}$ *and* $5\,515\,\mathrm{kg{\cdot}m^{-3}}$ *respectively. Also consider the Moon's orbit around the Earth to be circular.*

Tomáš Bárta imagining alternative universes.

Let us denote the gravitational field intensity by $\boldsymbol{K}$. We will treat the gravitational field in much the same way as we usually do with electromagnetic field. Gauss' law can be written as

$$\oint_S \boldsymbol{K} \cdot \mathrm{d}\boldsymbol{S} = 4\pi G M\,.$$

We choose a cylinder with radius r and length l as our gaussian surface. The mass enclosed by such a cylinder is $\pi R_\mathrm{E}^2 l \varrho_\mathrm{E}$. Rearranging the above expression for the flux of the gravitational field through the cylinder, we get

$$2\pi r l K(r) = 4\pi G \cdot \pi R_\mathrm{E}^2 l \varrho_\mathrm{E}\,,$$
$$K(r) = \frac{2}{r} G\pi\varrho_E R_\mathrm{E}^2\,.$$

Finally we equate the magnitudes of centripetal acceleration and the gravitational field intensity and we obtain

$$\frac{v^2}{r} = \frac{2}{r} G\pi\varrho_\mathrm{E} R_\mathrm{E}^2\,,$$
$$v = R_\mathrm{E}\sqrt{2\pi\varrho_\mathrm{E} G} \doteq 9\,700\,\mathrm{m{\cdot}s^{-1}}\,.$$

Therefore we have reached an interesting conclusion that the speed of the Moon in its orbit around cylindrical Earth does not depend on the distance of the Moon from the axis of the cylinder.

Problem FoL.29 ... pass me the hammer

An astronaut dropped a tool bag during one of his spacewalks, giving it an impulse. The bag then started receding from the spaceship along a straight line until it reached the distance $l = 180\,\mathrm{m}$ from the ship. At this distance, the speed of the bag relative to the spaceship was zero. How many days did it take the bag to get to that spot? The mass of the bag and the mass of the spaceship are known to be $m_1 = 50\,\mathrm{kg}$ and $m_2 = 500\,\mathrm{kg}$ respectively. You can neglect any effects due to gravity of other bodies. *Lukáš remembered the infamous ISS bag.*

The motion of our ill-fated bag will obey Kepler's laws of planetary motion. Note that straight line is only a special (degenerate) case of an ellipse, where the numerical eccentricity is equal to 1. Hence the distance l is twice the semi-major axis of such an ellipse. We see that it took the bag half of its orbit to reach the described point (apoapsis). Using third Kepler's law we have

$$\frac{\left(\frac{l}{2}\right)^3}{(2t)^2} = \frac{G(m_1 + m_2)}{4\pi^2}\,,$$

$$t = \sqrt{\frac{\pi^2 l^3}{8G(m_1 + m_2)}}\,.$$

Plugging in the numbers, we get $t \doteq 162\,\mathrm{days}$.

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Problem FoL.30 ... a messy one

A chemist was planning to prepare $100\,\mathrm{ml}$ of potassium permanganate solution with concentration $c_1 = 0.000\,5\,\mathrm{mol{\cdot}dm^{-3}}$, so he placed the required amount of potassium permanganate into a measuring flask, poured in distilled water and mixed the solution carefully. But then an accident happened and he spilled a part of the solution. He was very lazy and since nobody else saw him, he just topped the solution up to the original volume with distilled water and pretended that nothing happened. Then he started to feel bad about the whole incident, so he took a sample of his solution into a $1\,\mathrm{cm}$ long cuvette and put it into a spectrophotometer. From the subsequent measurement he found that having passed through the sample, the intensity of monochromatic light of wavelength $526\,\mathrm{nm}$ dropped by $90\,\%$ compared to its original value. What was the volume of the solution the chemist spilled assuming that his measurements were precise? Molar absorption coefficient of potassium permanganate for the above given wavelength is $2\,440\,\mathrm{cm^2{\cdot}mmol^{-1}}$. You should give your answer in millilitres.

Kiki will once become a real pharmacist.

In this problem, we will make use of Lambert–Beer law $A = \varepsilon c l$, where A is the absorbance, for which we can write $A = \log\left(I_0/I\right)$, where I_0 is the intensity of light before it passes through the sample, I is the intensity of light after it passes through the sample, c is the concentration of the given constituent in the sample, l is the length of the cuvette and ε is the molar absorption coefficient. Knowing this, we can compute the present concentration of potassium permanganate in the sample as

$$c_2 = \frac{\log \frac{I_0}{I}}{\varepsilon l}\,.$$

Now we can compute the amount of potassium permanganate in the solution as $m = nM = cVM$, where $M = 158\,\mathrm{g\cdot mol^{-1}}$ is the molar mass of potassium permanganate. Plugging in the numbers, we would obtain numerical values for m_1 (100%) and m_2 (x%). Now we can compute $(1 - x/100) \cdot 100\,\mathrm{ml}$, which will give us the volume of solution spilled (remember, the solution was perfectly mixed so we are allowed to use direct proportion), which corresponds to a volume of 18 ml.

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Problem FoL.31 ... valuing the decays

An isotope of gold ^{173}*Au has a half-life of* $T_{\mathrm{Au}} = 59.0\,\mathrm{ms}$. *It decays into an iridium isotope* ^{169}*Ir by emitting an alpha particle. This iridium isotope has a half-life of* $T_{\mathrm{Ir}} = 0.400\,\mathrm{s}$ *and decays into* ^{165}*Re. Initially we had a pure sample of gold isotope* ^{173}*Au. Find the time at which the amount of gold in our sample will be the same as the amount of iridium. You can assume that masses of isotopes are directly proportional to their nucleon numbers.* *Karel was thinking hard.*

Since we have modern technology (such as a computer) at our disposal, we can make use of spreadsheet software like Excel or Numbers. Since the theory of multiple decay is considered to be at university level, we will assume that the solution was attempted using numerical simulations and we will try to reach it here in the same way. Our numerical simulations was created in MS Excel 2007, but any other software or programming language could have been used just as well to complete the task. The simulation can be found in a file published on our website. We used Euler's method, which is the most primitive one, but the easiest one for implementation. Initially we only had gold ^{173}Au, with its maximal initial mass, $m_{\mathrm{Au}}(0)$ in whose multiples any subsequent result will be stated. The time is sampled by 0.01 ms and is stored in column A. We are computing the mass loss in every time step and store it in column D. It is given by

$$\Delta m_{\mathrm{Au}}(t + \Delta t) = m_{\mathrm{Au}}(t) - m_{\mathrm{Au}}(t + \Delta t) = m_{\mathrm{Au}}(t) \cdot \left(1 - 2^{-\frac{\Delta t}{T_{\mathrm{Au}}}}\right)\,.$$

The instantaneous mass of gold is computed as a difference between the initial mass and the mass lost. It is written in column C. Column E then represents the gain of mass of Iridium in the sample, which will be 169/173 times the mass of the gold lost. This fraction was introduced so that we took into account the loss of mass resulting from the emission of an α-particle during the decay. Column F is used to store the instantaneous mass of Iridium 169, which is calculated as a sum of the original value for this mass plus the gain minus the loss of mass due a the further decay to ^{165}Re. This further loss is computed in column G. Column H, which contains the ratio of instantaneous masses of iridium and gold, was introduced in order to simplify our search for the moment, when the amount of gold and iridium were the same. Hence we are looking for the moment, when the value of this ratio reaches 1, which happens between the times 0.062 64 s and 0.062 65 s.

The task can be also solved analytically. It is more time consuming, but also more precise. Let us start with the following two ordinary differential equations

$$\frac{\mathrm{d}N_{\mathrm{Au}}}{\mathrm{d}t} = -\lambda_{\mathrm{Au}} N_{\mathrm{Au}}\,,$$
$$\frac{\mathrm{d}N_{\mathrm{Ir}}}{\mathrm{d}t} = -\lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} - \frac{\mathrm{d}N_{\mathrm{Au}}}{\mathrm{d}t} = -\lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} + \lambda_{\mathrm{Au}} N_{\mathrm{Au}}\,,$$

where $\lambda_{\mathrm{Au}} = \ln 2 T_{\mathrm{Au}}^{-1}$ and $\lambda_{\mathrm{Ir}} = \ln 2 T_{\mathrm{Ir}}^{-1}$. The solution of the first equation is trivial and we can directly substitute it in the second equation which yields

$$\frac{\mathrm{d}N_{\mathrm{Ir}}}{\mathrm{d}t} + \lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} = \lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{-\lambda_{\mathrm{Au}} t}\,,$$

where N_{Au0} is the initial mass of gold. We will multiply the new equation by $\mathrm{e}^{\lambda_{\mathrm{Ir}} t}$ and rearrange so that we get

$$\frac{\mathrm{d}N_{\mathrm{Ir}}}{\mathrm{d}t} \mathrm{e}^{\lambda_{\mathrm{Ir}} t} + \lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} \mathrm{e}^{\lambda_{\mathrm{Ir}} t} = \lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t}\,,$$

where we make use of the property of exponential that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda_{\mathrm{Ir}} t}\right) = \lambda_{\mathrm{Ir}} \mathrm{e}^{\lambda_{\mathrm{Ir}} t}$, which together with Leibnitz rule yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(N_{\mathrm{Ir}} e^{\lambda_{\mathrm{Ir}} t}\right) = \lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t}\,,$$

which is an easily integrable equation. In the integrated equation

$$N_{\mathrm{Ir}} e^{\lambda_{\mathrm{Ir}} t} = \frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t}}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}} + c,$$

we need to determine the integration constant c. Using the initial condition $N_{\mathrm{Ir}}(0) = 0$, we get

$$c = -\frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}}}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}}\,.$$

Hence we can express the amount of iridium as it depends on time as

$$N_{\mathrm{Ir}} = \mathrm{e}^{-\lambda_{\mathrm{Ir}} t} \frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \left(\mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t} - 1\right)}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}}\,.$$

We are looking for the time when $m_{\mathrm{Au}} = m_{\mathrm{Ir}}$. In other words

$$M_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{-\lambda_{\mathrm{Au}} t} = M_{\mathrm{Ir}} \mathrm{e}^{-\lambda_{\mathrm{Ir}} t} \frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \left(\mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t} - 1\right)}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}}\,,$$

where M_{Au}, M_{Ir} are molar masses of gold and iridium. Expressed in terms of their half-lives, this becomes

$$\frac{\ln\left(1 - \frac{M_{\mathrm{Au}}}{M_{\mathrm{Ir}}}\left(\frac{T_{\mathrm{Au}}}{T_{\mathrm{Ir}}}\right)\right)}{\frac{\ln 2}{T_{\mathrm{Au}}} - \frac{\ln 2}{T_2}}\,.$$

Plugging in the numbers, we get $t \doteq 0.062\,64\,\mathrm{s}$. Answers lying in the interval from 0.062 5 s to 0.062 8 s were all considered correct since various values for the time sampling could be chosen.

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Problem FoL.32 ... Flash

After an attack, the superhero Flash does not bother to stop. Instead, he circles around the Earth, and attacks again. Initially, his speed was $v = 0.8c$. During the first attack, he looses half of his momentum. After repeating the run around the Earth with his new speed and engaging in the second attack, he looses half of his momentum again. What is the ratio E_1/E_2 of energies released during the first and the second attack? According to dc.wikia.com, Flash's rest mass is $m_0 = 89\,\mathrm{kg}$.

Mirek watching superhero programmes.

The energy transferred to the enemy during an attack is equal to the loss of Flash's kinetic energy. Because the relativistic effects are important we use Flash's total energy expressed using the magnitude of four-momentum as $E = \sqrt{m_0^2c^4 + p^2c^2}$. Therefore, the kinetic energy is $E_\mathrm{k} = \sqrt{m_0^2c^4 + p^2c^2} - m_0c^2$. The energy losses during the first and the second attack are then given by

$$\begin{aligned}E_1 &= \sqrt{m_0^2c^4 + p^2c^2} - \sqrt{m_0^2c^4 + p^2c^2/4}\,,\\ E_2 &= \sqrt{m_0^2c^4 + p^2c^2/4} - \sqrt{m_0^2c^4 + p^2c^2/16}\,.\end{aligned}$$

For the momentum p we can substitute $p = \gamma m_0 v = (m_0vc)/\sqrt{c^2 - v^2}$, and after some rearrangements we arrive at the final formula

$$\frac{E_1}{E_2} = \frac{1 - \sqrt{1 - \frac{3}{4}\beta^2}}{\sqrt{1 - \frac{3}{4}\beta^2} - \sqrt{1 - \frac{15}{16}\beta^2}}\,,$$

where $\beta = v/c$. Using $\beta = 0.8$, the wanted ratio is $E_1/E_2 \doteq \pi$.

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Problem FoL.33 ... we need to go deeper

What is the moment of inertia (relative to its axis of symmetry o) of the lamina shown in the figure (the lamina contains the shaded parts only)? The shape is constructed in the following way: given a semicircle, we cut out semicircular holes of it with their radii being half of the radius of the original semicircle. We then insert four four-times smaller semicircles (two into each hole) in these holes, again with semicircular holes cut out of them and in these we then insert smaller semicircles, continuing ad infinitum. The mass of the plate is $m = 7\,\mathrm{kg}$, and the radius of the largest semicircle is $R = 40\,\mathrm{cm}$.

Xellos was thinking about the old competition days.

First, let us calculate the mass of the object in the figure as it depends on R. Note that it is just a semicircle with two shapes cut out of it which are basically the same shape as the one whose mass we are trying to compute only twice as small. The object is two-dimensional which means that if the smaller shapes are twice as small, their mass is four times as small. Denoting the surface density of the object as σ, the mass of the semicircle with radius R is $m_0 = \pi R^2\sigma/2$. The mass of our object must satisfy

$$m = m_0 - \frac{m}{2}\,,$$
$$m = \frac{\pi R^2\sigma}{3}\,.$$

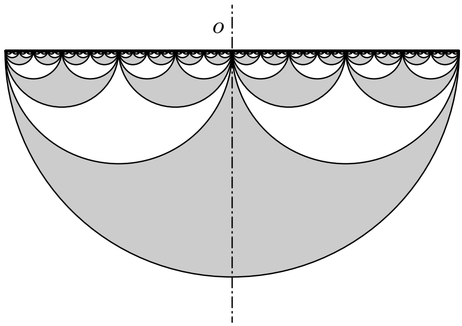

Fig. 5: Lamina

We can use similar reasoning to calculate its moment of inertia. First, the moment of inertia of the semicircle with radius R is $I_0 = m_0R^2/4$. Let us denote the wanted moment of inertia by I. Halving the radius, I decreases by a factor of 16, as it can be written as $I = KmR^2$, where K is an unknown constant. Using the parallel axis theorem, we find

$$\begin{aligned} I &= I_0 - 2\left(\frac{I}{16} + \frac{m}{4}\left(\frac{R}{2}\right)^2\right) = \frac{\pi R^4\sigma}{8} - \frac{I}{8} - \frac{\pi R^4\sigma}{24}\,,\\ I &= \frac{2\pi R^4\sigma}{27} = \frac{2mR^2}{9}\,. \end{aligned}$$

Plugging in the numbers, we can find the result to be $0.25\,\mathrm{kg\cdot m^2}$.

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Problem FoL.34 . . . little skewer

A thin rod of length $2l = 30\,\mathrm{cm}$ is placed into a hemispherical bowl of radius $R = 10\,\mathrm{cm}$. Assuming that the rod is at equilibrium, what is the angle α (in degrees) between the rod and the vertical? *f(Aleš) really liked a rod problem on the internet.*

Let us solve this problem using the principle of virtual work that can be stated as

$$\sum_{i=1}^{N} \boldsymbol{F_i}\cdot\delta\boldsymbol{r_i} = 0\,, \tag{2}$$

where $\boldsymbol{F}_i$ are the real forces acting on the system, and $\delta \boldsymbol{r}_i$ are the displacements allowed by the constraints. There is only one real force acting on the stick being the gravitational force $\boldsymbol{F}_\mathrm{G}$, which acts vertically downwards at the centre of mass of the stick. Let us choose our coordinate system so that the gravitational force acts along y-axis with x-axis perpendicular to the y-axis. Since there are no forces acting in the x direction, we can restrict our analysis to the y-component of motion.

The y coordinate of the center of mass can be written in terms of the angle α as

$$y = R\sin 2\alpha - l\cos\alpha,$$

and so the virtual displacement is

$$\delta y = \frac{\mathrm{d}y}{\mathrm{d}\alpha}\delta\alpha = (2R\cos 2\alpha + l\sin\alpha)\,\delta\alpha\,.$$

Equation (2) then states that

$$F\delta y = mg\,(2R\cos\alpha + l\sin\alpha)\,\delta\alpha = 0\,.$$

This equation has two solutions, but only the positive one is physically admissible. It is equal to

$$\sin\alpha = \frac{l}{8R} + \frac{1}{2}\sqrt{\frac{l^2}{16R^2} + 2}\,.$$

Plugging in the numbers we find $\alpha \doteq 67^\circ$.

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Problem FoL.35 ... tesseract

An expedition exploring the ocean floor has found a four-dimensional cube called tesseract or hypercube. Naturally, the explorers decided to investigate its physical properties, so they tried to melt it. They found out that it was made of an isotropic material with a large coefficient of linear thermal expansion which, in addition, was found to grow linearly with increasing temperature. Specifically, $\alpha_a(T_1) = 5\cdot 10^{-4}\,\mathrm{K}^{-1}$ and $\alpha_a(T_2) = 2\cdot 10^{-3}\,\mathrm{K}^{-1}$ for $T_1 = 300\,\mathrm{K}$ and $T_2 = 400\,\mathrm{K}$ respectively. Find the percentage increase in the 4-volume of the hypercube when heated from T_1 to T_2 given that at the temperature T_1, its edge length is $a = 10\,\mathrm{cm}$.

Mirek was thinking about the physics in Avengers.

The coefficient of linear thermal expansion is defined by

$$\alpha_a = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}T}$$

Similarly, for a change in volume we define

$$\alpha_V = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}T}\,.$$

The volume of a hypercube is $V = a^4$, and for very small changes in temperature we have

$$V + \mathrm{d}V = (a+\mathrm{d}a)^4 \approx a^4 + 4a^3\mathrm{d}a = V + 4V\frac{\mathrm{d}a}{a}.$$

Using $\mathrm{d}V = \alpha_V a^4 \mathrm{d}T$, $\mathrm{d}a = \alpha_a a \mathrm{d}T$, we arrive at

$$a^4 + a^4 \alpha_V \mathrm{d}T = a^4 + 4a^4 \alpha_a \mathrm{d}T\,,$$

so that $\alpha_V = 4\alpha_a$. The change in volume is obtained by a simple integration

$$\int_{V_1}^{V_2} \frac{\mathrm{d}V}{V} = \int_{T_1}^{T_2} \alpha_V(T)\mathrm{d}T\,,$$

$$\ln\frac{V_2}{V_1} = 2\left(T_2 - T_1\right)\left(\alpha_a(T_2) + \alpha_a(T_1)\right)\,.$$

The percentage increase is then

$$\frac{V_2 - V_1}{V_1} = \left(\mathrm{e}^{2(T_2 - T_1)(\alpha_a(T_2)+\alpha_a(T_1))} - 1\right) \cdot 100\,\% \doteq 64.9\,\%\,.$$

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Problem FoL.36 ... short hands

Mirek wanted to measure the dimensions of his room but all he could use was a 2 metres long folding rule. With each of his hands taking hold of one end of the ruler, he found out that to his surprise he could not straighten the ruler by spreading out his arms. What is the vertical distance between the lowest point of the sagged ruler and the horizontal line joining Mirek's hands? The ruler consists of four identical sections of length $l = 0.5\,\mathrm{m}$ and Mirek's arm span is $L = 1.8\,\mathrm{m}$. You should neglect friction and overlaps between the individual sections of the ruler. *Mirek discovered that height and the arm span are more or less the same.*

Exploiting the symmetry of the problem, it is sufficient to describe the system by angles α, β which the first and second section respectively make with the vertical. The total potential energy of the ruler can be expressed as

$$u(\alpha, \beta) = -2mg\frac{l\cos\alpha}{2} + 2mg\left(l\cos\alpha + \frac{l\cos\beta}{2}\right) = -mgl(3\cos\alpha + \cos\beta)\,,$$

where m is the mass of a single section of the ruler and we set the level of zero potential energy to coincide with the horizontal line connecting Mirek's hands. We would like to minimize potential energy subject to the length of the ruler being held fixed. This constraint can be expressed as

$$f(\alpha, \beta) = 2l(\sin\alpha + \sin\beta) - L = 0\,.$$

Using the method of Lagrange multipliers we see that the angles that minimize potential energy satisfy

$$\frac{\partial U}{\partial \alpha} - \lambda\frac{\partial f}{\partial \alpha} = 0\,,$$
$$\frac{\partial U}{\partial \beta} - \lambda\frac{\partial f}{\partial \beta} = 0\,,$$

where λ is the Lagrange multiplier. Expressing λ from these, equating both expressions obtained and taking derivatives we find

$$3\,\mathrm{tg}\,\alpha = \mathrm{tg}\,\beta\,.$$

A second equation for α and β follows from the constraint. This is a system of transcendental equations with the physically admissible solution $\alpha \approx 55.6\,°$, $\beta \approx 77.13\,°$. Using simple geometry, the wanted vertical distance of the lowest point can be expressed as

$$h = l(\cos\alpha + \cos\beta)\,,$$

which gives the result $h = 0.394\,\mathrm{m}$.

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Problem FoL.37 ... resistant Fibonacci

Let us consider a configuration of resistors with identical resistances $R = 1\,\Omega$ built as follows: first we connect two resistors in series, next to them we add two resistors connected in parallel followed by 3 resistors in parallel, then 5, 8, 13, ... resistors in parallel until we get a Fibonacci sequence of resistors. Find the resistance of such a configuration. *Mirek could not resist.*

Let us denote the n^{th} term of Fibonacci sequence by $F(n)$. The resistance of the n^{th} term of the sequence will therefore be $R/F(n)$. Hence the key question is to find an expression for $F(n)$.

The recurrence relation for Fibonacci sequence is as follows

$$F_n = F_{n-1} + F_{n-2}\,, \quad F_1 = 1\,, \quad F_2 = 1\,.$$

Solving the auxilliary equation $t^2 = t + 1$ we get two distinct roots $\varphi_+ = \left(1 + \sqrt{5}\right)/2$, $\varphi_- = \left(1 - \sqrt{5}\right)/2$. Having the initial conditions in mind, we know that the coefficients from our general solution must satisfy

$$c_1\varphi_+ + c_2\varphi_- = 1 c_1\varphi_+^2 + c_2\varphi_-^2 = 1\,.$$

Solving this system of equations for $c_{1,2}$, we obtain $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$, so we can write the solution as

$$F_n = \frac{\varphi_+ - \varphi_-}{\sqrt{5}}\,.$$

At this point we need to evaluate

$$R\sum_{i=1}^{\infty} \frac{1}{F_n}\,.$$

Resorting to numerical analysis and exploiting fast convergence of this series we get the rounded result as $3.36\,\Omega$.

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Problem M.1 ... stilettos

Which of the following bodies exerts a greater pressure on the ground and what is its value? A cube made of steel, its side 3 m *long or a woman weighing* $59.\bar{9}$ kg *wearing high heels with a diameter of* 5 mm*? Consider the situation when the entire weight of the woman rests on one heel.* *Monika stepped on a bug.*

Let us calculate the pressure exerted by the woman first. We have

$$p_1 = \frac{F_1}{S_1} = \frac{4m_1 g}{\pi d^2} \doteq 3.0 \cdot 10^7\,\mathrm{Pa}\,,$$

where we used $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. In order to calculate the pressure exerted by the cube, we need to know the density of steel ϱ_2. However, there are many varieties of steel, each with a different density. But note that for the cube to exert a pressure larger than p_1 we would need

$$\varrho_2 > \frac{p_1}{ag}\,,$$

where a is the side of the cube. Numerically we have $\varrho_2 > 1.0 \cdot 10^6\,\mathrm{kg{\cdot}m^{-3}}$, which differs from the density of ordinary metals by 3 orders of magnitude. Hence the pressure exerted by the woman must be larger.

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Problem M.2 ... traffic jam

Consider a car moving slowly in a traffic jam with a velocity of $v = 2\,\mathrm{m{\cdot}s^{-1}}$*. Find the rate of change of an angle under which a road sign placed* 2.5 m *above the ground is seen from the car at an initial distance of* $x = 30$ m *from the sign. The dimensions of the car and the road sign are not to be taken into account.* *Verča was observing the behaviour of drivers in a traffic jam.*

With respect to an observer in the car, the road sign moves with a relative horizontal velocity v. The observer sees the sign under the angle φ for which we can write

$$\sin\varphi = \frac{h}{\sqrt{h^2 + x^2}}\,.$$

Hence the component of the velocity of the road sign relative to the car, perpendicular to the line of sight of an observer in the car is

$$v_\mathrm{t} = v\ \sin\varphi\,.$$

Hence the rate of change of φ is

$$\omega = \frac{v_\mathrm{t}}{\sqrt{h^2 + x^2}} = \frac{v\ h}{h^2 + x^2} \doteq 5.5 \cdot 10^{-3}\,\mathrm{s^{-1}}\,.$$

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Problem M.3 ... very slow relativity

Consider two balls moving with velocities $0.4\,\mathrm{m\cdot s^{-1}}$ *and* $0.9\,\mathrm{m\cdot s^{-1}}$ *in the same direction. Before the collision, the slower ball moves ahead of the faster one and after the collision the balls stick together. Find out by how much the temperature of the balls will increase immediately after the collision, assuming that their temperatures before the collision were equal. The specific heat capacity of the material of the balls is* $c = 0.02\,\mathrm{mJ\cdot K^{-1}\cdot g^{-1}}$. *The balls are known to have identical masses.*

Janči was trying to symplify Lukáš's problem on relativity.

From the law of conservation of the momentum, we can get the resulting velocity of the balls as an aritmetic mean of their initial velocities. Then the change in the mechanical energy of the system (which will be entirely converted into heat) is

$$\Delta E = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 - m\left(\frac{v_1+v_2}{2}\right)^2 = \frac{1}{4}m(v_1-v_2)^2\,.$$

The change in temperature is equal to the added heat divided by the heat capacity i.e

$$\Delta t = \frac{\Delta E}{2mc} = \frac{(v_1-v_2)^2}{8c} \doteq 1.6\,^\circ\mathrm{C}\,.$$

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Problem M.4 ... splash

Find the minimal height above the water level at which we have to place the bottom end of a vertically oriented rod with length $l = 50\,\mathrm{cm}$ *in order for the entire rod to be submerged under the water when it is dropped. The density of the rod is half the density of water and the rod is slightly weighted at its bottom so that we don't have problems with stability.*

Lukáš was taking a bath.

We denote by ϱ, S, l, x the density of water, area of the cross-section of the rod, its total length and the submerged length respectively. The buoyancy force is then given by

$$F_\mathrm{b} = \varrho S\, xg\,.$$

Work done by this force during the fall of the rod is

$$W = \varrho S\, g\int_0^l x\,\mathrm{d}x = \frac{1}{2}\varrho S\, gl^2\,.$$

(It is actually the maximal potential energy of a spring with spring constant ϱSg.) Let us choose the level of zero gravitational potential energy just when the rod gets submerged entirely under water. Then it is true that

$$\frac{1}{2}\varrho S\, lgh = \frac{1}{2}\varrho S\, gl^2\,.$$

The left hand side of the equation represents the gravitational potential energy of the rod at height h relative to the above chosen level of zero gravitation potential energy. But from here it obviously follows that $h = l$, so initially, the bottom end of the rod has to be placed exactly at the water level.

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Problem E.1 ... flash

The battery in Janči's camera has the following specifications written on it: 3.6 V *and* 1 250 mAh. *The capacitor used in the flash has a capacity of* 90 µF *and it is charged to the voltage of* 180 V *each time the flash is used. Assume that during the process of charging the flash, exactly half of the energy is lost. How many pictures with flash can the camera shoot before the battery dies (assuming that initially, the battery was fully charged)? You can assume that the battery is ideal in the sense that its voltage does not decrease during its usage. A fair warning: the result should be an integer.* *Janči hates taking pictures with flash.*

There is an energy of $3.6\,\mathrm{V}\cdot 1.250\,\mathrm{Ah}\cdot 3\,600\,\mathrm{s{\cdot}h^{-1}}$ joules stored in the battery. One usage of flash requires (using the standard notation) the energy CU^2. The ratio between the energy needed for one flash and the total amount of energy available in the battery is 5 555 (rounded down). Hence we run out of the energy after this number of flashes.

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Problem E.2 ... tangled resistances

Find the current I through the source with voltage $U = 1\,\mathrm{V}$ assuming that each resistor has resistance $1\,\Omega$. *Janči was trying to think of a task, but not about the solution.*

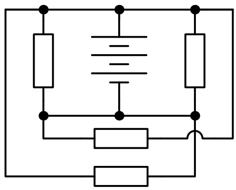

Fig. 6: Schematics

The lower two junctions where the three wires meet can be connected by another wire so that the meshed wires get disentangled. Redrawing the circuit, we see, that the source is connected in parallel to a pair of pairs resistors connected again in parallel. Hence the effective resistance of the circuit is one quarter of the resistance of one of the resistors. The current through the source is then 4 A.

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Problem E.3 ... aberration

Light with a wavelength of 400 nm *in vacuum has a wavelength of* 265 nm *when it passes through a lens, while light with a wavelength of* 700 nm *in vacuum has a wavelength of* 460 nm *when it passes through the same lens. Using blue light (vacuum wavelength* 400 nm*), the focal length of the lens is* 1 m. *By how much will the focal length change if we use red light (vacuum wavelength* 700 nm*) instead? Assume that the lens is thin.*

Janči trying to remember the parts of optics he actually liked.

Let us start with the lensmaker's equation for a thin lens

$$\frac{1}{f} = \frac{n - n_0}{n} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) ,$$

where n_0 is the refractive index of the lens' surroundings (in the case of vacuum we have $n_0 = = 1$) and n is the refractive index of the lens itself. We can write down two instances of this equation, one for each wavelength. Denoting the ensuing focal lengths by f_r (red light) and f_b (blue light) and dividing these two lensmaker's equations through one another, we obtain

$$f_\mathrm{r} = \frac{n_\mathrm{b} - 1}{n_\mathrm{r} - 1} f_\mathrm{b} ,$$

where the corresponding refractive indices n_r and n_b can be determined using $\lambda = \lambda_0/n$ where λ_0 is the vacuum wavelength and λ is the wavelength inside the lens. Hence the change in focal length is

$$f_\mathrm{r} - f_\mathrm{b} = f_\mathrm{b} \left(\frac{\lambda_\mathrm{r} \left(\lambda_\mathrm{b0} - \lambda_\mathrm{b} \right)}{\lambda_\mathrm{b} \left(\lambda_\mathrm{r0} - \lambda_\mathrm{r} \right)} - 1 \right) \doteq 0.024\,\mathrm{m} .$$

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Problem E.4 ... useless work

Consider a parallel-plate capacitor charged to a voltage of $1\,\mathrm{V}$*. In a direction perpendicular to the electric field lines between the plates, there is a magnetic field of* $2\,\mu\mathrm{T}$*. Distance between the two plates is* $d = 0.1\,\mathrm{mm}$*. Assume that an electron is emitted from the plate at lower potential with zero initial speed. Find the speed at which it arrives at the second plate.*

Xellos likes to be in the way.

Since magnetic field does not do any work the electron will gain kinetic energy $E = 1\,\mathrm{eV}$. In terms of speeds this means that

$$v = \sqrt{\frac{2E}{m_\mathrm{e}}} \doteq 593\,000\,\mathrm{m \cdot s^{-1}} .$$

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Problem X.1 ... let's spread out

At some temperature T, the cubic lattice of an iron crystal undergoes a transition from body-centred to face-centred phase. During this transition, the length of the unit cell edge increases by 22 %. Find the factor by which the density of the crystal will decrease.

Tomáš Bárta was thinking about iron.

In the body-centred phase, two atoms correspond to each cell while in the face-cetred phase, four atoms correspond to each. The mass corresponding to one cell will therefore double in magnitude and its volume will increase by the factor of 1.22^3. Hence the density of the crystal will decrease by the factor of $2/1.22 \cdot 10^3 \doteq 1.10$.

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Problem X.2 ... heating season

How many times do we have to let a bouncy ball with a mass of $m = 200\,\mathrm{g}$ fall on the ground (made of polyvinyl chloride with a density of $\varrho = 1380\,\mathrm{kg{\cdot}m^{-3}}$ and a specific heat capacity $c = 0.9\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$) from the height $h = 1\,\mathrm{m}$ so that a thermally well-isolated piece of the ground with area $S = 1\,\mathrm{m^2}$ and thickness $d = 1\,\mathrm{cm}$ would increase its temperature by one degree of Celsius? The coefficient of restitution k for collisions between the bouncy ball and the ground, defined as the ratio of the speed of the bouncy ball after a collision and its speed before the collision, is known to be 70 %. Use the following value for the acceleration due to gravity: $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Terka trying to warm her frozen feet by alternative means.

Before a collision with the ground, the bouncy ball has a total mechanical energy of mgh, which deacreases to $mghk^2$ after the collision. The amount of heat transferred to the ground during one collision is therefore equal to $mgh(1-k^2)$. If we denote by N the total number of collisions needed, we can write

$$Nmgh(1-k^2) = c\varrho S\, d\Delta T\,,$$

$$N = \frac{c\varrho S\, d\Delta T}{mgh(1-k^2)} \doteq 12\,413\,,$$

where we rounded the numerical value of N up to the nearest integer.

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Problem X.3 ... hexagonal sphere

The lattice of an alpha-boron nitride crystal consists of atomic layers in which the atoms are arranged in a hexagonal structure. We managed to acquire a sphere with radius $r = 1\,\mathrm{\mu m}$ made of this material. Find the maximum number of layers which can be intersected by a straight line intersecting the sphere. You are encouraged to look up any information needed.

Xellos having nightmares about chemistry.

The distance between individual layers can be found as $c = 6.66\,\text{Å}$ (e.g. `http://www.ioffe.rssi.ru/SVA/NSM/Semicond/BN/basic.html`). Clearly, the maximum number of layers will be intersected by a diameter of the sphere. Hence we find the answer to be

$$N = \frac{2r}{c} \doteq 3\,000\,.$$

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Problem X.4 ... firm or springy

Pistons in the fork of a bicycle have a cross-section of $S = 5\,\mathrm{cm}^2$ and height $h = 20\,\mathrm{cm}$. Assume that a manometer connected to the fork gives a reading of $p_1 = 100\,\mathrm{psi}$. By how much will the fork be compressed (in cm), when a $m = 60\,\mathrm{kg}$ man leans on it with all his weight? Assume ideal diatomic gas behaviour and that an adiabatic process takes place. Do not forget that the bike fork has two arms. *Mirek's been avoiding fixing his bike for some time already.*

First, let us write down the equation for adiabatic process with an ideal gas

$$p_1 V_1^{\kappa} = p_2 V_2^{\kappa}\,,$$

where κ is the Poisson's constant. Initially, the volume was $V_1 = Sh$ and after the compression, it changed to $V_2 = S(h - \Delta h)$, where Δh is the compression of the fork we are looking for. For the corresponding pressures we can write

$$p_2 = p_1 + \frac{mg}{2S}\,,$$

where the factor of a half comes from distributing the force into both arms of the fork. Substituting the relations for p_2, V_1, V_2 and calculating $1/\kappa$ power, we get

$$p_1^{1/\kappa} Sh = \left(p_1 + \frac{mg}{2S}\right)^{1/\kappa} S(h - \Delta h)$$

and rearranging

$$\Delta h = h\left(1 - \left(\frac{p_1}{p_1 + \frac{mg}{2S}}\right)^{1/\kappa}\right)\,.$$

Substituting $\kappa = 7/5$ and $1\,\mathrm{psi} = 6\,895\,\mathrm{Pa}$ yields $\Delta h \doteq 7.1\,\mathrm{cm}$.

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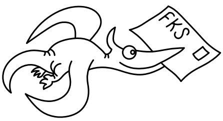

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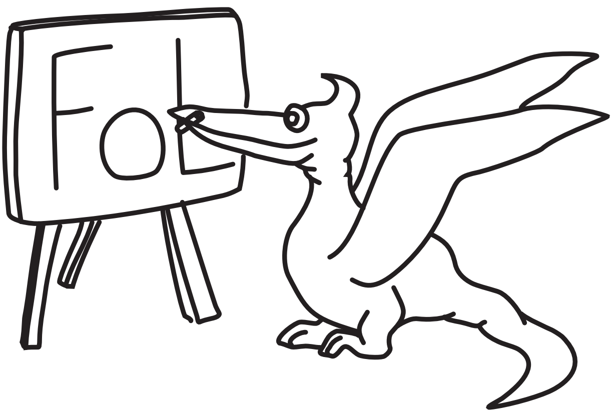
Solution of 4th Online Physics Brawl



Problem FoL.1 ... thoughtful

While sitting and thinking deeply, Aleš was throwing a little ball against a wall. Each time, he threw the ball with an initial speed $v_0 = 10\,\mathrm{m\cdot s^{-1}}$ at an angle $\alpha = 60^\circ$ from the vertical axis. Since he was sitting in a narrow corridor with width $d = 1.5\,\mathrm{m}$, the ball always bounced back and forth between the walls a couple of times. What is the maximum height (measured relative to ball's initial position) that the ball will reach along its trajectory? Consider all collisions to be perfectly elastic. The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek's mechanical mental image of periodic boundary conditions.

The ball leaves Aleš's hand at an elevation angle of $\pi/2-\alpha$. Initially, the ball follows a parabolic trajectory. When the ball hits the wall, the vertical component of its momentum is conserved and the horizontal one just flips its sign. Therefore, the ball will still follow a parabolic trajectory, which, however, will be mirrored at the point where the ball hits the wall. Therefore, the maximum height can be computed the same way as usually in the case without walls.

Yet, there is an easier path to follow, which makes use of conservation of energy. Assuming that the horizontal component of the ball's momentum is conserved, we can transform the problem into that of a projectile fired vertically upwards with initial speed $v_0 \sin(\pi/2-\alpha)$. The maximum height reached by the ball is therefore

$$h_{\max} = v_0^2 \sin^2(\pi/2-\alpha)/(2g) \doteq 1.27\,\mathrm{m}\,.$$

Hence, the ball climbs to the height $h_{\max} \doteq 1.27\,\mathrm{m}$ above its initial position.

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Problem FoL.2 ... Ded Moroz

Consider Mr Spherical (whose shape, incidentally, happens to be perfectly spherical) that likes to wear a tight-fitting jacket of negligible thickness (compared to his radius). Wearing his jacket, Mr Spherical went outside for a walk. Once the thermal equilibrium between Mr Spherical and his surroundings was established, the temperatures at the inner and outer surfaces of his jacket could be measured to be $25\,^\circ\mathrm{C}$ and $-5\,^\circ\mathrm{C}$, respectively. Find the temperature (in degrees Celsius), which is measured exactly in the middle of the layer of Mr Spherical's jacket (i.e. half-way through the bulk of the layer).

Sometimes, Lubošek feels really cold.

Since the thickness of the layer is negligible compared to Mr Spherical's radius, the boundary surfaces may be well approximated by planes, on which the temperature is kept constant. In such a case, the steady solution of the heat equation is linear in position inside the layer, so the temperature half-way through must be simply the arithmetic mean of the temperatures at both boundaries, i.e. $10\,^\circ\mathrm{C}$.

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Problem FoL.3 ... fiat nox

Consider a short-circuited two-conductor copper wire. At one end of the wire, a resistance of $R = 65.3\,\mathrm{m\Omega}$ was measured using an ohmmeter. What is the distance (along the wire) to the short-circuited section of the wire from the end where the measurement was performed?

The resistivity of copper is $\varrho = 1.71 \cdot 10^{-8}\,\Omega{\cdot}\mathrm{m}$, *the cross-sectional diameter of one conductor is* $d = 1.50\,\mathrm{mm}$.

A switch was tripped in Mirek's room.

The total resistance R of a conductor with length l, cross-sectional area $S = \pi d^2/4$ and resistivity ϱ is given by

$$R = \frac{\varrho l}{S}\,.$$

We must not forget that in our case, we are dealing with a two-conductor wire, so the distance along the wire is $l/2$. Hence, the distance between the short-circuited section and the place where the resistance was measured is

$$\frac{l}{2} = \frac{RS}{2\varrho} = \frac{R\pi d^2}{8\varrho} = 3.37\,\mathrm{m}\,.$$

Wanted distance is equal to 3.37 m.

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Problem FoL.4 ... the runnier, the better

Dominika is secretly preparing homeopathics at home. She uses a cup filled with a solution of volume $V_0 = 200\,\mathrm{ml}$ *and concentration of the active agent* $c_0 = 0.001\,\mathrm{M}$. *She then pours away a half of this volume and fills the cup with water to the original volume* V_0. *She repeats this process 70 times. Statistically speaking, how many molecules of the active agent are left in the cup at the end? Assume that the solution and water are always perfectly mixed.*

Mirek heard some gossip.

Assuming that the water and the solution are always perfectly mixed, the amount of the active agent in the solution is halved each time we repeat the process. Therefore, after i repetitions, the concentration of the solution is

$$c_i = \frac{c_0}{2^i}\,.$$

The amount of substance, given the known concentration and volume, can be written as

$$n_i = c_i V_0\,.$$

The number of molecules N_i is then given by multiplying this by the Avogadro constant N_A, i.e.

$$N_i = N_\mathrm{A} c_i V_0 = 2^{-i} N_\mathrm{A} c_0 V_0\,.$$

Finally, for $i = 70$ we get

$$N_{70} \doteq 0.10\,.$$

Hence, statistically speaking, only one tenth of a molecule of active agent is left in the cup.

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Problem FoL.5 ... brake failure

A truck experiences brake failure and needs to make an emergency stop. Luckily enough, it's just driving through a flood zone with lots of sand-filled bags. Assume that the mass of the truck is $M = 6\,\mathrm{t}$ and that one bag filled with sand weighs $m = 60\,\mathrm{kg}$. How many sand bags will the truck need to collide with until it slows down to $75\,\%$ of its initial speed $v_0 = 50\,\mathrm{km{\cdot}h^{-1}}$? Assume that the collisions are perfectly inelastic and that after colliding with the truck, the bags are pushed in front of it without friction.

Mirek was forced to replace people with sand bags.

Let n denote the number of collisions needed to slow the truck down. Also, denote the speed of the truck after n collisions by v_n and set $\alpha = v_n/v_0$. Conserving linear momentum, we can write

$$Mv_0 = (M+m)v_1 = (M+2m)v_2 = \ldots = (M+nm)v_n\,.$$

From this expression, we get

$$\frac{v_n}{v_0} = \alpha = \frac{M}{M+nm}\,,$$

where[1]

$$n = \left\lceil \frac{M}{m}\left(\frac{1}{\alpha} - 1\right)\right\rceil = 34\,.$$

Hence, the truck will slow down to $75\,\%$ of its initial speed after colliding with 34 sand bags.

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Problem FoL.6 ... fungus librariensis

In order to protect books from becoming damp, some libraries choose to keep the relative humidity at $\varphi_0 = 25\,\%$. Find the mass of water which would (after complete evaporation) turn such a library into a suitable environment for mould growth. You can assume that the ideal value of relative humidity for mould growth is $\varphi_p = 60\,\%$. The total volume of air in this particular library is $V = 180\,\mathrm{m^3}$, the (constant) ambient temperature is $t = 24\,^\circ\mathrm{C}$ and the saturated vapour density at $24\,^\circ\mathrm{C}$ is $21.8\,\mathrm{g{\cdot}m^{-3}}$.

Lydka likes to daydream about her private library.

Let us start with the defining relation for relative humidity,

$$\Phi = \varphi\Phi_{\mathrm{max}}\,,$$

where Φ is the absolute humidity and Φ_{max} is the mass of saturated vapour in a given volume at a given temperature. Also,

$$\Phi = \frac{m}{V}\,,$$

where m is the total mass of water vapour in volume V, so

$$m = \varphi\Phi_{\mathrm{max}}V\,.$$

[1] The notation $\lceil x\rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ is used for the ceiling function.

Denoting by m the mass of water required to increase the relative humidity from φ_0 to φ_p, we have $m = m_\mathrm{p} - m_0$, where m_0 and m_p are the total masses of water vapour at relative humidities φ_0 and φ_p, respectively. After substituting, we obtain

$$m = \Phi_\mathrm{max} V(\varphi_\mathrm{p} - \varphi_0) \doteq 1.37\,\mathrm{kg}\,.$$

Hence, in order to turn the given library into an ideal place for mould growth, we would need to evaporate 1.37 kg of water.

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Problem FoL.7 ... server not found

Consider computer X, which is connected by two optical fibres to two other computers A and B. The total length of these optical fibres together is $l = 3.0\,\mathrm{km}$. A signal from computer A, bearing a message that it was sent at the time $1\,413\,753\,899.213\,575\,3\,\mathrm{s}$ (Unix timestamp of the time counter at computer A) reached computer X at the time $1\,413\,753\,899.213\,592\,1\,\mathrm{s}$ (Unix timestamp of the time counter at computer X). Similarly, a signal from computer B was received by computer X at time $1\,413\,753\,906.459\,900\,4\,\mathrm{s}$, bearing the message that it was sent at the time $1\,413\,753\,906.459\,892\,5\,\mathrm{s}$. Assume that the time counters of computers A and B are synchronized, but that the time counter of computer X is either gaining or lagging behind. Luckily, this deviation stays constant in time. Also, do not take into account either the effects of relativity or the time to process the signals. Assuming that the speed of propagation of signals along the fibres is $c = 2.0 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$, find the length of the fibre between A and X.

Michal was trying to reduce the dimensions of GPS equations.

Let Δ be the deviation of the local time counter from the synchronized counters of computers A and B. Also, denote by s_b the time at which the signal was sent from computer B (measured by its time counter) and by r_b the time at which the signal was received from computer B (measured by the counter of the receiving computer). The distance x then satisfies

$$\begin{aligned} cr_a &= cs_a + x + c\Delta\,,\\ cr_b &= cs_b + l - x + c\Delta\,. \end{aligned}$$

Solving for x, we obtain

$$x = \frac{c(r_a - s_a - (r_b - s_b)) + l}{2}\,,$$

so, plugging in the numbers, we get $x \doteq 2\,390\,\mathrm{m}$.

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Problem FoL.8 ... Archimedes drowned

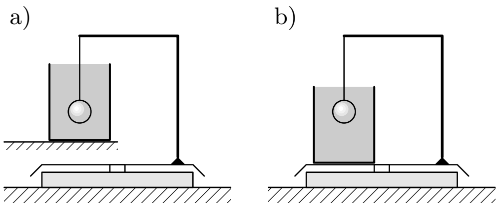
The diagram below shows schematically two situations dealing with an aluminium sphere, cylindrical flask with water and a holding apparatus from which the sphere can be suspended. In the two situations, we have that

a) the holder is placed on top of the scales with the sphere suspended from it. The sphere is fully submerged in the flask which is held off the scales,

b) the same as a), except in this case, the flask rests on the scales.
Find the difference $\Delta m = m_b - m_a$ of the readings m_b and m_a shown by the scales in the situation b) and a) respectively.

The radius of the (homogeneous) sphere is $r = 1.03\,\mathrm{cm}$, the inner radius of the flask is $R = 3.01\,\mathrm{cm}$. The height of the water column in the flask with the sphere fully submerged in it is $h = 5.10\,\mathrm{cm}$. The densities of water and aluminium are $\varrho = 996\,\mathrm{kg{\cdot}m^{-3}}$ and $\varrho_{Al} = 2\,700\,\mathrm{kg{\cdot}m^{-3}}$, respectively. The mass of an empty flask is $m_n = 124.5\,\mathrm{g}$. Assume that the suspension part of the holding apparatus is of negligible mass and volume. The rest of the apparatus has a mass of $m_d = 523.5\,\mathrm{g}$. The range of values that the scales can measure is sufficient for it to show both readings with accuracy up to $0.1\,\mathrm{g}$. The density of air should be regarded as negligible. State your answer in kilograms.

Karel wanted to set a simple-minded but computationally horrendous problem.



Let us first realise the effects to be considered in both situations. In the situation

a) there is a holder placed on top of the scales from which the sphere is suspended. However, since the sphere is submerged in water, buoyancy produces a small lift, which partially balances gravity. Hence, the reading in this case is

$$m_a = m_\mathrm{d} + (\varrho_\mathrm{Al} - \varrho)\,V_\mathrm{k}\,,$$

where $V_\mathrm{k} = 4\pi r^3/3$ is the volume of the sphere.

b) we have the holder, sphere and flask all resting on top of the scales, so the reading is

$$m_b = m_\mathrm{d} + \varrho_\mathrm{Al} V_\mathrm{k} + m_\mathrm{n} + \varrho V_\mathrm{v}\,,$$

where $V_\mathrm{v} = \pi h R^2 - 4\pi r^3/3$ is the volume of water in the flask.
It'd be more useful to express the volume of water in the flask in terms of the volume V_n of a cylinder with height h and radius R. Then we have $V_\mathrm{v} = V_\mathrm{n} - V_\mathrm{k}$, so

$$m_b = m_\mathrm{d} + \varrho_\mathrm{Al} V_\mathrm{k} + m_\mathrm{n} + \varrho V_\mathrm{n} - \varrho V_\mathrm{k}\,.$$

The required difference $\Delta m = m_b - m_a$ is then easily found to be

$$\Delta m = m_b - m_a = m_\mathrm{n} + \varrho V_\mathrm{n} = m_\mathrm{n} + \pi \varrho h R^2 \doteq 0.269\,1\,\mathrm{kg} = 269.1\,\mathrm{g}\,.$$

The difference of the readings shown by the scales in the two situations expressed in basic SI units is therefore $0.269\,1\,\mathrm{kg}$.

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Problem FoL.9 ... carriage on a blind track

Consider a railway carriage with mass $M = 20\,\mathrm{t}$, which moves on the tracks without friction. Inside the carriage, there is an elephant with mass $m = 4\,\mathrm{t}$, initially at rest. Find the distance by which the carriage moves, if the elephant walks a distance of $l = 12\,\mathrm{m}$ and then stops again. (The elephant can move only parallel to the tracks, because the carriage is quite narrow.)

Michal thinks that elephants are cool.

The key fact to realise is that the centre of mass of the combined system consisting of the carriage and the elephant does not move, since there are no external forces acting. The elephant's mass is $m/(m+M) = 1/6$ of the whole system's mass. When the elephant walks a distance l, the center of mass moves in the carriage's reference frame by $l/6 = 2\,\mathrm{m}$. Hence, for an external observer, the carriage moved by $2\,\mathrm{m}$ in the direction opposite to that of the elephant's movement.

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Problem FoL.10 ... driving in rain

Find the total volume of water (in litres) that falls down on a Karosa 700 *type bus as it travels for an hour at a speed of $v_\mathrm{b} = 50\,\mathrm{km{\cdot}h^{-1}}$. Assume the bus to be of a cuboid shape with height $h = 4\,\mathrm{m}$, width $w = 2.5\,\mathrm{m}$ and length $l = 12\,\mathrm{m}$. The rain falls vertically downwards with speed $v_\mathrm{r} = 10\,\mathrm{m{\cdot}s^{-1}}$ and flux (that is, the volume of water crossing a unit area perpendicular to the velocity of the falling rain, per unit time) $Q = 5\,\mathrm{mm{\cdot}h^{-1}}$.* *Xellos running from the the rain.*

Let us adopt the point of view of an observer, which is fixed in the reference frame connected with the falling droplets of rain. In this frame, the droplets are fixed and the bus gains an upward component of its velocity, equal to v_d. It is therefore an equivalent problem to consider the spatial volume swept out by the leading and the upper face of the bus over $t = 1\,\mathrm{h}$ in this frame and calculate the volume of water in this spatial volume.

To this end, considering (in the original frame connected with the ground) a cube with side-length a (through which the droplets pass in a time of a/v_d), which contains a total volume

$$Qa^2 \frac{a}{v_\mathrm{d}} = a^3 \frac{Q}{v_\mathrm{d}}$$

of water, we can deduce that in the frame, where the droplets are static, a spatial volume V contains a total volume VQ/v_d of water.

Clearly, the upper face of the bus sweeps out a rectangular parallelepiped with base surface area wl and height $v_\mathrm{d}t$, hence with volume $V_1 = wlv_\mathrm{d}t$. The leading face of the bus sweeps out another rectangular parallelepiped, this time with base surface area wh and height $v_\mathrm{a}t$, hence with volume $V_2 = whv_\mathrm{a}t$. Putting all results together, the volume of water that falls down on the bus in time t is

$$V_\mathrm{d} = (V_1 + V_2)\frac{Q}{v_\mathrm{d}} = Qw\left(l + \frac{v_\mathrm{a}}{v_\mathrm{d}}h\right)t \doteq 219\,\mathrm{l}\,.$$

Hence, as the bus travels for an hour, a volume of $219\,\mathrm{l}$ of water falls down on it.

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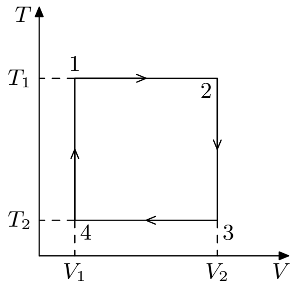
Problem FoL.11 ... on your own drive

Find the efficiency (in %) of a thermodynamic cycle consisting of two isothermal processes and two isochoric processes (see the figure). Assume that the cycle works between temperatures $T_1 = 373\,\mathrm{K}$ and $T_2 = 273\,\mathrm{K}$ and that the heat released during the isochoric process $2 \to 3$ is fed into the isochoric process $4 \to 1$. *Xellos spied on a hamster rolling his wheel.*



No work is done during an isochoric prosess, therefore the heat released by cooling the system from temperature T_2 to T_1 is the same as the heat needed to do the opposite process of heating the system from temperature T_1 to T_2. Hence, the contributions due to the two isochoric processes exactly cancel out and do not enter the calculation of efficiency. The work done by the system during the expansion process $1 \to 2$ (during which the volume increases from V_1 to V_2) is

$$W_1 = nRT_1 \ln \frac{V_2}{V_1}\,;$$

an equal amount of heat $Q = W_1$ must be supplied to the system. Also, the work done during the contraction process is $3 \to 4$ is $W_1 = -nRT_2 \ln(V_2/V_1)$ where this time, no additional heat is required. The efficiency can then be found as

$$\eta = \frac{W_1 + W_2}{Q_1} = \frac{T_1 - T_2}{T_1} \doteq 0.268 = 26.8\,\%\,.$$

The efficiency of our thermodynamic cycle is therefore equal to that of Carnot's cycle working between temperatures T_1 and T_2.

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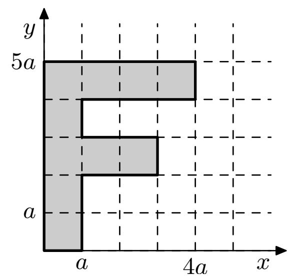
Problem FoL.12 ... the most important letter's centre of mass

Has it ever occurred to you that F *might actually be the most important letter of the alphabet? Find the y coordinate of the centre of mass of the shape of the letter* F*. You can assume that the shape is built up from 10 identical homogeneous square-shaped laminae with side length a. Express your answer to three significant figures as a multiple of a in the coordinate grid indicated in the figure.* *Karel contemplated on the importance of the letter* F.



The y coordinate of the centre of mass satisfies

$$m_{\mathrm{tot}} y = \sum m_i y_i\,,$$

where m_{tot} is the total mass of the shape. Taking into account homogeneity of each constitutive lamina, we can assert that the centre of mass of each of them lies at its geometrical centre (that is, at a distance of $0.5a$ from their sides). Denoting by $m = m_{\mathrm{tot}}/10$ the mass of each lamina, the above expression for y can be written as

$$10my = ma(1 \cdot 0.5 + 1 \cdot 1.5 + 3 \cdot 2.5 + 1 \cdot 3.5 + 4 \cdot 4.5)\,,$$
$$y = 3.1a\,.$$

Hence, the correct answer expressed to 3 significant figures is $y = 3.10a$. An analogous calculation yields $x = 1.4a$ for the x coordinate.

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Problem FoL.13 ... sulphuring the barrels

Sulphur dioxide was and still is used as a preservative for wine casks. One of the sulphuring techniques involves burning sulphur discs directly inside the casks. Consider a cask with volume $200\,\mathrm{l}$, *inside which two sulphur discs are placed, each containing* $2\,\mathrm{g}$ *of sulphur. The discs are then ignited and the cask is closed hermetically. What is the fraction of cask's volume that* $\mathrm{SO_2}$ *will occupy after both discs burn down completely? Assume that when all burning stops, the temperature inside the cask decreases immediately back to* $T = 20\,\mathrm{^\circ C}$, *the same as it was initially. Molar mass of sulphur is* $M = 32\,\mathrm{g{\cdot}mol^{-1}}$ *and the ambient atmospheric pressure is* $p = 101\,\mathrm{kPa}$. *Mirek knows a thing or two about physicists' views on chemical calculations.*

Sulphur burning is described by the chemical equation

$$\mathrm{S} + \mathrm{O_2} \longrightarrow \mathrm{SO_2}\,.$$

Therefore, one mole of sulphur becomes exactly one mole of sulphur dioxide. The volume occupied by one mole of gas under the given conditions is

$$V_\mathrm{m} = \frac{RT}{p}\,.$$

Also, the amount of $\mathrm{SO_2}$ can be written as

$$n_{\mathrm{SO_2}} = \frac{2m}{M}\,,$$

where m denotes the mass of sulphur in one disc. The fraction of cask's volume we are trying to express is simply the ratio of $V_{\mathrm{SO_2}}$ to the total volume V of the cask. Hence

$$\varphi = \frac{V_{\mathrm{SO_2}}}{V} = \frac{n_{\mathrm{SO_2}} V_\mathrm{m}}{V} = \frac{n_{\mathrm{SO_2}} RT}{pV} \doteq 0.015\,1$$

The wanted fraction of $\mathrm{SO_2}$ in cask is $0.015\,1$.

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Problem FoL.14 ... to jump, or not to jump

A horizontal plate performs simple harmonic motion with period $T = 0.5\,\mathrm{s}$ *in the vertical direction. Find the maximum value of the amplitude of oscillations* A *(in centimetres) such that a body lying on the plate would remain in contact with the plate throughout the motion. The acceleration due to gravity is* $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

f(Aleš) salvaged this from an ancient book.

The body lying on the plate will clearly stay in contact with the plate over the course of the motion, provided that the magnitude of acceleration of the plate at the upper turning point of its motion remains less than the magnitude of acceleration due to gravity. The vertical position y of the oscillating plate is given by

$$y = A \sin \omega t\,,$$

where we take the y axis to point upwards. The acceleration a of the plate at a given time t can be easily determined by differentiating the above expression for y twice with respect to time, i.e.

$$a = -\omega^2 A \sin \omega t\,.$$

We have $t = T/4$ at the upper turning point, so the condition for the body to remain in contact with the plate reads

$$-g = -\omega^2 A \sin \omega \frac{T}{4}\,.$$

Finally, substituting $\omega = 2\pi/T$ and $\sin \pi/2 = 1$ we have

$$A = \frac{gT^2}{4\pi^2} \doteq 6.2\,\mathrm{cm}\,.$$

Hence, the body remains in contact with the plate if the amplitude of oscillations does not exceed 6.2 cm.

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Problem FoL.15 ... tea at five

Náry decided to make himself a cup of mint tea. Reading the instructions „Place the teabag into boiling water.“ on the side of the box made him think about the following question: How much energy could be saved, if only he could prepare his tea at the summit of Triglav, the highest peak in Slovenia (2 864 meters above the sea level) instead of doing it at his home in Boskovice? You can assume that the ambient pressure and temperature in Boskovice are $p_\mathrm{B} = 101\,325\,\mathrm{Pa}$ and $t_\mathrm{B} = 25\,{}^\circ\mathrm{C}$ respectively. The temperature at Triglav is $t_T = 13\,{}^\circ\mathrm{C}$ and ideal gases there occupy 1.44 times the volume they occupy in Boskovice. Also, the boiling point of water t_v and the ambient pressure p are related (over a suitable range of values of p) as

$$t_\mathrm{v} = \tau + kp\,,$$

where $\tau = 71.6\,{}^\circ\mathrm{C}$ and $k = 2.8 \cdot 10^{-4}\,\mathrm{K{\cdot}Pa^{-1}}$. In order to prepare his tea, Náry needs to bring $m = 600\,\mathrm{g}$ of water with initial temperature $t_0 = 20\,{}^\circ\mathrm{C}$ (assumed to be the same in both cases) to a boil. All energy losses should be neglected. The specific heat capacity of water is $c = 4\,180\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. *Kiki and Tom pondering about their excessive consumption of tea.*

So as to find the boiling point of water at Triglav, let us first determine the ambient pressure p_T at Triglav. Using the Ideal Gas Law, we can write

$$\frac{p_\mathrm{B} V_\mathrm{B}}{T_\mathrm{B}} = \frac{p_\mathrm{T} V_\mathrm{T}}{T_\mathrm{T}}\,,$$

where the quantities denoted by subscripts B and T correspond to the values of pressure, volume and temperature in Boskovice and at Triglav, respectively. The pressure at Triglav may then be expressed as

$$p_\mathrm{T} = p_\mathrm{B} \frac{V_\mathrm{B}}{V_\mathrm{T}} \frac{T_\mathrm{T}}{T_\mathrm{B}} = \frac{p_\mathrm{B} T_\mathrm{T}}{p T_\mathrm{B}}\,.$$

The difference Δt between the boiling points t_{vB} and t_{vT} in Boskovice and at Triglav, respectively, is then proportional to the amount ΔQ of energy saved, since we are assuming that, in both situations, we boil water with the same initial temperature (irrespective of the fact that the ambient temperatures at the two sites are different) and we neglect all losses of energy. The answer can then be found as

$$\Delta Q = mc\Delta t\,,$$

where

$$\Delta t = t_{\mathrm{vB}} - t_{\mathrm{vT}} = k(p_{\mathrm{B}} - p_{\mathrm{T}}) = kp_{\mathrm{B}}\left(1 - \frac{T_{\mathrm{T}}}{pT_{\mathrm{B}}}\right)$$

and so

$$\Delta Q = mckp_{\mathrm{B}}\left(1 - \frac{T_{\mathrm{T}}}{pT_{\mathrm{B}}}\right)\,.$$

Numerically, we get $\Delta Q \doteq 23\,700\,\mathrm{J}$.

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Problem FoL.16 ... scary shopping

Kiki set off to London (which lies $s = 1\,500\,\mathrm{km}$ from her home) to buy some $^{18}\mathrm{F}$. Knowing that the half-life of this poor quality stuff was only $1.8\,\mathrm{h}$, she decided to buy $32\,\mathrm{g}$ right away. Find the speed (in $\mathrm{km{\cdot}h^{-1}}$) with which she had to fly back in order to bring home $10\,\mathrm{g}$ of $^{18}\mathrm{F}$ more than if she had opted for a 24 hour journey by bus. Dominika had to go and claim her money back already a third time this year, wondering why anyone else's shoes last so much longer.

Half-life is defined to be the time needed for a half of the total number of particles in a given sample to decay. The total mass m of $^{18}\mathrm{F}$ atoms in a sample after a time t (directly proportional to the total number N of atoms which have not decayed after time t) then satisfies

$$m = m_0 2^{-t/T}\,,$$

where m_0 is the total mass of $^{18}\mathrm{F}$ atoms in the sample at $t = 0$ and T is the half-life of $^{18}\mathrm{F}$. The difference $\Delta m = 10\,\mathrm{g}$ in the total mass of $^{18}\mathrm{F}$ atoms brought home (which is achieved by Kiki travelling faster) can be expressed as

$$\Delta m = m_0\left(2^{-t_{\mathrm{l}}/T} - 2^{-t_{\mathrm{b}}/T}\right)\,,$$

where t_{l} and t_{b} are the times it takes the plane and the bus respectively to travel $s = 1500\,\mathrm{km}$. Clearly $t_{\mathrm{l}} = s/v_{\mathrm{l}}$, where v_{l} is the quantity we are interested in. Substituting for $t_{\mathrm{l}} = s/v_{\mathrm{l}}$ into the above expression for Δm we get

$$v_{\mathrm{l}} = \frac{s}{T\log_2 \dfrac{1}{\dfrac{\Delta m}{m_0} + 2^{-t_{\mathrm{b}}/T}}}\,.$$

This can be simplified by realising that $2^{-t_{\mathrm{b}}/T} \ll \Delta m/m_0$. Then we can write

$$v_{\mathrm{l}} \approx \frac{s}{T\log_2 \dfrac{m_0}{\Delta m}} \doteq 500\,\mathrm{km{\cdot}h^{-1}}\,.$$

Hence, so as to bring home 10 g of ^{18}F more, the speed of the plane would have to be $v_l = 500\,\mathrm{km{\cdot}h^{-1}}$.

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Problem FoL.17 ... feed a castle

In their time (i.e. at the turn of the 19th and 20th century), the owners of the Kokořín castle were progressively thinking people – they introduced gas heating based on burning acetylene created by reacting calcium carbide with water

$$\mathrm{CaC_2} + 2\,\mathrm{H_2O} \longrightarrow \mathrm{C_2H_2} + \mathrm{Ca(OH)_2}\,.$$

Compare the energetic efficiency of this carbide-fuelled heating to that of a wood-fuelled alternative, i.e. find the ratio of energies gained from a unit mass of wood and a unit mass of carbide. Use the following data: density of calcium carbide $\varrho(\mathrm{CaC_2}) = 2\,200\,\mathrm{kg{\cdot}m^{-3}}$, molar mass of calcium carbide $M(\mathrm{CaC_2}) = 64\,\mathrm{g{\cdot}mol^{-1}}$, heating value of acetylene $H(\mathrm{C_2H_2}) = 48\,\mathrm{MJ{\cdot}kg^{-1}}$, molar mass of acetylene $M(\mathrm{C_2H_2}) = 26\,\mathrm{g{\cdot}mol^{-1}}$, heating value of wood $H_d = 14\,\mathrm{MJ{\cdot}kg^{-1}}$, density of wood $\varrho = 520\,\mathrm{kg{\cdot}m^{-3}}$. Assume that the castle has a sufficiently large water reservoir. The fact that the fuel needs to be transported over an altitude difference $h = 50\,\mathrm{m}$ from a nearby lying village uphill should be included in your energy balance considerations. The acceleration due to gravity is $g = 9.8\,\mathrm{m{\cdot}s^{-2}}$. *Mirek and Lukáš on a trip.*

Assume that we have a mass m of each of the two kinds of fuel at our disposal. Then, the net energies gained from each fuel are

$$E_\mathrm{d} = mH_\mathrm{d} - mgh$$
$$E(\mathrm{CaC_2}) = m\frac{M(\mathrm{C_2H_2})}{M(\mathrm{CaC_2})}H(\mathrm{C_2H_2}) - mgh\,,$$

where we already converted the mass of carbide to the corresponding mass of acetylene. Also, the energy needed to transport the fuel over the given altitude difference must be subtracted from the heat gained by burning in order to obtain the net energy gain. By inspecting the given data, it follows immediately that the difference in potential energy per unit mass associated with the given altitude difference is negligible compared to the energies gained by burning both wood and acetylene. Hence, we get

$$\frac{E_\mathrm{d}}{E(\mathrm{CaC_2})} = \frac{H_\mathrm{d} - gh}{\frac{M(\mathrm{C_2H_2})}{M(\mathrm{CaC_2})}H(\mathrm{C_2H_2}) - gh} \approx \frac{M(\mathrm{CaC_2})H_\mathrm{d}}{M(\mathrm{C_2H_2})H(\mathrm{C_2H_2})} \doteq 0.72\,.$$

By the above specified criteria, it is more efficient to use acetylene as fuel.

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Problem FoL.18 ... bullet

In a dark abandoned hall, there are a robber and a policeman standing against each other by the opposite sides of the hall. Having heard steps echoing in the darkness, the criminal points *his gun horizontally on the unsuspecting cop and is about to shoot. What he doesn't know is that a potential difference $U = 10\,\mathrm{kV}$ was brought between the floor and the ceiling of the hall, thus creating a homogeneous field acting (vertically downwards) on the fired bullet, since the bullet acquires a charge q as it travels through the muzzle. Find the least charge-to-mass ratio q/m of the bullet such that it hits the floor before it reaches the policeman (where by m, we denote the mass of the bullet). The robber and the policemen are separated by a distance of $D = 100\,\mathrm{m}$, the ceiling is $d = 8\,\mathrm{m}$ above the floor and the bullet is fired horizontally with muzzle velocity $v_0 = 400\,\mathrm{m\cdot s^{-1}}$. Assume that the robber holds his gun $h = 1.5\,\mathrm{m}$ above the floor and that the acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.*

Mirek wanted to point out the importance of triboelectric effect.

Clearly, the force acting on a charge q in a homogeneous electric field of magnitude $E = U/d$ has magnitude $F_\mathrm{e} = qE = Uq/d$. If we also take into account the force $F_g = mg$ acting on a particle with mass m in a homogeneous gravitational field, the magnitude of total acceleration of the particle can be expressed as

$$g' = \frac{F_\mathrm{g} + F_\mathrm{e}}{m} = g + \frac{Uq}{md}\,.$$

Recalling that the range of a horizontally projected particle can be written as

$$D = \sqrt{\frac{2h}{g'}}v_0$$

and substituting for g' from above, we obtain

$$D = \sqrt{\frac{2h}{g + \frac{Uq}{md}}}v_0\,,$$

which can be solved for q/m to yield

$$\frac{q}{m} = \left(\frac{2hv_0^2}{D^2} - g\right)\frac{d}{U} \doteq 0.031\,\mathrm{C\cdot kg^{-1}}\,.$$

The bullet needs to have charge-to-mass ratio of at least $0.031\,\mathrm{C\cdot kg^{-1}}$.

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Problem FoL.19 ... watermelon

Náry was sitting on a swing, which was at rest. Suddenly, he felt like eating a watermelon, so other FYKOS guys immediately threw him one with mass $m_m = 20\,\mathrm{kg}$. Assuming that the speed of the watermelon was $v_m = 10\,\mathrm{m\cdot s^{-1}}$, find the angle by which the swing with Náry moves. Náry (approximated by a point particle) weighs $m_n = 80\,\mathrm{kg}$ and the swing rope is $L = 4\,\mathrm{m}$ long. Also, assume that the watermelon hit Náry (who managed to catch it) horizontally in the plane of rotation of the swing. State your answer in degrees. Gravitational acceleration is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Lydka was eating a watermelon.

Conserving linear momentum, we can write

$$m_{\mathrm{m}} v_{\mathrm{m}} + m_{\mathrm{n}} v_{\mathrm{n}} = (m_{\mathrm{m}} + m_{\mathrm{n}})\, u\,,$$

where u is the speed of the combined system of Náry and the watermelon and $v_{\mathrm{n}} = 0\,\mathrm{m\cdot s^{-1}}$ is Náry's initial speed. Conserving mechanical energy, we have

$$\frac{1}{2}(m_{\mathrm{m}} + m_{\mathrm{n}})\, u^2 + 0 = 0 + (m_{\mathrm{m}} + m_{\mathrm{n}})\, hg\,,$$

where h is the maximum height, which Náry and the watermelon can reach (Náry's initial height is taken to be zero). These two expressions can be combined to give

$$h = \frac{(m_{\mathrm{m}} v_{\mathrm{m}})^2}{2\,(m_{\mathrm{m}} + m_{\mathrm{n}})^2\, g}\,.$$

So, the maximum deflection angle α of the swing from the vertical axis satisfies

$$\cos\alpha = \frac{L-h}{L} \quad \Rightarrow \quad \alpha \doteq 18.4^\circ\,.$$

Therefore, as a result of Náry catching his watermelon, the swing moves by 18.4°.

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Problem FoL.20 ... a superconducting one

Consider two superconducting capacitors with capacitances $C_1 = 10\,\mu\mathrm{F}$, $C_2 = 5.0\,\mu\mathrm{F}$ and a superconducting DC power supply with voltage $U = 10\,\mathrm{V}$. All these components are connected by superconducting wires. The superconductors are perfect, hence there is no resistance and therefore, Joule heating will not occur. First, let us connect the first capacitor to the DC supply and wait until the charge builds up. When it is fully charged, we disconnect the DC power supply and connect the second capacitor to the first one instead. Find the difference between the energy stored in the first capacitor alone and the total energy of the two connected capacitors.

Even the written exams can be educating for Mirek.

The charge on the first capacitor is $Q_1 = C_1 U$, so its energy is found to be

$$W_1 = \frac{Q_1^2}{2C_1} = \frac{1}{2} C_1 U^2 = 5 \cdot 10^{-4}\,\mathrm{J}\,.$$

After connecting the second capacitor and disconnecting the power supply, the total charge is conserved, but it is redistributed. The capacity of the two capacitors connected „in parallel“ is simply $C = C_1 + C_2$, so the total energy of the system is

$$W = \frac{Q_1^2}{2(C_1 + C_2)} \doteq 3.3 \cdot 10^{-4}\,\mathrm{J}\,.$$

Hence, the difference of the two energies is

$$\Delta W = W_1 - W = \frac{C_1 C_2 U^2}{2(C_1 + C_2)} \doteq 1.7 \cdot 10^{-4}\,\mathrm{J}\,.$$

In spite of the fact that Joule heating does not occur, the answer is not zero, because some energy is emitted in the form of electromagnetic waves emerging as a consequence of oscillating charge distribution.

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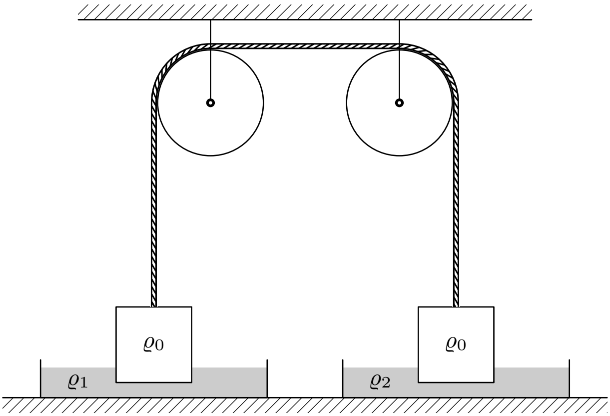
Problem FoL.21 ... buoyant swings with a pulley

Consider two identical cylinders (base radius 5 cm*, height* 10 cm *and density* $\varrho_0 = 2.2\,\mathrm{g\cdot cm^{-3}}$*) in equilibrium, partially submerged in two containers filled with liquids of densities* $\varrho_1 = 1.0\,\mathrm{g\cdot cm^{-3}}$ *and* $\varrho_2 = 0.8\,\mathrm{g\cdot cm^{-3}}$ *(see the diagram below). Find the period of small vertical oscillations of this system about its equilibrium. Neglect the changes in the level of liquids in the containers. Neglect also the mass of the strings. The acceleration due to gravity is* $g = 9.81\,\mathrm{m\cdot s^{-2}}$*.* *Due to Janči.*



The x components of equations of motion for the two cylinders are

$$m\ddot{x} = -T - Sx\varrho_1 g$$

and

$$m\ddot{x} = T - Sx\varrho_2 g\,,$$

where m denotes the mass of one cylinder, S its horizontal cross-section and T is the tension in the string. Let us choose to measure the x coordinate of the first cylinder vertically downwards and the acceleration of the second cylinder vertically upwards. We then obtain

$$m\ddot{x} = -\frac{1}{2}Sx(\varrho_1 + \varrho_2)g\,,$$

which is of the form of an equation of simple harmonic motion with period

$$T = 2\pi\sqrt{\frac{2m}{S(\varrho_1 + \varrho_2)g}} = 2\pi\sqrt{\frac{2V}{gS}\frac{\varrho_0}{\varrho_1 + \varrho_2}}\,.$$

Since V/S is simply the height of one cylinder, we get $T \doteq 0.99\,\mathrm{s}$.

Interestingly, the period depends only on the height and density of the cylinders. This can be interpreted considering a possibility of superposition of such oscillators.

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Problem FoL.22 ... on the head

A new notice board needs a new nail (with length $l = 80\,\mathrm{mm}$), with which it should be nailed to the wall using our hammer with mass $m = 0.4\,\mathrm{kg}$. Five hits in total were needed to do this. During each hit, the speed of the hammer just before it hit the head of the nail was $v = 8.0\,\mathrm{m\cdot s^{-1}}$ (assume that upon hitting the nail, the hammer transfers all its kinetic energy into the nail). However, having nailed it down, we noticed that the board hangs askew. What is the magnitude of the force, with which we will need to pull the nail out of the wall? Assume that the mass of the nail is negligible. Also, assume that the magnitude of the frictional force is directly proportional to the length of the part of the nail, which is inside the board, and that it is the same during both pulling the nail out and nailing it in.

f(Aleš) was cleaning his notice board.

The work W needed to hammer the nail down is

$$W = 5\frac{mv^2}{2}\,.$$

Taking into account that the magnitude of the frictional force is directly proportional to the length of the part of the nail inside the board, W can also be expressed as

$$W = \frac{1}{2}F_\mathrm{o} l\,,$$

where F_o is the maximum magnitude of the frictional force (which is attained when the nail is fully inside the board).

Whence the magnitude of the frictional force (and so the magnitude of the force with which we will need to pull the nail out of the wall) is easily read off to be

$$F_\mathrm{o} = \frac{2W}{l} = \frac{5mv^2}{l} = 1\,600\,\mathrm{N}\,.$$

Force needed to pull the nail is equal to $1\,600\,\mathrm{N}$.

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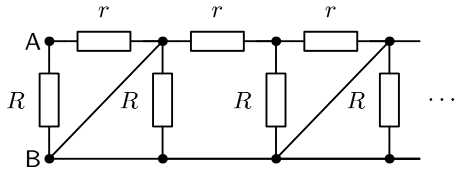
Problem FoL.23 ... an infinite circuit

Find the resistance between the terminals A *and* B *as shown on the diagram below. Put $R = 5\,\Omega$, $r = 1.5\,\Omega$ and neglect the resistance of leads.*

Xellos found solving the infinite resistor ladder too easy.



Everything apart from the two resistors which are closest to A is connected in parallel with a wire of zero resistance and therefore can be ignored. We are left with two resistors R and r in parallel which gives us the combined resistance as

$$\frac{Rr}{R+r} \doteq 1.15\,\Omega\,.$$

Note that the name of the question was misleading, we did not really need to consider any infinite series of resistors.

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Problem FoL.24 ... bowling with a physicist

Once, when Olda went to play bowling, an intriguing thing happened. He threw the bowling ball of mass $7.25\,\mathrm{kg}$ *and radius* $0.108\,\mathrm{m}$ *with speed* $12\,\mathrm{m\cdot s^{-1}}$ *and gave it such a spin that the ball stopped and stayed put at rest before reaching the skittles. What was the angular speed (in radians per second) which Olda had to give the ball for this to occur? Gravitational acceleration is* $9.81\,\mathrm{m\cdot s^{-2}}$*. Air resistance and rolling friction can be neglected.*

Organizers went bowling.

First we need to think about the mechanism which makes the ball stop. The only thing we were not told to neglect is the force due to friction between the ball and the floor whose magnitude we denote by F_T. Of course since mechanical energy is dissipated in all processes involving friction, we will not be able to make use of conservation of energy.

To solve the problem, we will revert to fundamental equations of Newtonian mechanics, namely the Newton's second law

$$F_\mathrm{T} = \frac{\mathrm{d}p}{\mathrm{d}t}$$

which in our case can be recast into

$$F_\mathrm{T} t = m v_0\,,$$

where v_0 is the initial speed, m mass of the ball and t the time from the moment the ball was thrown until it stops. Also, we can write down the rotational analogue of the Newton's second law, i.e.

$$F_\mathrm{T} r = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}J\omega}{\mathrm{d}t}$$

which, again, can be rewritten as

$$F_\mathrm{T} r t = J\omega_0\,,$$

where J is the moment of inertia of the ball, r its radius and ω_0 its initial angular speed.

Finally, eliminating t and substituting for the moment of inertia, we obtain

$$\omega_0 = \frac{F_\mathrm{T} r t}{J} = \frac{5F_\mathrm{T} r}{2mr^2}\frac{mv_0}{F_\mathrm{T}} = \frac{5v_0}{2r}\,.$$

Plugging in the numbers, we get $\omega_0 \doteq 280\,\mathrm{rad{\cdot}s^{-1}}$. A value this high could be anticipated by looking at the initial value of the translational kinetic energy.

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Problem FoL.25 ... waves and wavelets

As it known, propagation of surface waves on a lake is not significantly influenced by gravity in the short-wavelength limit. Nevertheless there is a strong effect of surface tension which needs to be considered. Using dimensional analysis, determine the angular frequency of surface waves as a function of surface tension σ, water density ϱ and the wave number $k = 2\pi/\lambda$, where λ is the wavelength. Assume that the dimensionless coefficient is equal to one in this case. Also, derive an expression for the group velocity and substitute the values $\sigma = 73\,\mathrm{mN{\cdot}m^{-1}}$, $\varrho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$, $\lambda = 2\,\mathrm{cm}$. *Mirek trying to catch short waves.*

We aim to write down a relation of form $\omega = \sigma^\alpha \varrho^\beta k^\gamma$. Decomposing the unit of newton as $\mathrm{kg{\cdot}m{\cdot}s^{-2}}$, we obtain the following system of linear equations for α, β, γ

$$\begin{aligned} 0 &= \alpha + \beta\,,\\ 0 &= -3\beta - \gamma\,,\\ -1 &= -2\alpha\,. \end{aligned}$$

We can immediately read off that $\alpha = 1/2$. Substituting this into the remaining two equations, we get $\beta = -1/2$ and $\gamma = 3/2$. Hence the angular frequency is given by (putting the dimensionless coefficient equal to 1, as we are told)

$$\omega = \sqrt{\frac{\sigma k^3}{\varrho}}\,.$$

Using the definition of the group velocity, we can write

$$v_\mathrm{g} = \frac{\partial \omega}{\partial k} = \frac{3}{2}\sqrt{\frac{\sigma k}{\varrho}}\,.$$

Substituting $k = 2\pi/\lambda$ and the rest of data given, we get

$$v_\mathrm{g} = \frac{3}{2}\sqrt{\frac{2\pi\sigma}{\varrho\lambda}} \doteq 0.23\,\mathrm{m{\cdot}s^{-1}}\,,$$

which is the magnitude of group velocity.

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Problem FoL.26 ... rolling cylinder

As a proper engineer, Dominika often indulges herself with rolling homogeneous cylinders down a hill. At one such occasion (having a moment of clarity) she came up with an alternative method of determining the number of turns the cylinder completes before it falls off an inclined plane. First, she measured its mass ($m = 123.8\,\mathrm{g}$) and also its moment of inertia $J = 1.807\cdot 10^{-5}\,\mathrm{kg\cdot m^2}$. It is widely known that Dominika's inclined plane is $l = 1.02\,\mathrm{m}$ long and that it makes an arbitrary angle α with horizontal. How many turns (real number, to three significant figures) does the cylinder complete before it reaches the end of the inclined plane? Assume that the cylinder rolls precisely from one end of the plane to the other and that it moves without slipping in the direction of maximum slope.

Karel wondered about Dominika's future in the field of engineering.

Moment of inertia of a solid homogeneous cylinder is $J = mr^2/2$. Knowing J and also the mass m we can calculate the radius of Dominika's cylinder as

$$r = \sqrt{\frac{2J}{m}} \doteq 1.71 \cdot 10^{-2}\,\mathrm{m}\,.$$

The number of turns N completed can then be obtained as the ratio of total distance l covered and circumference $o = 2\pi r$ of the cylinder. Thus

$$N = \frac{l}{o} = \frac{l}{2\pi r} = \frac{l}{2\pi}\sqrt{\frac{m}{2J}} \doteq 9.50\,.$$

Hence the cylinder completes 9.50 turns before it reaches the end of the inclined plane.

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Problem FoL.27 ... Zdeněk's sphere

As a proper soon-to-be-engineer, Zdeněk (like Dominika) likes to roll various objects down an inclined plane. Nevertheless, unlike Dominika, who is rather more into cylinders, Zdeněk prefers to play with his sphere. He is not a fan of ordinary experimental methods, so in order to find out the density of his sphere, he decided to measure its mass $m = 123.8\,\mathrm{g}$ (which is incidentally the same as that of Dominika's cylinder) and its moment of inertia $J = 3.48 \cdot 10^{-5}\,\mathrm{kg\cdot m^2}$. Your task is to help Zdeněk with his calculation and determine the density of his solid homogeneous sphere.

Karel fantasizing about BUT...

Moment of inertia of a solid homogeneous sphere with mass a and radius r is $J = 2mr^2/5$. The radius can then be expressed as

$$r = \sqrt{\frac{5J}{2m}}\,.$$

Density ϱ of the sphere can be found as the ratio of its mass m and its volume $V = 4\pi r^3/3$, i.e.

$$\varrho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi r^3} = \frac{3m}{4\pi}\left(\frac{2m}{5J}\right)^{\frac{3}{2}} = \frac{3m^{\frac{5}{2}}}{\sqrt{2}5^{\frac{3}{2}}\pi J^{\frac{3}{2}}} \doteq 1.59 \cdot 10^3\,\mathrm{kg\cdot m^{-3}}\,.$$

Hence the density of Zdeněk's sphere is $1\,590\,\mathrm{kg\cdot m^{-3}}$.

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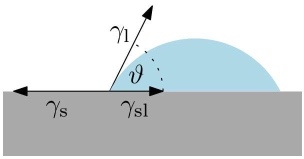
Problem FoL.28 ... energetic surface

What is the contact angle (in degrees) of the interface of air, water (surface tension $\gamma_l = 74\,\mathrm{mN\cdot m^{-1}}$) and an unknown solid with surface energy (sc. the solid-air interfacial energy) $\gamma_s = 53\,\mathrm{mN\cdot m^{-1}}$? The solid-water interfacial energy is $\gamma_{sl} = 25\,\mathrm{mN\cdot m^{-1}}$.

A by-product of Terka's work on a FYKOS experimental task.

The interfacial energy (that is to say the surface tension – for a pure liquid, these two notions coincide) gives the magnitude of a force acting on a line element on the interface divided by the length of that line element. The vector of this force, which is perpendicular to the line element under consideration, lies in the plane which is tangent to the interface.



A droplet of water placed onto our mysterious material takes a steady axisymmetric form. In particular, this means that the horizontal components of forces along the line of contact must sum to zero. Mathematically, this is expressed by what is normally referred to as the Young's equation

$$\gamma_\mathrm{sl} + \gamma_\mathrm{l}\cos\vartheta = \gamma_\mathrm{s}\,,$$

where ϑ denotes the contact angle. Whence

$$\vartheta = \arccos\frac{\gamma_\mathrm{s} - \gamma_\mathrm{sl}}{\gamma_\mathrm{l}} \doteq 68^\circ\,.$$

Hence the contact angle is 68°. This is consistent with our mysterious material being e.g. nylon.

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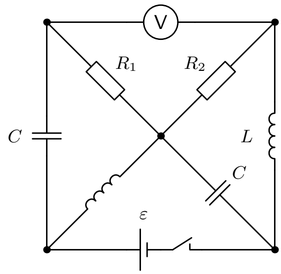
Problem FoL.29 ... pyramid's circuits vol. 2

In the circuit shown below we first put the switch on and wait for the currents to come to a steady state. Then we put the switch back off. What is the magnitude of the potential difference (in volts) measured by the voltmeter immediately after the switch was turned off? Take the capacitors and coils to be ideal and assume that the voltmeter provides an infinite resistance. Put $\varepsilon = 1\,\mathrm{V}$ and $R_1 = 3R_2$.

Xellos found out about the Estonian-Finnish Olympiad in Physics.



We'll calculate currents through the resistors from the top vertex of the pyramid (junction in the center of the picture).

When the switch is on and the currents reach steady state, the capacitors and voltmeter can be thought of as elements with infinite resistance. On the other hand, the coil behaves like a conductor with no resistance. Hence the currents through R_1 and R_2 are $I_1 = 0$ and $I_2 = \varepsilon/R_2$ respectively.

The properties of capacitors and coils are such that it is not possible to create a temporal discontinuity in potential difference across a capacitor and similarly with current through a coil. Therefore, after the switch is put off, there is a current $I_2 = \varepsilon/R_2$ through the resistor R_2. Also, the current through the resistor R_1 is now $I_3 = I_2$, so the potential difference across the two resistors is

$$|R_1 I_3 - R_2 I_2| = 2\varepsilon = 2\,\mathrm{V}\,.$$

Hence the reading shown by the voltmeter is $2\,\mathrm{V}$.

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Problem FoL.30 ... gassed joules

Náry once passed by a pressure vessel containing sulphur dioxide to which a number of meters were attached and some data were printed on its mantle. Namely, he managed to read off that the volume of the vessel was $V = 1\,\mathrm{m}^3$ and that the mass of an empty vessel would be $m = 100\,\mathrm{kg}$. The meters were showing the temperature of the gas inside, its pressure $p = 5.3\,\mathrm{MPa}$ as well as the combined mass of the gas and the vessel $M = 152\,\mathrm{kg}$. However, being myopic, Náry was unable to read off the value of the temperature. Luckily, he had 4 hours of free time so he set off to calculate the temperature assuming there was an ideal gas inside the vessel. Deeming the value he obtained too low and having another 6 hours free, he decided to recalculate the temperature under the assumptions of van der Waals theory. What is the magnitude of the difference of the two results he got (in kelvins)?

f(Aleš) remembered his Thermodynamics example classes.

We want to calculate the difference $\Delta T = T_\mathrm{v} - T_\mathrm{i}$ of the temperatures yielded by the two models. Hence we need to calculate the temperature in each model first, i.e. we need to find an expression for the temperatures T_v and T_i of a van der Waals and an ideal gas, respectively. Let us start with ideal gas, just as Náry did. From the Ideal Gas Law

$$pV = nRT_\mathrm{i}\,,$$

the temperature T_i can be easily expressed as

$$T_\mathrm{i} = \frac{pV}{nR}\,.$$

In order to find out the temperature of a van der Waals gas, we need to know what the van der Waals equation of state looks like. We have

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT_\mathrm{v}\,,$$

where $v = V/n$ is the molar volume. We then obtain

$$T_\mathrm{v} = \frac{1}{nR}\left(p + \frac{n^2 a}{V^2}\right)(V - nb)\,.$$

It remains to subtract the two derived expressions and to calculate the amount n of gas inside the vessel as

$$n = \frac{M - m}{M_{\mathrm{SO}_2}}\,,$$

where M_{SO_2} is the molar mass of sulphur dioxide. Hence

$$\Delta T = \frac{M_{SO_2}}{(M-m)\,R}\left[\left(p + \frac{(M-m)^2 a}{V^2 M_{SO_2}^2}\right)\left(V - \frac{M-m}{M_{SO_2}} b\right) - pV\right] .$$

Finally we recall that the universal gas constant is $R = 8.314\,\mathrm{J\cdot K^{-1}\cdot mol^{-1}}$, that the coefficients of the van der Waals equation for sulphur dioxide are $a = 0.680\,\mathrm{J\cdot m^3\cdot mol^{-2}}$, $b = 5.64\cdot 10^{-5}\,\mathrm{m^3\cdot mol^{-1}}$ and that the molar mass of sulphur dioxide is $M_{SO_2} = 6.41\cdot 10^{-2}\,\mathrm{kg\cdot mol^{-1}}$. We then obtain $\Delta T \doteq 28\,\mathrm{K}$.

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Problem FoL.31 ... elusive lens

During a clean-up day in the FYKOS room, a thin lens together with a circular-aperture light source with diameter $d = 5\,\mathrm{mm}$ were found. First consider placing the lens in a distance $s_1 = 12\,\mathrm{cm}$ from a screen. If we then put the source in a large distance from both the screen and the lens, we will observe a point-like image on the screen. If on the other hand we place the lens and the source in a distance s_2 and l_2 respectively from the screen, we will observe a sharp circular image with diameter $d_2 = 2\,\mathrm{cm}$. Find the distance l_2 (in cm).

Outrageously, Xellos prefers doing experiments rather than helping with clean- up.

It is clear from the first case that the focal length of the lens is $f = s_1$. In the second case an image is projected onto the screen with linear magnification

$$Z = \frac{d_2}{d} = \frac{s_2}{l_2 - s_2} = 4\,,$$

whence

$$s_2 = l_2 \frac{Z}{1+Z}\,.$$

Finally, using the thin lens formula, we have

$$\begin{aligned}
\frac{1}{s_2} + \frac{1}{l_2 - s_2} &= \frac{1}{f}\,,\\
\frac{1+Z}{Zl_2} + \frac{1+Z}{l_2} &= \frac{1}{f}\,,\\
\frac{(1+Z)^2}{Zl_2} &= \frac{1}{f}\,,\\
l_2 &= \frac{(1+Z)^2}{Z} f \doteq 75\,\mathrm{cm}\,.
\end{aligned}$$

Hence in the second case the source was placed 75 cm from the screen.

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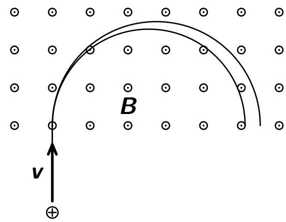
Problem FoL.32 ... playing in the field

Maťko and Kubko, two neon isotopes with mass numbers $A_M = 20$ *and* $A_K = 22$*, came up with a new game, where Maťko simply tries to copy all moves of Kubko. Carrying a charge of* $+e$*, Kubko entered a homogeneous magnetic field with magnitude* $B = 0.24\,\mathrm{T}$ *in the direction perpendicular to the field lines. Maťko, who also carries a charge of* $+e$*, entered the field at the same point and in the same direction as Kubko did. Find the distance* d *of the points where the respective isotopes left the field, provided they both have the same kinetic energy* $E_K = 6.2 \cdot 10^{-16}\,\mathrm{J}$*. Take the values of the atomic mass unit and the elementary charge to be* $m_u = 1.660\,57 \cdot 10^{-27}\,\mathrm{kg}$ *and* $e = 1.602 \cdot 10^{-19}\,\mathrm{C}$*.* *Lydka likes to play with fridge magnets.*



Let us denote by r_M and r_K the radii of circular trajectories followed by Maťko and Kubko. The constraints of the problem then imply $d = 2|r_\mathrm{K} - r_\mathrm{M}|$. First, let us find a general expression for the radius of the trajectory followed by a charged particle in a homogeneous magnetic field. Equating the magnitude of the centripetal force $F_\mathrm{c} = mv^2/r$ with the magnitude of the Lorentz force $F_\mathrm{m} = Bev\sin\alpha$ (where in our case we actually have $\alpha = 90^\circ$) we get

$$r = \frac{mv}{Be}\,.$$

Next,

$$E_\mathrm{K} = \frac{mv^2}{2} \quad \Rightarrow \quad v = \sqrt{\frac{2E_\mathrm{K}}{m}}\,.$$

Also, the mass of a particle with mass number A clearly satisfies $m = Am_\mathrm{u}$. Combining the two expressions for r and v above gives

$$r = \frac{\sqrt{2Am_\mathrm{u}E_\mathrm{K}}}{Be}$$

and so d is as follows

$$d = 2\frac{\sqrt{2A_\mathrm{K}m_\mathrm{u}E_\mathrm{K}}}{Be} - 2\frac{\sqrt{2A_\mathrm{M}m_\mathrm{u}E_\mathrm{K}}}{Be} = 0.016\,3\,\mathrm{m}\,.$$

Thus wanted distance of the points is 0.0163 m.

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Problem FoL.33 ... planet Fykosia

Fykosia is an as yet undiscovered planet of our Solar System. Find the ratio a/r *(where* r *is the radius of Fykosia and* a *is its distance from the Sun) and submit your answer as a multiple of* π*. Assume that the trajectories along which the planets orbit the Sun are circles and that the Sun lies in their common centre. The orbital period of Fykosia is* $T = 3\sqrt{3}\,\mathrm{yr}$ *and its density is* $\varrho = 5\,000\,\mathrm{kg{\cdot}m^{-3}}$*. Also, assume that the magnitude of the force due to gravity acting on a body with mass* $m = 1\,\mathrm{kg}$ *placed on the surface of Fykosia is* $F_G = 69\,\mathrm{N}$*. The gravitational constant is* $G = 6.67 \cdot 10^{-11}\,\mathrm{N{\cdot}m^2{\cdot}kg^{-2}}$*.* *Fykosie, my homeland!*

On one hand, using the third Kepler's law we have

$$\frac{T^2}{a^3} = \frac{T_Z^2}{a_Z^3}\,,$$

where $T_Z = 1\,\mathrm{yr}$ is the orbital period of Earth and $a_Z = 1\,\mathrm{AU}$ is the distance between the Earth and the Sun. Whence

$$a = \left(\frac{T}{T_Z}\right)^{2/3} a_Z\,.$$

On the other hand, the Newton's law of gravity says that

$$F_G = G\frac{Mm}{r^2}\,,$$

where M is the mass of Fykosia, for which we can write

$$M = V\varrho = \frac{4}{3}\pi\varrho r^3\,.$$

Substituting into the Newton's law and handling the units carefully, we finally arrive at

$$r = \frac{3F_G}{4\pi\varrho G m}\,.$$

Therefore the ratio is $a/r \doteq 2\,900\pi$.

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Problem FoL.34 ... feather-light

Imagine yourself in the role of an observer situated in the direction of north pole of a galaxy called Fykopaedia, which you observe rotating clockwise. Once you have managed to identify what appears to be the galactic centre, you also spot a star orbiting the centre along a circular trajectory with radius $r = 7\,500\,\mathrm{pc}$ at the very edge of the galaxy. Having obtained a spectrum of this star, you notice that a hydrogen line with laboratory wavelength $\lambda_{emit} = 587.49\,\mathrm{nm}$ is transverse Doppler-shifted to $\lambda_{obs} = 589.89\,\mathrm{nm}$. Your task is now to find the mass of Fykopaedia in the units of solar mass. You should disregard any effects due to possible presence of dark matter or dark energy. Also, assume that the observed spectral shift is solely due to transverse Doppler effect and that the mass distribution in the galaxy is spherically symmetric.

f(Aleš) complaining about his heavy bag.

Given the distance of the star from the galactic centre and its speed, it is easy to infer its orbital period. We can then apply the Third Kepler's law to obtain the mass of the galaxy, i.e.

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{rv^2}{G}\,,$$

where v is the orbital speed and G is the gravitational constant. Hence we are left to find the orbital speed, for which we may write

$$\frac{\lambda_{obs}}{\lambda_{emit}} = \frac{1}{\sqrt{1 - v^2/c^2}}\,,$$

which follows from the formalism of transverse Doppler effect. The speed v can then be expressed as

$$v^2 = c^2 \left(1 - \left(\frac{\lambda_{\mathrm{emit}}}{\lambda_{\mathrm{obs}}}\right)^2\right) .$$

Substituting into the expression for the mass derived earlier, we get

$$M = \frac{rc^2 \left(1 - \left(\frac{\lambda_{\mathrm{emit}}}{\lambda_{\mathrm{obs}}}\right)^2\right)}{G} .$$

Numerically $M \doteq 2.5 \cdot 10^{45}\,\mathrm{kg} \doteq 1.3 \cdot 10^{15} M_{\mathrm{S}}$, where the values $M_{\mathrm{S}} = 1.99 \cdot 10^{30}\,\mathrm{kg}$ and $G = 6.67 \cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}$ were used.

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Problem FoL.35 ... an accelerated one

Consider a proton travelling inside a synchrotron with a total energy of $E_{\mathrm{p}} = 1.00\,\mathrm{GeV}$. The proton collides with an alpha particle with a total energy of $E_\alpha = 3.75\,\mathrm{GeV}$. Given the rest masses of a proton and an alpha particle $m_{\mathrm{p}} = 0.938\,\mathrm{GeV}/c^2$ and $m_\alpha = 3.727\,\mathrm{GeV}/c^2$, respectively, determine the ratio v_{p}/v_α of the speeds of the particles before they collide.

Mirek likes getting results which resemble fundamental mathematical constants.

It follows from the principles of the theory of special relativity that total energy of a particle is given by

$$E = \gamma mc^2 ,$$

where the speed of the particle is hidden inside the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and thus can be expressed as

$$v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2} .$$

Therefore the ratio v_{p}/v_α satisfies

$$\frac{v_{\mathrm{p}}}{v_\alpha} = \frac{E_\alpha}{E_{\mathrm{p}}}\sqrt{\frac{E_{\mathrm{p}}^2 - (m_{\mathrm{p}}c^2)^2}{E_\alpha - (m_\alpha c^2)^2}} \doteq 3.13 .$$

The ratio of velocities is equal to 3.13.

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Problem FoL.36 ... resistance upside down

We want to build a circuit with total resistance $R = (57\,688/1\,807)\,\Omega$, but we only have resistors with resistance $r = 1\,\Omega$ at our disposal. On top of that, we want to build our circuit so that in each step (starting with one resistor) we connect only one resistor to an already existing part of the circuit, either in series or in parallel. What is the smallest number of resistors we need to reach our goal?

Xellos recalled a physically motivated problem from a programming competition.

Starting with a resistance $R/r = a/b$ we can make it either into a resistance $(a + b/b$ by connecting one resistor in series, or, into a resistance $a/(b + a)$ by connecting one resistor in parallel. In other words, resistance a/b can be built starting with either $(a - b)/b$ (if $a \geq b$) or $a/(b-a)$ (if $b > a$) and with nothing else. Therefore the number of resistors needed to build a/b is unique and can be calculated by subtracting the numerator from the denominator or vice versa. Hence we find out that getting to $a/b = 0$ takes 68 steps.

There is a faster approach, which makes use of the fact that a/b can be built using the same number P of steps as b/a, hence

$$P\left(\frac{a}{b}\right) = P\left(\frac{b}{a}\right) ,$$

and for $a \geq b$ have

$$P\left(\frac{a}{b}\right) = \left\lfloor\frac{a}{b}\right\rfloor + P\left(\frac{a \bmod b}{b}\right) .$$

Hence the answer is $31 + 1 + 12 + 3 + 2 + 19 = 68$.

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Problem FoL.37 ... nomen omen

During a meeting of the greatest villains of all time, The Emperor was laughed at: his Death Star has a word "star" in its name, yet it does not shine. He could not possibly let such insolence pass and decided to fix the problem. His Death Star now emits light as if it was a gray body with emissivity $\varepsilon = 0.8$ across all wavelengths. (Emissivity of a body is defined to be the ratio of intensity of radiation emitted by the body to the intensity of radiation emitted by a black body at the same temperature.) The radius of the Death Star is $r = 450\,\mathrm{km}$ and an inspecting spaceship, being at a distance of $d = 10\,000\,\mathrm{km}$ from the Star, measured intensity $I = 33\,\mathrm{kW\cdot m^{-2}}$ of radiation coming from the Star. What is the wavelength (in nanometres) at which the Death Star radiates most of its energy? Stefan-Boltzmann constant is $\sigma = 5.67 \cdot 10^{-8}\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}$, Wien's displacement constant is $b = 2.90 \cdot 10^{-3}\,\mathrm{m\cdot K}$.

Mirek lives by the philosophy that there is never enough Star Wars themed problems.

Intensity of radiation decreases with the square of the distance. Therefore, we can use the Stefan-Boltzmann law to find that

$$I(d/r)^2 = \varepsilon\sigma T^4 .$$

We then use Wien's displacement law to calculate the wavelength at which the Death Star radiates most of its energy. We obtain

$$\lambda_{\max} = \frac{b}{T} = \frac{b}{\sqrt[4]{d^2 I/(\varepsilon\sigma r^2)}} \doteq 666\,\mathrm{nm} .$$

The Death Star radiates most of its energy at wavelength 666 nm.

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Problem FoL.38 ... pendulum

Once upon a time there lived an infinite superconducting horizontal plane with surface mass density $\sigma = 2 \cdot 10^{10}\,\mathrm{kg{\cdot}m^{-2}}$ *and a pivot placed* $d = 5\,\mathrm{m}$ *above the plane. And so it came to pass that somebody came up with the idea to suspend an electron on a string with length* $l = 1\,\mathrm{m}$ *from the pivot. Find the period of small oscillations (in seconds) of the electron about its equilibrium.*

Assume that charge is free to move within the plane arbitrarily fast. Neglect any effects due to magnetic field. *Xellos walking an electron on a dog lead.*

We think of the electron as being a point charge $-e$ with mass m_e. There are two forces acting on it, both in the direction downwards perpendicular to the plane: electrostatic force with magnitude F_e and gravitational force $F_\mathrm{g} = m_\mathrm{e} g$.

Gauss's law of electrostatics may be (in an analogy) used to determine the magnitude of the acceleration due to gravity g. First, the field intensity is defined as a force per unit mass, which directly corresponds to acceleration in the case of gravitational field. Next, comparing Coulomb's and Newton's inverse square laws for fields due to point sources, we find that the factor of $1/\varepsilon_0$ needs to be replaced by $4\pi G$. Hence the Gauss's law of gravity can be written as

$$\oint_{\partial V} g\,\mathrm{d}S = 4\pi G m\,,$$

where m is total mass enclosed by volume V. Thus, choosing a suitable Gaussian surface, we obtain

$$g = 2\pi G\sigma\,.$$

Next, we use the method of images to find the magnitude F_e of the electrostatic force acting on the electron: given a charge $-e$ situated in a distance h above a conducting plane, the force acting on the charge is the same as if the plane was replaced by a mirror charge $+e$ situated in a distance h below the plane. Coulomb's law then gives

$$F_\mathrm{e} = \frac{e^2}{4\pi\varepsilon_0 (2h)^2}\,.$$

The distance $h \approx d - l$ between the electron and the plane stays roughly constant during the period of oscillations (which are assumed to be small). We can therefore conclude that there is a constant force with magnitude $F = F_\mathrm{e} + F_\mathrm{g} = 1.12 \cdot 10^{-29}\,\mathrm{N}$ acting on the electron. Next, upon disturbing the electron form its equilibrium sideways be an angle α, there appears a restoring torque $\tau \approx Fl\alpha$. Finally, given that the moment of inertia of the electron relative to the pivot is clearly $I = m_\mathrm{e} l^2$, the angular frequency ω of the oscillations satisfies

$$\omega^2 = \frac{4\pi^2}{T^2} = \frac{\tau}{I\alpha} = \frac{F}{m_\mathrm{e} l}\,,$$

whence $T = 2\pi\sqrt{m_\mathrm{e} l/F} \doteq 1.8\,\mathrm{s}$.

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Problem FoL.39 ... magnetic focusing

Assume that we fire a beam of protons from a given location on the x axis. In the beam, every proton has a speed of $v = 10^5\,\mathrm{m{\cdot}s^{-1}}$, whose direction deviates from that of the x axis by at most $\Delta\alpha = 1^\circ$. Assume that there is an electric field pointing in the x direction with magnitude $E = 12\,\mathrm{N{\cdot}C^{-1}}$, together with a magnetic field with magnitude $B = 1\,\mathrm{mT}$, also pointing in the x direction. The protons will first come back to the x axis at a distance from the firing point which lies in an interval of the form $(L, L + \Delta L)$. Find the ratio $(\Delta L)/L$. Do not take into account the protons fired precisely along the x axis. *Xellos was playing Bang!*

Consider a single particle of mass m and charge q, whose direction deviates from the x-axis by a small angle α. Its velocity in the direction perpendicular to x-axis is $v_\perp = v \sin\alpha$, which is a constant. Therefore the particle will move along a circular path of radius

$$R = \frac{v_\perp m}{qB}\,,$$

and it will intersect the x-axis again in time

$$T = \frac{2\pi R}{v_\perp} = \frac{2\pi m}{qB}\,.$$

The initial velocity in the direction of the x-axis is $v_\parallel = v\cos\alpha$ and the acceleration is $a = Eq/m$, so in the time T the particle travels in the direction of x-axis distance

$$L = v_\parallel T + \frac{Eq}{2m}T^2 = \frac{2\pi m v_\parallel}{qB} + \frac{2\pi^2 mE}{qB^2}\,.$$

It's obvious that $\Delta L = L(\alpha = 0) - L(\alpha = \Delta\alpha)$ and since the angle α is small, i.e. $\alpha \le \Delta\alpha \ll 1$, we can use the approximation $\cos\alpha \approx 1 - \alpha^2/2$ and write

$$\Delta L \approx \frac{\pi m v (\Delta\alpha)^2}{qB}\,.$$

A less accurate approximation $\cos\alpha \approx 1$ yields

$$L \approx \frac{2\pi m v}{qB} + \frac{2\pi^2 mE}{qB^2}\,,$$

$$\frac{\Delta L}{L} \approx \frac{(\Delta\alpha)^2}{2 + \frac{2\pi E}{vB}} \doteq 1.1\cdot 10^{-4}\,,$$

($\Delta\alpha$ is expressed in radians). The ratio $\Delta L/L$ equals $1.1\cdot 10^{-4}$.

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Problem FoL.40 ... slingshooty

The Czech mythical genius Jára Cimrman is fabled for his construction of the so called Čakovice slingshot. Being curious, we loaded the slingshot with a projectile and fired. Having grossly underestimated its power, not only did we find that the projectile did not land in Čakovice (now a district of Prague), we discovered that it basically did not land at all. It went through *the Earth's atmosphere, across the Solar system, all the way out of the Galaxy, even escaping the local group, passing through the Virgo cluster… until it reached the very borders of the visible universe and ended up in an alternative one.*

The physical laws of this alternative universe are quite similar to those of our universe, except that the gravitational force between the projectile and rest of the matter in this alternative universe, while still derived from the Newton's law, acts repulsively. Consider the projectile approaching one of local stars. If there was no interaction between the projectile and the star, the projectile would miss the star at distance $b = 10^{10}\,\mathrm{m}$. However, as a result of the above described interaction, the trajectory of the projectile will be deflected. Assuming that the mass of the projectile is $m = 1\,\mathrm{kg}$, that the mass of the star is $M = 10^{30}\,\mathrm{kg}$ and that the speed of the projectile far away from the star is $v_0 = 10^5\,\mathrm{m{\cdot}s^{-1}}$, find the minimum separation of the projectile and the star (both of which are to be regarded as point particles) during their interaction. The gravitational constant is $G = 6.67\cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}$.

Mirek applied Rutherford's experiment in the macroworld.

We will approach the problem as if it was the Rutherford's gold foil experiment. In a planetary system angular momentum and total mechanical energy are always conserved. It is convenient to express these two conserved quantities in polar coordinates. We will set them equal to their initial values

$$
\begin{aligned}
L &= mv_0 b = mr^2\dot{\varphi}\,,\\
E &= \frac{1}{2}mv_0^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{GMm}{r}\,.
\end{aligned}
$$

Now we express the time derivative $\dot{\varphi}$ from the first equation and plug it into the second equation. After some algebraic manipulation we arrive at

$$v_0^2 = \left(\dot{r}^2 + \frac{L^2}{m^2r^2}\right) + \frac{2GM}{r}\,.$$

The projectile reaches its closest point to the sun when the radial distance r is minimal, which implies $\dot{r} = 0$ and the equation simplifies to

$$v_0^2 - \frac{2GM}{r} - \frac{L^2}{m^2r^2} = 0\,.$$

It remains to plug in the initial value of angular momentum and the minimum distance can be obtained by solving the quadratic equation in r. The only physically meaningful root is

$$r_{\mathrm{min}} = \frac{GM}{v_0^2} + \sqrt{\frac{G^2M^2}{v_0^4} + b^2} \doteq 1.87\cdot 10^{10}\,\mathrm{m}\,.$$

The minimum distance is $1.87\cdot 10^{10}\,\mathrm{m}$.

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Problem FoL.41 ... little gifts

Having left the student housing, Mirek's roommate left behind a number of interesting objects, amongst them a 10 m *long rope,* 170 cm *long iron rod and a bag full of cobble stones. Now, let's consider the following situation: first, let's tie the rope to one end of the rod and fix the bag with cobble stones to the same end. Then throw the rod with the bag out of the window, while holding the other end of the rope firmly. The rod lands on the pavement (which is $d = 11\,\mathrm{m}$ below the window) in such a way that its free end rests on the pavement exactly below the point where the other end of the rope is held. Assuming that the rod can rotate freely about the end on which it rests, but that the end itself cannot move (there is an infinite friction between the pavement and the rod), find the tension in the rope in equilibrium. The bag with cobble stones has mass $m_k = 8\,\mathrm{kg}$, the rod weighs $m_t = 5\,\mathrm{kg}$ and the stiffness of the rope is $k = 200\,\mathrm{kg{\cdot}s^{-2}}$. Neglect the mass of the rope.* *And don't forget to fix the road afterwards! (Mirek).*

Let us denote by L and l the length of the rod and the length of the rope, respectively. Define

$$M = \frac{m_\mathrm{t}}{2} + m_\mathrm{k}\,.$$

The total potential energy of the system is then given by

$$V = MgL\cos\alpha + \frac{k}{2}\left(\sqrt{L^2 + d^2 - 2dL\cos\alpha} - l_0\right)^2\,,$$

where l_0 is the rest length of the rope. Clearly, the first term in the above equation has the meaning of gravitational potential energy whereas the second term is nothing but the elastic potential energy $k(\Delta l)^2/2$. The total potential energy must have a minimum at equilibrium, hence

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = 0\,.$$

Differentiating the above expression for V and equating the result to zero yields

$$l = \frac{l_0}{1 - Mg/kd}\,,$$

which we, in turn, substitute into the Hooke's law. We obtain

$$F = k(l - l_0) = \frac{kl_0}{kd/Mg - 1} \doteq 98.2\,\mathrm{N}\,.$$

The tension in the rope is $F \doteq 98.2\,\mathrm{N}$.

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Problem FoL.42 ... van de Graaff

Consider a conducting sphere with radius $R = 7\,\mathrm{cm}$ charged to a potential $\varphi = -5\,\mathrm{V}$ (taken relative to a point at infinity) and covered with a thin layer of dielectric. The sphere is placed into a chamber with ionised helium He^+ where we assume that only the ions with negligible kinetic energy stick to the surface of the sphere. Finally, the sphere is moved into an evacuated chamber with volume $V = 10\,\mathrm{m}^3$ and quickly discharged. Find the equilibrium temperature (in kelvins) of the ion gas thus created in the chamber. *Xellos loves electroshocks.*

The electrostatic potential on the surface of a homogeneously charged sphere is equal to the potential of a point charge $-Q$ placed in the center of the sphere

$$\varphi = -\frac{Q}{4\pi\varepsilon_0 R}\,.$$

The ions which stick to the sphere have the same magnitude of charge, so the sphere appears to be neutral and does not attract more ions.

After discharging, the force holding the ions to the surface of the sphere will cease to exist and all the ions will spread into the surrounding environment. The container is reasonably big, so we can safely assume that the potential energy will be transformed into kinetic energy. Now we will recall the equipartition theorem

$$E_\mathrm{k} = \frac{3}{2}k_\mathrm{B}T\,.$$

Therefore we need to compute electrostatic energy of one ion. That can be done for example by increasing the radius of the sphere with a charge Q and then computing the work done by the electric force on one ion, which is in fact equal to the difference of the energies with the electrostatic energy at infinity equal to zero.

As the next step we compute the electrical intensity acting upon a infinitesimal surface $\mathrm{d}S$ of continuously distributed charge on the sphere with the radius r. The intensity at the surface of the sphere, using Gauss's law, is

$$E_\mathrm{g} = \frac{Q}{4\pi\varepsilon_0 r^2}\,.$$

From this result we must subtract the intensity E_s from the surface $\mathrm{d}S$ itself. The chosen surface is small, therefore we can treat it as a short cylinder and find (Gauss's law again)

$$E_\mathrm{s} = \frac{Q}{8\pi\varepsilon_0 r^2}\,,$$

the force acting upon charged ion is

$$F = e(E_\mathrm{g} - E_\mathrm{s}) = \frac{eQ}{8\pi\varepsilon_0 r^2}\,.$$

Work is obtained as the integral

$$W = \int_R^\infty F\mathrm{d}r = \frac{eQ}{8\pi\varepsilon_0 R}\,,$$

and the temperature is

$$T = \frac{2W}{3k_B} = \frac{eQ}{12\pi\varepsilon_0 R k_B} = \frac{-e\varphi}{3k_B} \doteq 19\,300\,\mathrm{K}\,.$$

The gas in the container will have temperature $T \doteq 19\,300\,\mathrm{K}$.

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Problem M.1 ... electrical shopping cart

Being incredibly lazy, the organisers of FYKOS decided to construct a small self-propelled cargo vehicle. It remains to fine tune the engine. Having a total mass of $m = 70\,\mathrm{kg}$, the vehicle is known to accelerate with $a = 1\,\mathrm{m{\cdot}s^{-2}}$ when it moves horizontally. Find the vehicle's (uniform) acceleration on a 10% grade slope assuming that the same thrust is applied in both cases. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *f(Aleš) transporting the competition prizes.*

The slope grade can be related to the angle of inclination α as

$$\operatorname{tg}\alpha = \frac{1}{10}\,.$$

The magnitude of acceleration a_s of the vehicle on the inclined surface can then be calculated as

$$a_\mathrm{s} = \frac{F_\mathrm{s}}{m}\,.$$

It remains to find an expression for the magnitude F_s of the force acting on the vehicle parallel to the inclined surface in terms of the magnitude F_v of the force which propelled the vehicle when it moved horizontally. We have

$$F_\mathrm{s} = F_\mathrm{v} - F_G = ma_\mathrm{v} - mg\sin\alpha\,,$$

because in the second case there is a non-zero component of gravitational force on the vehicle parallel to the inclined plane. Thus

$$a_\mathrm{s} = a_\mathrm{v} - g\sin\alpha\,.$$

Hence we get $a_\mathrm{s} \doteq 0.024\,\mathrm{m{\cdot}s^{-2}}$.

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Problem M.2 ... Náry's academic quarter

Most people's reactions can be virtually immediate but not in the case of Náry. Even when it comes to mere releasing of things to fall freely under gravity, it takes him $\tau = 6\,\mathrm{s}$ to react on a signal. Find the time (in seconds) which elapses until the distance between a body released by Náry and a body released immediately on signal grows to $l = 200\,\mathrm{m}$. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ and we start measuring time at the exact moment when the first body is released. *f(Aleš) lamenting Náry's late arrivals...*

Let us denote by t the time elapsed since we released the first body. Clearly, the time elapsed since Náry released his body is then $t - \tau$. The distances through which the bodies fall after time t can be expressed as

$$s(t) = \frac{1}{2}gt^2\,,$$
$$s_\mathrm{N}(t) = \frac{1}{2}g\,(t-\tau)^2\,\Theta(t-\tau)\,,$$

where Θ is the Heaviside step function which is defined to be zero at negative values and 1 elsewhere. The distance $l = s - s_\mathrm{N}$ happens to be greater than $s(\tau)$, so $t > \tau$ and

$$l = g\tau\left(t - \frac{1}{2}\tau\right)\,,$$

thus

$$t = \frac{l}{g\tau} + \frac{\tau}{2}\,.$$

Hence we get $t \doteq 6.4\,\mathrm{s}$.

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Problem M.3 ... truly solid lecture notes

As we all know, some books do not really count as light-weights, so instead of carrying them in a bag let's try and drag them on the floor on a string. Consider a book lying on the floor and a string with length $l = 2.0\,\mathrm{m}$ whose one end is attached to the centre of the upper side of book's cover. The other end of the string is held in height $h = 1.2\,\mathrm{m}$ above the floor. Find the coefficient of friction between the book and the floor knowing that the book which is of mass $m = 1\,\mathrm{kg}$ is dragged at constant speed with force of magnitude $F = 5\,\mathrm{N}$. Assume that the book does not open. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

f(Aleš)'s bag is getting heavier and heavier.

The coefficient of friction is can be calculated as a ratio of magnitudes of the force due to friction and the net force on the book normal to the floor. The magnitude F_f of the force due to friction must be equal to magnitude of the horizontal component of the force applied on the string, i.e.

$$F_x = F\cos\alpha = F\frac{\sqrt{l^2 - h^2}}{l}\,,$$

where α is the angle between the string and the horizontal and the Pythagorean theorem was used to express $\cos\alpha$ in terms of l and h.

The force on the book normal to the floor is found to be the book's weight from which the vertical component of the force applied to the string was subtracted. Hence

$$F_\mathrm{N} = F_\mathrm{G} - F_y = mg - F\sin\alpha = mg - F\frac{h}{l}\,.$$

The coefficient of friction f is then

$$f = \frac{F_\mathrm{f}}{F_\mathrm{N}} = \frac{F\sqrt{l^2 - h^2}}{mgl - Fh}\,.$$

Hence we get $f \doteq 0.59$.

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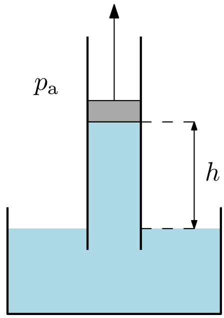
Problem M.4 ... all in vain

Consider a long pipe whose one end is plunged into water. Inside the pipe, there is a piston which perfectly fits between the walls. Having squeezed the air out, we start pulling the piston up, thus lifting a column of water in the pipe under the piston head. What is the maximum height h in which we can lift the water assuming that the ambient atmospheric pressure is $p_a = 760\,\mathrm{mmHg}$*?*

f(Aleš) was short of water.

The water column under the piston head will continue to rise until the hydrostatic pressure will balance the ambient atmospheric pressure. Since we are given the ambient atmospheric pressure as the length of a mercury column, we can obtain the correct numerical answer by first multiplying its value by density of mercury $13\,595\,\mathrm{kg{\cdot}m^{-3}}$ and then dividing by density of water $1\,000\,\mathrm{kg{\cdot}m^{-3}}$. Hence the height at which the water column breaks is $h \doteq 10.3\,\mathrm{m}$.



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Problem E.1 ... the long one and the wide one

Consider two homogeneous cylindrical copper wires of equal mass, one of them being thrice as long as the other. Find the ratio of electrical resistances along the two wires, longer to the shorter.

Lukáš likes to steal lab equipment.

Given resistivity of the material ϱ, length of the wire l and its cross-sectional area S, the electrical resistance along the wire can be found as

$$R = \frac{\varrho l}{S}\,.$$

The ratio of lengths of the two wires is 3. Since the two wires have the same mass, their volumes must also be the same and so the ratio of their cross-sectional areas must be 1/3. Since the cross-sectional area appears in the denominator of the expression for the resistance, we get the ratio to be 9. Hence the resistance of the longer wire is nine times as large as that of the shorter wire.

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Problem E.2 ... one, two, three,...

Náry was asked to find out the number of turns in his brand new copper coil. Too lazy to count them one by one, he resorted to his multimeter and good old data tables. Having brought a potential difference $U = 7.8\,\mathrm{V}$ *across the coil, he measured a current* $I = 2\,\mathrm{A}$ *flowing in the coil. Also, he measured the cross-sectional diameter of the wire to be* $S = 1\,\mathrm{mm^2}$*. He then used his data tables to find the electrical resistance of copper to be* $\varrho = 1.8\cdot 10^{-7}\,\Omega{\cdot}\mathrm{m}$*. He also measured the diameter of the turns and obtained an average value* $d = 6\,\mathrm{cm}$*. Assuming that the total length of leads is* $l_0 = 18\,\mathrm{cm}$*, find the number of turns in Náry's coil.*

f(Aleš) wanted to set a black-box problem.

We can use Ohm's law to find the resistance along the wire. This resistance is then directly proportional to the length of the wire, which may then be used to find the number of turns. We have

$$N = \frac{l - l_0}{2\pi \frac{d}{2}} = \frac{\frac{RS}{\varrho} - l_0}{\pi d} = \frac{\frac{US}{I\varrho} - l_0}{\pi d}\,.$$

Hence, we get $N \doteq 114$. It can be shown that the error we made by neglecting the helicity of the coil is on third decimal place.

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Problem E.3 ... experimental afternoon

The power outlets in a building where a FYKOS camp was held provide voltage $U_0 = 230\,\mathrm{V}$. However, some of the experiments had to be performed outside, especially those which involved shooting with an air gun. To this end, an extension lead of length $l = 101\,\mathrm{m}$ and resistance $R_0 = 1.4\,\Omega$ was used to bring the power supply where it was needed. Consider connecting a load to the extension lead. The properties of the load are such that if we bring a potential difference $U_s = U_0$ to it, the load provides power $P_s = 1\,000\,\mathrm{W}$ to an appliance (e.g. a heating spiral). Assuming that voltage $U_s = U_0$ is brought to the other end of the lead, find the potential drop across the load. *f(Aleš) had a moment of nostalgia.*

The whole circuit can essentially be thought of as consisting of two components: the load and the lead. The current flowing through both of these must be the same, so the Ohm's law yields

$$I = \frac{U_0}{R_0 + R_\mathrm{s}}\,,$$

where R_s is the resistance of the load. Clearly, we must also have

$$I = \frac{U}{R_\mathrm{s}}\,,$$

where U is the potential drop across the load and R_s is the resistance of the load. Equating the two expressions for I yields an equation for U. It remains to find an appropriate expression for R_s. Clearly, this is provided by knowing the power supplied by the load, when a potential difference U_0 is brought to it. We have

$$P_\mathrm{s} = \frac{U_0^2}{R_\mathrm{s}}\,.$$

Combining the two equations we derived, we obtain

$$U = \frac{U_0}{1 + \frac{R_0}{R_\mathrm{s}}} = \frac{U_0^3}{U_0^2 + R_0 P_\mathrm{s}}\,.$$

Plugging in the numbers, we get $U \doteq 224\,\mathrm{V}$.

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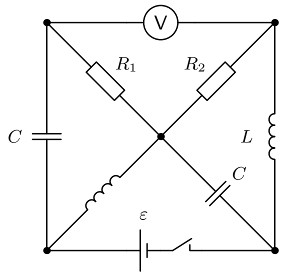
Problem E.4 ... pyramid's circuits

In the circuit shown below we first put the switch on and wait for the currents to come to a steady state. Find the magnitude of the potential difference (in volts) measured by the voltmeter. Take the capacitors and coils to be ideal and assume that the voltmeter provides an infinite resistance. Put $\varepsilon = 1\,\mathrm{V}$. The resistance $R_1 = 3R_2$.

Xellos found out about the Estonian-Finnish Olympiad in Physics.



We'll calculate currents through the resistors from the top vertex of the pyramid (junction in the center of the picture).

After the currents reach steady state, the capacitors and the voltmeter can be thought of as being elements with infinite resistance. On the other hand, the coil behaves like a conductor with no resistance. Hence the currents through R_1 and R_2 are $I_1 = 0$ and $I_2 = \varepsilon/R_2$ respectively, so the potential difference across the two resistors is

$$|R_1 I_1 - R_2 I_2| = \varepsilon\,.$$

The reading shown by the voltmeter is therefore $1\,\mathrm{V}$.

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Problem X.1 ... uncertain bacteria

Use a quantum uncertainty principle to find the uncertainty Δv_x in the component v_x of the velocity of a bacterium with mass $m = 1.0\cdot 10^{-15}\,\mathrm{kg}$ and linear size $l = 1.0\cdot 10^{-6}\,\mathrm{m}$. Assume that the position of the bacterium can be determined with uncertainty $\Delta x = l/100$ and that $v_x = 3.0\cdot 10^{-5}\,\mathrm{m\cdot s^{-1}}$. Submit your answer as $\log_{10}(\Delta v_x/v_x)$.

Mirek was trying to figure out why it is so difficult to determine the origin of some diseases.

The Heisenberg uncertainty principle relating the bacterium's position and momentum

$$\Delta x \Delta p_x = \frac{\hbar}{2}\,,$$

can obviously be recast as

$$\Delta x \Delta v_x = \frac{\hbar}{2m}\,.$$

Hence the ratio $\log_{10}(\Delta v_x/v_x)$ can be expressed as

$$\log_{10}\frac{\Delta v_x}{v_x} = \log_{10}\frac{\hbar}{2m\Delta x v_x} \doteq -7\,.$$

The order of magnutude of the uncertainty is -7.

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Problem X.2 ... a lecture on relativity

Two professors pass each other in a corridor at the Institute of Theoretical Physics, going in opposite directions. One is coming back from a lecture about special theory of relativity and his speed is $v_s = 1\,\mathrm{m\cdot s^{-1}}$, while the second one is going to a lecture about general relativity and his speed is $v_g = 2\,\mathrm{m\cdot s^{-1}}$. Determine the error that we will make by using classical physics instead of relativity to calculate the speed of the second professor in the reference frame of the first professor. *Mirek trying to entertain himself while waiting for a lecture on relativity.*

The classical result on addition of velocities says that

$$u = v_\mathrm{s} + v_\mathrm{g}\,,$$

whereas the corresponding relativistic expression can be written as

$$w = \frac{v_\mathrm{s} + v_\mathrm{g}}{1 + \frac{v_\mathrm{s} v_\mathrm{g}}{c^2}}\,.$$

The error made by using classical physics instead of relativity is therefore

$$u - w = (v_\mathrm{s} + v_\mathrm{g})\left(1 - \frac{1}{1 + \frac{v_\mathrm{s} v_\mathrm{g}}{c^2}}\right) = (v_\mathrm{s} + v_\mathrm{g})\frac{1}{\frac{c^2}{v_\mathrm{s} v_\mathrm{g}} + 1} \approx \frac{v_\mathrm{s} v_\mathrm{g}(v_\mathrm{s} + v_\mathrm{g})}{c^2} \doteq 6.7\cdot 10^{-17}\,\mathrm{m\cdot s^{-1}}\,.$$

Error made by considering classical theory equals $6.7\cdot 10^{-17}\,\mathrm{m\cdot s^{-1}}$.

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Problem X.3 ... hot grating

Determine the temperature needed to observe diffraction of a mono-energetic $^{20}\mathrm{Ne}$ beam (produced by an energy filter) on a grating with slit spacing $d = 0.5\,\mathrm{nm}$. The condition for diffraction to be observed is taken to be $\lambda = d$ where λ is the de Broglie wavelength of the atoms in the beam. *Mirek skimming through a quantum physics textbook.*

Kinetic energy of monoatomic particles is

$$E = \frac{3}{2}kT\,,$$

where $k = 1.38\cdot 10^{-23}\,\mathrm{J\cdot K^{-1}}$ is the Boltzmann constant. By considering de Broglie relation

$$p = \frac{h}{\lambda}$$

we can write

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}\,,$$

where m is the atomic mass of neon, which can be calculated as a product of atomic mass unit $a_\mathrm{u} = 1.67\cdot 10^{-27}\,\mathrm{kg}$ and the relative atomic mass of neon $A_\mathrm{Ne} = 20$. Hence the temperature is given as

$$T = \frac{h^2}{3mkd^2} \doteq 1.3\,\mathrm{K}\,.$$

The needed temperature to see the difraction is $T \doteq 1.3\,\mathrm{K}$.

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Problem X.4 ... beta decay

Consider a beam of neutrons, where every neutron in the beam has kinetic energy $E_k = 0.05\,\mathrm{eV}$. Take the half-life and the rest energy of a neutron to be $T_{1/2} = 640\,\mathrm{s}$ and $m_n c^2 = 940\,\mathrm{MeV}$, respectively. Find the fraction of the number of neutrons which will decay before the beam travels a distance of $d = 10\,\mathrm{m}$. *Mirek got inspired by a textbook about particle physics.*

It is obvious that $m_\mathrm{n}c^2 \gg E_\mathrm{k}$, therefore we can treat this problem classically. In such a case, speed of a particle can be written as

$$v = \sqrt{\frac{2E_\mathrm{k}}{m_\mathrm{n}}}$$

and so the time in which it travels distance d is

$$t = \frac{d}{v} = \frac{d}{c}\sqrt{\frac{m_\mathrm{n}c^2}{2E_\mathrm{k}}}\,.$$

It remains to apply the law of radioactive decay in the form

$$N(t) = N_0 \mathrm{e}^{-(t\ln 2)/T_{1/2}}$$

using which we find the fraction of neutrons which will decay, i.e.

$$\frac{N_0 - N}{N_0} = 1 - \mathrm{e}^{-(t\ln 2)/T_{1/2}} = 1 - \mathrm{e}^{-\frac{d\ln 2}{T_{1/2}c}\sqrt{\frac{m_\mathrm{n}c^2}{2E_\mathrm{k}}}} \approx \frac{d\ln 2}{T_{1/2}c}\sqrt{\frac{m_\mathrm{n}c^2}{2E_\mathrm{k}}} \doteq 3.5\cdot 10^{-6}\,.$$

Only $3.5\cdot 10^{-6}$ of the neutrons will decay.

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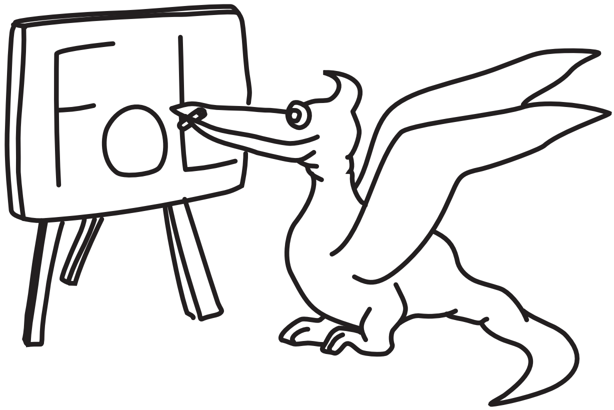
Solutions of 5th Online Physics Brawl



Problem FoL.1 ... misbehaved students

On a planet far, far away, some misbehaved students threw an object out of a window, located $h = 40\,\mathrm{m}$ above the ground. It landed after $t = 4\,\mathrm{s}$. How much would this object weigh on Earth, if the gravitational force acting upon the object on the distant planet is $F = 55\,\mathrm{N}$? Consider the gravitational field to be constant on both planets and the air drag to be negligible.

Olda was having plenty of time while Náry was cooking.

To solve this problem, we must be aware of the difference between the weight and the mass of an object, i.e. that its mass is independent of its location and the intensity of the gravitational field. Our first task is to find the gravitational acceleration on the distant planet, and then, using Newton's second law, to get the mass of the object (which is equal to its weight on Earth divided by g).

The equation for free fall is

$$h = \frac{1}{2}at^2\,,$$

which we can rearrange to express the acceleration

$$a = \frac{2h}{t^2} = 5\,\mathrm{m{\cdot}s^{-2}}\,.$$

To find the weight, we divide the force $F = 55\,\mathrm{N}$ by the gravitational acceleration

$$m = \frac{F}{a} = \frac{Ft^2}{2h} = 11\,\mathrm{kg}\,.$$

The object weighs 11 kg.

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Problem FoL.2 ... plastered block

Kuba is boasting that he can lay a perfectly smooth cuboid on an inclined plank in such a way that the cuboid won't slip down. The plank is inclined at an angle $\alpha = 20^\circ$ to the horizontal plane. Find out what Kuba's acceleration has to be when he holds the plank, so that the cuboid would remain at rest (with respect to the plank). *Mirek doesn't believe such silly tricks...*

It's a straightforward analysis of forces. The cuboid is pulled down by gravity mg, where m is its mass and g is the acceleration due to gravity. When projected onto the plank, it's $mg\sin\alpha$. Kuba's acceleration $\boldsymbol{a}$ will result in a reaction force ma in the opposite direction to Kuba's acceleration. Its projection onto the plank is $ma\cos\alpha$. The cuboid won't move if both projections are equal in magnitude

$$mg\sin\alpha = ma\cos\alpha\,,$$

from which we can express

$$a = g\,\mathrm{tg}\,\alpha\,.$$

After substitution $g = 9.8\,\mathrm{m{\cdot}s^{-2}}$, we get a numeric result $a = 3.6\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.3 ... we're on a roll

A truncated cone is rolling on a table in such a way, that a point on the perimeter of its smaller base moves at a speed $v_1 = 1.5\,\mathrm{m\cdot s^{-1}}$ with respect to the base's center and a point on the perimeter of its greater base moves at a speed $v_2 = 1\,\mathrm{m\cdot s^{-1}}$ (again, with respect to the base's center). The slant height is $l = 0.1\,\mathrm{m}$. How long will it take the cone to return to the point from which it started rolling?

Tom dropped a pestle.

The greater base's point of contact with the table follows a circle of a radius $R = l + L$, where L is the slant height of a cone which we would have to cut off from a normal cone in order to obtain the truncated one. Both bases move at the same angular velocity, thus

$$\frac{v_1}{r_1} = \frac{v_2}{r_2}\,,$$

from which we isolate

$$\frac{r_1}{r_2} = \frac{v_1}{v_2}\,.$$

Then, from the similarity of triangles, we have

$$\frac{r_2}{L} = \frac{r_1}{L+l}\,,$$

so

$$1 + \frac{l}{L} = \frac{r_1}{r_2} = \frac{v_1}{v_2}\,.$$

For the unknown variable L, we can write

$$L = \frac{l}{\frac{v_1}{v_2} - 1}\,.$$

The truncated cone returns to the starting point when the greater base makes a whole circle of the radius $R = L + l$, so the time is

$$t = \frac{2\pi(L+l)}{v_1} = \frac{2\pi l}{v_1}\left(1 + \frac{1}{\frac{v_1}{v_2} - 1}\right) \doteq 1.26\,\mathrm{s}\,.$$

It takes the cone $t \doteq 1.26\,\mathrm{s}$ to return to the point from which it started rolling.

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Problem FoL.4 ... cooling

Consider a Peltier cell with cooling power $P = 10\,\mathrm{W}$, placed on a person's wrist. Let's assume that all of its power is spent on cooling blood flowing through a vein with volumetric flow rate $Q = 1.6\,\mathrm{ml\cdot s^{-1}}$. Blood density is $\varrho = 1\,025\,\mathrm{kg\cdot m^{-3}}$, its heat capacity $c = 4.2\,\mathrm{kJ\cdot kg^{-1}\cdot K^{-1}}$. How much cooler is the blood after flowing through the wrist?

Filip had to improvise in the summer.

The mass flow rate of blood through the vein is ϱQ. Let's express the cooling power as heat taken away over a time element Δt,

$$c\varrho Q\Delta t = P\,,$$
$$\Delta t = \frac{P}{c\varrho Q}\,,$$

which leads to $\Delta t \doteq 1.5\,^\circ\mathrm{C}$.

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Problem FoL.5 ... wet road

How much does the shortest braking distance increase when it rains, if we do not want the car to skid? The car is moving at a speed of $v = 90\,\mathrm{km{\cdot}h^{-1}}$, the coefficient of friction between a dry road and the tires is $f_1 = 0.55$ and it changes to $f_2 = 0.30$ in wet conditions. The weight of the car is $m = 1\,500\,\mathrm{kg}$ and the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. Consider only kinetic energy of the car's translational motion. *Pikoš barely managed to stop.*

For the car to not skid, the braking force mustn't exceed the friction force, which is given by $F_\mathrm{t} = fF_\perp$, where $F_\perp$ is the normal force from the car acting upon the road - in our case, it's equal to the force of gravity. Therefore, $F_\perp = mg$, which means $F_\mathrm{t} = fmg$. If this force is exerted over a distance of l, it changes the kinetic energy by $\Delta E = E_\mathrm{k}$.

The kinetic energy of the car in the beginning is $E_\mathrm{k} = mv^2/2$; after the car stops, it's zero, so the work needed to stop the car is $\Delta E = E_\mathrm{k}$, from which we can find the braking distance

$$l = \frac{v^2}{2fg}\,.$$

The difference between the braking distance on a wet and a dry road is then

$$\Delta l = \frac{v^2}{2g}\left(\frac{1}{f_2} - \frac{1}{f_1}\right) \doteq 48.3\,\mathrm{m}\,.$$

The braking distance increases by about 48.3 m, which means it almost doubles.

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Problem FoL.6 ... high five!

Tomáš and Michal have met on a cycleway and want to high five while riding. Both of them have the same mass $m = 70\,\mathrm{kg}$, speed $v = 8\,\mathrm{m{\cdot}s^{-1}}$ and their arms are stretched horizontally at a distance of $l = 1\,\mathrm{m}$ from the centres of mass of their bodies. They are moving along parallel lines, but in opposite directions. The distance between their centres of mass at the moment when they are passing by is exactly $2l$. Determine the frequency (in Hz) with which they would rotate around their common centre if they grabbed each other's hands. Neglect the mass of their bicycles and energy loss during the act of catching each other. Consider the cyclists to be mass points. *Mirek making friends with a birch while riding on a bicycle.*

Since we consider the cyclists to be mass points, it is obvious that their angular momentum and mechanical energy are conserved if the peripheral speed of each cyclist during the rotation corresponds to translational speed at the beginning. Let's denote the initial state as 0 and the final one as 1. The equations for energies and angular momenta are

$$
\begin{aligned}
E_0 &= \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2\,,\\
E_1 &= \frac{1}{2}I\omega^2 + \frac{1}{2}I\omega^2 = I\omega^2 = ml^2\omega^2\,,\\
L_0 &= 2mvl\,,\\
L_1 &= 2m\omega l^2\,.
\end{aligned}
$$

where we used the definition of the moment of inertia for a mass point $I = ml^2$. The frequency is

$$f = \frac{\omega}{2\pi} = \frac{v}{2\pi l} = 1.3\,\mathrm{Hz}\,.$$

The cyclists will rotate with this frequency until they fall to the ground (which will happen quickly).

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Problem FoL.7 ... (non)broken thermometer

We have a particular thermometer with which we measured the temperature of melting ice as $t_0 = -0.3\,^\circ\mathrm{C}$ and the temperature of water vapor as $t_\mathrm{v} = 101.4\,^\circ\mathrm{C}$. What is the real boiling point τ of methanol at atmospheric pressure, if the thermometer shows a temperature $t = 65.5\,^\circ\mathrm{C}$? Assume that the real temperature of melting ice is $\tau_0 = 0\,^\circ\mathrm{C}$ and of water vapor $\tau_\mathrm{v} = 100\,^\circ\mathrm{C}$. Also assume that the number of ticks on the broken thermometer is proportional to the real temperature in Celsius degrees. *Lydka played with a thermometer during laboratory practice.*

The given values imply that the temperature difference of $100\,^\circ\mathrm{C}$ matches 101.7 ticks on the thermometer. Using the assumptions that the capilary of the thermometer is divided by the ticks into parts of equal volume and that the scale reading is a linear function of the real temperature, we obtain

$$\frac{\tau_\mathrm{v} - \tau_0}{\tau - \tau_0} = \frac{t_\mathrm{v} - t_0}{t - t_0}\,.$$

For $\tau_0 = 0\,^\circ\mathrm{C}$,

$$\tau = \frac{t - \tau_0}{t_\mathrm{v} - t_0}\tau_\mathrm{v} = 64.7\,^\circ\mathrm{C}\,,$$

which is the temperature which a correctly working thermometer would show.

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Problem FoL.8 ... line of apsides

Consider a terrestrial planet which orbits its maternal solar-type star in such a way, that we can neglect other bodies in the universe and the eccentricity the of planet's orbit is $e = 0.150\,0$. Determine the ratio $v_\mathrm{p}/v_\mathrm{a}$ of the speed of the planet at periapsis v_p (the nearest point to the star) to the speed at apoapsis v_a (the farthest point from the star) . Karel exercising.

We use the 2nd Kepler law, which tells us that a line segment joining the planet and the star sweeps out equal areas during equal intervals of time. Hence, we have an equation

$$v_\mathrm{a} r_\mathrm{a} = v_\mathrm{p} r_\mathrm{p}\,, \tag{1}$$

where r_p is the distance from the star to the planet at periapsis and analogously r_a is the distance to the planet at apoapsis. We can understand easily from the geometry of an ellipse that $r_\mathrm{p} = a\,(1-e)$ and $r_\mathrm{a} = a\,(1+e)$, where a is length of the semi-major axis of the ellipse. After plugging this into the equation (1) and rearranging it, we get the ratio

$$\frac{v_\mathrm{p}}{v_\mathrm{a}} = \frac{1+e}{1-e} \doteq 1.353\,.$$

The ratio of speeds is 1.353.

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Problem FoL.9 ... the fire rises

What is the lowest temperature of the air (in °C) inside a balloon, which will make it float? The volume of the balloon is $V = 6\cdot 10^6\,\mathrm{l}$ and its weight (without the air inside) is $m = 700\,\mathrm{kg}$. The density of surrounding air is $\varrho = 1.2\,\mathrm{kg{\cdot}m^{-3}}$ and the atmospheric pressure is $p = 101\,\mathrm{kPa}$. Xellos was baneposting.

For the balloon to float, its density must be equal to the density of the surrounding air. We know the volume of the balloon, so we can find the weight of the air inside it m_v

$$\varrho = \frac{m+m_\mathrm{v}}{V} \quad \Rightarrow \quad m_\mathrm{v} = \varrho V - m = 6.5\,\mathrm{t}\,.$$

Using the ideal gas law, we can find the temperature of the air, knowing its weight (the pressure inside is still equal to the atmospheric pressure)

$$T = \frac{pV}{Rn} = \frac{pVM_\mathrm{v}}{Rm_\mathrm{v}} = \frac{pVM_\mathrm{v}}{R\,(\varrho V - m)} = 325\,\mathrm{K}\,,$$

where $M_\mathrm{v} = 28.97\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of air. This temperature is equal to $52\,^\circ\mathrm{C}$.

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Problem FoL.10 ... sunny

Mirek decided to do something for his health, so he took his schoolbooks outdoors for fresh air. The sky was clear and the sun was exactly at the south, $\alpha = 50^\circ$ above the horizon. If Mirek places his book horizontally, then it receives radiation power P from the sun. Determine the lowest angle β by which Mirek must tilt the book towards himself (he looks at it from the north) so that the incident radiation power drops to $P/2$. *Mirek's paper was too white.*

Incident power is proportional to the surface on which light is incident. If a book has an area S_0, then the area of its projection onto a plane perpendicular to the rays is

$$S_1 = S_0 \sin\alpha\,.$$

We seek an area

$$S_2 = S_0 \sin(\alpha - \beta)\,,$$

that satisfies

$$\frac{S_2}{S_1} = \frac{\sin(\alpha-\beta)}{\sin\alpha} = \frac{1}{2}\,.$$

We can either solve this equation numericaly or rewrite it as follows:

$$2\sin\alpha\cos\beta - 2\sin\beta\cos\alpha = \sin\alpha\,.$$

Now we choose $\sin\beta$ as the unknown variable; we get the sines over to the right side, square both sides of the equation, using the Pythagorean theorem we substitute cosines for sines and divide the equation by $cos^2\alpha$, transforming $sin^2\alpha$ to $tg^2\alpha$. After these operations, we get

$$(1 + \mathrm{tg}^2\,\alpha)\sin^2\beta + \mathrm{tg}\,\alpha\sin\beta - \frac{3}{4}\,\mathrm{tg}^2\,\alpha = 0\,.$$

We seek the lowest possible angle. Therefore, we choose the positive root of the above equation, which means that further tilting the book decreases the angle at which the incoming rays are incident on the book. The result is

$$\sin\beta = 0.461\,4\ldots \quad \Rightarrow \quad \beta \doteq 27^\circ\,.$$

The book must be tilted approximately by 27°.

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Problem FoL.11 ... the potassium foil

When Kiki was young and used to not know anything about quantum physics (she only knew the classical view that energy is delivered continuously by a light beam), her home was lit with an isotropic light source with an output of $P = 1.5\,\mathrm{W}$. At a distance $R = 3.5\,\mathrm{m}$, she placed a foil made of potassium with work function $\varphi = 2.2\,\mathrm{eV}$. Help Kiki determine how many seconds it'd take for the foil to absorb enough energy to emit an electron. Assume that the foil absorbs all the incident energy, and that an emitted electron absorbs the energy incident on a circular surface of radius $r = 5\cdot 10^{-11}\,\mathrm{m}$, given by the typical radius of an atom.

Dominika gazed into the past.

Let's assume that the energy emitted by the source is distrbuted evenly in an expanding spherical wavefront centered at the source. The intensity at a distance R from a point source is given as

$$I = \frac{P}{4\pi R^2}\,.$$

If light of intensity I is incident on a surface $S = \pi r^2$ over time t, an energy $E = ISt$ is absorbed. If only one electron recieves all the 2.2 eV of energy, it needs to absorb it over time

$$t = \frac{\varphi}{IS} = \frac{4\pi\varphi R^2}{PS} \doteq 4\,600\,\mathrm{s}\,.$$

In fact, the emission of electrons doesn't have to occur, since we don't know the energy (wavelength) of incident photons.

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Problem FoL.12 ... tuned

A source plays a tone with frequency $f_0 = 440\,\mathrm{Hz}$ (a'). What is the speed of a car moving straight towards the source, if the driver hears b' flat ($f = 466\,\mathrm{Hz}$)? The speed of sound in the air is $v = 343\,\mathrm{m{\cdot}s^{-1}}$, the effects of the material of the car can be neglected.

Meggy wanted a musical problem.

According to Doppler's law for a stationary source,

$$f = f_0 \frac{v + v_\mathrm{o}}{v}\,,$$

where v_o is the observer's speed. Expressing the speed, we get

$$v_\mathrm{o} = v\left(\frac{f}{f_0} - 1\right)\,,$$

which gives us the numerical value $v_\mathrm{o} \doteq 20.3\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.13 ... pitch-black

The retina of the human eye is the most sensitive to the yellow-green colour $\lambda = 550\,\mathrm{nm}$, for which the sensitivity threshold is $P = 1.7 \cdot 10^{-18}\,\mathrm{W}$. What is the minimal number of photons that have to hit the retina over one second for the light to be percepted?

Verča had total visual blackout.

The energy of one photon is $E = hf$, which (since we know the wavelength) can be rewritten as $E = hc/\lambda$. For the light to be percepted, the number of photons N that hit the retina in $t = 1\,\mathrm{s}$ has to satisfy the following equation:

$$P = N\frac{E}{t} = N\frac{hc}{\lambda t}\,.$$

From that, we can find the formula for N:

$$N = \left\lceil \frac{P\lambda t}{hc} \right\rceil \doteq 5\,.$$

We can see that 5 photons are enough to stimulate the retina.[1]

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Problem FoL.14 ... pyramid

A school model of a regular four-sided pyramid was made of wood with mass density $\varrho_1 = 600\,\mathrm{kg{\cdot}m^{-3}}$. The pyramid had mass $m_1 = 300\,\mathrm{g}$ and the ratio between the length of its base edge and the height from the base to the apex was $2:3$. One day, naughty students broke into physics room and sawed off the poor pyramid's apex. The section was made parallel to the base. The teacher made a new apex with the same size for the pyramid; however, it was made of wood with mass density $\varrho_2 = 900\,\mathrm{kg{\cdot}m^{-3}}$ and then joined with the truncated pyramid. Although the pyramid looks the same now, its mass is $m_2 = 309.6\,\mathrm{g}$. Determine the distance from the base (in cm) at which the section was made. *Meggy remembered solid geometry.*

Let us convert the mass densities into $\mathrm{g{\cdot}cm^{-3}}$, so $\varrho_1 = 0.6\,\mathrm{g{\cdot}cm^{-3}}$ and $\varrho_2 = 0.9\,\mathrm{g{\cdot}cm^{-3}}$.

We can easily compute the volume of the pyramid from its mass and mass density.

$$V_1 = \frac{m_1}{\varrho_1} = 500\,\mathrm{cm}^3\,.$$

Let's denote the length of the base edge by a and its height by v. The latter can be expressed from their ratio

$$v = \frac{3}{2}a$$

and plugged into the formula for the volume of a pyramid

$$V_1 = \frac{1}{3}a^2 v = \frac{a^3}{2}\,.$$

We get the length of the base edge

$$a = \sqrt[3]{2V_1} = 10\,\mathrm{cm}$$

and from that, we easily get that $v = 15\,\mathrm{cm}$.

Let us denote the mass of the cut off apex by m' and the mass of the added apex by m''. The volumes of both are the same, denoted by V'. We know the difference of the two complete pyramids' masses (before the section and after adding the new apex) and therefore, we also know the difference between the masses of the apex parts

$$m_2 - m_1 = m'' - m' = 9.6\,\mathrm{g}\,.$$

We can express the masses again using volumes and mass densities

$$\varrho_2 V' - \varrho_1 V' = 9.6\,\mathrm{g}\,.$$

[1] The number of photons can only be a natural number, that's why we have to take the ceiling function.

When we substitute for the mass densities, we get

$$V' = 32\,\mathrm{cm}^3\,.$$

Since the cut off apex and the pyramid itself are similar, the apex base edge is denoted by a' and its height by v', we can do what we have already done for the whole pyramid and get $a' = 4\,\mathrm{cm}$ a $v' = 6\,\mathrm{cm}$.

Hence, the section was done at the height $v - v' = (15 - 6)\,\mathrm{cm} = 9.0\,\mathrm{cm}$ from the base of the complete pyramid.

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Problem FoL.15 ... battering ram vol. 2

As a part of their offensive, the orcs of Mordor want to break through the gate of Minas Tirith. In a parallel-universe Middle-earth, no progress is made even after calling for Grond, a fire spitting battering ram. Feeling hopeless, Sauron eventually decides to prop up another battering ram horizontally between the gate and a newly built perfectly rigid wall. He then orders to heat the ram so as to exploit thermal expansion properties of its material to break the gate. Find out by how much (in Kelvins) the ram needs to be heated. Assume that the ram has the shape of a cylinder with axis perpendicular to the gate. The material of the ram has linear expansion coefficient $\alpha = 1.2 \cdot 10^{-5}\,\mathrm{K}^{-1}$, Young's modulus of compression $E = 211\,\mathrm{GPa}$ and the maximum compressive stress that the gate endures is $\sigma_{\max} = 400\,\mathrm{MPa}$. Consider the thermal expansion process to be fully linear and assume that there is no deformation of the gate until it yields. *Ondra watched El Señor de los Anillos.*

Obviously, there are two main phenomena affecting the length of the battering ram which need to be considered: thermal expansion and compressive deformation (which is described by Hooke's law). If we denote the relative expansions corresponding to the two above mentioned effects by ε_1 and ε_2 respectively, we have

$$\varepsilon_1 = \alpha \mathit{\Delta} t\,, \quad \varepsilon_2 = -\frac{\sigma}{E}\,.$$

Assuming that the gate stays rigid until it breaks, the length of the ram needs to stay constant during the heating process. This is facilitated by increasing the compressive stress inside the ram. The condition of keeping the length of the ram constant then translates to

$$\varepsilon_1 + \varepsilon_2 = 0 \quad \Rightarrow \quad \mathit{\Delta} t = \frac{\sigma}{\alpha E}\,.$$

The increase in temperature is therefore directly proportional to an increase in compressive stress. Thus, since we know that the gate yields when $\sigma = \sigma_{\max}$, the required change of temperature is

$$\mathit{\Delta} t = \frac{\sigma_{\max}}{\alpha E} \doteq 158\,\mathrm{K}\,.$$

In order to break through the gate, the orcs need to increase the temperature of the ram by $\mathit{\Delta} t = 158\,\mathrm{K}$.

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Problem FoL.16 ... refracter

A ray of white light is incident on a thin glass wall of a hollow prism filled with carbon disulphide. The angle of incidence (measured from the normal to the interface) is $\varepsilon_1 = 50°$. The walls of the prism are made from thin plane-parallel plates. The apex angle of the prism is $\varphi = 60°$. Compute the angular width of the light exiting the prism (in degrees). The refractive index of the glass is $n_{\mathrm{cs}} = 1.518$ for red light and $n_{\mathrm{fs}} = 1.599$ for violet light; the refractive index of carbon disulphide for red light is $n_{\mathrm{c}} = 1.618$ and for violet light $n_{\mathrm{f}} = 1.699$. Assume that the surrounding air has refractive index $n = 1$ for all wavelengths.

Faleš struggled with old physics exercise books.

We can compute the desired angle α as the difference between the angles of refraction of red light and violet light. Each ray of light going through the prism refracts four times – twice while entering the prism and twice while exiting it. However, we can simplify the situation when we come to realize that the pairs of refractions can be considered as only one refraction directly from the air to the carbon disulphide because the plates are plane-parallel. We could show this by repeatedly using Snell's law.

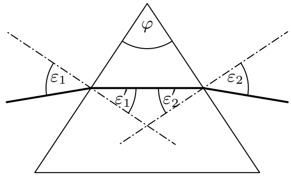

Fig. 1: Geometry of the prism.

Let us denote the final angle of refraction for a particular wavelength δ. The geometry tells us

$$\delta = \varepsilon_1 + \varepsilon_2 - \varphi\,,$$

where ε_2 is the angle of incidence for the second wall (the one through which the light escapes the prism). The angle α can be expressed as

$$\alpha = \delta_{\mathrm{f}} - \delta_{\mathrm{c}}\,.$$

The indices f and c denote violet and red light, respectively. The angles ε_1 and φ are the same for all wavelengths and therefore, they cancel out in the formula. We have

$$\alpha = \varepsilon_{2\mathrm{f}} - \varepsilon_{2\mathrm{c}}\,.$$

We find this angle as follows

$$\sin\varepsilon_{2i} = \frac{\sin\varepsilon_2' n_i}{n}\,,$$

where the primed angles are those inside the prism and the index i is f or c, depending on the wavelength. For ε_2', it follows that

$$\varepsilon_{2i}' = \varphi - \varepsilon_{1i}'\,,$$
$$\sin\varepsilon_{1i}' = \frac{\sin\varepsilon_{1i} n}{n_i}\,.$$

Finally,

$$\alpha = \arcsin\left(\frac{\sin\left(\varphi - \arcsin\frac{\sin\varepsilon_1 n}{n_{\mathrm{f}}}\right) n_{\mathrm{f}}}{n}\right) - \arcsin\left(\frac{\sin\left(\varphi - \arcsin\frac{\sin\varepsilon_1 n}{n_{\mathrm{c}}}\right) n_{\mathrm{c}}}{n}\right)\,.$$

After numerical computations, we have $\alpha \doteq 10.1°$.

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Problem FoL.17 ... the Long Night

What amount of work must be done to extend a day by 4 hours? Consider the Earth to be a homogeneous solid sphere with radius $R = 6\,378\,\mathrm{km}$ *and mass* $M = 5.972 \cdot 10^{24}\,\mathrm{kg}$.

Kuba would like to have more time to sleep.

In order to extend the day on Earth, we must slow down its rotation. In other words, we're asking by how much the kinetic energy of the Earth's rotation needs to be changed. We know that rotational kinetic energy is given by the relation $E_\mathrm{r} = \frac{1}{2}J\omega^2$, where J is the moment of inertia of the Earth and ω is the angular velocity of its rotation, which can be calculated from the rotation period (i.e. length of the day). The moment of inertia of a solid sphere is $J = \frac{2}{5}MR^2$. Then,

$$W = \frac{1}{2}J\left(\left(\frac{2\pi}{T_1}\right)^2 - \left(\frac{2\pi}{T_2}\right)^2\right) = \frac{1}{5}MR^2\left(\left(\frac{2\pi}{T_1}\right)^2 - \left(\frac{2\pi}{T_2}\right)^2\right),$$

where T_1 is the original length and T_2 is the new length of the day. Numerical evaluation gives us $W = 6.817 \cdot 10^{28}\,\mathrm{J}$.

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Problem FoL.18 ... thick glass

Lukáš wants to admire the Black rock from his dormitory window, but another building is standing in the way. Lukáš is $d = 400\,\mathrm{m}$ *away from the building and looks (approximately) right at it, so the building appears as a rectangle with height* $h = 70\,\mathrm{m}$ *and width* $w = 50\,\mathrm{m}$. *Determine the solid angle (in steradians) covered by the building.*

Mirek realized there's landmark next to the dormitory.

Notice that the distance d is about one order of magnitude bigger than h, w. Then, the solid angle Ω can be sufficiently approximated by the ratio of the visible surface area of the building and a sphere of radius d, multiplied by the full solid angle 4π. Written as a formula,

$$\Omega = 4\pi\frac{wh}{4\pi d^2} = \frac{wh}{d^2} \doteq 0.021\,9\,.$$

The exact result can be obtained by integration

$$\int \frac{\mathrm{d}S}{r^2}\cos\alpha\,,$$

where $\mathrm{d}S$ is an area element, r its distance from the observer and α is the angle between the normal $\boldsymbol{n}$ of the area element and the vector $\boldsymbol{r}$ from the observer to it. Computing this in Cartesian coordinates,

$$\int_{-w}^{w}\int_{-h}^{h} \frac{d}{(d^2+x^2+y^2)^{3/2}}\,\mathrm{d}x\,\mathrm{d}y = 4\,\mathrm{arctg}\,\frac{\frac{wh}{2d}}{\sqrt{4d^2+w^2+h^2}} \doteq 0.022\,.$$

We can see that the approximation was accurate enough.

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Problem FoL.19 ... wrong slope

Construction workers don't always follow the blueprints precisely. After one "hard night", they read the plans incorrectly and built a road curve tilted to the other side than it is usually done. By how many kilometers per hour did the maximum speed, with which a car may cross the curve without slipping, decrease? The radius of the curve is $r = 100\,\mathrm{m}$, the angle by which it's tilted is $\alpha = 5^\circ$ and the static friction coefficient between a tyre and the road surface is $f = 0.55$. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Pikoš was watching construction workers.*

Consider the problem in the reference frame of the car. There are four forces acting on the car when it's crossing the curve:

- *force of gravity* $\boldsymbol{F}_g = m\boldsymbol{g}$, where m is the mass of the car and $\boldsymbol{g}$ the acceleration due to gravity,
- *centrifugal force* $\boldsymbol{F}_\mathrm{o} = mv^2\hat{r}/r$, where v is the speed of the car, r is the radius of the curve and $\hat{r}$ is the horizontal unit vector pointing away from the center of curvature of the curve,
- *reaction force from the road* $\boldsymbol{F}_\perp = F_\perp(-\sin\alpha, \cos\alpha)$, which is normal to the road, and
- *friction force* $\boldsymbol{F}_\mathrm{t} = F_\mathrm{t}(-\cos\alpha, -\sin\alpha)$, which is tangent to the road and whose magnitude mustn't exceed $fF_\perp$, where f is the static friction coefficient. Provided that the centrifugal force is large enough, the friction force will act against it. Since we're looking for the maximum speed, which is limited just by the maximum magnitude of this force, let's suppose $F_\mathrm{t} = fF_\perp$.

The total force is $\boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{F}_\mathrm{o} + \boldsymbol{F}_\perp + \boldsymbol{F}_\mathrm{t}$. Since the car doesn't move in its own reference frame, we must have $\boldsymbol{F} = \boldsymbol{0}$, so

$$m(0,-g) + \frac{mv^2}{r}(1,0) + F_\perp(-\sin\alpha, \cos\alpha) + fF_\perp(-\cos\alpha, -\sin\alpha) = \boldsymbol{0}\,.$$

We can express $F_\perp$ from both components and set those expressions equal; from that equality, we can express the speed

$$v = \sqrt{gr\frac{\sin\alpha + f\cos\alpha}{\cos\alpha - f\sin\alpha}}\,.$$

For $\alpha = 5^\circ$, we find out that the maximum speed is approximately $92.3\,\mathrm{km{\cdot}h^{-1}}$; for $\alpha = -5^\circ$, it's $74.9\,\mathrm{km{\cdot}h^{-1}}$, so the maximum speed decreased by $17.4\,\mathrm{km{\cdot}h^{-1}}$.

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Problem FoL.20 ... minimal force

Consider three particles with positive charges $Q_1 = 1\,\mathrm{C}$, $Q_2 = 2\,\mathrm{C}$ and $Q_3 = 4\,\mathrm{C}$ placed in vacuum. Find the minimal possible magnitude of the force between Q_1 and Q_3, if you know that the force between Q_1 and Q_2 is $F_{12} = 1\,\mathrm{N}$ and the force between Q_2 and Q_3 is $F_{23} = 4\,\mathrm{N}$. *Náry was trying to minimise his repulsiveness.*

We use Coulomb's law

$$F_{AB} = \frac{1}{4\pi\varepsilon}\frac{Q_A Q_B}{d^2}\,,$$

where d is the distance between two charges Q_A, Q_B and ε is the permittivity of the surrounding material. Using this formula, we start by calculating the distances d_{12} and d_{23}

$$d_{12} = \sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{F_{12}}}\,,$$

$$d_{23} = \sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_2Q_3}{F_{23}}}\,.$$

Because the repulsive force between particles decreases with their distance, we get the minimal repulsive force for the maximal distance between charges Q_1 and Q_3, which is $d_{13} = d_{12} + d_{23}$. Coulomb's law then gives us

$$F_{13} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_3}{\left(\sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{F_{12}}} + \sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_2Q_3}{F_{23}}}\right)^2}\,,$$

$$F_{13} = \frac{Q_1Q_3}{\left(\sqrt{\frac{Q_1Q_2}{F_{12}}} + \sqrt{\frac{Q_2Q_3}{F_{23}}}\right)^2}\,.$$

Plugging in the numbers, we get the value $0.50\,\mathrm{N}$.

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Problem FoL.21 ... laser pointer

A monochromatic laser points perpendicularly at a solid screen. We mark the point on the screen at which the laser is pointing when there is nothing but air in between. Now, we place a transparent plate made from an optically denser material with thickness $d = 2\,\mathrm{cm}$ between the laser and the screen. The laser beam enters the plate at an angle $\alpha = 30^\circ$ from the normal and emerges on the opposite side of the plate. The laser is now pointing at a point which is shifted from the original one by $\delta = 0.6\,\mathrm{cm}$. What is the refractive index n_2 of the plate? The refractive index of air is $n_1 = 1$. *Meggy tried geometry.*

We'll project our scene onto one plane, in which the laser beam propagates. Both before entering and after exiting the plate, the laser beam points in the same direction, which means that the shift had to occur within the plate. According to Snell's law, the relation

$$\frac{n_1}{n_2} = \frac{\sin\beta}{\sin\alpha}$$

holds, where β is the angle of refraction within the plate. Imagine the propagation of the beam without the plate. Let us call this beam p and the real refracted beam q. Both rays enter the plate at one point C, the beam p emerges from the plate at point P and the beam q at another point Q. The distance δ is the length of the perpendicular from point Q to the beam p; we denote its foot by R and the point opposite to C by D (then, $|\mathsf{CD}| = d$). The triangles PQR and PCD are similar.

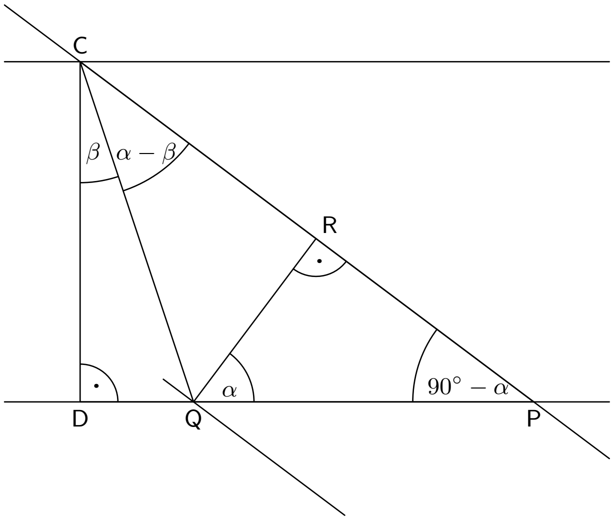

Fig. 2: Geometry of refraction of the laser.

We can see that

$$|\mathsf{PQ}| = \frac{\delta}{\cos\alpha} \doteq 0.692\,8\,\mathrm{cm}\,,$$
$$|\mathsf{PD}| = \frac{d}{\operatorname{tg}\left(\frac{\pi}{2} - \alpha\right)} = d\operatorname{tg}\alpha \doteq 1.154\,7\,\mathrm{cm}\,,$$
$$\operatorname{tg}\beta = \frac{|\mathsf{QD}|}{d} = \frac{|\mathsf{PD}| - |\mathsf{PQ}|}{d} \doteq 0.231\,.$$

From that, we get

$$\beta \doteq 13.0°\,.$$

Thus, we can express

$$n_2 = \frac{n_1 \sin\alpha}{\sin\beta} \doteq 2.22\,.$$

The refractive index of the screen is 2.22.

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Problem FoL.22 ... when to stop a catapult

Consider a massless rod in Earth's homogeneous gravity field. The rod rotates about an axis that is perpendicular both to the gravitational acceleration and to the rod. At a distance $R = 1\,\mathrm{m}$ from the axis, we place a weight $M = 10\,\mathrm{kg}$, and on the other side, we place a projectile with mass $m = 500\,\mathrm{g}$ at a distance $r = 10\,\mathrm{m}$ from the axis. Afterwards, we set the catapult (rod) to horizontal position. When the catapult is ready, we release the rod and let it rotate until it reaches an angle of φ with the horizontal, where it's stopped abruptly. Find the angle φ

that maximizes the range of the catapult. To simplify the calculations, we assume that the projectile hits the ground at the same height as the height at which it was released when the rod was stopped. Also, the projectile cannot slide on the rod.

Lubošek wanted to shoot at his roommate.

First, let us show on which variables the range d depends. Elementary ballistics states that

$$d = v_x t = \frac{2}{g} v_y v_x = \frac{2}{g} v^2 \sin\varphi \cos\varphi \,,$$

where v is the initial velocity of the projectile, v_x and v_y are its components and g is the gravitational acceleration. At the moment of release of the projectile, we have

$$d = \frac{2}{g} r^2 \omega^2 \sin\varphi \cos\varphi \,,$$

where ω denotes the angular velocity of the rod right before the release.

In the next step, we use the law of conservation of mechanical energy before and after release

$$g(MR - mr)\sin\varphi = \frac{1}{2}(MR^2 + mr^2)\omega^2 \,.$$

Substituting into the equation for the range, we obtain

$$d = \frac{4r^2(MR - mr)}{MR^2 + mr^2} \sin^2\varphi \cos\varphi \,.$$

It is worth noticing that the range does not depend on gravitational acceleration and all the mechanical parameters of the catapult stand only for a multiplicative factor.

In the last step, we use the fact that the range is zero for $\varphi = 0^\circ$ and $\varphi = 90^\circ$ and positive for any value in this interval. From that, it follows that if the function $d(\varphi)$ has only one stationary point (point with zero derivative), then it'll also be the maximum of that function. The condition

$$\frac{\mathrm{d}d}{\mathrm{d}\varphi} = 0$$

leads us to the equation

$$2\cos^2\varphi - \sin^2\varphi = 0 \,.$$

Finally, through trigonometric identities, we get $\varphi = \arccos\left(1/\sqrt{3}\right) \doteq 54.74^\circ$.

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Problem FoL.23 ... alternating-direct 1

Consider the circuit in the figure 3, the DC source has voltage $U_\mathrm{j} = 4.5\,\mathrm{V}$, the AC source has voltage amplitude $U_\mathrm{s} = 5\,\mathrm{V}$ and frequency $f = 50\,\mathrm{Hz}$. The resistor's resistance is $R = 100\,\mathrm{k\Omega}$ and the capacitor's capacity is $C = 10\,\mathrm{nF}$. What's the expected charge on the capacitor, in nC?

Xellos doesn't like broccoli, that's why he made a problem without it.

The two sources are a trap – all elements of the circuit are linear, so we can view it as a superposition (sum) of two circuits – one with the DC and the other with the AC source. The charge on the capacitor is a superposition of the charges in both cases; however, the expected

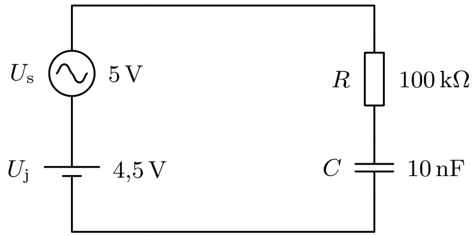

Fig. 3: Circuit diagram.

value with the AC source is 0, so the result is the same as if we only had the DC source. In that case, there's no current passing through the resistor, so we can discard it and the result is $Q = CU_\mathrm{j} = 45\,\mathrm{nC}$.

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Problem FoL.24 ... tick tock

There are two clocks in a box. One is a balance wheel clock (like a usual watch; alternatively, a digital clock) and the other a pendulum clock – its pendulum consists of a rod (length $l = 30\,\mathrm{cm}$ and mass $m = 300\,\mathrm{g}$) with a disk at its lower end (diameter $d = 10\,\mathrm{cm}$ and area density $\sigma = 7.5 \cdot 10^{-2}\,\mathrm{kg{\cdot}m^{-2}}$). The disk is attached lengthwise in the only direction in which the pendulum can move (oscillate). A small rocket lifts the box with acceleration $a = 5g$ (g is the gravitational acceleration of the Earth) up to the height $h = 30\,\mathrm{km}$, where it lets the box move with unchanged inertia in Earth's gravity. Assume that the gravitational field is homogeneous with $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ and neglect the drag force. By how much will the times on the two clocks differ (in absolute value) at the highest point of the box's trajectory?

Fales remade the relativistic paradox.

We need to realize that unlike a clock with a balance wheel (digital clock), the pendulum clock depends on the gravitational acceleration. During the ascent of the rocket, it is effectively six times greater $(1 + 5)$ than the acceleration due to Earth's gravity. On the contrary, after being released, the box will be in a weightless state – it will be in free fall (with initial velocity v pointing upward).

The digital clock remains unaffected, which means that the time it shows is the sum $t = t_1 + t_2$, where t_1 is the time during which the box was being lifted and t_2 is the time during which it was in a weightless state since being released till reaching the highest point. They can be obtained as

$$t_1 = \sqrt{\frac{2h}{a}}\,,$$

$$t_2 = \frac{v}{g} = \frac{t_1 a}{g} = \frac{\sqrt{2ha}}{g}\,.$$

The period of the pendulum clock is computed as

$$T = 2\pi\sqrt{\frac{I + ml^2}{mgl}} = 2\pi\sqrt{\frac{l + \frac{I}{ml}}{g}} = 2\pi\sqrt{\frac{l_{\mathrm{red}}}{g}}\,,$$

where I is the moment of inertia of the system of the rod and disc about the axis crossing its center of mass, whose distance from the actual axis of rotation is l. Using the Parallel axis theorem, we get the moment of inertia about the rotational axis as $I + ml^2$. However, the only important thing is that the original expression can be transformed to the same form as the one for a mathematical pendulum with some new length l_{red}, the numerical value of which isn't important for us, since we found out that the period is proportional to $g^{-\frac{1}{2}}$. When the acceleration gets six times greater, the time of ascent measured by the pendulum clock will be $t_1^{\mathrm{k}} = \sqrt{6}t_1$ (the period will be $\sqrt{6}$-times smaller, so the measured time will be bigger). The clock stops in the weightless state, so the time measured by it in this state will be $t_2^{\mathrm{k}} = 0\,\mathrm{s}$.

The desired time difference τ is

$$\tau = \left|\left(1 - \sqrt{6}\right)\sqrt{\frac{2h}{a}} + \frac{\sqrt{2ha}}{g}\right|\,.$$

After numerical evaluation, we get $\tau \doteq 124\,\mathrm{s}$.

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Problem FoL.25 ... sleepyhead

Exactly at 6:30 in the morning, a solution of a radiopharmaceutical containing radioactive ^{18}F with volumetric activity $330\,\mathrm{MBq\cdot ml^{-1}}$ was prepared for a patient. $1\,\mathrm{ml}$ of this solution should have been applied at exactly 8:00 on the same day. However, the patient was late and he took his shot at 8:40. What volume of the solution of the radiopharmaceutical (in ml) should the patient get so that the dose would be equivalent to the originally planned one? The half-life of ^{18}F is $109\,\mathrm{min}$ and its decay product is stable oxygen.

Wake up early and don't miss your deadlines.

The activity of the radiopharmaceutical at 8:00 will be

$$A(t) = A_0 \exp\left(-\frac{\ln 2}{T}t\right)\,,$$

where A_0 is the original activity, T is the half-life and $t = 90\,\mathrm{min}$ is the time since preparation. Similarly, we can calculate the activity of the solution at 8:40, if we just substitute a different time $t' = 130\,\mathrm{min}$. If we know that at 8:00, one milliliter of the solution would be enough, then with the current activity at 8:40, we will need a proportionally larger volume

$$\frac{V'}{V} = \left(A_0 \mathrm{e}^{-\frac{\ln 2}{T}t}\right) / \left(A_0 \mathrm{e}^{-\frac{\ln 2}{T}t'}\right) = \exp\left(\frac{\ln 2\,(t' - t)}{T}\right) = 2^{\frac{t'-t}{T}} = 1.29\,,$$

and therefore, the needed volume will be $V' = 1.29\,\mathrm{ml}$ of the solution.

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Problem FoL.26 ... Martian

What is the maximal pressure (in μPa*) that a spherical balloon the size of Mars can hold, if it's made from monolayer graphene? The radius of Mars is* $R_{\mathrm{M}} = 3\,390\,\mathrm{km}$*, the maximal tension in one layer of graphene is* $\sigma_{\max} = 42\,\mathrm{N{\cdot}m^{-1}}$*. Neglect the gravitation of the gas.* *Elon Musk.*

Divide the balloon (with gas) into two equal hemispheres. Each one is pushed away from their (fictitious) contact circle by the pressure of the gas and they're held together by the tension on the circumference of this circle. The maximal force from this tension is $2\pi R_{\mathrm{M}}\sigma_{\max}$. This force has to be $\geq$ than the force exerted by pressure on their contact circle, which is equal to $\pi R_{\mathrm{M}}^2 p$. Expressing the pressure and plugging in the numbers, we get

$$p_{\max} = \frac{2\sigma_{\max}}{R_{\mathrm{M}}} \doteq 25\,\mu\mathrm{Pa}\,.$$

The maximal pressure that can be held in the balloon is $p_{\max} \doteq 25\,\mu\mathrm{Pa}$.

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Problem FoL.27 ... nitrogen suffocates the audience

What volume V would the nitrogen from a $V_{\mathrm{D}} = 7.0\,\mathrm{l}$ Dewar flask take after boiling away in standard conditions, i.e. $t = 20\,^{\circ}\mathrm{C}$ and $p_{\mathrm{a}} = 1\,013\,\mathrm{hPa}$? We are interested only in the volume of the evaporated nitrogen, so you can also consider an equivalent problem of determining the volume of a room which contained only vacuum and liquid nitrogen at the beginning and only N_2 gas at standard temperature and pressure after the nitrogen evaporated. The density of liquid nitrogen is $\varrho_{\mathrm{L}} = 808\,\mathrm{kg{\cdot}m^{-3}}$, the molar mass of nitrogen is $M_{\mathrm{m}} = 28.0\,\mathrm{g{\cdot}mol^{-1}}$, the molar gas constant is $R = 8.31\,\mathrm{J{\cdot}K^{-1}{\cdot}mol^{-1}}$ and the density of nitrogen gas under standard conditions is $\varrho_{\mathrm{G}} = 1.16\,\mathrm{kg{\cdot}m^{-3}}$. *Karel was wondering what would have to happen to kill the audience of a liquid nitrogen experiment.*

The simplest solution is to use the fact that the mass of nitrogen m is the same before and after the evaporation. We have, "using the rule of three"

$$m = \varrho_{\mathrm{L}} V_{\mathrm{D}} = \varrho_{\mathrm{G}} V \quad \Rightarrow \quad V = V_{\mathrm{D}} \frac{\varrho_{\mathrm{L}}}{\varrho_{\mathrm{G}}} \doteq 4.9\,\mathrm{m}^3\,.$$

We got our solution very quickly: the nitrogen would, after evaporation, take the volume $4.9\,\mathrm{m}^3$.

However, the problem was set in a little bit tricky way in order to make you want to use a slower path, which also leads to the right solution. Let's also show that solution. We start from the equation of state for an ideal gas, because we know the pressure and the temperature of the final state:

$$p_{\mathrm{a}} V = nRT\,,$$

where p_{a} and R are given, V is to be determined and $T \doteq 293\,\mathrm{K}$ can be expressed from $t = 20\,^{\circ}\mathrm{C}$. The amount of nitrogen n can then be expressed as

$$n = \frac{m}{M_{\mathrm{m}}} = \frac{\varrho_{\mathrm{L}} V_{\mathrm{D}}}{M_{\mathrm{m}}}\,.$$

We get an expression for the volume as

$$p_\mathrm{a} V = \frac{\varrho_\mathrm{L} V_\mathrm{D}}{M_\mathrm{m}} RT \quad \Rightarrow \quad V = \frac{\varrho_\mathrm{L} V_\mathrm{D}}{p_\mathrm{a} M_\mathrm{m}} RT \doteq 4.9\,\mathrm{m}^3\,.$$

Thus, within the required precision, we obtain the same result. There is a difference in the next significant digits, but it's caused by rounding and doesn't have physical meaning.

If any of you are wondering whether experiments with liquid nitrogen would suffocate the audience, it would be necessary to reduce the oxygen content in the room at least under 11 % (and probably under 8 %). As rooms are usually ventillated, it is not easy to suffocate someone.

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Problem FoL.28 ... exoplanet

The exoplanet Kepler-138c has radius $R_\mathrm{K} = 1.2\,R_\mathrm{E}$, where R_E is the radius of the Earth, and density equal to that of the Earth. The inhabitants dug a tunnel straight through the center of the planet (along the diameter). How much longer will it take to free-fall through this tunnel (without friction or propulsion) than through a similar tunnel on Earth? Give your answer in seconds. Assume that both planets are homogeneous. *Filip was digging in his garden.*

From Gauss's law, we know that the mass "above" the falling object has no influence on it. This means that the force acting on an object at a distance r from the center is

$$m\ddot{r} = -\frac{4\pi r^3 \varrho G m}{3r^2}\,,$$
$$\ddot{r} = -\frac{4}{3}\pi \varrho G r\,.$$

From this, we can see that the period of the oscillations depends only on the density, so the difference in times is zero.

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Problem FoL.29 ... a hideout on the Moon

Consider two people (let's call them Iva and Radek) standing on the surface of the Moon. Find their smallest possible separation needed so that a voyeur (let's call him Aleš), standing on the summit of Mt. Palomar in the distance $r = 3.8\cdot 10^5\,\mathrm{km}$ from the pair, would be able to resolve them from one another using a telescope with aperture diameter $5\,\mathrm{m}$. Take Iva and Radek to be two point-like sources emitting light with wavelength $500\,\mathrm{nm}$. *Dominika making up gossips.*

Light is known to be diffracted by individual components of the optical system we happen to be using – lenses, apertures etc. Thus the image of a point-like source consists of multiple diffraction rings. The Rayleigh criterion then says that two point-like sources can be distinguished when the first diffraction minimum of the image of one source coincides with the maximum of another. Expressing this mathematically for the given telescope, we get

$$\alpha = \frac{1.22\lambda}{d}\,,$$

where λ is the wavelength, d is the aperture diameter and the factor of 1.22 follows from the intensity profile of the diffraction pattern which can be expanded into Bessel's function of the first kind. Denoting the separation of Iva and Radek by x, we clearly have $x \ll r$, so the corresponding small angle approximation $\operatorname{tg} \alpha \approx \sin \alpha \approx \alpha$ may be invoked. We then have

$$\alpha \approx \frac{x}{r}\,.$$

and so

$$x = 1.22\frac{\lambda r}{d} \doteq 46.4\,\mathrm{m}\,.$$

It is much to Aleš's displeasure that he will be able to distinguish the two only after they move away to a distance $x = 46.4\,\mathrm{m}$ from each other.

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Problem FoL.30 ... melted down thermistor

A thermistor is a semiconductor component, the resistance of which decreases significantly with temperature according to the formula $R = R_0 \exp\left(c\left(1/T - 1/T_0\right)\right)$, where the temperatures T, T_0 are substituted in K, $R_0 = 120\,\Omega$ is its resistance at room temperature $T_0 = 25\,^\circ\mathrm{C}$ and $c = 3.0 \cdot 10^3\,\mathrm{K}$. This formula loses validity at the temperatur $T_\mathrm{t} = 150\,^\circ\mathrm{C}$, when the thermistor starts to melt down. The power output of heat transfer from the thermistor at a temperature T to its surroundings at the temperature T_0 is $P_\mathrm{chl} = k(T - T_0)$, where $k = 4.5 \cdot 10^{-4}\,\mathrm{W{\cdot}K^{-1}}$. Determine the highest voltage to which the thermistor can be connected without being heated up to the temperature T_t or higher (after a sufficiently long time).

Xellos remembered his laboratory practice.

We want to find the maximal voltage at which the thermistor reaches thermal equilibrium – there exists a temperature at which all the Joule heat is being transferred to the surroundings. Then, it holds true that

$$k(T - T_0) = \frac{U^2}{R}\,,$$
$$U = \sqrt{k(T - T_0)R_0 \mathrm{e}^{c\left(\frac{1}{T} - \frac{1}{T_0}\right)}}\,.$$

The expression under the square root grows linearly with T in the limit of infinite temperatures. However, on the scale of thousands of kelvins (much more than T_t), it has only one local maximum. Therefore, we can search for a point with zero first derivative:

$$\mathrm{e}^{c\left(\frac{1}{T} - \frac{1}{T_0}\right)} + (T - T_0)\mathrm{e}^{c\left(\frac{1}{T} - \frac{1}{T_0}\right)} c\left(-\frac{1}{T^2}\right) = 0\,,$$

$$T^2 = c(T - T_0)\,,$$
$$T = \frac{c \pm \sqrt{c^2 - 4cT_0}}{2} \doteq 336\,\mathrm{K}$$

for the minus sign (the plus sign leads to a minimum at a temperature around $2\,700\,\mathrm{K}$). The voltage at which the termistor has this equilibrium temperature is $U = 0.81\,\mathrm{V}$.

Another option is to read out the maximum from a graph of $U(T)$.

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Problem FoL.31 ... rotation rotation

Verča had a fancy party hat on, but she moved too fast and it fell to the ground and began to roll. The conical hat has a slant height of $R = 30\,\mathrm{cm}$, the radius of the base circle is $r = 10\,\mathrm{cm}$ and during its movement, the cone rotates about its stationary apex with an angular velocity ω_y. The cone also rotates about its rotational axis with an angular velocity $\omega_o = 5.0\,\mathrm{rad{\cdot}s^{-1}}$. What is the angular velocity of any point on the surface of the cone with respect to the immediate axis of rotation? The cone rolls on a rough surface. *Mirek combined rotations.*

The angular velocity ω, which we are supposed to find, is given by vector addition of the angular velocities ω_o and ω_y. The vectors are shown in the figure 4.

Since the movement takes place on a rough surface, the cone cannot slip, so it moves circularly about its apex. We can immediately see the simple relation between ω_y and ω_o

$$R\omega_y = r\omega_o\,. \tag{2}$$

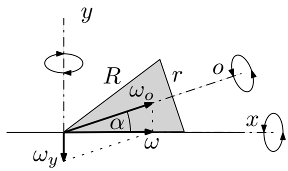

Fig. 4: Analysis of the rotational motion.

We introduce the angle α between ω and ω_o; using the cosine law, we can write

$$\omega_y^2 = \omega^2 + \omega_o^2 - 2\omega\omega_o\cos\alpha\,. \tag{3}$$

The angle α is the angle at the apex and satisfies the relation

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{r^2}{R^2}} = \frac{\omega}{\omega_o}\,.$$

Substituting into the cosine law (3), we obtain

$$\omega^2 = \omega_o^2 - \omega_y^2\,;$$

that is the Pythagorean theorem, thus ω must lie in the ground plane, as the picture hints. Using the relation (2), we can finally express the magnitude of the angular velocity ω

$$\omega = \omega_o\sqrt{1-\frac{r^2}{R^2}} = 4.7\,\mathrm{s^{-1}}\,.$$

The angular velocity is $\omega = 4.7\,\mathrm{s^{-1}}$.

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Problem FoL.32 ... superpowers

If Aleš were to pick a superpower, he would choose the ability to see the intensities of electric and magnetic field. Let's assume that he has such an ability. Imagine that he stands $1.8\,\mathrm{m}$ *away from a point source of light with power* $P = 250\,\mathrm{W}$*. What is the effective value of the magnetic field (magnetic induction) which he feels?*

Dominika thinks of the consequences of the organizers' wishes.

In this case, the energy of electromagnetic waves is conserved – if we construct a sphere with radius r centered at the light source, all the energy from the source must pass through its surface. The energy that passes through that surface in a unit of time must be the same as the energy produced by the source over the same time, i.e. the power P. The intensity I at the surface of the sphere is

$$I = \frac{1}{c\mu_0} E_{\mathrm{ef}}^2 = c\varepsilon_0 E_{\mathrm{ef}}^2 \,,$$

where c is the speed of light, μ_0 is vacuum permeability and ε_0 vacuum permittivity; the relation between those constants is $c^2\varepsilon_0\mu_0 = 1$. Next, we use the relation between magnetic induction B_{ef} and electric intensity E_{ef}: $E_{\mathrm{ef}} = cB_{\mathrm{ef}}$ (thus, I is the density of EM energy multiplied by the speed of the wave). Using these relations, we isolate

$$B_{\mathrm{ef}} = \sqrt{\frac{P\mu_0}{4\pi r^2 c}} = 1.6 \cdot 10^{-7}\,\mathrm{T} \,.$$

Aleš feels a magnetic field with induction $1.6 \cdot 10^{-7}\,\mathrm{T}$.

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Problem FoL.33 ... ball-ey

A small homogeneous spherical ball with radius $r = 3\,\mathrm{cm}$ *is placed in a fixed spherical hole with radius* $R = 10\,\mathrm{cm}$*. The ball can roll without slipping. We move the ball from its equilibrium position a little bit. What will be the period of its oscillations?*

Guljočka v jamočke, Tom kept repeating.

Let's forget about the ball not rolling on a flat surface. If the centre of the ball moves with a speed v, then the ball will rotate with the angular speed $\omega = v/R$. Then, its kinetic energy will be

$$E_{\mathrm{k}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \,,$$

where m is the mass and $I = \frac{2}{5}mr^2$ is the moment of inertia of the ball. After some simplifications, we get

$$E_{\mathrm{k}} = \frac{1}{2}mv^2 + \frac{1}{5}mr^2\omega^2 = \frac{7}{10}mv^2 \,.$$

If the ball is displaced by an angle φ (measured with respect to the centre of the hole; in the equilibrium position, $\varphi = 0$), we may express the speed of its centre as $v = (R - r)\mathit{\Omega}$, where $\mathit{\Omega} = \mathrm{d}\varphi/\mathrm{d}t$ is the angular speed of the centre of the ball with respect to the centre of the hole. The potential energy of the ball (with zero in the equilibrium position) is

$$E_{\mathrm{p}} = mg(R - r)\left(1 - \cos(\varphi)\right) \,.$$

Since φ is very small, we can approximate it as $\cos(\varphi) \approx 1 - \varphi^2/2$. Then, the law of energy conservation is as follows:

$$\frac{7}{10} m(R-r)^2 \Omega^2 + mg(R-r)\frac{\varphi^2}{2} = \mathrm{const}\,.$$

It describes a harmonic oscillator, whose solution is

$$\varphi(t) = \varphi_0 \sin\left(\frac{2\pi}{T} t\right)\,,$$

where φ_0 is the amplitude and T is the period

$$T = 2\pi\sqrt{\frac{7(R-r)}{5g}} \doteq 0.63\,\mathrm{s}\,.$$

The ball oscillates with period $T \doteq 0.63\,\mathrm{s}$.

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Problem FoL.34 ... diodes

Consider two diodes satisfying the ideal Shockley diode law

$$I(U) = I_\mathrm{S}\left(\exp\left(\frac{U}{V_\mathrm{T}}\right) - 1\right)\,,$$

connected in series with a DC source with voltage $V = 1\,\mathrm{V}$ *and internal resistance* $100\,\Omega$*. The saturation current of one diode is* $I_\mathrm{S} = 1\cdot 10^{-11}\,\mathrm{A}$*, the saturation current of the other one is twice as large. The thermal voltage is* $V_\mathrm{T} = 26\,\mathrm{mV}$*. What is the total electric power (in* mW*) produced on the diodes?*

Hint *Since we only want a numerical result, it's sufficient to solve the equations numerically.*

Janči was thinking about diodes in series.

Denote the potentials as in the figure; the zero potential will be on the negative pole of the source. Using Ohm's and Shockley's law, we write

$$V - \varphi_2 = RI\,,$$
$$I_{\mathrm{S2}}(\mathrm{e}^{\frac{\varphi_2-\varphi_1}{V_\mathrm{T}}} - 1) = I\,,$$
$$I_{\mathrm{S1}}(\mathrm{e}^{\frac{\varphi_1}{V_\mathrm{T}}} - 1) = I\,.$$

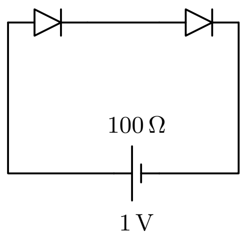

Fig. 5: Circuit diagram.

Then, we express $\varphi_2 = V - RI$ and $\exp(\varphi_1/V_\mathrm{T}) = I/I_{\mathrm{S1}} + 1$ and plug into the second equation

$$I_{\mathrm{S2}}\left(\mathrm{e}^{\frac{V-IR}{V_\mathrm{T}}}\left(\frac{I}{I_{\mathrm{S1}}} + 1\right)^{-1} - 1\right) = I\,.$$

Notice that the currents I_{S1} and I_{S2} have to be very small, compared to the current I: if it weren't the case, the voltages on the diodes would be comparable to V_T and the voltage on the resistor would then have to be similar to V, almost 1 V. That would, however, mean that the current I is about V/R, which is a contradiction.

This allows us to neglect the small currents with respect to I and, after simplifying a bit, we get

$$I = \sqrt{I_{S1} I_{S2}}\, \mathrm{e}^{\frac{V - RI}{2V_T}} .$$

This equation can be solved numerically for the current. The result is

$$I \doteq 0.75\,\mathrm{mA} ,$$

where we see that the error due to our approximation is about $I_S/I \doteq 10^{-8}$.

Because the current I is the same for both diodes and the total voltage drop across them is φ_2, the total power is

$$P = \varphi_2 I = (V - RI)I \doteq 0.69\,\mathrm{mW} .$$

The power produced on the diodes is $P = 0.69\,\mathrm{mW}$.

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Problem FoL.35 ... gardener's problem

An amateur gardener is watering his garden. He took a cylindrical bucket with base area $S = 420\,\mathrm{cm}^2$ and filled it with water up to a height $h = 40\,\mathrm{cm}$. Afterwards, he made a hole with area $s = 2\,\mathrm{cm}^2$ in the bottom base and started watering. What's the longest row of plants which he's able to water, if his speed is $u = 1\,\mathrm{m{\cdot}s^{-1}}$?
Hint *The speed with which water flows out of the hole is given by the formula $v = \mu s\sqrt{2gx}$ with a coefficient $\mu = 0.6$. where x is the current water level height.*

Marek was watering his garden.

We need to compute the time t needed for all the water to flow out of the bucket. It's given by

$$t = \int_0^h \frac{S}{v}\,\mathrm{d}x .$$

After substituting the formula for speed v, we get

$$t = \int_0^h \frac{S}{\mu s\sqrt{2gx}}\,\mathrm{d}x ,$$

and after integration, we can find the time

$$t = \frac{2S\sqrt{h}}{\mu s\sqrt{2g}} .$$

Substituting the given values, we get $t \doteq 99.9\,\mathrm{s} \doteq 100\,\mathrm{s}$, which gives approximately $l = 100\,\mathrm{m}$ for the given speed of the gardener.

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Problem FoL.36 ... a merry encounter

In a vast empty space, there are six electrons very far from each other, located in the vertices of a regular hexagon. Each of them is moving towards the center of this hexagon with the speed $v = 1\,000\,\mathrm{m{\cdot}s^{-1}}$. How close to each other will two adjacent electrons get before the repulsive force separates them forever? *Mirek was inspired by Náry's struggles.*

If we consider the electrons to initially be very distant, we can neglect the forces between them, which means that their initial potential energy will be zero and the only present component of energy will be the kinetic energy of the electrons, which is

$$E_\mathrm{k} = 6 \cdot \frac{1}{2} m_\mathrm{e} v^2 \,,$$

where $m_\mathrm{e} = 9.1 \cdot 10^{-31}$ kg is the rest mass of an electron. The moment at which the electrostatic forces start to be significant is not really important for us, we only need to know that the forces affecting each electron will be of the same magnitude and pointing outwards from the center. Therefore, at some point, the electrons will stop in the vertices of a regular hexagon, which, obviously, will be smaller than the one in the beginning. There, their kinetic energy will be zero, and their potential energy will reach its maximum.

The potential energy of a system of point charges is equal to the sum of the potentials over each pair of charges. Generally,

$$E_\mathrm{p} = \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{j=1, j\neq i}^{n} \frac{k q_j}{r_{ij}} \,,$$

where $k = (4\pi\varepsilon_0)^{-1}$ is Couloumb's constant, q is the point charge, r_{ij} is the distance between charges q_i and q_j, and n is the number of charges. If we label the distance between adjacent electrons r_1, the length of the shorter diagonal r_2 and the length of the longer diagonal r_3, and consider all q_i to be elementary charges, we can write

$$E_\mathrm{p} = 6 \cdot \frac{1}{2} k e^2 \left(\frac{2}{r_1} + \frac{2}{r_2} + \frac{1}{r_3} \right) .$$

The distances written as multiples of the side length r of the hexagon is

$$r_2 = \sqrt{3} r \,, \quad r_3 = 2r \,.$$

The potential energy has to be equal to the initial kinetic energy, which gives us the equation

$$\frac{3ke^2}{r} \left(2 + \frac{2}{\sqrt{3}} + \frac{1}{2} \right) = 3 m_\mathrm{e} v^2 \,, r = \frac{ke^2}{m_\mathrm{e} v^2} \left(2 + \frac{2}{\sqrt{3}} + \frac{1}{2} \right) .$$

Using the given values and the values of physical constants, we can evaluate the equations and get the result $r \doteq 9.3 \cdot 10^{-4}$ m.

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Problem FoL.37 ... Achilles and the tortoise

Consider an optical fibre which consists of a core, with diameter $d = 50\,\mathrm{\mu m}$ and refractive index $n_1 = 1.460$, which is surrounded by cladding with refractive index $n_2 < n_1$. The rays propagate inside the core due to total internal reflection. A measurement was performed on a fibre with length $l = 500\,\mathrm{m}$ and it was estimated that the time dispersion (which is due to the rays propagating along different paths) at the receiving end was $\Delta t = 20\,\mathrm{ns}$. Determine the refractive index n_2 of the cladding. Assume that the speed of light in vacuum is $c = 2.997\,9 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$ and that light is not dispersed inside the fibre. *Michaleus.*

The ray of light that arrives first propagates along a straight path in the core, which corresponds to time $n_1 l/c$. The last ray is reflected from the boundary between the core and the cladding at the critical angle α. Considering the geometry of the problem, we can write $l' = l/\sin\alpha$. Let's consider the case of total reflection; then, $n_2 = n_1 \sin\alpha$, so if we know Δt, we can write

$$n_2 = n_1 \frac{l n_1}{c\Delta t + l n_1}\,.$$

Plugging in the numbers, we arrive at $n_2 \doteq 1.448$.

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Problem FoL.38 ... this problem sucks

Find the radius (in millimeters) of the largest lead ball which we are able to suck up using a vacuum cleaner with suction power $200\,\mathrm{W}$. The machine is equipped with a plain hose with a circular cross-section of diameter $6\,\mathrm{cm}$. To simplify matters, assume that the ball is already hovering in the air high above the floor. Take the densities of air and lead to be $\varrho = 1.2\,\mathrm{kg{\cdot}m^{-3}}$ and $\varrho_{\mathrm{Pb}} = 11\,340\,\mathrm{kg{\cdot}m^{-3}}$, respectively. *Kuba lost his pen-drive in a peculiar way.*

First, let us relate the suction power P of the vacuum cleaner to the speed v of the air entering the hose. If we denote by ϱ the density of the air and if we let $A = \pi d^2/4$ be the cross-sectional area of the hose (where d is the cross-sectional diameter), then the engine of the vacuum cleaner works to accelerate a mass $\mu = \varrho A v$ of the air per unit time, from rest to the speed of v. We then have

$$P = \frac{1}{2}\mu v^2 = \frac{1}{2}\varrho A v^3\,,$$

so

$$v = \left(\frac{2P}{\varrho A}\right)^{\frac{1}{3}}\,.$$

It remains to equate the magnitude of quadratic drag on the ball (we can check that the value of Re indeed suggests using quadratic drag) and gravity acting on the ball with density ϱ_{Pb}, radius r and drag coefficient C (since the ball hovers in the air far from any other objects, the situation is well approximated by turbulent flow past a rigid sphere with far-field velocity v). Therefore

$$\frac{4}{3}\pi\varrho_{\mathrm{Pb}} g r^3 \leq \frac{1}{2}\varrho C \pi r^2 \left(\frac{8P}{\varrho\pi d^2}\right)^{\frac{2}{3}}\,,$$

so we obtain

$$r \leq r_{\max} = \frac{3}{8g} C \left(\frac{8P}{\varrho \pi d^2} \right)^{\frac{2}{3}} \frac{\varrho}{\varrho_{\mathrm{Pb}}} \,.$$

Using $C = 0.5$, we have $r_{\max} \doteq 4.9\,\mathrm{mm}$.

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Problem FoL.39 ... bursting with rage

Kiki found a hair band and didn't know what to do with it, so she put it around Náry's head. The length of the unstretched band is $l_0 = 15\,\mathrm{cm}$, its length when put around Náry's head is $l = 55\,\mathrm{cm}$. The cross-section of the band has a square shape with a constant side length of $w = 2\,\mathrm{mm}$. Let us naively assume that the band is behaving linearly according to Hooke's law, and has Young's modulus $E = 50\,\mathrm{MPa}$. What is the pressure (in kPa) crushing Náry's head? Take the head to be perfectly spherical and the band to be lying on its great circle.

Watermelon gore by Mirek.

We can use the very simplifying assumption of linearity of the band – the force of tension in the band is

$$F = w^2 E \frac{l - l_0}{l_0} \,.$$

Now we need to find how much a small length element dl pushes on Náry's head. The angle corresponding to this element is ϑ. Both ends of this element are pulled tangentially with force $\boldsymbol{F}$, so these two forces add up vector-wise and the resulting force pushing on the head is

$$\mathrm{d}F_{\mathrm{r}} = 2F \sin \frac{\vartheta}{2} \,.$$

In polar coordinates, we can write the element length as $\mathrm{d}l = R\vartheta$, where R is the radius of the circle with circumference l. The contact area element is $\mathrm{d}S = wR\vartheta$, and using the circumference instead of the radius, we have

$$\mathrm{d}S = \frac{\vartheta l w}{2\pi} \,.$$

The pressure can be computed as the limit over a very small area, and it suffices to send the angle ϑ to 0

$$p = \lim_{\vartheta \to 0} \frac{\mathrm{d}F_{\mathrm{r}}}{\mathrm{d}S} = \lim_{\vartheta \to 0} \frac{2F \sin(\vartheta/2)}{\vartheta l w/(2\pi)} = \frac{2\pi F}{lw} = \frac{2\pi w^2 E (l - l_0)}{l l_0 w} = 2\pi w E \left(\frac{1}{l_0} - \frac{1}{l} \right) \,.$$

After plugging in the numbers, we get $p \doteq 3.05\,\mathrm{MPa}$. It is, however, reasonable to doubt the correctness of this result, since the material properties of the hair band are probably more complicated.

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Problem FoL.40 ... alternating-direct 2

Consider the circuit in the figure, the DC source has voltage $U_\mathrm{j} = 4.5\,\mathrm{V}$*, the AC source has voltage amplitude* $U_\mathrm{s} = 5\,\mathrm{V}$ *and frequency* $f = 50\,\mathrm{Hz}$*. The resistor's resistance is* $R = 100\,\mathrm{k\Omega}$ *and the capacitor's capacity is* $C = 10\,\mathrm{nF}$*. What's the maximum charge on the capacitor, in* nC*?*

Xellos still doesn't like broccoli.

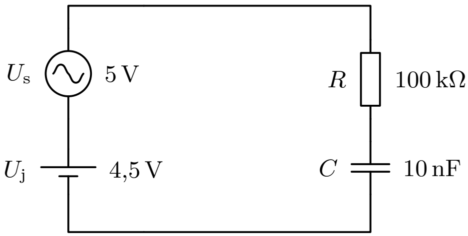

Fig. 6: Circuit diagram.

The two sources are again a trap – all elements of the circuit are linear, so we can view it as a superposition (sum) of two circuits – one with the DC and the other with the AC source. The charge on the capacitor is a superposition of charges in both cases. As we already know, in the circuit with the DC source, the capacitor carries a constant charge $Q_\mathrm{j} = CU_\mathrm{j}$.

The circuit with the AC source is harder to analyse, because we can't just ignore the resistor. In order to find the maximum charge on the capacitor, we need to find the maximum voltage on it. We'll express the immediate voltage u_c on the capacitor from the 2$^\mathrm{nd}$ Kirchhoff's law using the immediate voltages on the AC source u_s and on the resistor u_r as $u_\mathrm{c} = u_\mathrm{s} - u_\mathrm{r} = u_\mathrm{s} - jR$, where j is the immediate current through the circuit. The impedance of the circuit is $Z = R + 1/(\mathrm{i}\omega C)$ and we know that $j = u_\mathrm{s}/Z$, so

$$u_\mathrm{c} = u_\mathrm{s}\left(1 - \frac{R}{Z}\right) = u_\mathrm{s}\frac{1}{1 + \mathrm{i}R\omega C} = u_\mathrm{s}\frac{1}{1 + 2\mathrm{i}\pi f RC}$$

and the maximum value of U_c is

$$U_\mathrm{c} = U_\mathrm{s}\left|\frac{1}{1 + 2\mathrm{i}\pi f RC}\right| = \frac{U_\mathrm{s}}{\sqrt{1 + (2\pi f RC)^2}}\,.$$

Therefore, the maximum charge on the capacitor is

$$Q = C\left(U_\mathrm{j} + \frac{U_\mathrm{s}}{\sqrt{1 + (2\pi f RC)^2}}\right) \doteq 92.7\,\mathrm{nC}\,.$$

Note that in the circuit with the AC source, there's a phase shift between u_c and u_s, which causes the maximum of u_c to occur at a different time than the maximum of u_s.

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Problem FoL.41 ... ambulance

Verča wanted to pretend being an ambulance. She took a honking siren and started running from Karel towards the door at a speed $v = 7\,\mathrm{km{\cdot}h^{-1}}$. Karel, who watches the situation with interest, hears interference beats at a frequency $f = 6\,\mathrm{Hz}$. What is the frequency f_0 of the siren? The speed of sound in the air is $v_\mathrm{s} = 340\,\mathrm{m{\cdot}s^{-1}}$.

Verča is starting to go crazy from all that physics.

The beats that Karel is hearing are a superposition of two waves – one direct and one reflected from the door. The movement of the source causes Doppler's effect. Therefore, both frequencies differ from the frequency of the siren. The frequency f_1 is lower (the source is moving away from the observer) according to the relation for $v \ll v_\mathrm{s}$

$$f_1 = f_0 \left(1 - \frac{v}{v_\mathrm{s}}\right) .$$

The amplitude A which Karel is hearing can be described as

$$A(t) = A_0 \left(\cos\left(2\pi f_1 t\right) + \cos\left(2\pi f_2 t + \varphi\right)\right) ,$$

where A_0 is the amplitude of the original wave and φ is the phase shift of the first wave against the second. Using the trigonometric identity for the sum of cosines, we get

$$A(t) = 2A_0 \cos\left(\frac{2\pi(f_1 + f_2) + \varphi}{2} t\right) \cos\left(\frac{2\pi(f_1 - f_2) - \varphi}{2} t\right) .$$

The first cosine's frequency is $(f_1 + f_2)/2 = f_0$, which is too high; that means the beats come from the second cosine. The human ear can only perceive the absolute value of the amplitude. During one period of the cosine, two beats are generated (with opposite phase). Therefore, the frequency of beats is twice the frequency $|f_1 - f_2|/2$ of the second cosine. Hence

$$f = f_2 - f_1 = f_0 \frac{2v}{v_\mathrm{s}} ,$$

from which we isolate f_0

$$f_0 = f \frac{v_\mathrm{s}}{2v} .$$

For the given numeric quantities, we get $525\,\mathrm{Hz}$.

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Problem FoL.42 ... transmittors

Imagine that we have two plane-parallel mirrors and we know that each mirror transmits only a fraction of incident light. The first mirror transmits $t_1 = 2/3$ of incident light, the second only $t_2 = 1/3$. What fraction T of incident light passes through our system of two parallel mirrors? No energy is lost in the mirrors.

Karel pondered what "showing someone a mirror" meant in the former Communist Bloc.

First, we observe that we have been given transmission coefficients t_i and the reflection coefficients are simply $1 = t_i + r_i \Rightarrow r_i = 1 - t_i$, where i indicates the respective mirror.

Next, we mustn't forget that the light may bounce from one mirror to the other infinitely many times (theoretically), until its energy decreases to zero. To obtain the total coefficient of reflection, we must evaluate a geometric series

$$T = t_1t_2 + t_1r_2r_1t_2 + t_1r_2r_1r_2r_1t_2 + \cdots = t_1t_2\sum_{n=0}^{\infty}(r_1r_2)^i = t_1t_2\frac{1}{1-r_1r_2}\,.$$

The formula implies that the result does not depend on the order of the mirrors (which should not surprise us). Now, we can rewrite the reflection coeffcient using $r_i = 1 - t_i$ and obtain the coefficient

$$T = \frac{t_1t_2}{t_1+t_2-t_1t_2} = \frac{1}{\frac{1}{t_1}+\frac{1}{t_2}-1} = \frac{2}{7} \doteq 0.286\,.$$

The transmission coefficient of the system of two mirrors is 0.286.

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Problem FoL.43 ... homemade maglev

Mišo adores trains and desperately wants to take a ride on a maglev - a train that uses magnetic levitation to move without touching the ground. In order to avoid travelling far abroad, he decided to build his own maglev at home. He plans to use a set of electromagnets in the shape of the letter U *to create the required magnetic field. The inductor is wound on a core and has* $N = 100$ *turns. The cross section of the core is* $S = 5\,\mathrm{cm}^2$ *and it's constant along its whole length. The length of the central line of force inside the inductor is* $l = 20\,\mathrm{cm}$*. The core has relative permeability* $\mu_\mathrm{r} = 1\,000$*. To test the strength of the magnets, Mišo places a steel girder next to both ends of one magnet, then connects the inductor to a current source and tries to pull the girder away. How large will be the current passing through the inductor if he needs a force of* $F = 100\,\mathrm{N}$ *to pull the girder away? Consider the permeability of air to be approximately the same as the permeability of vacuum* $\mu_0 = 4\pi\cdot 10^{-7}\,\mathrm{H{\cdot}m^{-1}}$*. Neglect the weight of the girder.*

Mirek tried to find a "practical" use for excercise problems.

We can express the force needed to pull a girder away as

$$F = \frac{\mathrm{d}E}{\mathrm{d}x}\,,$$

where E is the energy of the magnetic field of the inductor. The magnitude of magnetic induction can be derived from Ampère's law; we get

$$B = \mu_\mathrm{r}\mu_0\frac{NI}{l}\,,$$

where I is the current passing through the inductor. The energy density of magnetic field w outside the inductor is given by

$$w = \frac{B^2}{2\mu_0}\,.$$

Let's express a volume element as $\mathrm{d}V = 2s\mathrm{d}x$, then

$$\mathrm{d}E = w\mathrm{d}V = 2Sw\mathrm{d}x\,.$$

After dividing by dx

$$F = 2Sw = \frac{SB^2}{\mu_0}\,.$$

We are left with substituting for the magnetic field and expressing the current

$$I = \frac{l}{N\mu_\mathrm{r}}\sqrt{\frac{F}{\mu_0 S}} = 0.798\,\mathrm{A}\,.$$

The current in the inductor has to be $I = 0.798\,\mathrm{A}$.

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Problem FoL.44 ... accretion is too mainstream

According to some theories, there used to be a 10th planet of the Solar System called Phaeton between Mars and Jupiter. Its distance from the Sun is supposed to have been approximately $D_\mathrm{P} = 2.5\,\mathrm{au}$ and its radius $r_\mathrm{P} = 1\,000\,\mathrm{km}$. Determine the surface temperature of Phaeton after a radiation equilibrium is established, if its Bond albedo is similar to that of Earth, $A = 0.3$ (and so is emissivity, which is approximately 1). Phaeton is supposed to have existed at a time when the Sun had surface temperature $T_\mathrm{S} = 5\,000\,\mathrm{K}$ and radius $r_\mathrm{S} = 6\cdot 10^5\,\mathrm{km}$.

Mirek has been thinking about moving somewhere else.

The main source of radiated energy in the Solar System is the Sun. We'll compute its radiative power using the Stefan-Boltzmann law

$$P_\mathrm{S} = 4\pi r_\mathrm{S}^2\sigma T_\mathrm{S}^4\,,$$

where $\sigma = 5.7\cdot 10^{-8}\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}$ is the Stefan-Boltzmann constant. The power incident on Phaeton is proportional to the surface covered by the planet on a sphere of radius D_P, which is

$$P_\mathrm{inc} = P_\mathrm{S}\frac{\pi r_\mathrm{P}^2}{4\pi D_\mathrm{P}^2} = \pi r_\mathrm{S}^2\sigma T_\mathrm{S}^4\frac{r_\mathrm{P}^2}{D_\mathrm{P}^2}\,.$$

The Bond albedo determines the fraction of power that's reflected from the surface of the planet, so only power

$$P = P_\mathrm{inc}(1-A)$$

is actually absorbed. In the state of radiation equilibrium, the absorbed power must be equal to the emitted power

$$P_\mathrm{P} = 4\pi\sigma r_\mathrm{P}^2 T_\mathrm{P}^4\,,$$

radiated by the planet. We neglected the emissivity of Earth here, since it's sufficiently close to 1. We get the equation

$$\pi r_\mathrm{S}^2\sigma T_\mathrm{S}^4\frac{r_\mathrm{P}^2}{D_\mathrm{P}^2}(1-A) = 4\pi\sigma r_\mathrm{P}^2 T_\mathrm{P}^4\,,$$

from which we'll express the desired temperature

$$T_\mathrm{P} = T_\mathrm{S}\sqrt[4]{\frac{r_\mathrm{S}^2}{4D_\mathrm{P}^2}(1-A)}\,.$$

Numerically, we get $T_\mathrm{P} \doteq 130\,\mathrm{K}$.

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Problem FoL.45 ... wavy electron

What wavelength will an electron have, if we speed it up from rest with electric field over a potential difference of $60\,\mathrm{kV}$*?* *Kiki and her introduction to quantum physics for dogs.*

Let us use de Broglie's relation $\lambda = h/p$, where h is the Planck constant and p the momentum of the electron, which we have to determine. The kinetic energy which the electron obtains when crossing the given potential difference $\Delta\varphi$ is $E_\mathrm{k} = e\Delta\varphi$, where e is the elementary charge. The total energy of the electron with rest mass m_e will then be $E = E_0 + E_\mathrm{k}$, where $E_0 = m_\mathrm{e}c^2$ is its rest energy. The relation between energy E and momentum p is $(cp)^2 + E_0^2 = E^2$, from which we can express $p = \sqrt{(m_\mathrm{e}c + e\varphi/c)^2 - (m_\mathrm{e}c)^2}$ and then the wavelength as

$$\lambda = \frac{h}{\sqrt{\left(m_\mathrm{e}c + \frac{e\varphi}{c}\right)^2 - (m_\mathrm{e}c)^2}} = 4.87 \cdot 10^{-12}\,\mathrm{m}\,.$$

The electron is almost located at a point and its wavelength decreases with increasing voltage; maybe in contradiction with intuition, a faster particle isn't "wider".

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Problem FoL.46 ... rising bubble

Having had a great time doing our laundry in a sink, we managed to release a number of soap bubbles into the air. Some of them have burst after flying around for a short while, but a number of them happened to ascend relatively high up – the most successful guy reached the altitude of $H = 1\,500\,\mathrm{m}$ *above ground level. How many times did the volume of this bubble increase during its ascent? Use the ideal gas law and the equation of hydrostatic equilibrium applied to the atmosphere in order to find the pressure as a function of altitude above ground level. Assume that there is a linear relation of the form* $T(h) = T(0) - kh$ *between the temperature and the altitude, where* $k = 0.006\,5\,\mathrm{K{\cdot}m^{-1}}$*. The atmospheric pressure at ground level is* $p(0) = 101\,\mathrm{kPa}$*, the temperature at ground level is* $T(0) = 25\,^\circ\mathrm{C}$*. The molar mass of air is* $M = 29.0\,\mathrm{g{\cdot}mol^{-1}}$*. Assume that the pressure inside and outside the bubble is the same.*

Mirek doing calculations and laundry at the same time.

Substituting from the ideal gas law

$$\varrho = \frac{pM}{RT}$$

into the differential form of the equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}h} = -\varrho g\,,$$

we get a separable first-order ODE

$$\frac{\mathrm{d}p}{p} = -\frac{gM}{RT}\mathrm{d}h = -\frac{gM}{R(T(0) - kh)}\mathrm{d}h\,.$$

Integrating (and taking into account the appropriate boundary conditions), we obtain an expression for the ambient pressure p as a function of altitude h above ground

$$p = p(0)\left(1 - \frac{kh}{T(0)}\right)^{gM/Rk}\,.$$

Finally, using the ideal gas law $pV/T = \text{const}$, we find

$$\frac{V(h)}{V(0)} = \frac{T(h)p(0)}{T(0)p(h)} = 1.15\,.$$

The volume of the bubble increased 1.15 times.

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Problem FoL.47 ... a charged spiral

Consider a charged non-conducting spiral parametrised in 2D cartesian coordinates as

$$\{[l_0 t \sin \ln t, l_0 t \cos \ln t], t > 0\}\,.$$

Assume that the linear charge density ϱ along the spiral can be expressed as a function of t, namely $\varrho(t) = \varrho_0 t \exp(-t^2)$. Find the value of electric potential at the origin and express it as a multiple of ϱ_0/ε_0. The spiral is placed in vacuum. *Due to Janči.*

Let us first find an expression for the length of an infinitesimal element of the spiral corresponding to a change $\mathrm{d}t$ in the parameter t. Using the Pythagorean theorem, we obtain

$$\begin{aligned}\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} &= l_0 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\ &= l_0 \sqrt{(\sin \ln t + \cos \ln t)^2 + (\cos \ln t - \sin \ln t)^2}\mathrm{d}t = \sqrt{2} l_0 \,\mathrm{d}t\,.\end{aligned}$$

The distance between the origin and a point on the spiral corresponding to a given value of t is simply $r(t) = l_0 t$, so the potential can be found by direct integration as

$$\varphi = \int_0^\infty \frac{1}{4\pi\varepsilon_0} \frac{\varrho(t)\mathrm{d}l}{r(t)} = \frac{\sqrt{2}\varrho_0}{4\pi\varepsilon_0} \int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{\sqrt{2}\varrho_0}{4\pi\varepsilon_0}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{2}}{8\sqrt{\pi}} \approx 0.099\,7\frac{\varrho_0}{\varepsilon_0}\,.$$

The potential in the desired units is $0.099\,7\,\varrho_0/\varepsilon_0$.

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Problem FoL.48 ... washing the dishes

Verča likes to wash the dishes. When she's done, she takes a little boat and places it on the surface of water mixed with detergent. The boat is a small wooden board with thickness $b = 0.5\,\mathrm{cm}$, length $l = 10\,\mathrm{cm}$ and width $w = 5\,\mathrm{cm}$. Its length is parallel to the axis x. The surface tension in the x-direction is given by $\sigma(x) = \sigma_0 + xs$ and in the y-direction by $\sigma(y) = \sigma_0$, where $\sigma_0 = \mathrm{const}$ and s is the gradient of surface tension. Determine the intial acceleration of boat after being placed on the water surface. The density of wood is $\varrho = 800\,\mathrm{kg{\cdot}m^{-3}}$, $\sigma_0 = 30\,\mathrm{mN{\cdot}m^{-1}}$, $s = 80\,\mathrm{mN{\cdot}m^{-2}}$. If the acceleration is in the direction of increasing surface tension, it's positive; if it's in the opposite direction, it's negative. The contact angle of wood and water is $\beta = 45^\circ$. The inclination of the boat is negligible.

Mirek watched students washing the dishes during the summer camp.

The problem is heavily simplifed by the fact that the boat is rectangular and its length is parallel to the gradient of the surface tension. Thus, the resulting force acting on the boat is simply the difference of forces acting on the front and the rear side of the ship (the lateral forces compensate for each other). The forces must be multiplied by the factor $\sin\beta$, because their vertical components compensate for the gravity of the boat.

The surface tension represents the propelling force per element of length; in our case, it is the length of the shorter side w. The magnitude of the propelling force is

$$(F(l) - F(0))\sin\beta = ((\sigma_0 + (x+l)s) - (\sigma_0 + xs))\, w\sin\beta = wls\sin\beta\,,$$

the result is positive, so the boat moves in the direction of increasing surface tension. The mass of the boat is $m = \varrho wlb$, so

$$a = \frac{s\sin\beta}{\varrho b}\,.$$

Numerically, $a \doteq 0.014\,\mathrm{m\cdot s^{-2}}$.

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Problem FoL.49 ... amplifying

Find the gain of the circuit in the figure. The gain is defined as $g = U_{\mathrm{output}}/U_{\mathrm{input}}$ (including the sign, the voltages are calculated with respect to the ground). All resistors in the circuit have resistivity $R = 10\,\mathrm{k\Omega}$. Assume that the op-amp is ideal.

Hint *To solve this problem, you will only need Ohm's law, Kirchhoff's first law and the following information: The triangle symbol in the circuit is an op-amp (operational amplifier). It has two inputs (− and +) and one output (the third vertex of the triangle), and follows these rules:*

- *there is no current going in or out of any of the inputs,*
- *the op-amp sets the voltage at its output so that the voltage difference between its inputs is zero.*

Pikoš likes playing with complicated circuits.

The input − is called inverting, the input + non-inverting. This use of the op-amp (connecting the output back to the input) is called feedback.

First, we need to find the voltages on the two inputs of the op-amp. Since no current flows into them, the current through R_5 is zero and, using Ohm's law, so is the voltage across it. We write $U_5 = 0\,\mathrm{V}$. This makes the voltage with respect to the ground (junction E) on the inverting input zero. Because the op-amp wants to keep the voltage difference between its inputs equal to zero, the voltage on the non-inverting input (again, with respect to the ground) is also zero.

Now we know that the voltage between the junctions A and E is zero, the voltage across the resistor R_1 is $U_1 = U_{\mathrm{input}}$ and the current through it is $I_1 = U_1/R_1 = U_{\mathrm{input}}/R_1$. Because from the junction A, no current can flow into the op-amp, the current through the resistor R_2 is $I_2 = I_1$. We can then calculate the voltage across the resistor R_2 as $U_2 = R_2 I_2 = U_{\mathrm{input}} R_2/R_1$. Because the voltage between the junction A and the ground is zero, the voltage between the junction B and the ground is $U_4 = U_{\mathrm{input}} R_2/R_1$, and the potential is lower in the junction B.

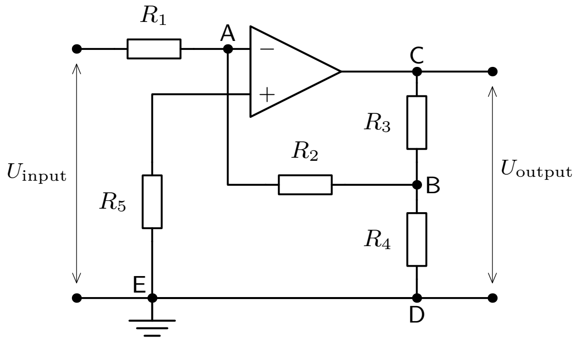

Fig. 7: Circuit scheme.

Because the potential in the junction B is lower that the potential in the junction D, the current flows from D to B and its magnitude is $I_4 = U_4/R_4 = U_\mathrm{input}R_2/(R_1R_4)$.

The currents flowing into the junction B are I_2 through the resistor R_2 and I_4 through the resistor R_4. Therefore, using the first Kirchhoff's law, the outgoing current, through the resistor R_3, is $I_3 = I_2 + I_4 = U_\mathrm{input}[1/R_1 + R_2/(R_1R_4)]$, the voltage across it is $U_3 = R_3I_3 = U_\mathrm{input}R_3[1/R_1 + R_2/(R_1R_4)]$ and, because of the direction of the current, the potential is lower in C than in B.

The voltage on the output is now easily computed as $U_\mathrm{output} = -(U_4 + U_3)$ (the minus sign is there because the potential is lower in C than in B, which has lower potential than D), so

$$U_\mathrm{output} = -U_\mathrm{input}\left[\frac{R_2}{R_1} + R_3\left(\frac{1}{R_1} + \frac{R_2}{R_1R_4}\right)\right],$$

and the gain is

$$g = \frac{U_\mathrm{output}}{U_\mathrm{input}} = -\frac{R_2R_4 + R_3R_4 + R_2R_3}{R_1R_4};$$

if all the resistors have the same resistances, the gain is $g = -3$.

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Problem FoL.50 ... watering

In a pool full of water, there is a vertical tube with internal cross-section $S_1 = 4\,\mathrm{cm^2}$*. On its top, it opens into a cylindrical container with a few small holes on the perimeter. The whole construction rotates and the cylindrical container is radially partitioned, so that the water rotates with the container. The cylinder has radius* $r = 10\,\mathrm{cm}$ *and the total cross-section of the holes is* $S_2 = 5\,\mathrm{mm^2}$*. At first, the holes are closed and the whole tube and cylinder are filled with water and rotated with an angular speed* $\omega = 25\,\mathrm{rad{\cdot}s^{-1}}$*. Then, the holes are opened, so that water can flow out. Determine the speed* v *of the water flowing out of the holes with respect to the laboratory reference frame (which doesn't rotate with the container). The air pressure is* $p_0 = 1\,015\,\mathrm{hPa}$*, the acceleration due to gravity is* $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ *and the holes in the cyliner are* $h = 20\,\mathrm{cm}$ *above the surface of the pool. Consider the density of water to* be $\varrho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$.

Faleš was inspired by the Estonian-Finnish olympiad, which had a partially wrong solution.

First, let's determine the pressure (with respect to the ambient atmospheric pressure) that will be pushing the water out of the container. Gravity will pull the water with pressure $p_\mathrm{g} = -\varrho g h$. The rotation of the cylinder will, however, push the water out, increasing the pressure near the perimeter and decreasing it near the center, sucking more water in. The centrifugal force acting on an element with a mass m at a distance r from the center is

$$F_\mathrm{o} = m\omega^2 r\,.$$

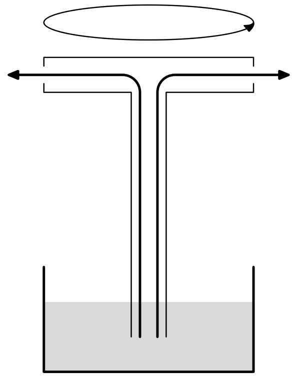

Fig. 8: Scheme of the water pump.

The pressure difference is effectively the potential energy change of a unit volume of water. Thus, we need to find the potential to this force, which we do by integrating (we may neglect the radius of the tube, so we start at the radius 0)

$$U_\mathrm{c} = \int_0^r m\omega^2 r \mathrm{d}r = \frac{1}{2} m\omega^2 r^2\,.$$

The pressure is then

$$p_\mathrm{c} = \frac{1}{2}\varrho\omega^2 r^2\,.$$

The total pressure difference from the top of the tube to the perimeter of the cylinder is then

$$\begin{aligned} p &= p_\mathrm{g} + p_\mathrm{c} \\ &= -\varrho g h + \frac{1}{2}\varrho\omega^2 r^2\,. \end{aligned}$$

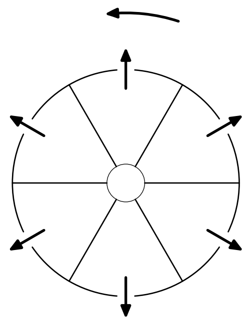

Fig. 9: Top view on the water reservoir.

The speed of the water that's flowing out is given by Bernoulli's equation

$$\frac{1}{2}\varrho v_\mathrm{r}^2 = p = \frac{1}{2}\varrho\omega^2 r^2 - \varrho g h\,.$$

So

$$v_\mathrm{r}^2 = \omega^2 r^2 - 2gh$$

is the square of the speed in the frame of reference rotating with the container. To get to the laboratory frame, we need to transform back. Luckily, the velocity is given by

$$\boldsymbol{v}_\mathrm{i} = \boldsymbol{v}_\mathrm{r} + \boldsymbol{\omega} \times \boldsymbol{r}$$

and the two terms on the RHS are perpendicular, so we can just use the Pythagorean theorem to add the velocities

$$v_\mathrm{i}^2 = v_\mathrm{r}^2 + (\omega r)^2\,,$$

so

$$v_\mathrm{i} = \sqrt{2\left(\omega^2 r^2 - gh\right)}\,.$$

Numerically, this is $v_\mathrm{i} \doteq 2.9\,\mathrm{m{\cdot}s^{-1}}$.

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Problem M.1 ... submarine

Michal is planning a journey from water level at the Mariana Trench (0 m above sea level) down to the seabed (10 971 m below sea level). What difference in hydrostatic pressure (in MPa) will Michal's submarine measure? Assume that water density doesn't change with depth. Consider the constants $\varrho_{\mathrm{H_2O}} = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$ and $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Zuzka was thinking about a submarine.

Hydrostatic pressure p in depth h satisfies

$$p = \varrho g h\,,$$

where ϱ is water density. The pressure at the water level is p_2 and the pressure at the seabed is p_1. Then, the pressure difference satisfies

$$\Delta p = p_1 - p_2 = \varrho g h_1 - \varrho g h_2 = \varrho g (h_1 - h_2)\,.$$

After substitution, we get that the magnitude of change in hydrostatic pressure is 110 MPa.

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Problem M.2 ... wet FYKOS-bird

The FYKOS-bird is riding on a tricycle with speed $v = 10\,\mathrm{km{\cdot}h^{-1}}$, when, to his displeasure, it starts raining. Not only is he getting wet, but he also has to increase his power output in order to keep moving at the same speed. Assume that the surface exposed to the uniform rain is $S = 0.5\,\mathrm{m^2}$ and that on average, 2 droplets fall on each $\mathrm{cm^2}$ over time $\Delta t = 1\,\mathrm{s}$. One droplet weighs $m = 0.1\,\mathrm{g}$. The rain falls straight down, the bird is moving in a straight horizontal line at constant speed. Neglect the fact that the FYKOS-bird has to move with respect to the tricycle. By how much does he need to increase his output power? *Faleš got wet.*

The bird needs to compensate for the momentum he transfers to the droplets in the horizontal direction. Let's denote by S_0 the area element $1\,\mathrm{cm^2}$, on which $n = 2\,\mathrm{s^{-1}}$ droplets per time fall.

The momentum which the bird has to transfer to the droplets over time Δt then is

$$\Delta p = v \Delta M = v n m \frac{S}{S_0} \Delta t\,.$$

Power is work over time, where we can express work as force over distance and power as force times distance over time, or as force times speed

$$\Delta P = \frac{\Delta W}{\Delta t} = \frac{sF}{\Delta t} = v \frac{\Delta p}{\Delta t} = v^2 n m \frac{S}{S_0}\,.$$

Numerically, we get $\Delta P \doteq 7.7\,\mathrm{W}$.

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Problem M.3 ... water pump

A tube of diameter $r = 2\,\mathrm{cm}$, bent to the shape of the letter L*, has one end submerged in a water tank. Its lower part is parallel to the water surface and the other part sticks out from the water vertically. The tube moves in the direction of the line connecting the bend and the submerged end, with speed $v = 2\,\mathrm{m{\cdot}s^{-1}}$. How high above the water surface (in* cm*) in the tank will the water in the tube rise? Consider the acceleration due to gravity to be $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.*

Faleš saw a nice picture of a bent tube.

By Bernoulli's equation

$$\frac{1}{2}\varrho v^2 + p = \mathrm{const}\,.$$

The pressure against the water flowing into the tube caused by the water rise in the tube is

$$p = -h\varrho g\,.$$

Overall, we get the elevation h as

$$h = \frac{v^2}{2g}\,.$$

Numerically, $h \doteq 20\,\mathrm{cm}$.

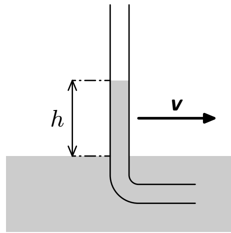

Fig. 10: Scheme of the pipe.

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Problem M.4 ... rocket bird

The Fykos-bird's wings hurt, so he decided to use physics and build a rocket-propelled backpack. During ignition, the backpack consumes $100\,\mathrm{g}$ of fuel per $1\,\mathrm{s}$ and exerts a $50\,\mathrm{N}$ propulsion force. What's the speed of exhaust gases with respect to the backpack? The bird with the jet-pack is much heavier than the consumed fuel.

Faleš's legs hurt.

The law of momentum conservation gives for the momentum changes of the bird and gas over time Δt

$$m\frac{\Delta v}{\Delta t} = -\frac{\Delta m}{\Delta t}v_{\mathrm{r}}\,,$$

where v_{r} is the speed of gases with respect to the backpack, m is the mass of the propelled body (Fykos-bird + the backpack), Δv is the change of its speed, Δm the mass of gases emitted over time Δt. The change of momentum over time is also the propulsion force

$$F = m\frac{\Delta v}{\Delta t}\,,$$

so we find the speed of gases as

$$v_{\mathrm{r}} = \frac{F\Delta t}{\Delta m}\,.$$

Numerically, $v_{\mathrm{r}} = 500\,\mathrm{m{\cdot}s^{-1}}$.

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Problem E.1 ... trinity

The resistance R_x is equal to the resistance of the whole circuit. The resistances R_1 and R_2 are the same. Compute the value of R_x as a multiple of $R = R_1 = R_2$.

Faleš liked the golden ratio.

We just have to write down the formula for the combination of resistors, or

$$xR = \frac{R\,(xR + R)}{R + (xR + R)}\,,$$

where x is the sought multiple of R, i. e., $xR = R_x$.

Solving this quadratic equation, we get two roots, and the positive one is $x \doteq 0.62$ (the golden ratio).

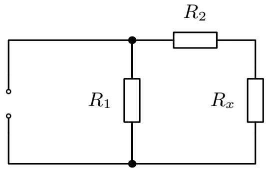

Fig. 11: Circuit diagram.

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Problem E.2 ... Náry brawls with numbers

An electromagnetic plane wave is travelling through vacuum. We have measured the values of electric intensity $\boldsymbol{E}$ and magnetic induction $\boldsymbol{B}$ at a particular place in space. The resulting values are $\boldsymbol{E} = (2, 3, 1)\,\mathrm{V{\cdot}m^{-1}}$, $\boldsymbol{B} = (5, -3, -1)\,\mathrm{T}$. We are interested in the direction in which the wave is travelling. Compute this direction, normalize the direction vector (multiply it by a positive number such that the resulting vector has unit length) and send us sum of its three components.

Náry was rotating while being normalized by field theory knowledge.

Since it's a plane wave, the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal. Moreover, both of them are orthogonal to the direction $\boldsymbol{n}$ in which the wave travels. In three dimensions, it holds true that $\boldsymbol{E} \times \boldsymbol{B} = \alpha \boldsymbol{n}$, where α is a positive number in the units of $\mathrm{m^{-1}{\cdot}s^5{\cdot}kg^{-2}{\cdot}A^2}$ – the number inverse to the normalization constant. The cross product gives us a vector $(0, 7, -21)\,\mathrm{m^{-1}{\cdot}s^5{\cdot}kg^{-2}{\cdot}A^2}$. We are left with finding the normalization constant; it is the inverse to the length of our vector – to the value $\sqrt{7^2 + (-21)^2}$. We divide our vector by this number and get the vector $\left(0, 1/\sqrt{10}, -3/\sqrt{10}\right)$. The sum of its components after rounding is -0.63.

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Problem E.3 ... Náry works hard

When he was supposed to move a telescope, Náry rather moved a charge $Q = 2\,\mathrm{C}$ from infinity to a place with electric potential $\varphi = 2\,\mathrm{kV}$. How much work (in mJ) did he do? Consider the potential at infinity to be zero.

Náry and Faleš at the theoretical physics camp.

Because the potential at infinity is zero and work is given as the difference of potentials times the charge, the work done by Náry is $W = Q\varphi \doteq 4.0 \cdot 10^6\,\mathrm{mJ}$.

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Problem E.4 ... capacitOr

A dielectric medium between two circular plates of a capacitor has two layers. The first one is air, with thickness $d_1 = 2\,\mathrm{mm}$, and the second is acrylic glass with thickness $d_2 = 4\,\mathrm{\mu m}$. Determine the capacity of the capacitor (in pF), if the area of each plate is $S = 2\,\mathrm{dm}^2$. The permittivity of vacuum is $\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{C^2{\cdot}N^{-1}{\cdot}m^{-2}}$.

Faleš couldn't read a number in his old notebook, so he changed the dimensions.

This arrangement is equivalent to two capacitors in series, where capacities combine through inverse values

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}\,.$$

From this, we can express

$$C = \frac{C_1 C_2}{C_1 + C_2}\,.$$

The capacity of one capacitor is

$$\frac{\varepsilon_0 \varepsilon_\mathrm{r} S}{d}\,,$$

where ε_r is the relative permittivity of the medium in it. The permittivity of air is almost exactly equal to the permittivity of vacuum, so its relative permittivity is 1. The resulting capacity is then

$$C = \frac{\varepsilon_0 \varepsilon_\mathrm{r,p} S}{\varepsilon_\mathrm{r,p} d_1 + d_2}\,.$$

It looks like we are missing the relative permittivity of acrylic glass $\varepsilon_\mathrm{r,p}$. However, if we realize that d_2 is smaller than d_1 by three orders of magnitude, we find out that the capacity of the acrylic glass capacitor has only a small effect on the result (especially when rounded to one significant digit). The resulting capacity is therefore given by the capacity of the air capacitor, which is something we could have concluded at the beginning

$$C \approx C_1 = \frac{\varepsilon_0 S}{d_1}\,.$$

Numerically, we have $C \doteq 90\,\mathrm{pF}$.

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Problem X.1 ... we're expanding

What amount of work (in kJ) is done by an ideal gas during isothermal expansion, if its initial volume is $V_1 = 10\,\mathrm{dm}^3$ and initial pressure is $p_1 = 1.0\,\mathrm{MPa}$? The final pressure of the gas is $p_2 = 100\,\mathrm{kPa}$.

Marek doesn't remember where he got it from.

The gas expands to a certain volume V_2. The work done by the gas can be expressed as

$$W = \int_{V_1}^{V_2} p\,\mathrm{d}V\,. \tag{4}$$

An isothermal process is described by Boyle's law, so

$$p_1 V_1 = pV \quad \Rightarrow \quad p = \frac{p_1 V_1}{V}\,. \tag{5}$$

Substituting from (5) to (4) yields

$$W = \int_{V_1}^{V_2} \frac{p_1 V_1}{V}\, \mathrm{d}V = p_1 V_1 \ln \frac{V_2}{V_1}\,. \tag{6}$$

Now, we will replace V_2/V_1 with the pressure ratio obtained using Boyle's law

$$\frac{V_2}{V_1} = \frac{p_1}{p_2}\,. \tag{7}$$

Finally, with the aid of equation (7), we express the work and compute its numerical value

$$W = p_1 V_1 \ln \frac{p_1}{p_2} \doteq 23\,\mathrm{kJ}\,.$$

The amount of work done by the gas is $W = 23\,\mathrm{kJ}$.

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Problem X.2 ... fiat lux

Every time Domča gets all sad and weary in the evening, all she needs to do is switch her lamp on and the incoming photons always cheer her up. What is the number of photons emitted by a 100-watt lamp over a period of one second? Assume that the lamp emits yellow light with a wavelength of $580\,\mathrm{nm}$ *with* $10\,\%$ *efficiency and the photons with other wavelengths are considered losses.* *Kiki hates transitions to daylight saving.*

The total energy E_1 available for conversion into yellow photons is $E_1 = Pt\eta$, where $P = 100\,\mathrm{W}$ is the input power given in the problem, $t = 1\,\mathrm{s}$ is the time interval over which we count the emitted photons and $\eta = 0.1$ is the efficiency of converting the input power into radiative power of yellow photons. The energy of one photon is well-known to be $E = hf = hc/\lambda$, where f is its frequency, h is the Planck's constant, c is the speed of light and $\lambda = 580\,\mathrm{nm}$ is the wavelength which we assume that our photons have. The number of photons radiated by the lamp over an interval of one second is therefore determined as

$$N = \frac{Pt\eta\lambda}{hc}\,,$$

which gives approximately $2.9 \cdot 10^{19}$ photons.

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Problem X.3 ... Christmas

For a three-dimensional object, a line `o` *is called an* n*-fold axis of symmetry if the object does not change upon rotation around this axis by an angle* $2\pi/n$*. For example, a regular heptagon (regular planar polygon with 7 edges) has one 7-fold axis of symmetry, perpendicular to the heptagon and passing through its center.*

Janči was decorating cookies for Christmas. One of them was shaped like a sphere. Janči divided the surface into eight equal parts and coloured them alternately, such that the coloured *parts of the surface touched only in vertices. How many 3-fold axes of symmetry does this cookie have?* *Jančí was helping in the kitchen.*

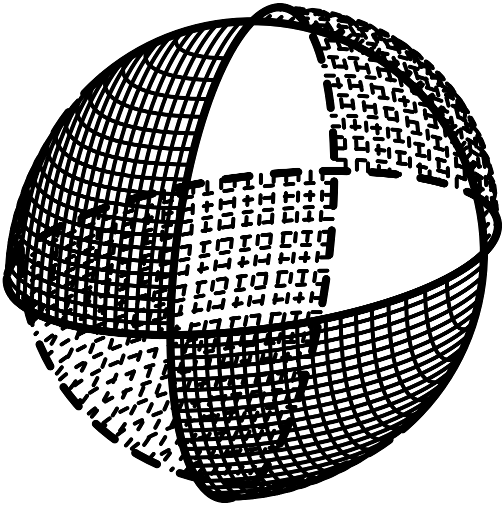

Fig. 12: Christmas cookie

Each axis has to pass through the center of the sphere, because otherwise, the rotation would move the sphere around.

Look at the picture – the desired axes have to pass through centres of opposite *curved triangles*, which means there are 4 distinct axes. More precisely, the sphere belongs to the point group T_d.

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Problem X.4 ... antichemical

One of the rules governing the filling of electron orbitals in atoms is Pauli's exclusion principle, which says that two electrons can never be in the same state. This gives us the usual rule of two electrons in one orbital (with the same quantum numbers n, l and m), because they can differ by the orientations of their spins (this is the fourth quantum number m_s).

Imagine that we changed this rule and now, two electrons can be in the same state, but not three. How many valence *electrons would a neutral sulphur atom have, if the orbitals were still being filled in the same order as for normal electrons.* *Lada was thinking about fermions.*

The energetic order of orbitals is $1s$, $2s$, $2p$ (the other ones are empty in our sulphur with modified electrons). The first orbital contains 4 electrons, with spins

$$1s: \uparrow\uparrow\downarrow\downarrow\ .$$

The valence electrons are those in orbitals with $n = 2$, i.e. all the remaining ones. Neutral sulphur has 16 electrons in total, which gives us 12 valence electrons.

More precisely, the orbital $2s$ would look the same as $1s$ and (three) orbitals $2p$ would contain the remaining 8 electrons, although $3 \times 4 = 12$ electrons in total could fit into them.

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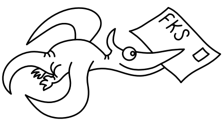

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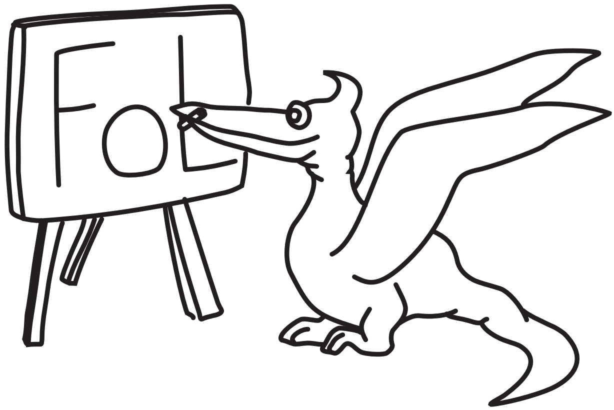
FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.

Solutions of 6th Online Physics Brawl



Problem FoL.1 ... librarian's problem

A stack of 40 books lies on the ground. Each of them is $d = 2\,\mathrm{cm}$ *thick and has mass* $m = 1\,\mathrm{kg}$. *We want to place the books into 4 shelves, with each shelf containing a stack of 10 books lying on the side and the shelves' are at heights* $100\,\mathrm{cm}$, $130\,\mathrm{cm}$, $160\,\mathrm{cm}$ *a* $190\,\mathrm{cm}$. *What work do we have to perform to move the books?* *Mirek was unable to move full bookshelves.*

This problem is simple, we just need to pick the right approach and avoid dealing with each book individually, but only with the center of mass of all books before and after the process. Initially, the center of mass of the books is at the height $h = 40\,\mathrm{cm}$ (in the middle of the homogeneous stack); after they are moved to the shelves, the center of mass is at a height

$$h' = \frac{1}{4}\Big(100\,\mathrm{cm} + 5d + 130\,\mathrm{cm} + 5d + 160\,\mathrm{cm} + 5d + 190\,\mathrm{cm} + 5d\Big) = 155\,\mathrm{cm}\,.$$

The work is then given as the change in potential energy

$$W = \Delta E_\mathrm{p} = 40mg(h' - h) \doteq 450\,\mathrm{J}\,.$$

In order to place the books on the shelves, we need to perform work $W = 450\,\mathrm{J}$.

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Problem FoL.2 ... tough slice of bread

A square wheat field covers an area of $16\,\mathrm{ha}$. *A harvesting company owns a combine harvester whose reel is* $8\,\mathrm{m}$ *wide. The combine moves with velocity* $5\,\mathrm{km{\cdot}h^{-1}}$ *and it consumes 10 litres of fuel per kilometer when harvesting. The company buys the fuel for* 28.40 *CZK per litre. How much will harvesting the whole field cost the company (rounded to integer CZK), if they have to pay the combine driver* $85\,\mathrm{CZK}$ *for each started hour of work? Neglect any delays of the combine due to turning or fires in the field.* *Meggy was harvesting part-time.*

One side of the field has length $400\,\mathrm{m}$. We have $400/8 = 50$, so the combine has to cross the whole field along one side 50 times. That way, the length of its path will be at least $20,000\,\mathrm{m}$, it will consume $200\,\mathrm{l}$ of fuel and take $4\,\mathrm{h}$ to do so. The company has to pay $5,680\,\mathrm{CZK}$ for the fuel and $340\,\mathrm{CZK}$ to the driver. The total cost of harvesting is $6,020\,\mathrm{CZK}$.

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Problem FoL.3 ... curve

Olda is driving a car with velocity $v = 20\,\mathrm{m{\cdot}s^{-1}}$; *he keeps an air freshener on his rear view mirror. When crossing a curve, his air frshener tilted by* $\alpha = 35^\circ$. *What's the radius of the curve Olda was crossing? The acceleration due to gravity is* $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.
Olda was driving through a curve.

First of all, we need to realise that gravity is perpendicular to the (flat) road. In addition, there's the centrifugal force given by

$$F_\mathrm{c} = ma_\mathrm{c} = \frac{mv^2}{r}\,.$$

The force of gravity is given by

$$F_{\mathrm{G}} = mg\,.$$

Simple reasoning leads us to the conclusion that

$$\begin{aligned}\operatorname{tg}\alpha &= \frac{F_{\mathrm{c}}}{F_{\mathrm{G}}}\,,\\ mg\operatorname{tg}\alpha &= \frac{mv^2}{r}\,,\\ r &= \frac{v^2}{g\operatorname{tg}\alpha}\,.\end{aligned}$$

The radius of the curve is $r = 58.23\,\mathrm{m}$.

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Problem FoL.4 ... astral

Dr Strange is an impenitent pragmatist who does not believe in out-of-body experiences. A nun wants to correct his mistake by hitting him in the chest with a constant force and knocking the astral body (with mass identical to that of Dr Strange, $m = 80\,\mathrm{kg}$) out of him. If the impact (contact of the fist and the chest) took $t = 0.1\,\mathrm{s}$ and the astral body was moving with velocity $v = 5\,\mathrm{m{\cdot}s^{-1}}$ afterwards, what force did the nun hit with? The body of Dr Strange itself does not move, violation of mass conservation is not our problem.

Mirek was relaxing and watching trailers.

This is a simple problem on the topic of impulse

$$F = \frac{p}{t}\,.$$

The force we are looking for is therefore

$$F = \frac{mv}{t} = 4{,}000\,\mathrm{N}\,.$$

The nun hits Dr Strange with force $F = 4\,\mathrm{kN}$.

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Problem FoL.5 ... don't mind the tune

Meggy brought a flute to a campfire. The flute is made of the ABS polymer with thermal expansion coefficient $\alpha = 9\cdot 10^{-5}\,\mathrm{K^{-1}}$. When she tried playing it indoors at $20\,^{\circ}\mathrm{C}$, its lowest tone had frequency $349\,\mathrm{Hz}$ and the flute had length l. Near the fire, she placed the flute on a bench. Assuming that the period of sound waves forming each tone is proportional to the length of the flute, determine the frequency of its lowest tone, if the flute got heated up to $25\,^{\circ}\mathrm{C}$ in the meantime.

Meggy was creating problems during a Fykos camp.

The elongation of the flute is computed easily, $\Delta l = l\alpha\Delta t$. After substituting ($\alpha$ in $\mathrm{K^{-1}}$ and Δt in K), we get $\Delta l = 4.5\cdot 10^{-4} l$. Let us denote the frequency of the tune at $20\,^{\circ}\mathrm{C}$ by f_1 and the frequency at 25 °C by f_2. Since a period is just the inverse value of frequency, we obtain a formula for the new frequency: $(1 + 4.5 \cdot 10^{-4})f_2 = f_1$, so $f_2 = 348.843\,\mathrm{Hz}$.

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Problem FoL.6 ... little cubes

Assume the molecules of water to be cubes in close contact (the volume of water is completely filled by the cubes). Determine the number of molecules that fit in $V' = 1\,\mathrm{nm}^3$ *of volume. Consider the water to be at standard conditions (*$T \approx 300\,\mathrm{K}$, $p \approx 1\,\mathrm{atm}$*), look up the necessary values of water density, molar mass and Avogadro's number (if you don't know them by heart).*

Mirek didn't see the point of including such primitive problems in master's studies.

We know the density of water ϱ, its molar mass M and Avogadro's number N_A. The mass of one molecule is $m = M/N_\mathrm{A}$, its volume is $V = a^3$, so

$$a = \sqrt[3]{\frac{m}{\varrho}} = \sqrt[3]{\frac{M}{N_\mathrm{A}\varrho}} \doteq 3.1 \cdot 10^{-10}\,\mathrm{m}\,.$$

Therefore, the number of molecules in $V' = 1\,\mathrm{nm}^3$ is

$$N = \frac{V'}{a^3} \doteq 33\,.$$

Here, we used well-known approximate values $\varrho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$, $N_\mathrm{A} = 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}$ and $M = 18\,\mathrm{g{\cdot}mol^{-1}}$.

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Problem FoL.7 ... methane

At the bottom of, for example, the South China Sea, is a substance know as methane clathrate. It's a crystal of water ice with gaps in which methane molecules are enclosed. These gaps are present in common ice as well, but they aren't filled by any substance. The chemical formula of this clathrate is $\mathrm{CH_4 \cdot 5.75H_2O}$*. How many* m^3 *of methane can be released to the atmosphere at temperature* $0\,^\circ\mathrm{C}$ *and pressure* $100\,\mathrm{kPa}$ *from* $1\,\mathrm{m}^3$ *of the clathrate? Water ice density is* $917\,\mathrm{kg{\cdot}m^{-3}}$.

Sweet memories of Katka's high school years.

Since we're dealing with water ice with density of $917\,\mathrm{kg{\cdot}m^{-3}}$, one cubic metre contains 917 kg of ice. Since one mole of water weighs 18 g, the amount of ice in one cubic metre corresponds to 50.9 kmol. There's 5.75 times less methane, which is 8.86 kmol. To determine the volume of methane, we can use the ideal gas law

$$V = \frac{nRT}{p}\,.$$

Numerically, $V \doteq 201\,\mathrm{m}^3$.

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Problem FoL.8 ... do it tomorrow

Compute how long would day on Earth have to be in order to cause weightlessness at the equator. The Earth's radius is $R = 6,400\,\mathrm{km}$, its mass is $M = 6 \cdot 10^{24}\,\mathrm{kg}$, the gravitational constant is $G = 6.7 \cdot 10^{-11}\,\mathrm{kg^{-1}{\cdot}m^3{\cdot}s^{-2}}$. Assume the Earth to be spherical.

Mirek was sitting by the window watching time fly.

The gravitational acceleration at the equator is

$$a_\mathrm{g} = \frac{GM}{R^2}$$

and it has to be balanced by centrifugal acceleration at the equator

$$a_\mathrm{c} = \omega^2 R$$

in order to yield zero net acceleration. Thus, we get

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{GM}{R^3}}\,,$$

from which

$$T = 2\pi\sqrt{\frac{R^3}{GM}} \doteq 5,100\,\mathrm{s} \doteq 1.4\,\mathrm{h}\,.$$

The day would have to last 1.4 h.

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Problem FoL.9 ... Toricelli reloaded

We are on a space station with standard atmospheric pressure $p_\mathrm{a} = 101\,\mathrm{kPa}$ inside, performing the Toricelli experiment. We take a bottle of mercury, stick a long test tube inside and when it is full, place it vertically in such a way that the open end is submerged under the surface of mercury in the bottle. During the experiment, the space station is sufficiently far from the gravitational influence of any celestial bodies and is accelerating in the direction of the closed end of the test tube (after the above described manipulation) with acceleration $a = 20\,\mathrm{m{\cdot}s^{-2}}$. How high will the mercury column in the test tube be? The density of mercury is $\varrho = 13\,600\,\mathrm{kg{\cdot}m^{-3}}$.

Mirek was disappointed by an easy lecture, so he upgraded the problems.

The hydrostatic pressure in the test tube after the experiment ends has to balance the surrounding pressure, which is $p_\mathrm{a} \doteq 101\,\mathrm{kPa}$. We can compute the hydrostatic pressure using the formula

$$p = h\varrho a\,,$$

replacing the usual acceleration due to gravity g by the acceleration of the space station a. From the formula

$$p_\mathrm{a} = p = h\varrho a$$

we express

$$h = \frac{p_\mathrm{a}}{\varrho a} \doteq 0.37\,\mathrm{m}\,.$$

The column of mercury is approximately half as high as would be on Earth, since the acceleration of the space station is approx. twice as large.

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Problem FoL.10 ... effectiveness evaluation

Karel and Aleš have one homogeneous brick each. The bricks have cuboid shapes with edge lengths $a \times a \times 2a$, where $a = 5\,\mathrm{cm}$. The mass of each brick is $m = 1\,\mathrm{kg}$. Karel puts his brick down and starts flipping it in one direction always about the shorter edge (looking from the side, the longer edge and the shorter edge alternate in touching the ground), until the brick crosses a distance $d = 150\,\mathrm{cm}$. Aleš is also flipping his brick until it crosses the distance d, but he is flipping it about the longer edges only (only the square base of the cuboid is ever visible from the side). The bricks do not slip and collisions with the ground are inelastic. How much more work will Karel perform? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek was evaluating the others' performances.

For Karel to move the brick from lying down state to standing state, he has to move the center of mass from the height $a/2$ to the height $a\sqrt{5}/2$, with the brick standing on an edge and the center of mass is the highest. The rest of the rotation is managed by gravity. Then, he has to lift it again from a to $a\sqrt{5}/2$, so that it would be able to fall back to its lying down state. During these actions, the brick moves by $3a = 15\,\mathrm{cm}$, so both flips have to be performed 10 times. The total energy that Karel used, and thus the work done is

$$W_\mathrm{K} = 10\left(a\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) + a\left(\frac{\sqrt{5}}{2} - 1\right)\right)mg = 10\left(\sqrt{5} - \frac{3}{2}\right)mga\,.$$

Aleš has to flip the brick 30 times about an edge, moving the center of mass from $a/2$ to $a\sqrt{2}/2$, so his total work is

$$W_\mathrm{A} = 30\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)mga\,.$$

The difference in work performed by Karel and Aleš is

$$W_\mathrm{K} - W_\mathrm{A} = \left(10\sqrt{5} - 15\sqrt{2}\right)mga \doteq 0.56\,\mathrm{J}\,.$$

Karel performed 0.56 J more work than Aleš.

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Problem FoL.11 ... 'murican

Náry does not like unpleasant surprises. How far from the muzzle of an air gun does he have to stand in order to hear the shot exactly one second before he would get hit? The speed of sound is $334\,\mathrm{m{\cdot}s^{-1}}$, the exit velocity of the bullet is $152\,\mathrm{m{\cdot}s^{-1}}$, the elevation angle of the shot is $10\circ$. Neglect air resistance. The term "heard" is considered equivalent to Náry being reached by the sound wave. The time the bullet spends inside the muzzle is also negligible. Compute the answer to metre precision.

Kiki would shoot.

The x-component of the path crossed by the bullet will be the same as the length of sound's path. Knowing the time delay of the shot, we can write this fact as

$$v_0 t \cos\alpha = v(t-1)\,,$$

where v_0 is the initial velocity of the shot, t is the time the shot takes to hit Náry, α is the elevation angle and v is the speed of sound. The time t can be expressed as

$$t = \frac{v}{v - v_0\cos\alpha}$$

and substituted into the formula for the x-component of the bullet's displacement

$$x = v_0 \frac{v}{v - v_0\cos\alpha}\cos\alpha\,.$$

After plugging in the numerical values, we get $x \doteq 271\,\mathrm{m}$. Náry would theoretically have to stand approximately 271 m away from the air gun's muzzle.

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Problem FoL.12 ... from Prague to Brno

You are driving from Prague to Brno with velocity $v = 150\,\mathrm{km{\cdot}h^{-1}}$ on the D1 motorway. The length of this path is $d = 196\,\mathrm{km}$. How many times will you pass by a RegioJet bus (also heading to Brno) during driving, if you know that every half hour, a bus leaves Prague for Brno with velocity $v_\mathrm{b} = 80\,\mathrm{km{\cdot}h^{-1}}$ and you leave Prague at the same time as one bus? (Don't count that bus, However, if you meet a bus when arriving at Brno, do count that one.) Assume that there are no traffic jams or closed sections of the motorway due to construction.

Dominika had an idea when travelling back from a FYKOS camp.

Let us say that you pass by the first bus at a distance s from Prague. When you left Prague, the bus was already

$$s(t_0) = v_\mathrm{b} t_0$$

($t_0 = 0.5\,\mathrm{h}$) away from you. Until you meet, the bus crosses an additional distance s_b. It's clear that

$$s(t_0) + s_\mathrm{b} = s\,.$$

The time you took to cross the distance s and the bus took to cross the distance s_b must be identical:

$$\frac{s_\mathrm{b}}{v_\mathrm{b}} = \frac{s}{v}\,.$$

From the above, the distance s can be computed easily as

$$s = \frac{v_\mathrm{b} v}{v - v_\mathrm{b}} t_0\,.$$

As soon as you pass by that bus, your situation is the same with respect to the next bus in front of you, so it is enough to divide the length of the motorway by the distance s and we get the number of buses you meet:

$$\left\lfloor \frac{d(v - v_\mathrm{b})}{v_\mathrm{b} v t_0} \right\rfloor\,.$$

After plugging in the given values, the result is 2.

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Problem FoL.13 ... and the Sun in Ophiuchus means that...

What is the maximum force that Jupiter can exert on a FYKOS member on Earth? The FYKOS member has mass $m_{\mathrm{F}} = 85.6\,\mathrm{kg}$, *look up the remaining parameters. Assume that the orbits of Jupiter and Earth are circular with radii equal to the lengths of their semi-major axes and coplanar.* *Karel heard that this is good argument for weak believers in astrology.*

Let us look up the lengths of semi-major axes of Earth's orbit $a_{\mathrm{Z}} = 1.50 \cdot 10^{11}\,\mathrm{m}$ and Jupiter's orbit $a_{\mathrm{J}} = 7.78 \cdot 10^{11}\,\mathrm{m}$. The planets will be farthest or closest from each other exactly when they are collinear with the Sun. Since we are interested in the maximum force between the FYKOS member and Jupiter, we consider the minimum distance between the planets. Relative to that distance, the exact position of the FYKOS member is negligible and the minimum distance FYKOS member – Jupiter is simply $r_{\min} = a_{\mathrm{J}} - a_{\mathrm{Z}}$. The force between them is

$$F_{\mathrm{g}} = G\frac{m_{\mathrm{F}} m_{\mathrm{J}}}{r_{\min}^2}\,,$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{N{\cdot}kg^{-2}{\cdot}m^2}$ is the gravitational constant and $m_{\mathrm{J}} = 1.90 \cdot 10^{27}\,\mathrm{kg}$ is the mass of Jupiter. When we plug it into the formula, we get the maximum force $F \doteq 2.75 \cdot 10^{-5}\,\mathrm{N}$. That's really very little.

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Problem FoL.14 ... bounce from zero

In the xy plane, there is a perfectly reflecting mirror represented by the curve $x^2 + y^2 = 1$, $y < 0$. *We send a light ray towards the concave side of the mirror along the line* $x = \sqrt{3}/2$. *How many times will the ray reflect before flying back to the half-plane* $y > 0$?

Mirek was too lazy to turn.

The angle between the ray and the tangent to the half-circle at the incidence point is simply

$$\alpha = \arccos\left(\frac{\sqrt{3}}{2}\right) = 30^\circ\,,$$

since the circle has unit radius. Next, we know that the incidence angle is equal to the reflection angle (here, we work with the 90° complement) and that the central angle of a half-circle is 180°. Since the incident ray is parallel to the y axis, the central angle corresponding to the segment connecting two successive reflection points is equal to 2α. For the number of reflections n, we can write an inequality

$$\alpha + 2(n-1)\alpha < 180^\circ\,,$$

where the left hand side represents the angle by which the ray has turned (or equivalently the angle, "at which we see" the n-th reflection point from the center of the half-circle). We can look for the largest integer n satisfying the inequality. After expressing n,

$$n < \frac{1}{2}\left(\frac{180^\circ}{\alpha} + 1\right) = \frac{7}{2}\,,$$

so the sought for number of reflections is $n = 3$. This result can be seen almost instantly if we realise that the segments connecting successive reflection points are just sides of a regular hexagon, so the ray turns around after three reflections.

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Problem FoL.15 ... St. Elmo's sphere

What's the minimum radius r of a conducting sphere such that there's no spontaneous discharge, if the charge of the sphere is $Q = 1\,\mathrm{C}$? In air, spontaneous discharge occurs if the electric field reaches $E_{\mathrm{max}} = 25\,\mathrm{kV{\cdot}cm^{-1}}$. *Erik likes easy problems.*

The electric field outside a charged ball is the same as for a point charge. Therefore, the el. field at the surface of the sphere is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} < E_{\mathrm{max}}\,.$$

From that inequality, we can express the radius

$$r > \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{Q}{E_{\mathrm{max}}}} \doteq 60.0\,\mathrm{m}\,.$$

The radius of the sphere has to be at least $60\,\mathrm{m}$.

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Problem FoL.16 ... under pressure

We put a metal rod of length $l = 1\,\mathrm{m}$ into a hydraulic press. The ends of the rod are held in a way that creates a small stress σ_0. The rod's thermal expansion coefficient is $\alpha = 1.1{\cdot}10^{-5}\,\mathrm{K^{-1}}$ and its Young modulus is $E = 200\,\mathrm{GPa}$. We heat up the rod by $\Delta T = 1\,\mathrm{K}$. Calculate the change in stress in the rod if the rams of the press do not move and are not thermally conducting.
Mirek was watching a hydraulic press.

If the rod was not held in the press, heating it would cause elongation according to the formula

$$\varepsilon = \frac{\Delta l}{l} = \alpha \Delta T\,,$$

where ε is the strain. Since the rod cannot extend in the press, it has to be compressed by this ε. The initial stress is irrelevant as long as we are still in the linear range of dependence of strain on stress (which is implied by the initial stress σ_0 being small). Hooke's law says that the stress increases by

$$\Delta\sigma = \varepsilon E = \alpha \Delta T E \doteq 2.2 \cdot 10^6\,\mathrm{Pa}\,.$$

The stress in the rod increases by $\Delta\sigma = 2.2\,\mathrm{MPa}$.

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Problem FoL.17 ... long drop

We throw a polystyrene ball of radius $r = 1.0\,\mathrm{cm}$ and density $\varrho = 0.06\,\mathrm{g\cdot cm^{-3}}$ vertically down into a ventillation shaft. In the shaft, hot warm air with density $\varrho_\mathrm{a} = 1.1\,\mathrm{kg\cdot m^{-3}}$ flows up with velocity $v_0 = 10\,\mathrm{m\cdot s^{-1}}$. Determine the magnitude of terminal velocity of the ball with respect to the shaft, if the drag coefficient of the ball is $C = 0.50$ and the acceleration due to gravity is $g = 9.8\,\mathrm{m\cdot s^{-2}}$. *Mirek couldn't work during backup, so he was creating problems.*

Gravity and air drag have to balance at terminal velocity,

$$mg = \frac{1}{2} C \varrho_\mathrm{a} S v^2 \,;$$

for a ball with mass $m = (4/3)\varrho S r$, we can determine the terminal velocity with respect to static air

$$v = \sqrt{\frac{8 g r \varrho}{3 C \varrho_\mathrm{a}}} \doteq 5.3\,\mathrm{m\cdot s^{-1}}\,.$$

This is the terminal velocity of the ball with respect to air, so it will move up with velocity $v' = |v - v_0| = 4.7\,\mathrm{m\cdot s^{-1}}$ with respect to the shaft until it exits the shaft.

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Problem FoL.18 ... microrotation

The molar mass of water is $M = 18\,\mathrm{g\cdot mol^{-1}}$. Its molecules have a certain rotational energy with characteristic scale $\varepsilon = 650\,\mathrm{J\cdot mol^{-1}}$. The linear dimension of one molecule (its diameter) is $l = 310\,\mathrm{pm}$. Using dimensional analysis, determine the characteristic scale for the period of rotation. The answer is the base-10 logarithm of this period in seconds.
Mirek didn't see the point of including such primitive problems in master's studies.

From the given energy and length scales and the mass of one molecule, dimensional analysis lets us construct a formula

$$t = l\sqrt{\frac{M}{\varepsilon}}\,.$$

Simply plugging in the given values after conversion to basic SI units, we get

$$t \doteq 1.6 \cdot 10^{-12}\,\mathrm{s} = 1.6\,\mathrm{ps}\,.$$

The characteristic time scale of rotation of a water molecule is in the order of picoseconds.

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Problem FoL.19 ... perfectly balanced

Consider a thin homogeneous disk of radius r, supported in the middle. Initially, there were two point masses m placed on the perimeter of the disk opposite each other so that the system remained balanced. We want to replace one of the masses by a pair of masses $3m/5$ and $4m/5$, both placed on the perimeter of the disk, so that the system still remained balanced. How far *(in multiples of r) from each other do the two replacement masses have to be?*

Meggy was bored during an algebra lecture.

A mass $3m/5$ placed at the perimeter of the disk has the same effect as a mass m placed at a distance $3r/5$ from the middle. Let's denote the vector from the middle to this mass by $\boldsymbol{r}_1$. Similarly, a mass $4m/5$ at the perimeter has the same effect as a mass m at 4/5-ths of the radius; let's denote the vector to it by $\boldsymbol{r}_2$. These two masses together have the same effect as one mass m placed at the sum of vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Therefore, these two vectors have to sum up to one vector of length r. We've got a triangle with sides of length $3r/5$, $4r/5$ and r, which is right-angled according to the Pythagorean theorem. The angle between vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is a right angle and the masses were supposed to be placed at the perimeter of the disk, so their distance is, again by the Pythagorean theorem, $\sqrt{2}\,r$.

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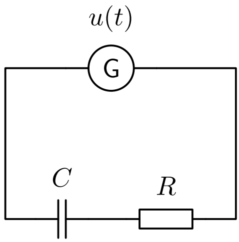
Problem FoL.20 ... sine



We connect a tone generator producing a sine wave voltage $u(t)$ into a series RC circuit. The resistor in the circuit has resistance $R = 200\,\Omega$, the capacitor has capacity $C = 50\,\mu\mathrm{F}$, the maximum voltage of the source is $U = 5\,\mathrm{V}$. What's the mean charge on the capacitor?

Kuba simplified his first circuit.

The generator can be replaced by a DC source of voltage

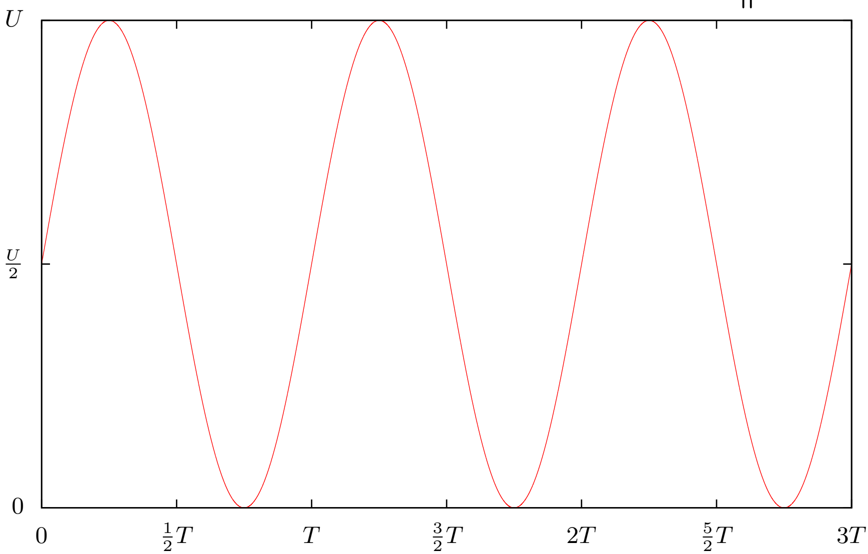


Fig. 1: Sine wave voltage.

$$U_1 = U/2$$

and an AC source of voltage

$$U_2(t) = \frac{U}{2}\sin\left(\frac{2\pi t}{T}\right)$$

connected in series.

All elements of the circuit are linear, so the current can be viewed as a superposition of two circuits, where each of them contains just one of these sources. In the DC circuit, the charge on the capacitor is constant

$$Q_1(t) = CU_1 = \frac{CU}{2}\,.$$

In the AC circuit, the charge on the capacitor harmonically changes (with some phase shift with respect to the voltage U_2) and its mean value is zero.

The total mean charge is a sum of mean charges in both circuits. Therefore, we get

$$\langle Q\rangle = Q_1 = \frac{CU}{2} = 125\,\mathrm{\mu C}\,.$$

The mean charge on the capacitor is $125\,\mathrm{\mu C}$.

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Problem FoL.21 ... chain reaction

We placed a large number of balls of radius $r = 1\,\mathrm{cm}$ in a line. The distance of centers of any two successive balls is $d = 5\,\mathrm{cm}$. Each ball has mass $m = 20\,\mathrm{g}$ and rolling friction coefficient $\xi = 0.6\,\mathrm{mm}$. The first ball in the line is given a velocity $v = 1\,\mathrm{m\cdot s^{-1}}$ in order to start a series of central collisions of the balls. Which ball will be the last one to move? All collisions are perfectly elastic and the balls don't slip. The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek was strolling through the St. Wenceslas square.

Since the collisions are perfectly elastic, they occur without energy losses. The balls have equal masses, which means that when the first ball collides with the second one, the first one stops and the second one starts moving with velocity equal to the velocity of the first ball before the collision. The balls are completely identical, so we can view each collision as an instantaneous jump (displacement) of the incident ball by $2r$. Total kinetic energy of the first ball is

$$E_\mathrm{k} = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2\,.$$

Using $\omega = v/r$, $J = 2mr^2/5$ we get

$$E_\mathrm{k} = \frac{7}{10}mv^2\,.$$

The ball stops when the work done by the floor on the ball becomes equal to the initial kinetic energy. Since the rolling drag force is constant, we easily obtain

$$mg\frac{\xi}{r}s = \frac{7}{10}mv^2$$

and

$$s = \frac{7v^2 r}{10\xi g}\,,$$

where s is the distance covered by the first ball, neglecting other balls. There is one collision for every $d - 2r$ of length, thus the total number of collisions will be

$$\left\lfloor \frac{7v^2 r}{10(d-2r)\xi g} \right\rfloor = 39\,.$$

Since there are 39 collisions, the last ball to move is ball number 40.

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Problem FoL.22 ... ready steady

We placed two identically charged particles with masses $m = 1\,\mathrm{g}$ and charges $q = 1\,\mathrm{\mu C}$ at a height $h = 2\,\mathrm{m}$ above the ground. Afterwards, we release them and measure the velocity of one of the particles just before colliding with the ground to be $v = 10\,\mathrm{m\cdot s^{-1}}$. How much did the electrostatic potential energy decrease between the start of the movement and this moment? The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek was not so fondly remembering children's games.

Both particles are acted on by the vertical gravity and horizontal electrostatic force – it's horizontal because the particles are identical, so neither one can fall faster than the other. In order to find the horizontal velocity of a particle before impact, we need to determine the vertical component of the velocity. That's well-known:

$$v_{\perp} = \sqrt{2hg}\,,$$

so the horizontal component is

$$v_{\parallel} = \sqrt{v^2 - v_{\perp}^2}$$

and it appears in the law of energy conservation for the system in the form

$$\Delta E_{\mathrm{e}} = \Delta E_{\mathrm{k}\parallel} = 2 \cdot \frac{1}{2} m v_{\parallel}^2 = m\left(v^2 - 2hg\right);$$

This formula can be seen based on the law of energy conservation for vertical components. Numerically, the electrostatic potential energy change is $\Delta E_{\mathrm{e}} = 0.061\,\mathrm{J}$.

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Problem FoL.23 ... it hangs – it hangs

In a laboratory, a polystyrene ball of mass $m_1 = 1\,\mathrm{kg}$ and density $\varrho = 1\,020\,\mathrm{kg\cdot m^{-3}}$ hangs on a massless rope. The molar heat capacity of polystyrene is $c = 1\,400\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$, its thermal expansion coefficient is $\alpha = 7\cdot 10^{-5}\,\mathrm{K^{-1}}$. On another rope, we hang another ball made of the same material and with mass $m_2 = 0.5\,\mathrm{kg}$. To each ball, we transfer heat $Q = 5\,\mathrm{kJ}$, which increases the energy of the first ball by E_1 and of the second ball by E_2. Find the difference

$E_2 - E_1$. The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek was pondering textbook problems.

The balls are located in the gravity of Earth. After we add heat Q to a ball, its temperature increases by $\Delta T = Q/(mc)$ and its radius increases by

$$\Delta r = r_0 \alpha \Delta T\,.$$

The original radius can be determined using the mass and density to be

$$r_0 = \sqrt[3]{\frac{3m}{4\pi\varrho}}\,.$$

The change in radius also tells us how much the center of mass of a ball changed (it was originally at distance r_0 from one end of the respective rope, now it's at distance $r = r_0 + \Delta r$). The potential energy of a ball decreased by

$$mg\Delta r = mg\alpha\Delta T \sqrt[3]{\frac{3m}{4\pi\varrho}} = \frac{\alpha Q g}{c}\sqrt[3]{\frac{3m}{4\pi\varrho}}\,.$$

Since the same heat was transferred to each ball, the difference of energy changes is

$$E_2 - E_1 = m_2 g\Delta r_2 - m_1 g \Delta r_1 = \frac{\alpha Q g}{c}\left(\sqrt[3]{\frac{3m_2}{4\pi\varrho}} - \sqrt[3]{\frac{3m_1}{4\pi\varrho}}\right) \doteq 3.1\cdot 10^{-5}\,\mathrm{J}\,.$$

Let's note that if the balls were standing on a thermally insulating surface, the result would have the opposite sign.

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Problem FoL.24 ... eccentric travelling companion

Given the ratio of velocities of a planet on an eliptical trajectory around a star in the pericenter (the point of the trajectory closest to the star) and in the apocenter (the point farthest from the star) $v_\mathrm{p}/v_\mathrm{a} = K = \sqrt{2}$, what's the numerical (relative) eccentricity ε of that trajectory?

Karel likes problems that deal with Kepler's 2nd law.

We'll use Kepler's 2nd law, which says that the area swept out per unit time is constant, so

$$w = \frac{v_\mathrm{a} a_\mathrm{a}}{2} = \frac{v_\mathrm{p} a_\mathrm{p}}{2}\,,$$

where a_a is the distance of the planet and sun in the apocenter and a_p is the distance in the pericenter. These distances can be expressed using its major semiaxis and eccentricity as $a_\mathrm{a} = a(1+\varepsilon)$ a $a_\mathrm{p} = a(1-\varepsilon)$. Substituting in the equation for the swept out areas and expressing the ratio of velocities, we obtain

$$v_\mathrm{a} a(1+\varepsilon) = v_\mathrm{p} a(1-\varepsilon) \quad\Rightarrow\quad \frac{v_\mathrm{p}}{v_\mathrm{a}} = K = \frac{1+\varepsilon}{1-\varepsilon}\,.$$

Now we have a relatively simple equation, containing only K and the unknown ε, from which we easily express

$$\varepsilon = \frac{K-1}{K+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2} \doteq 0.172\,.$$

The relative eccentricity of the trajectory is 0.172.

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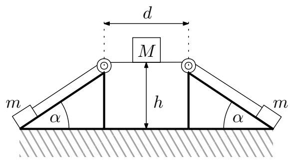
Problem FoL.25 ... cuboids up in the air



In the figure, you see a large cuboid of mass $M = 1\,\mathrm{kg}$, pulling two smaller cuboids of masses $m = M/4$ through pulleys along inclined planes with inclination angles $\alpha = 35^\circ$. Before the small cuboids reach the pulleys, the large cuboid will collide with the ground inelastically. What is the maximum height the small cuboids reach? We know the lengths marked in the figure: $d = 0.8\,\mathrm{m}$, $h = 0.6\,\mathrm{m}$; neglect the sizes of the cuboids and pulleys. Both the dynamic and static friction coefficients between the cuboids and the inclined plane are $f = 0.5$. We measure the height we are asking for with respect to the initial heights of the cuboids, the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. There is no friction in the pulleys.

Mirek's ingenious construction.

The length of the rope between the mass M and a pulley is initially $d/2$. After the mass drops to the ground, its length will be

$$\sqrt{h^2 + \left(\frac{d}{2}\right)^2}\,.$$

At that time, a cuboid on an inclined plane will have moved up along the inclined plane by

$$l = \sqrt{h^2 + \left(\frac{d}{2}\right)^2} - \frac{d}{2}\,.$$

The potential energy of the large cuboid will decrease by Mgh, while the potential energy of a small cuboid increased by $mgl\sin\alpha$. The energetic balance of the system is

$$Mgh = \frac{1}{2}MV^2 + 2\left(\frac{1}{2}mv^2 + mgl\sin\alpha + W_\mathrm{t}\right)\,,$$

where v is the velocity of a small cuboid at the time of impact of the large cuboid, V is the velocity of the large cuboid just before the impact, and W_t is the energy dissipated due to friction during the movement of one small cuboid, given by

$$W_\mathrm{t} = mgfl\cos\alpha\,.$$

During the first phase of the movement, a small cuboid moves with the speed with which the length of the rope between the large cuboid and the respective pulley increases. From that, we obtain an expression for velocities of the large and small cuboids before the impact

$$v = V\frac{h}{\sqrt{h^2 + \left(\frac{d}{2}\right)^2}}\,.$$

Now, we're able to determine the velocity of a small cuboid

$$v = \sqrt{\frac{Mgh - 2mgl\sin\alpha - 2W_\mathrm{t}}{\frac{1}{2}\frac{Mh^2}{h^2 + \left(\frac{d}{2}\right)^2} + m}}\,.$$

With this velocity, the small cuboid is moving against gravity + friction with net acceleration $a = fg\cos\alpha + g\sin\alpha$. The distance travelled during movement with constant deceleration until stopping is $v^2/(2a)$, so together with the distance l travelled until the impact, we get the vertical distance by which a small cuboid moved

$$s = \left(l + \frac{Mh/2 - ml\sin\alpha - mfl\cos\alpha}{\left(m + \frac{Mh^2}{2\left(h^2 + (d/2)^2\right)}\right)(f\cos\alpha + \sin\alpha)} \right)\sin\alpha\,.$$

Numerically, $s \doteq 0.40\,\mathrm{m}$.

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Problem FoL.26 ... tuning

In the middle of a field, there's Pepa with a tuning fork emitting a sound with frequency $f_1 = 440\,\mathrm{Hz}$. You're driving away from Pepa with velocity $v = 70\,\mathrm{km{\cdot}h^{-1}}$. What will be the frequency f of the sound you hear? The speed of sound in air is $v_\mathrm{s} = 330\,\mathrm{m{\cdot}s^{-1}}$.

Olda made it.

The problem statement immediately suggests Doppler effect, which describes the change in received signal for a moving source or receiver. When the receiver, in this case us in a car, moves away from the source, it will detect a lower frequency than emitted from the source. Its magnitude can be found using the formula

$$f = f_1\frac{v_\mathrm{s} - v}{v_\mathrm{s}}\,,$$

where v_s is the speed of sound. After plugging in the numbers, we get $f = 414\,\mathrm{Hz}$.

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Problem FoL.27 ... aloha snackbar

The US President's airplane has range approximately 12 600 km. Once upon a time, it was hijacked by terrorists shortly after takeoff, with the president on board, and vanished from the radars. A thorough search of both land and ocean was launched immediately. What area (in km^2*) has to be searched if the terrorists can't refuel anywhere?* *Mikulas was kidnapped.*

First of all, we need to realise that at this range, curvature of Earth plays a significant role, so we can't work in Euclidean geometry. However, we can approximate Earth by a sphere. Even though formulae for the area of a spherical cap can be found easily, we'll use elemental integration. Let's denote the radius of Earth by R, the range of the plane by D and work in spherical coordinates with the center of Earth at the origin and the z axis passing through the place where the plane was kidnapped. Then, the area is given by a surface integral $\int \mathrm{d}S$. In our case, $\mathrm{d}S = R^2 \sin(\vartheta)\,\mathrm{d}\varphi\,\mathrm{d}\vartheta$, so we may write

$$S = \int \mathrm{d}S = R^2 \int_0^{2\pi} \int_0^{D/R} \sin\vartheta \,\mathrm{d}\vartheta\,\mathrm{d}\varphi =$$
$$= R^2 \int_0^{2\pi} 1 - \cos\frac{D}{R}\,\mathrm{d}\varphi = 2\pi R^2 \left(1 - \cos\frac{D}{R}\right) .$$

After plugging in our values, we obtain the correct area: approx. $3.6 \cdot 10^8\,\mathrm{km}^2$. If we used the formula πD^2, we'd get approx. $5 \cdot 10^8\,\mathrm{km}^2$, which is considerably different.

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Problem FoL.28 ... burst

Into how many smaller droplets could a larger water droplet break, if it fell from a $h = 50\,\mathrm{cm}$ height? Consider all droplets to be spherical. The radius of the original droplet is $r = 1\,\mathrm{cm}$ (with its center of mass at height h) and the droplets it breaks into should have equal sizes. The surface tension of water is $\sigma = 73\,\mathrm{mN{\cdot}m^{-1}}$ and its density is $\varrho = 1.00\,\mathrm{g{\cdot}cm^{-3}}$. We're only looking for an estimate of an upper bound of the number of droplets based on the energetic balance at the beginning and at the end. *Karel was watching a falling droplet.*

According to the problem statement, the larger droplet has to break into n smaller ones. Assuming that the volume V of water is conserved, the radius of each of them will be

$$r_n = r \sqrt[3]{\frac{1}{n}} .$$

The potential energy that can be converted to surface energy is given by the formula

$$\Delta E = mgh = \frac{4}{3}\pi r^3 \varrho g h ,$$

where g is the acceleration due to gravity and $m = \varrho V$ the mass of one droplet. Here, we used the approximation $r \ll h$. The difference in surface energy between the situation when there's only one droplet and the situation with many smaller droplets is

$$\Delta E = \sigma \Delta S = \sigma \left(n S_n - S_1\right) = \sigma \left(4\pi r_n^2 n - 4\pi r^2\right) = 4\pi\sigma r^2 \left(\sqrt[3]{n} - 1\right) .$$

Now, it's sufficient to equate those two expressions for energy

$$\frac{4}{3}\pi r^3 \varrho g h = 4\pi\sigma r^2 \left(\sqrt[3]{n} - 1\right) \qquad \Rightarrow \qquad n = \left(\frac{\varrho g h r}{3\sigma} + 1\right)^3 \doteq 1.14 \cdot 10^7 .$$

The large droplet can break into at most 11 million smaller droplets. This is really just an upper bound, neglecting e.g. that the smaller droplets need to pass through non-spherical shapes while forming or that 100 % of the potential energy can't be converted to surface energy of the droplets.

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Problem FoL.29 ... damped oscillation

There's a mass point that oscillates along the y axis. The oscillation is described by the equation $y = y_\mathrm{m}\mathrm{e}^{-bt}\sin(\omega t + \varphi_0)$, where y_m is the initial amplitude, $b = 0.05\,\mathrm{s^{-1}}$ is the damping coefficient, $\omega = 200\,\mathrm{Hz}$ is the angular frequency and φ_0 is the initial angular displacement. When will the maximal amplitude decrease below $y_\mathrm{m}/2$?

Karel was watching the oscillation of a strongly damped spring.

It's important to realise that the amplitude is given by the term $y_\mathrm{m}\mathrm{e}^{-bt}$ and multiplication by a trigonometric function gives oscillation, but doesn't affect the amplitude. Therefore, we're only solving the equation

$$y_\mathrm{m}\mathrm{e}^{-bt} = \frac{y_\mathrm{m}}{2}\,, \qquad \Rightarrow \qquad \mathrm{e}^{-bt} = \frac{1}{2}\,, \qquad \Rightarrow \qquad t = -\frac{\ln\frac{1}{2}}{b} = \frac{\ln 2}{b} \doteq 13.9\,\mathrm{s}\,.$$

The amplitude of the mass point is cut in half after 13.9 s.

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Problem FoL.30 ... don't stick your fingers in!

A large ceiling fan may be approximated as a disk of radius $r = 10\,\mathrm{cm}$ and mass $M = 0.5\,\mathrm{kg}$, with three rods attached, pointing away from the centre of the disk. Each rod has length $l = 40\,\mathrm{cm}$ and mass $m = 0.2\,\mathrm{kg}$. The fan rotates with frequency $f = 1\,\mathrm{Hz}$. What work do we need to perform to stop the fan? All parts of the fan are homogeneous, neglect any friction.

Mirek was tickling the blades.

The approach to this problem is quite straightforward: we compute the rotational kinetic energy of the fan, which is equal to the work that needs to be performed to stop it. We need to know the moments of inertia of each part. The moment of inertia of a disk rotating around its axis is

$$I_0 = \frac{1}{2}Mr^2\,.$$

The moment of inertia of a rod rotating around the centre of the disk is

$$I_1 = \frac{1}{12}ml^2 + m\left(\frac{l}{2} + r\right)^2 = m\left(\frac{1}{3}l^2 + lr + r^2\right)\,.$$

The total moment of inertia of the fan (three rods plus one disk) is

$$I = I_0 + 3I_1 = \frac{1}{2}Mr^2 + m\left(l^2 + 3lr + 3r^2\right)\,.$$

The kinetic energy is computed as

$$E_\mathrm{k} = \frac{1}{2}I\omega^2 = \frac{1}{2}I(2\pi f)^2 = \left(Mr^2 + 2m(l^2 + 3lr + 3r^2)\right)\pi^2 f^2 \doteq 1.27\,\mathrm{J}\,.$$

In order to stop the fan, we need to perform 1.27 J of work.

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Problem FoL.31 ... dark planet

What's the radius of a sphere with Earth-like density $\varrho = 5.50\,\mathrm{g{\cdot}cm^{-3}}$, but escape velocity equal to the speed of light? We're interested in the first-order approximation, i.e. neglecting relativistic effects. Assume that the planet's density is constant (the planet is homogeneous). ***Give the result as a multiple of the radius of Earth*** $R_\mathrm{Z} = 6,371\,\mathrm{km}$.
Karel likes to talk about relativistic effects, but doesn't like actually solving relativistic problems.

The escape velocity of a planet with mass M and radius R can be looked up in formula lists or computed from equality of centrifugal and gravitational force. It's

$$v = \sqrt{\frac{2GM}{R}}\,,$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{N{\cdot}kg^{-2}{\cdot}m^2}$ is the gravitational constant. The mass M can be expressed as $M = \varrho V$, where V is the volume of the planet – in our case, $V = 4\pi R^3/3$. We substitute that into the formula for escape velocity, which we set equal to the speed of light $v = c = 299,792,458\,\mathrm{m{\cdot}s^{-1}}$.

$$c = \sqrt{\frac{8\pi G R^3 \varrho}{3R}}\,, \qquad \Rightarrow \qquad R = \sqrt{\frac{3}{8\pi G\varrho}}\,c \doteq 1.7 \cdot 10^{11}\,\mathrm{m} \doteq 27,000\,R_\mathrm{Z}\,.$$

Our mystical, hypothetical planet's radius would have to be 27,000 times larger than the radius of Earth.

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Problem FoL.32 ... crash

The Moon is an approximately homogeneous sphere of mass $M = 7.3 \cdot 10^{22}\,\mathrm{kg}$ and radius $R = 1,700\,\mathrm{km}$, rotating with a period $T = 27\,\mathrm{d}$. A megameteorite of mass $m = M/1\,000$ collides with the Moon's equator under an incidence angle $\vartheta = 45^\circ$ into the lunar soil, against the direction of rotation. The impact velocity is $v = 10\,\mathrm{km{\cdot}s^{-1}}$. How long will the lunar day be after the impact, if the debris from the meteorite spreads evenly over the surface of the Moon? The meteorite and the Moon are both made of the same material. Neglect mutual gravitational influence of the bodies before the impact. Give the result as a positive number in the units of (Earth's) days. *Mirek wanted the Moon to have a moon.*

The meteorite, which should collide with the Moon's surface under the angle $\vartheta = 45^\circ$, has an angular momentum with respect to the Moon's surface

$$L_\mathrm{m} = \eta M v R \frac{\sqrt{2}}{2}\,,$$

where we denoted $\eta = 1/1\,000$. Since it is an impact against the original direction of rotation, this angular momentum has to be subtracted from the original angular momentum of the Moon, so

$$L' = L - L_{\mathrm{m}} \,,$$

where L' is the new angular momentum of the Moon and L is its original ang. momentum. The moment of inertia of the spherical Moon around its center is

$$I = \frac{2}{5} M R^2 \,.$$

After the impact,

$$I' = \frac{2}{5} \left(M + \eta M\right) \left(R \sqrt[3]{1+\eta}\right)^2 \,.$$

Since angular momentum can be written as $L = I\omega$, where ω is the angular velocity $\omega = 2\pi/T$, the new angular velocity ω' will be

$$\omega' = \frac{I}{I'}\omega - \frac{L_{\mathrm{m}}}{I'} \,,$$

which can be written using rotation periods as

$$T' = \frac{1}{\frac{I}{I'}\frac{1}{T} - \frac{L_{\mathrm{m}}}{2\pi I'}} \,.$$

After simplifying and substituting, we obtain the new period

$$T' = \frac{(1+\eta)^{5/3}}{\frac{1}{T} - \frac{5\eta}{4\pi}\frac{v}{R}\frac{\sqrt{2}}{2}} \doteq -9.5\,\mathrm{d} \,.$$

The new period is 9.5 d, but the Moon is rotating in the opposite direction.

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Problem FoL.33 ... just a little closer

There are two hooks on two opposite walls with distance $d = 1\,\mathrm{m}$. On one hook, there's a charge $q = 1\,\mathrm{\mu C}$. On the other hook, there's no charge, but a thin non-conducting massless rope of length $l = d$, with a ball with mass $m = 2\,\mathrm{g}$ and charge $-q$ on its other end. Find the smallest angle (in °) between the rope and the wall such that the ball is at rest.

Mirek was reaching for his notebook.

Let's denote the unknown angle by α, the angle between the line connecting the ball with the other hook and the vertical by β. The ball splits the distance between the walls into two parts of lengths $l \sin\alpha$ and $l(1 - \sin\alpha)$. The vertical distance of the ball from the hooks is $l\cos\alpha$. The angle β can then be expressed using the angle α as

$$\operatorname{tg}\beta = \frac{1 - \sin\alpha}{\cos\alpha} \,.$$

The electromagnetic force can be written in the form $F_\mathrm{e} = a/r^2$, where

$$a = \frac{q^2}{4\pi\varepsilon_0}$$

and r is the distance of the ball from the other hook. For the ball to be at rest, the net force of $F_g = mg$ and F_e has to point in the direction of the rope. This geometric fact can be expressed by the equation

$$\operatorname{tg}\alpha = \frac{F_\mathrm{e}\sin\beta}{F_g - F_\mathrm{e}\cos\beta} = \frac{\operatorname{tg}\beta}{\dfrac{mgr^2}{a\cos\beta} - 1}$$

We may compute from the Pythagorean theorem

$$r = l\sqrt{2(1-\sin\alpha)}$$

and also

$$\cos\beta = \frac{l\cos\alpha}{r} = \frac{\cos\alpha}{\sqrt{2(1-\sin\alpha)}}\,.$$

Substituting for $\cos\beta$ and $\operatorname{tg}\beta$ in the expression for $\operatorname{tg}\alpha$, we get after several simplifications and substituting for r

$$1 - \operatorname{tg}\alpha\frac{mgl^2}{a}\big(2(1-\sin\alpha)\big)^{3/2} = 0\,.$$

The smallest positive root of this expression is $\alpha \doteq 0.239 \doteq 13.7^\circ$.

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Problem FoL.34 ... chain reaction II

We placed a large number of identical ice cuboids with square bases of side $a = 2\,\mathrm{cm}$ in a line on an ice rink. The distance between centers of any two successive cuboids is $d = 5\,\mathrm{cm}$. Each cuboid has mass $m = 20\,\mathrm{g}$ and the friction coefficients (both static and dynamic) are $f = 0.03$. The first cuboid in the line is given a velocity $v_0 = 1\,\mathrm{m\cdot s^{-1}}$ in order to start a series of central collisions of the cuboids. Which cuboid will be the last one to move? All collisions are perfectly inelastic, the cuboids don't tilt and are placed in such a way that they have parallel sides. The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek was strolling through the St. Wenceslas square again.

During the first cuboid's movement, there's a frictional force acting on it, so its acceleration is $a_0 = fg$ opposite to the direction of its movement. Denoting the distance of successive cuboids' near sides by $\delta = d - a$; over this distance, the cuboid's velocity decreases from v_0 to $\sqrt{v_0^2 - 2a_0\delta}$. After a perfectly inelastic collision, the mass is doubled (the cuboids stick together) and due to momentum conservation, the velocity after the first collision will be

$$v_1 = \frac{1}{2}\sqrt{v_0^2 - 2a_0\delta}\,.$$

In the next collisions, the mass will keep increasing, so the velocity is multiplied by $n/(n+1)$ after the n-th collision. Together with the formula for the decrease of velocity between the collisions, we find the velocity after the n-th collision to be

$$v_n = \frac{1}{n+1}\sqrt{v_0^2 - 2a_0\delta\sum_{k=1}^{n} k^2} = \frac{1}{n+1}\sqrt{v_0^2 - \frac{n(n+1)(2n+1)fg\delta}{3}}\,.$$

The largest integer n, for which the above formula still has physical meaning, is $n = 5$. The last cuboid to move will therefore be the sixth one.

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Problem FoL.35 ... Newton's rings

Olda was watching Newton's glass. Compute the distance of the 2^{nd} minimum of the interference circles from the center, if we know that the light incident on it has wavelength $\lambda = 500\,\mathrm{nm}$, the refraction index of glass is $n_\mathrm{s} = 1.5$ and the refraction index of air is $n_\mathrm{v} = 1$. The radius of curvature of the upper glass is $r = 20\,\mathrm{cm}$. *Olda was looking at Newton's ring.*

The phase difference of interfering rays is caused by travelling through air from the upper glass to the bottom one and back, and from a phase shift of π when reflected from an optically denser medium – in total, we get

$$\delta\varphi = 2yn_\mathrm{v}k_0 - \pi\,,$$

where k_0 is the wave number (we have $k_0 = 2\pi/\lambda$) and y is the height of the upper, round glass above the lower one.

For the following calculations, we need to determine y depending on the distance from the middle of the round glass. From the Pythagorean theorem, we have

$$x^2 + (y-r)^2 = r^2\,,$$
$$x^2 + y^2 - 2ry = 0\,.$$

In the approximation $y \ll r$, we may neglect the term y^2. Then, the formula for y is

$$y = \frac{x^2}{2r}\,.$$

After substituting in the expression for $\delta\varphi$, we have

$$\delta\varphi = k_0 n_\mathrm{v}\frac{x^2}{r} - \pi\,.$$

The interference minima happen when the phase difference of interfering rays is

$$\delta\varphi = 2m\pi - \pi\,,$$

where m is the number of the minimum (in the center, we have the 0^{th} minimum). Combining these equations, we get the condition for calculating the minimum

$$x = \sqrt{\frac{mr\lambda}{n_\mathrm{v}}} = \sqrt{2r\lambda} \doteq 4.47\cdot 10^{-4}\,\mathrm{m}\,.$$

In the solution, we assumed that the rays only refract negligibly. We can see that $x \ll r$, so the assumption was reasonable and we also get $y \ll r$.

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Problem FoL.36 ... short circuit

There's a simple electric circuit with two elements: a DC source with voltage $U = 1\,\mathrm{V}$ and internal resistance $R = 1\,\Omega$, and a coil with $N = 200$ turns of copper winding (circular cross-section), length $l = 30\,\mathrm{cm}$, inner radius $r_1 = 1.90\,\mathrm{cm}$ and outer radius $r_2 = 2.05\,\mathrm{cm}$. What's the maximum current passing through the coil after we switch on the source? The temperature in the room is $20\,^\circ\mathrm{C}$, neglect any heating of the winding.

Punchline redacted to avoid people whining about sexism.

Even though the source has internal resistance, we need to take into account the resistance of the coil's winding as well:

$$R_\mathrm{L} = \varrho_\mathrm{Cu} \frac{4l_\mathrm{d}}{\pi(r_2 - r_1)^2}\,,$$

where l_d is the length of the wire, $r_2 - r_1$ is the diameter of the wire and $\varrho_\mathrm{Cu} = 1.68 \cdot 10^{-8}\,\Omega{\cdot}\mathrm{m}$ is copper resistivity at the given temperature. The length of the wire is anything but equal to l; in addition, the wire isn't wound in a circle, but slanted a bit. We can view it this way: the length of the wire along the axis of the coil is l and the length "normal to the axis" is $\pi(r_1 + r_2)N$, so the total length of the wire

$$l_\mathrm{d} = \sqrt{l^2 + \left(\pi(r_1 + r_2)N\right)^2}$$

according to the Pythagoren theorem. (In this case, only the latter term is relevant and $l_\mathrm{d} \approx \pi(r_1 + r_2)N$.)

Next, we know that if there's direct current passing through the coil, there's no voltage induced in it and the coil acts as a simple wire. Intuitively, it would make sense for the current to be maximum with direct current, which gives

$$I_\mathrm{max} = \frac{U}{R + R_\mathrm{L}} \approx \frac{U}{R + \varrho_\mathrm{Cu}\dfrac{4(r_1 + r_2)N}{(r_2 - r_1)^2}} \doteq 0.809\,\mathrm{A}\,.$$

The exact expression for $I(t)$ after an instantaneous switch-on can be calculated from the differential equation $U = (R + R_\mathrm{L})I + L\dot{I}$, whose solution for $I(0) = 0$ has the form $I = A(1 - \mathrm{e}^{-\lambda t})$; plugging it back in the diff. equation gives $\lambda = (R + R_\mathrm{L})/L$ and $A = U/(R + R_\mathrm{L})$. The current will be approaching the previously computed maximum I_max, but never reach (or exceed) it.

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Problem FoL.37 ... exoplanet

An exoplanet of radius $r_\mathrm{p} = 8.0 \cdot 10^3\,\mathrm{km}$ is in circular orbit around a distant star with mass $M = 1.0 \cdot 10^{30}\,\mathrm{kg}$. The orbit and the observer are coplanar. A transit of the planet across the star *was observed. The time from the first decrease in brightness until brightness returned to its initial value was $T = 2.0$ hours and the maximum decrease in brightness was $0.3\,\%$. Determine the orbital radius.* *Filip read about newly discovered exoplanets.*

Let's denote the radius of the star by R. If we look at the star and the planet from above, we can see that since the star is very distant, the light rays we see from Earth are almost parallel. The brightness starts decreasing when the planet's edge first touches the line between the Earth and an edge of the star and returns to its original value again when it touches the line between the Earth and the other edge of the star from the outside. We can assume that the radius of the star is much smaller than the radius of the planets orbit and much larger than the radius of the planet. Then, the distance the planet has to travel between these events is $Tv \approx 2R + 2r \approx 2R$. The planet moves at a constant orbital velocity

$$v = \sqrt{\frac{GM}{r}}\,,$$

The planet blocks a part of light proportional to the blocked surface area of the star as visible from Earth, so $p = r_p^2/R^2$. Putting it all together:

$$T\sqrt{\frac{GM}{r}} = 2\frac{r_p}{\sqrt{p}}$$

$$r = \frac{pT^2GM}{4r_p^2}\,.$$

Therefore, $r \doteq 4.05 \cdot 10^{10}\,\mathrm{m}$. By evaluating the orbital velocity at this distance we can easily verify that our assumption was correct; it takes the planet $0.39\,\mathrm{s}$ to travel the distance $2r$, which is only $0.005\,\%$ of the total transit time.

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Problem FoL.38 ... tse-tse

Little rabbit has a prescription for sleeping pills. When he takes $1/3$ of a pill, he sleeps for 5 hours. When he takes $2/3$ of a pill, he sleeps for 9 hours. Once, little rabbit went nuts and took 100 pills. How many hours did he sleep, if we know that little rabbit sleeps only when the medicine's concentration in blood exceeds a certain value? Assume that the rate at which the medicine is broken down is proportional to its concentration. Neglect any medicine left over in blood from the previous days or little rabbit's death. *Mikulas couldn't sleep.*

Solving the differential equation

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\alpha c\,,$$

we find that the medicine's concentration has to decrease exponentially according to the formula $c = c_0 \exp(-\alpha t)$ with an unknown constant α. We'll determine that based on our knowledge of little rabbit's sleep times. Let's denote the concentration after taking one pill by c_1, the concentration, at which little rabbit wakes up, by c_x, and consider all times to be in hours (so α will be in h^{-1}). We know that

$$\frac{1}{3}c_1\mathrm{e}^{-5\alpha} = c_x \,,$$
$$\frac{2}{3}c_1\mathrm{e}^{-9\alpha} = c_x \,.$$

Comparing the two equations and expressing the exponent, we get

$$\alpha = \frac{\ln 2}{4}\,\mathrm{h}^{-1} \,.$$

Comparing with the exponential containing the unknown time t, we get the equation

$$100\,\mathrm{e}^{-\frac{\ln 2}{4}t} = \frac{1}{3}\mathrm{e}^{-5\frac{\ln 2}{4}} \,,$$

whose solution is

$$t = \frac{5\frac{\ln 2}{4} + \ln 300}{\frac{\ln 2}{4}} \,,$$

numerically approx. 38 hours. That's not so much if we consider that little rabbit took many times more than a day's dose. In practice, of course, liver enzymes get saturated at high concentrations and the breakdown rate stops being dependent on concentration, decreasing linearly instead.

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Problem FoL.39 ... throwing peas at the wall

Mirek was sitting on an office chair with wheels at work, looking at a wall deep in thoughts. In order to do something more useful, he took a sack of peas and started throwing them at a wall. There are $n = 3,600$ peas in the sack, each of them weighs $m_\mathrm{h} = 0.2\,\mathrm{g}$ and Mirek is throwing them with a frequency $f = 0.5\,\mathrm{s}^{-1}$. The peas are thrown horizontally with velocity $v_\mathrm{h} = 10\,\mathrm{m{\cdot}s}^{-1}$. What will Mirek's velocity be after emptying the sack? The mass of Mirek + chair is $m_\mathrm{M} = 60\,\mathrm{kg}$, the chair moves without friction. *Mirek needed to move on.*

The principle is the same as for acceleration of a rocket, so let us use the same approach as when deriving Tsiolkovsky's equation (we will consider the throwing of peas to be continuous). Let Mirek, the chair and the sack in total weigh $m(t)$ at time t and move in the x-direction, with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, $m(0) = m_\mathrm{M} + nm_\mathrm{h}$. Mirek's momentum along with the chair and sack change according to the formula

$$\frac{\mathrm{d}(m\dot{x})}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}\dot{x} + m\ddot{x} \,.$$

In order to work with the net velocity of the system, we have to add the momentum of the thrown peas

$$-\frac{\mathrm{d}m}{\mathrm{d}t}v_\mathrm{o} \,,$$

where the velocity v_o of a pea is measured in the rest frame, so $v_\mathrm{o} = \dot{x} - v_\mathrm{h}$ (we consider all velocities to be positive). The net momentum change has to be zero, so

$$0 = \frac{\mathrm{d}m}{\mathrm{d}t}\dot{x} + m\ddot{x} - \frac{\mathrm{d}m}{\mathrm{d}t}v_\mathrm{o}\,,$$

from which we get

$$m\ddot{x} = -\frac{\mathrm{d}m}{\mathrm{d}t}\left(\dot{x} - v_\mathrm{o}\right) = -v_\mathrm{h}\frac{\mathrm{d}m}{\mathrm{d}t}\,.$$

Integrating the right hand side with respect to mass gives Mirek's velocity

$$v_\mathrm{M} = v_\mathrm{h}\ln\frac{m(0)}{m(n/f)}\,,$$

where $t = n/f$ is the time it takes Mirek to empty the sack. After plugging in the values,

$$v_\mathrm{M} = v_\mathrm{h}\ln\frac{m_\mathrm{M} + nm_\mathrm{h}}{m_\mathrm{M}} \doteq 0.1193\,\mathrm{m{\cdot}s^{-1}}\,.$$

If you think it's a small number, try to compute the distance Mirek passed.

Let us also note that if we realise at the start that $m_\mathrm{M} \gg nm_\mathrm{h}$, we may linearise the problem and obtain the result in the form

$$v_\mathrm{M} = v_\mathrm{h}\frac{nm_\mathrm{h}}{m_\mathrm{M} + nm_\mathrm{h}/2}\,,$$

which is close to the Taylor series of the formula above.

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Problem FoL.40 ... wire

There's a thin, perfectly black wire with circular cross-section of radius $r = 0.3\,\mathrm{mm}$ in thermal equilibrium at room temperature ($T_0 = 20\,^\circ\mathrm{C}$) in vacuum. Under these conditions, the resistance of the wire per unit length is $R_0 = 5\,\Omega/\mathrm{m}$. If the thermal coefficient of resistance is $\alpha = 0.004\,\mathrm{K}^{-1}$, determine the current that must pass through the wire so that its temperature stabilised at $220\,^\circ\mathrm{C}$. *Filip burnt his finger.*

As the wire is in vacuum, the only way it can exchange energy with its environment is through radiative transfer. The radiated power is given by the Stefan-Boltzmann law $P = 2\pi rl\sigma\left(T^4 - T_0^4\right)$, where l is the length of the wire and σ is the Stefan-Boltzmann constant. Then, we only have to equate this power to the power delivered by electric current $P = I^2R = I^2R_0l\left(1 + \alpha(T - T_0)\right)$, from which

$$I = \sqrt{\frac{2\pi r\sigma\left(T^4 - T_0^4\right)}{R_0\left(1 + \alpha\left(T - T_0\right)\right)}}$$

and plugging in the given values, $I \doteq 0.78\,\mathrm{A}$.

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Problem FoL.41 ... hydrophobe

There's a water droplet on the horizontal part of an umbrella. The contact angle between the droplet and the umbrella is quite large, $\vartheta_c = 120^\circ$. The surface tension between water and air is $\sigma_{lg} = 73\,\mathrm{mN\cdot m^{-1}}$. We add a bit of detergent to the droplet, decreasing the surface tension to $\sigma'_{lg} = 56\,\mathrm{mN\cdot m^{-1}}$ and the contact angle to $\vartheta'_c = 90^\circ$. How much did the surface tension between the surface of the umbrella and the liquid have to change? Compute the result in units $\mathrm{mN\cdot m^{-1}}$! *Mirek was inspecting his broken umbrella.*

The forces of surface tension act between the liquid and the surface of the umbrella (surface tension σ_{sl}), between air and the umbrella (σ_{sg}) and between liquid and air (σ_{lg}). For the forces to be balanced, their vector sum has to be zero. Since σ_{sg} and σ_{sl} act in opposite directions along the surface of the umbrella and the force between air and liquid is inclined by ϑ_c from that surface, the following scalar equalities for liquid without/with the detergent hold

$$\sigma_{sg} - \sigma_{sl} - \sigma_{lg}\cos\vartheta_c = 0\,,$$
$$\sigma'_{sg} - \sigma'_{sl} - \sigma'_{lg}\cos\vartheta'_c = 0\,.$$

The surface tension between the umbrella and the air doesn't change, so $\sigma'_{sg} = \sigma_{sg}$. Subtracting the equations, we easily find the change in surface tension

$$\sigma_{sl} - \sigma'_{sl} = \sigma'_{lg}\cos\vartheta'_c - \sigma_{lg}\cos\vartheta_c \doteq 36.5\,\mathrm{mN\cdot m^{-1}}\,.$$

The surface tension between the umbrella material and the liquid decreased after adding the detergent by $36.5\,\mathrm{mN\cdot m^{-1}}$.

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Problem FoL.42 ... distributed

When describing plasma in magnetic field, it's often useful to work in cylindrical coordinates related to magnetic field lines. In these coordinates, we can model an anisotropic velocity distribution using a so-called bi-Maxwell distribution function

$$f(v_\perp, v_\parallel) = \exp\left(-\frac{\left(v_\parallel - \mu_\parallel\right)^2}{2\sigma_\parallel^2}\right)\exp\left(-\frac{(v_\perp - \mu_\perp)^2}{2\sigma_\perp^2}\right)\frac{1}{(2\pi)^{3/2}\sigma_\perp\sigma_\parallel}\,.$$

The indices $\parallel$ and $\perp$ denote components parallel to the magnetic field and perpendicular to it; the angular component has been removed by integration (due to axial symmetry). The parameters of the velocity distribution are known to be $\sigma_\parallel = 10^6\,\mathrm{m\cdot s^{-1}}$, $\sigma_\perp = 4\cdot 10^5\,\mathrm{m\cdot s^{-1}}$ and $\mu_\parallel = 4\cdot 10^4\,\mathrm{m\cdot s^{-1}}$, $\mu_\perp = 3\cdot 10^4\,\mathrm{m\cdot s^{-1}}$. Determine the mean velocity of a particle described by the distribution $f(v_\perp, v_\parallel)$. *Mirek submitted a practical problem.*

If we draw or imagine the said distribution, it's immediately clear that it's composed of two independent distributions in the axis parallel to the field and the axis perpendicular to it. In the graph with axes $v_\perp$ and $v_\parallel$, the maximum of the function $f(v_\perp, v_\parallel)$ (and, due to symmetry, the mean value as well) lies at $[\mu_\parallel, \mu_\perp]$. We obtain

$$\mu = \sqrt{\mu_\parallel^2 + \mu_\perp^2} = 5\cdot 10^4\,\mathrm{m\cdot s^{-1}}$$

simply by using the Pythagorean theorem. The mean velocity of particles is $\mu = 5 \cdot 10^4\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.43 ... merry-go-round

There's a box of mass $m = 5\,\mathrm{kg}$ and negligible size standing on a merry-go-round. The box is $r = 10\,\mathrm{m}$ far from the center of the merry-go-round; the static friction coefficient between them is $f_0 = 1$. The merry-go-round starts rotating from rest with linearly increasing angular acceleration $\varepsilon = kt$, where $k = 1\,\mathrm{s^{-3}}$. How long since the start of the rotation will it take for the box to fly away from its initial position? *Kuba wanted a problem where you need Coriolis...*

The friction force F acts in the reference frame of the merry-go-round, where its magnitude is just big enough to keep the box at rest (standing on the merry-go-round). The net force acting against friction is the vector sum of the centrifugal force $F_\mathrm{c} = mr\omega^2$ in the radial direction and the Euler force $F_\mathrm{E} = mr\varepsilon$ in the angular direction. We can thus write a force balance in the form

$$F = \sqrt{F_\mathrm{c}^2 + F_\mathrm{E}^2} = mr\sqrt{\omega^4 + \varepsilon^2} = mkr\sqrt{t^2 + \frac{k^2t^8}{16}}\,.$$

The maximum value of the friction force is

$$F \leq mgf_0\,.$$

For the critical time τ, we obtain a quartic equation

$$\frac{k^2\tau^8}{16} + \tau^2 - \left(\frac{gf_0}{kr}\right)^2 = 0\,,$$

which can be solved numerically. The equation has only one positive root, which is $\tau \doteq 0.96\,\mathrm{s}$.

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Problem FoL.44 ... twin paradox

One of two twins leaves in a rocket for a planet $s = 5\,\mathrm{ly}$ far from the Earth, while the other twin stays at their home planet. The rocket moves with velocity $v = 0.2c$. At half the distance s, the cosmonaut twin enters a fast module, which starts moving with respect to the original rocket with velocity $u = 0.1c$ towards the original goal of the rocket. How many years younger will the cosmonaut twin be compared to the twin at the home planet? Neglect any effects caused by acceleration or deceleration of the space vessels (as if they only passed by each other and gravity did not exist). In order to avoid trouble with relativity of simultaneity, compare the times spent by each twin waiting for the arrival of the fast module at the goal.

All special relativity problems are washed out.

Let's measure the distances in light years, the times in years and velocities in multiples of light speed.

The first half of the trip in the home planet's reference frame takes

$$t_1 = \frac{s}{2v}\,.$$

Since the first half of the trip is local in the rocket's reference frame, we may use the simple formula for time dilation in the rocket's reference frame

$$t_1' = \frac{s}{2v}\sqrt{1 - v^2}\,.$$

In order to be able to use the same formula[1] for the second half of the trip, we need to transform the velocity u from the rocket's reference frame to the reference frame of the home planet. We get

$$w = \frac{u + v}{1 + uv}\,.$$

Since the second half of the trip is local with respect to the module, we may write

$$t_2 = \frac{s}{2w}\,,$$

$$t_2' = \frac{s}{2w}\sqrt{1 - w^2} = \frac{s}{2}\frac{\sqrt{1 - v^2}\sqrt{1 - u^2}}{v + u}\,.$$

Now, we can express the twins' years in the form

$$\tau = \frac{s}{2}\left(\frac{1}{v} + \frac{1 + uv}{u + v}\right)\,,$$

$$\tau' = \frac{s}{2}\left(\frac{\sqrt{1 - v^2}}{v} + \frac{\sqrt{1 - v^2}\sqrt{1 - u^2}}{u + v}\right)\,.$$

The age difference between the twins is therefore

$$\Delta\tau = \frac{s}{2}\left(\frac{1 - \sqrt{1 - v^2}}{v} + \frac{1 + uv - \sqrt{1 - v^2}\sqrt{1 - u^2}}{u + v}\right) \doteq 0.63\,\mathrm{yr}\,.$$

The sought for difference in the twins' ages is 0.63 years.

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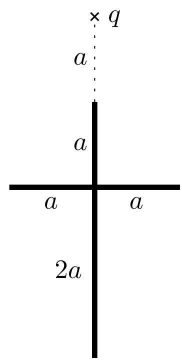
[1] The duration of the second half of the trip could be determined alternatively by viewing it from the rocket, accounting for length contraction (from the rocket to the goal planet – speed v) and then for time dilation (from the rocket to the module – length u).

Problem FoL.45 ... vampirectrical



There's a charged non-conducting cross and a point charge, placed as shown in the figure. We know $a = 20\,\mathrm{cm}$, $q = 1\,\mathrm{\mu C}$ and the linear charge density of the cross $\lambda = q/a$. What's the magnitude of the electrostatic force acting on the charge? *Mirek was defending himself from diabolical problems.*

Let's denote the direction of the shorter rod by x, the direction of the longer rod by z and the normal to the figure by y. It's clear that there's no force acting in the y-direction. The origin of the coordinate system is the point of contact of the rods.

Let's take a look at the shorter rod. Since the testing charge is placed in the middle above it, the force acting in the x-direction cancels out and we only have the z-component of the force remaining. An element of length $\mathrm{d}x$ in the point $[x, 0, 0]$ contributes to the intensity of the electric field by

$$\mathrm{d}E_{1z} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,\mathrm{d}x\cos\alpha}{(2a)^2 + x^2}\,,$$

where $\cos\alpha$ projects the intensity vector to the z-axis and

$$\cos\alpha = \frac{2a}{\sqrt{(2a)^2 + x^2}}\,.$$

Overall, E_{1z} from the shorter rod can be found by integrating

$$E_{1z} = \int_{-a}^{a}\mathrm{d}E_{1z} = \int_{-a}^{a}\frac{1}{4\pi\varepsilon_0}\frac{2\lambda a\,\mathrm{d}x}{\left(4a^2 + x^2\right)^{3/2m}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{x}{2a\sqrt{4a^2 + x^2}}\right]_{-a}^{a} = \frac{q}{4\pi\varepsilon_0 a^2}\frac{1}{\sqrt{5}}\,.$$

In the case of the vertical rod, the situation is even simpler. Due to symmetry, the only non-zero component is E_{2z} again and the element with distance z contributes to the intensity of the el. field by

$$\mathrm{d}E_{2z} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,\mathrm{d}z}{z^2}\,.$$

Integrating gives

$$E_{2z} = \int_{a}^{4a}\mathrm{d}E_{2z} = \int_{a}^{4a}\frac{1}{4\pi\varepsilon_0}\frac{\lambda\,\mathrm{d}z}{z^2} = \frac{1}{4\pi\varepsilon_0}\left[-\frac{\lambda}{z}\right]_{a}^{4a} = \frac{q}{4\pi\varepsilon_0 a^2}\frac{3}{4}\,.$$

The electrostatic force is

$$F = q(E_{1z} + E_{2z}) = \frac{q^2}{4\pi\varepsilon_0 a^2}\left(\frac{3}{4} + \frac{1}{\sqrt{5}}\right)\,.$$

After plugging in the given values, we get $F \doteq 0.27\,\mathrm{N}$.

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Problem FoL.46 ... focus

On one side of a thin biconvex lens with identical radii of curvature $R = 20\,\mathrm{cm}$ is a spherical mirror placed coaxially in such a way that a ray that passes through the lens is reflected back towards the lens. What is the (positive) focal length of such a system? The refractive index of glass is $n = 1.5$. *Damn optics again!*

A focus is a place where collinear rays gather, so we are solving a pair of thin lens equations for light from a source at $a_0 = \infty$.

A thin lens has optical power

$$\varphi = \frac{1}{f} = \frac{2}{R}(n-1)$$

and a spherical mirror has focal length

$$f' = \frac{R}{2}\,.$$

One by one, we go through the chain of images: image of the source a_0 using the lens is at a_1, then its image using the mirror is at a_2 and then the image from a_2 again using the lens is at a_3,

$$\begin{aligned}\varphi &= \frac{1}{a_1}\,,\\ \frac{2}{R} &= -\frac{1}{a_1} + \frac{1}{a_2}\,,\\ \varphi &= -\frac{1}{a_2} + \frac{1}{a_3}\,,\end{aligned}$$

where we used the following sign convention: an object before the lens and an image behind the lens have positive positions, and both an object and an image before a mirror (real) have positive positions.

Solving this system of equations, we obtain the focal length of the whole system in the form of the position of the final image a_3

$$f = a_3 = \frac{R}{4n-2} = 5\,\mathrm{cm}\,.$$

Focal length of the whole system is $5\,\mathrm{cm}$.

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Problem FoL.47 ... Sisyphean labour

Sisyphos is standing at the bottom of a hemispherical hole of radius $R = 100\,\mathrm{m}$. His task is to push an approximately spherical, bumpy rock of mass $M = 1\,000\,\mathrm{kg}$ and radius $r = 50\,\mathrm{cm}$. The effective rolling friction coefficient of the rock is $\xi = 4\,\mathrm{cm}$. Sisyphos is pushing the rock up with negligible constant velocity while exerting a force F_s, horizontally towards the center of mass of the rock. Sisyphos runs out of stamina after performing work $W_\mathrm{s} = 200\,\mathrm{kJ}$. By how much will he lift the center of mass of the rock before it happens? Neglect changes in potential energy of Sisyphos, the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek tried to resolve all unread e-mails.

Let us view the situation in the plane in which Sisyphos pushes the rock. The rolling friction force, acting at the point of contact of the rock and the earth against the direction of the rock's movement, can be expressed as

$$F_\mathrm{o} = F_\mathrm{n} \frac{\xi}{r} \,,$$

where F_n is the normal force, pushing the rock down towards the ground. It is composed of gravity and Sisyphos's force, so

$$F_\mathrm{n} = F_g \cos\alpha + F_\mathrm{s} \sin\alpha \,,$$

where α is the angle between the direction of the rock's movement and the vertical plane, and $F_g = Mg$. Using the angle α, we can also express the change in potential energy of the rock

$$\Delta E_\mathrm{p} = Mg(R-r)(1-\cos\alpha) \approx MgR(1-\cos\alpha)\,.$$

The total work performed by Sisyphos is

$$W_\mathrm{s} = W_\mathrm{o} + Mg(R-r)(1-\cos\alpha)\,,$$

where W_o is the work performed by friction. In order to determine the magnitude of the friction force, we will use the scalar force balance

$$F_\mathrm{s}\cos\alpha = F_g \sin\alpha + F_\mathrm{o}\,,$$

which gives

$$F_\mathrm{o} = \frac{\frac{\xi}{r} Mg}{\cos\alpha - \frac{\xi}{r}\sin\alpha}\,.$$

after plugging into the equation for rolling resistance. Now, we will assume $\cos\alpha \gg \xi/r \sin\alpha$ (which we will verify later), so we may write

$$F_\mathrm{o} \approx \frac{\xi}{r} Mg \frac{1}{\cos\alpha}\,.$$

The work of the rolling friction then is

$$W_\mathrm{o} \approx \frac{\xi}{r} RMg \int_0^\alpha \frac{\mathrm{d}\varphi}{\cos\varphi} = \frac{\xi}{r} RMg \ln\left(\mathrm{tg}\,\alpha + 1/\cos\alpha\right)\,.$$

The formula for the angle α, at which Sisyphos runs out of stamina, can then be written in an approximate form ($r \ll R$)

$$\frac{W_\mathrm{s}}{MgR} \approx \frac{\xi}{r} \ln\left(\mathrm{tg}\,\alpha + 1/\cos\alpha\right) + 1 - \cos\alpha\,.$$

Now, all that remains is to numerically determine the physically meaningful root $\alpha \doteq 0.566$ and (without any approximations now) compute

$$\Delta y = (R-r)(1-\cos\alpha) \doteq 15.5\,\mathrm{m}\,.$$

Sisyfos will roll the rock up to the height $\Delta y \doteq 15.5\,\mathrm{m}$. We may verify that the approximation $\cos\alpha \gg \xi/r \sin\alpha$ was good, because $0.84 \gg 0.04$. A more accurate computation (integrating without approximations) gives the result $\Delta y \doteq 14.4\,\mathrm{m}$, which is also accepted, of course, but it cannot be written in a nice form. The result $\Delta y \doteq 20\,\mathrm{m}$, obtained by neglecting friction, was not accepted.

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Problem FoL.48 ... spin it

There is a bicycle on a bicycle stand. We know that its rear wheel's moment of inertia around its axis is $I = 0.5\,\mathrm{kg{\cdot}m^2}$ and its radius is $r = 30\,\mathrm{cm}$. We speed up the wheel by repeatedly hitting it with a hand tangentially on its circumference in the direction of its rotation. The speed of each hit in the rest reference frame is $w = 5\,\mathrm{m{\cdot}s^{-1}}$, the mass of the hand is $m = 1.5\,\mathrm{kg}$, the hand loses all its velocity in the reference frame of the point of impact after the impact *(the hand is moving around an elbow, but we neglect rotation of the forearm). What will the velocity of a point on the perimeter of the wheel after 10 hits be?*

Mirek tampered with something again.

The hand is a point mass which has a moment of inertia with respect to the center of the wheel mr^2 at the time of each impact. The angular momentum transferred from the hand to the wheel in the n-th hit (after the hit, the hand is at rest with respect to the point of impact) is $mr(w - v_n)$, where v_n is the velocity of the point of impact after the n-th hit. The angular momentum of the wheel after the n-th hit is therefore

$$L_n = mr(w - v_n) + L_{n-1}\,.$$

That gives a recurrent formula for the velocities of the point of impact based on the formula $L_n = I\omega_n$

$$Av_n = w - v_n + Av_{n-1}\,,$$

where we used the substitution $A = I/(mr^2)$.

Now, we will try to compute the first few terms of the progression. We get

$$v_1 = \frac{w}{1+A}\,, \quad v_2 = \frac{w}{1+A}\left(1 + \frac{A}{1+A}\right)\,, \quad v_3 = \frac{w}{1+A}\left(1 + \frac{A}{1+A} + \left(\frac{A}{1+A}\right)^2\right)\,, \quad \dots$$

The formula for v_n will clearly be

$$v_n = \frac{w}{1+A}\left(1 + \frac{A}{1+A} + \dots + \left(\frac{A}{1+A}\right)^{n-1}\right)\,.$$

Summing up a geometric series and simplifying, we obtain

$$v_n = w\left(1 - \left(\frac{A}{1+A}\right)^n\right)\,.$$

The velocity is $v_{10} \doteq 4.54\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.49 ... all day long

Imagine that the observable universe is spherical with diameter $d = 10^{27}$ m and filled with stars placed in the lattice points of a cubic grid with side length of the unit cell $a = 10^{19}$ m. The stars are identical and their radiation is isotropic with luminosity $L = 10^{27}$ W and radius $r = 10^9$ m. The stars don't block other stars' light; there is no absorption. What's the incident power per square metre near the centre of the universe in the middle of a body diagonal of a unit cell? Compute the base-10 logarithm of this value! *Mirek was trying to see through a dense forest.*

A star's radiance at its surface is $\mathcal{L} = L/(4\pi r^2)$. The observer's distance from the stars in the corners of his unit cell is $a\sqrt{3}/2$. When looking at higher layers, i.e. cubes with side lengths $3a$, $5a$, $7a$ etc., centered at the observer's location, we may notice that the number of stars on each layer increases approx. quadratically with distance, since the number of stars is proportional to the area of the layer. At the same time, we know that a star's radiance decreases quadratically with distance, generally

$$L(r') = \frac{r^2}{r'^2} L(r)\,.$$

Introducing a density of stars $n = 1/a^3$, we're able to express the incident power as an integral

$$\mathcal{L}_{\mathrm{tot}} = \int_0^{d/2} \int_0^{2\pi} \int_0^{\pi} \mathcal{L} \frac{r^2}{\varrho^2} n \sin\vartheta \varrho^2 \,\mathrm{d}\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\varrho = 4\pi \mathcal{L} r^2 \frac{d}{2} = \frac{d}{2a^3} L \doteq 5 \cdot 10^{-4}\,\mathrm{W{\cdot}m^{-2}}\,.$$

This result, however, could have a large error due to the contribution from nearby stars being inaccurately approximated – but we see that even the closest stars are far enough. The contribution from the nearest eight stars is

$$\mathcal{L}_8 = 8 \left(\frac{r}{a\sqrt{3}/2} \right)^2 \mathcal{L} \doteq 8 \cdot 10^{-12}\,\mathrm{W{\cdot}m^{-2}}\,,$$

which is completely negligible compared to $\mathcal{L}_{\mathrm{tot}}$ (we just need to realise that if the number of stars in a layer increases quadratically with the layer's dimensions and the radiance decreases quadratically as well, each layer must contribute approximately equally. It's possible to verify numerically that the approximation is accurate enough.[2]

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Problem FoL.50 ... comfy by the pond

It's summer and we're resting on a bank of a deep pond. The sun is shining, the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$, the surface tension of water is $\sigma = 72.8\,\mathrm{mN{\cdot}m^{-1}}$, water density is $\varrho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$. At a distance $d = 30.0$ m from the bank, a backswimmer appears and creates circular waves on the surface. When we focus on one wave, we can see that it's moving with speed $c = 1.00\,\mathrm{m{\cdot}s^{-1}}$. How long will it take for the wavefront of the backswimmer's waves to reach the bank? You may use the formula for phase velocity of deep-water waves $c = \sqrt{\frac{g}{k} + \frac{\sigma k}{\varrho}}$, where k is the wave number. *Mirek confused gravity waves for gravitational waves.*

[2] For the given values, it's not easy to perform the computation in reasonable time; it's better to test this assumption e.g. for $d = 10^3 a$.

The phase velocity is defined as the ratio of angular frequency and wave number,

$$c = \frac{\omega}{k}\,.$$

From this formula, we can express

$$\omega(k) = \sqrt{gk + \frac{\sigma k^3}{\varrho}}\,.$$

In order to determine how long it takes for the wavefront to reach us, we need to know the group velocity

$$c_\mathrm{g} = \frac{\partial \omega}{\partial k} = \frac{g + \frac{3\sigma k^2}{\varrho}}{2\sqrt{gk + \frac{\sigma k^3}{\varrho}}}\,.$$

The wave number k can be determined from the formula for phase velocity

$$c^2 = \frac{g}{k} + \frac{\sigma k}{\varrho}\,,$$

$$0 = k^2 - \frac{\varrho c^2}{\sigma} k + \frac{\varrho g}{\sigma}\,.$$

There's no need to substitute for k generally in the formula for group velocity. It's sufficient to solve the quadratic equation and substitute the right root (the one with minus – for small σ, k shouldn't approach infinity) into the expression for group velocity numerically. The group velocity can be plugged into the formula for the sought for time $t = d/c_\mathrm{g}$ to get

$$t = d\left(c - \frac{c}{2}\sqrt{1 - \frac{4\sigma g}{\varrho c^4}}\right)^{-1} \doteq 59.9\,\mathrm{s}\,.$$

Note that this result differs from the value $t = 60\,\mathrm{s}$, obtained by neglecting surface tension. In addition, it's very likely that the waves in a real pond would be damped before reaching the bank.

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Problem FoL.51 ... toast

We'd like to make a toast using a very strange glass. It's a hollow cube of inner volume $V = 1\,\mathrm{l}$, which is standing on one corner (a body diagonal is vertical). Through a small opening near the top vertex, we fill the cube with $V' = 1/3\,\mathrm{l}$ of wine. How high above the ground (i.e. the vertex the cube is standing on) will the wine surface be? Neglect the thickness of the cube's sides. Compute the result as a multiple of the cube's edge length!

Mirek remembered stereometry and had mixed feelings.

In order to solve the problem, it will be useful to first derive some formulae in a regular three-sided pyramid with right angles between pairs of edges at the main vertex. Let an edge of the base have length b, an edge at the main vertex have length a and its height be v. The relation between b and a is found easily from the Pythagorean theorem on one side of the pyramid to be

$$b(a) = \sqrt{2}\,a\,.$$

Next, we'll use the height of the base v_b, which is again found using the Pythagorean theorem for the triangle (base vertex)-(base vertex)-(foot of a height of the base) to be

$$v_\mathrm{b} = \frac{\sqrt{3}}{2}b\,.$$

The next is an expression for v using a and b. Let's denote the distance of a base vertex from the barycenter of the base by s. The barycenter splits each median into two parts with length ratio $2:1$. In addition, for a equilateral triangle (like our base), medians and heights coincide, so the barycenter and orthocenter also coincide. The distance s is

$$s = \frac{2}{3}v_\mathrm{b} = \frac{\sqrt{3}}{3}b\,.$$

Let's realise that in our pyramid, the foot of the height coincides with the base's orthocenter. Now, if we consider the Pythagorean theorem for the triangle (base vertex)-(main vertex)-(orthocenter), we get a formula for the height v of the pyramid:

$$v = \sqrt{a^2 - s^2} = \sqrt{\frac{1}{2}b^2 - \frac{1}{3}b^2} = \frac{\sqrt{6}}{6}b\,.$$

We'll also need the area S of the base, which is

$$S = \frac{1}{2}bv_\mathrm{b} = \frac{\sqrt{3}}{4}b\,.$$

Now, we can finally compute the volume V of the pyramid:

$$V = \frac{1}{3}vS = \frac{1}{3}\frac{\sqrt{3}}{4}\frac{\sqrt{6}}{6}b^3 = \frac{\sqrt{2}}{24}b^3\,,$$

but an expression using a will be more useful here:

$$V = \frac{1}{6}a^3$$

…and in v, it's

$$V(v) = \frac{\sqrt{3}}{2}v^3\,.$$

Let's return to our problem. If the volume of the wine inside the cube was at most $16\,\mathrm{l}^3 1\,\mathrm{l} = a^3$ the wine would be shaped like the above described pyramid, so we'd be pretty much done. The same goes for volume greater than $56\,\mathrm{l}$, where it's sufficient to compute the height of the empty part and subtract it from the body diagonal length $v_\mathrm{T} = \sqrt{3}\,a$. The critical volume is $16\,\mathrm{l}$ exactly because that's the volume of a pyramid obtained if we cut off the cube at its three vertices with smallest non-zero height.

[3] ,

Unfortunately, our volume really is in the range $\left\langle \frac{1}{6}, \frac{5}{6} \right\rangle$. Let's extend the edges of the cube from the corner on the ground upwards. Now, let's consider the plane parallel to the ground and at distance v from the ground, where $\frac{\sqrt{3}}{3}a \leq v \leq \frac{2\sqrt{3}}{3}a$, i.e. at other heights than in the cases described above. The corner on the ground and the intersection points of the extended edges are vertices of a regular three-sided pyramid, whose edges meet at right angles at the main vertex and whose height is v. However, this pyramid reaches out of the cube. Let's consider the three vertices of the cube that lie at the same height below the plane, i.e. just those three vertices which lie on one extended edge each. Let's consider the intersection points of edges of the cube that lead upwards from these vertices (6 points in total), in pairs respectively for each vertex. Each of these three points, together with its two intersection points and the intersection point of the extended edge with the plane, forms a pyramid. This pyramid has the same shape as above, its main vertex is the corresponding vertex of the cube and its height is $v - \frac{\sqrt{3}}{3}a$, because $\frac{\sqrt{3}}{3}a$ is the height of this corner of the cube above the ground. Those three pyramids fill the whole volume of the original large pyramid that extends beyond the cube. If we want to find the volume that's filled with the surface of wine at height v, it will be

$$V_{\mathrm{clk}}(v) = V(v) - 3V\left(v - \frac{\sqrt{3}}{3}a\right).$$

If we perform homogenisation $w = v/a$ and substitute for $V(v)$, we get

$$V_{\mathrm{clk}}(w) = \frac{\sqrt{3}}{2}\left(w^3 - 3\left(w - \frac{\sqrt{3}}{3}\right)^3\right).$$

You can test that by substituting $w = \frac{\sqrt{3}}{2}$, i.e. half the body diagonal, we get $V_{\mathrm{clk}} = \frac{1}{2}$, and by substituting critical heights $\frac{\sqrt{3}}{3}$ and $\frac{2\sqrt{3}}{3}$, we get the correct critical volumes $\frac{1}{6}$ and $\frac{5}{6}$, respectively. Now, we just need to set $V_{\mathrm{clk}} = \frac{1}{3}$ and solve the equation. This is a cubic equation with an analytical solution that can be expressed through the Cardano formulae, but in the contest, we only need the numerical result, which can be found efficiently e.g. using the service Wolfram Alpha; we need to pick the right root – the one that lies between the critical heights $\frac{\sqrt{3}}{3}$ and $\frac{2\sqrt{3}}{3}$. Numerically, this height is at 0.7347 of the cube's edge length.

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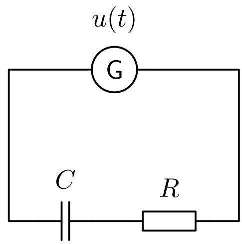
Problem FoL.52 ... saw

We connect a tone generator producing a sawtooth wave voltage $u(t)$ into a series RC circuit. The resistor in the circuit has resistance $R = 200\,\Omega$, the capacitor has capacity $C = 50\,\mu\mathrm{F}$, the maximum voltage of the source is $U = 5\,\mathrm{V}$. For the period of voltage $T = RC$, calculate the current passing through the circuit when the voltage of the source is at a maximum (specifically, instantly after the source voltage reaches the maximum) a long time after the generator is switched on.

Kuba thought there weren't any decent circuit problems yet.



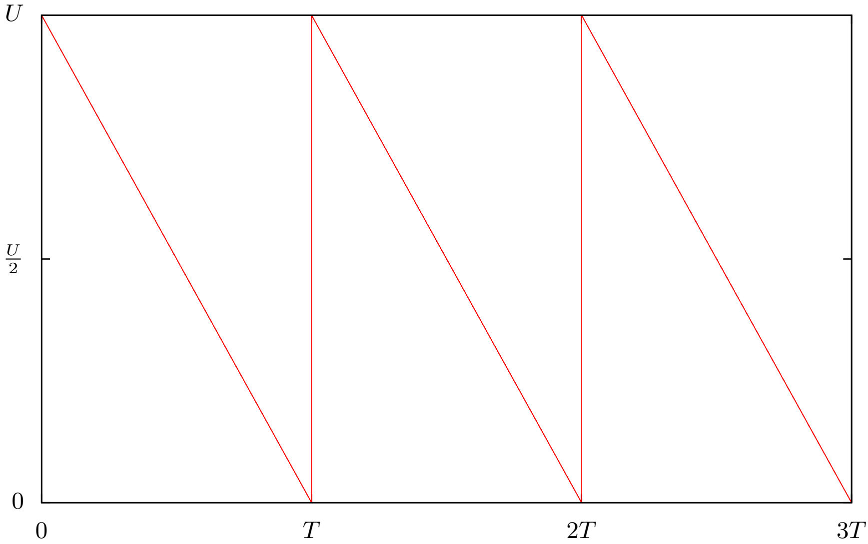


Fig. 2: Sawtooth wave voltage.

During one period, the voltage is

$$u(t) = U\left(1 - \frac{t}{T}\right)$$

and at the end, it discontinuously jumps back to $u(T) = U$.

Let's write the 2nd Kirchhoff law for this circuit during one period,

$$u(t) = RI(t) + \frac{q(t)}{C}\,, \tag{1}$$

where $I(t)$ is the current passing through the circuit at a time $t \in (0, T)$ and $q(t)$ is the charge of the capacitor. The polarity of this change is chosen so that it fits the equation (1) and we have

$$\dot{q}(t) = I(t)\,.$$

Differentiating (1) and substituting for $\dot{u}(t)$ and $\dot{q}(t)$, we obtain a differential equation

$$-\frac{U}{T} = R\dot{I}(t) + \frac{I(t)}{C}\,,$$

the solution of which is the current evolution in the form

$$I(t) = I_0 \exp\left(-\frac{t}{RC}\right) - \frac{CU}{T}\,, \tag{2}$$

with an unknown constant I_0, which we need to express.

Since the circuit is in stationary (periodically changing) state, the charge of the capacitor has to remain the same at the beginning and end of each cycle. The only difficulty is in thinking through what happens to the charge when voltage changes discontinuously. The change in charge, expressed as a current integral, will be zero in that (infinitely short) time interval, since the change in the current will be finite. Therefore, the net change in charge during the phase when voltage decreases has to be zero as well.

We may express the charge from (1), where we substitute for $I(t)$. We get

$$q(t) = CU\left(1 - \frac{t}{T}\right) + \frac{RC^2U}{T} - RCI_0 \exp\left(-\frac{t}{RC}\right) .$$

The zero change in charge between the beginning and end of the cycle is

$$0 = \Delta q = q(T) - q(0) = -CU + RCI_0\left[1 - \exp\left(-\frac{T}{RC}\right)\right] .$$

From that, we get

$$I_0 = \frac{U}{R\left[1 - \exp\left(-\frac{T}{RC}\right)\right]} .$$

Now, we may express the required current – in the equation (2), we substitute for I_0, set $t = 0$ and substitute the period $T = RC$ from the problem statement.

$$I(0) = \frac{U}{R\left[1 - \exp\left(-\frac{T}{RC}\right)\right]} - \frac{CU}{T} = \frac{U}{R}\frac{1}{\mathrm{e} - 1} \doteq 14.5\,\mathrm{mA} .$$

The current passing through the circuit in the given moments is $14.5\,\mathrm{mA}$.

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Problem FoL.53 ... you shall not transmit!

Consider a simple model of a static one-dimensional atmosphere, denoting its relevant coordinate (elevation) by h. If we took a thin slice of this atmosphere with thickness Δh, its reflectance would be $r(h)\Delta h$. What fraction of incident light is transmitted by the atmosphere if $r(h) = r_0\mathrm{e}^{-kh}$, where $k = r_0 = 0.1\,\mathrm{m}^{-1}$? *Kuba, an old organiser, decided to make a problem.*

First of all, let's consider an atmosphere of finite length L, with radiant flux I_0 perpendicularly incident on it from one side. Let's denote the coordinate normal to the atmospheric layer by x, so $x = 0$ on the side where the flux I_0 is incident and $x = L$ on the other side. Inside a layer, there are photons moving in both directions. Let's denote by $I_+(x)$ the flux in the direction of increasing x and by I_- the flux in the other direction. Then, it's clear that $I_+(0) = I_0$, $I_-(0) = RI_0$, $I_+(L) = TI_0$ and $I_-(L) = 0$, where R is the reflectance and T the transmittance of the atmosphere.

Considering a thin layer of thickness Δx within this atmosphere, we can write

$$\begin{aligned} I_+(x + \Delta x) &= \big(1 - r(x)\Delta x\big)I_+(x) + r(x)\Delta x I_-(x + \Delta x) , \\ I_-(x) &= \big(1 - r(x)\Delta x\big)I_-(x + \Delta x) + r(x)\Delta x I_+(x) , \end{aligned}$$

from which we express $I_+(x + \Delta x)$ and $I_-(x + \Delta x)$,

$$\begin{aligned} I_+(x + \Delta x)\big(1 - r(x)\Delta x\big) &= \big(1 - 2r(x)\Delta x\big)I_+(x) + r(x)\Delta x I_-(x) , \\ I_-(x + \Delta x)\big(1 - r(x)\Delta x\big) &= I_-(x) - r(x)\Delta x I_+(x) . \end{aligned}$$

If we expand this to the first order in Δx, we get in the limit $\Delta x \to 0$

$$\frac{\mathrm{d}I_+}{\mathrm{d}x} = r(x)(I_- - I_+)\,,$$
$$\frac{\mathrm{d}I_-}{\mathrm{d}x} = r(x)(I_- - I_+)\,.$$

This means $\mathrm{d}(I_- - I_+)/\mathrm{d}x = 0$, so $I_-(x) - I_+(x) = \mathrm{const} = -I_+(L) = -I_0T$ and after substituting in the previous formula, $\mathrm{d}I_+/\mathrm{d}x = r(x)(I_- - I_+) = -r(x)I_0T$, from which we may derive

$$-I_0T\int_0^L r(x)\,\mathrm{d}x = I_+(L) - I_+(0) = (T-1)I_0\,.$$

From this, we can compute

$$T = \left(1 + \int_0^L r(x)\,\mathrm{d}x\right)^{-1}\,.$$

In our case, $r(x) = r_0\mathrm{e}^{-k(L-x)}$ and $L \to \infty$, so

$$T_{\mathrm{atm}} = \lim_{L\to\infty}\left(1 + r_0\mathrm{e}^{-kL}\int_0^L \mathrm{e}^{kx}\,\mathrm{d}x\right)^{-1} = \frac{k}{r_0+k} = \frac{1}{2}\,.$$

The atmosphere will transmit 50 % of incident light.

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Problem FoL.54 ... transit

Consider a system in which infinitely many discrete states exist, indexed by an index $i \in \mathbb{N}$ (positive integers). The state of the whole system can be described by a state vector $\boldsymbol{f} = (f_1, f_2, \dots)$, where f_i denotes the number of objects in state i. The time evolution of the system occurs in discrete steps. The probability of transition of an object from a state i to the state $i+1$ in one step is $P_{i,i+1} = 1/i^2$, the probability of transition from a state i to the state $i-1$ is $P_{i,i-1} = 1 - P_{i,i+1}$. It follows from basic properties of probability that transitions between non-adjacent states or remaining in the same state isn't allowed. Initially, all objects are in state f_1. Find the stationary state the system converges to and compute the number of objects in the first state as a multiple of the total number of objects in the system N. Assume that N is a very large number. The total number of objects remains constant during the time evolution.

Mirek was transitioning from the dorm to school.

The time evolution of the system can be described by a recurrence formula

$$\boldsymbol{f}_{t+1} - \boldsymbol{f}_t = A\boldsymbol{f}_t$$

which actually represents an infinite system of linear difference equations with coefficients determined by the matrix A. That matrix looks like this:

$$A = \begin{pmatrix} -1 & P_{21} & 0 & 0 & \cdots \\ P_{12} & -1 & P_{32} & 0 & \cdots \\ 0 & P_{23} & -1 & P_{43} & \cdots \\ 0 & 0 & P_{34} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} -1 & 3/4 & 0 & 0 & \cdots \\ 1 & -1 & 8/9 & 0 & \cdots \\ 0 & 1/4 & -1 & 15/16 & \cdots \\ 0 & 0 & 1/9 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can express the solution of this equation as a matrix exponential, but expressing it explicitly would require solving an infinite-dimensional eigenvalue problem. We can avoid that by using the simple form of the matrix and "seeing" the solution.

Let us denote the stable vector $\boldsymbol{f}^0$. W.l.o.g., let's take its first component to be $f_1^0 = 1$ (we can re-normalise it later). For this vector to remain constant when acted on by the matrix A, the number of objects to move from the second state to the first one must be 1, since objects can't move to the first state from any other. Thus, we necessarily have $f_2^0 = 4/3$, because $P_{21} = 3/4$. At the same time, we know that the number of objects that move to the second state from the first one is $P_{12} f_1^0 = 1$, so the number of objects that have to move from the third state to the second one is $4/3 - 1 = 1/3$. We know P_{32}, so it's easy to compute $f_3^0 = 3/8$. Using this method, we can compute the other components

$$f_4^0 = 2/45\,,\; f_5^0 = 5/1728\,,\; f_6^0 = 1/8400\,,\; \dots\,.$$

Generally, the component f_n^0 has to represent $1/(1-1/n^2)$ of the term that represents $1/(n-1)^2$ of the previous component f_{n-1}^0 (this can be derived by comparing the number of objects that pass in each direction between states n and $n-1$). We obtain a recurrence formula

$$f_n^0 = \frac{n^2}{n^2-1}\frac{1}{(n-1)^2} f_{n-1}^0\,.$$

This formula can be written using just the first element of the stable state vector $f_1^0 = 1$ (for $n > 1$)

$$f_n^0 = n^2 \prod_{k=0}^{n-2} \frac{1}{(n-k)^2 - 1}\,.$$

The total number of objects in the system can be expressed as

$$N = \sum_{n=0}^{\infty} f_n^0 = 1 + \sum_{n=0}^{\infty} n^2 \prod_{k=0}^{n-2} \frac{1}{(n-k)^2-1}\,.$$

Since this series converges quickly, it suffices to sum up the first few terms to get a fairly accurate result. Computer algebra systems tell us that the exact result is $N = 4 I_2(2) f_1^0 \doteq 2.756$, where I_j is the modified Bessel function of the j-th order. The problem asks for the inverse of this number, which is $1/(4 I_2(2)) \doteq 0.3629$.

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Problem FoL.55 . . . megamagnet

There's a cylindrical coil of length $l = 100\,\mathrm{km}$ and base diameter $d = 1\,\mathrm{km}$ with $N = 10^6$ turns. There's a current $I = 1\,\mathrm{kA}$ passing through the coil. Determine the electric field felt when flying out of the coil parallel to its axis at a distance $r = 1\,\mathrm{m}$ from that axis with velocity $v = 1{,}000\,\mathrm{km{\cdot}s^{-1}}$. Neglect any effects of general relativity. *Xellos was watching Red Dwarf.*

Since $v \ll c$, we can neglect special relativity as well. We can also neglect Maxwell's current in the Ampere law and assume that the magnetic field sensed when moving is the same as at rest (if you don't agree, you can use the relativistic transform of the EM field).

The problem can now be solved using Maxwell's equations – specifically, the electric field has to satisfy Gauss's law div $\boldsymbol{E} = 0$ (there are no free charges) and Faraday's law of induction

$$\operatorname{rot} \boldsymbol{E} = -\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = -v\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}z}$$

where we replaced the time changes of mag. field by movement along the axis. The magnetic field inside a long coil is almost parallel to the axis; at the end of the coil, it's starting to diverge from the axis, but only negligibly for $r \ll d$, so let's say that $\boldsymbol{B} = (0, 0, B_z(z))$ (we work in cylindrical coordinates (r, φ, z)).

The described system of equations is still too complex, especially since the el. field has 3 unknown components whose derivatives mix quite a lot. However, nothing stops us from guessing $E_z = 0$ (indeed, the field in the direction parallel to $\boldsymbol{v}$ doesn't change at all in the exact relativistic transform). Now, axial symmetry + Gauss's law give $E_r = 0$, since both E_φ and E_r only depend on r. Faraday's law simplifies (in the integral form) to $2\pi r E_\varphi = v\frac{\mathrm{d}B_z}{\mathrm{d}z}\pi r^2$ – imagine a coil with radius r sliding on the axis of the coil, then the left hand side gives the induced voltage in the coil and the right hand side the time change of mag. flux.

We still have to find $\frac{\mathrm{d}B_z}{\mathrm{d}z}$ near the axis at one end of the coil. "Near the axis" is important, because we know that the field near the axis for one coil centered at $(r = z = 0)$ with current I is given by Biot-Savart's law:

$$B_z(z) = \frac{\mu_0 I (d/2)^2}{2(z^2 + d^2/4)^{3/2}} \, .$$

We can imagine the coil to be infinite at the other end – it should have negligible effect on the mag. field – and imagine it to be composed of $\frac{xN}{l}$ thin current loops per x metres of length. The mag. field can be expressed as an integral over these loops:

$$B_z = \int_0^\infty \frac{\mu_0 I d^2}{(4z^2 + d^2)^{3/2}} \frac{N}{l} \mathrm{d}z \, .$$

We don't have to solve the integral (but we could, it gives $\mu_0 NI/2l$), since we're looking for its derivative by z – and that's simply equal to the integrand for $z = 0$. The resulting el. field is

$$E = \frac{\mu_0 N I v r}{2ld} \doteq 6.3\,\mathrm{V{\cdot}m^{-1}} \, .$$

We can see that E is proportional to both v and r, which fits the expectations that $E = 0$ when $v = 0$ or $r = 0$.

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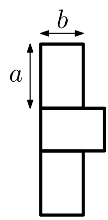
Problem M.1 ... tower builder

We have glued together a tower from three identical blocks with edge lengths a, b (see the picture). Find the largest possible ratio a/b for which the tower would still be stable. *Náry and Lego.*



The condition for stability is that the centre of mass must lie above the base of the tower. To maximise a for given b, we want to put the centre of mass to its rightmost position – and that is b because there the base of the lower block ends.

Since the centre of mass of the lower and the upper block is at $b/2$ from the left edge, physics of the lever tells us that the center of mass of the middle block must be at $2b$. So the length of the block turns out to be $4b$ and the ratio in the question is $4:1$.

For completeness, the equation for the lever system is

$$2Mg\left(x-\frac{b}{2}\right)=3Mg(x-b)\,,$$

where Mg is the weight of one block and x is the distance of middle block's center of mass from the left edge.

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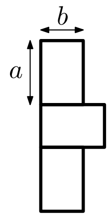
Problem M.2 ... tower builder returns

We have built a tower from three identical blocks with edge lengths a, b. Find the largest possible ratio a/b for which the whole tower would not collapse immediately.
Náry and Lego.



We have to be more careful than in the previous task because we cannot use the argument about the center of gravity and the base of the tower which said that the stability is ensured if and only if the center is above the base. The blocks here can affect each other only by pressure, they can not pull each other, some of the forces and moments of force between them may not add to zero.

We shall inspect stability for each of the blocks separately. Stability of the bottom block is ensured automatically because its pressure to the ground is always compensated by the ground. The uppermost block is influenced by gravitation and pressure from the middle block, which is affected by both bottom and uppermost blocks together with gravitation. Let Mg denote the gravitational force acting on the uppermost block. The forces and torques (with respect to any point) must add to zero. The action and reaction principle says that a force with the same magnitude acts on the middle block at a point $b/2$ from its left side, where b is the width of the block.

Moreover, the middle block creates a force on the bottom one. This force is equal to the sum of gravitational forces of the uppermost and middle blocks, which are held up by the bottom one. Hence, the force is $2Mg$. In order to maximise the length of the edge a, we must maximise the distance from the center of gravity of the middle block to the place where the uppermost block acts. This is ensured if the bottom block acts on the one above it at its edge at distance b from the left side. We can use the law of the lever which leads us to conclude that the center of gravity of the middle block is $3b/2$ from its left side. Hence, the full length of the block is $3b$. The ratio is $3:1$.

For completeness, the condition for the lever (the law of the lever) has the form

$$2Mg(x-b)=Mg\left(x-\frac{b}{2}\right)$$

where x is the distance from the center of gravity of the middle block to its left side.

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Problem M.3 ... collision imminent

Calculate the lowest possible energy loss resulting from a perfectly inelastic central collision of two rigid balls with masses $m = 3\,\mathrm{kg}$ and $M = 2\,\mathrm{kg}$. The speeds before the collision are $5\,\mathrm{m\cdot s^{-1}}$ and $6\,\mathrm{m\cdot s^{-1}}$ (with respect to a static observer). *Náry, life experience.*

First we should realize that the task of minimizing the energy loss is equivalent to the task of maximizing the speed after the collision. Clearly, this can be achieved by sending both balls in the same direction.

Because the change of total kinetic energy with time is invariant under Galilean transform, it doesn't matter whether the heavier ball is the faster one or the slower one. The invariance will become obvious after the following calculation.

Let's denote v_1 the speed of the ball with mass m, v_2 the speed of the ball with mass M and w the final velocity. Conservation law for linear momentum implies

$$mv_1 + Mv_2 = (m + M)w\,,$$

and thus

$$w = \frac{mv_1 + Mv_2}{m + M}\,.$$

The decrease in total kinetic energy ΔE_k is given as the difference between total kinetic energy before and after the collision, that is

$$\frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 = \Delta E_\mathrm{k} + \frac{1}{2}(m + M)\left(\frac{mv_1 + Mv_2}{m + M}\right)^2\,.$$

Isolating ΔE_k we get

$$\Delta E_\mathrm{k} = \frac{1}{2}\frac{mM}{m + M}(v_1 - v_2)^2\,.$$

Now we can see that the energy loss is equal to the square of the velocity of one ball relative to the other. Therefore we can choose arbitrarily $v_1 = 5\,\mathrm{m\cdot s^{-1}}$, $v_2 = 6\,\mathrm{m\cdot s^{-1}}$. For given values of speeds and masses we get $\Delta E_\mathrm{k} \doteq 0.6\,\mathrm{J}$.

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Problem M.4 ... stick like no other

Rigid homogeneous stick of length $l = 1\,\mathrm{m}$ has been thrown upwards in a uniform gravitational field. During its flight, there was a moment when the translational kinetic energy equaled the rotational kinetic energy. We know that in that precise moment the speed of the stick's center of mass was $9\,\mathrm{m\cdot s^{-1}}$. Find the instantaneous angular velocity of the stick upon its release.

Everybody once met a rotating stick.

Since we are in a uniform gravitational field, the angular velocity, and by extension the rotational kinetic energy, are constant. When both components of the kinetic energy become equal, we can write

$$\frac{1}{2}mv^2 = \frac{1}{2}J\omega^2\,.$$

Notation: m is the mass of the stick, v is the translational velocity, J is the moment of inertia $ml^2/12$, l is the length of the stick, ω is the angular velocity. Through simple algebraic manipulation we obtain

$$\omega = \frac{2\sqrt{3}\,v}{l}\,.$$

Now we plug in the numbers and get $\omega \doteq 31.2\,\mathrm{rad{\cdot}s^{-1}}$.

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Problem E.1 ... charged Fykosák

Electrically neutral Fykos bird has been endowed with 10^{12} electrons. It then flies into a homogeneous electric field with speed $v_0 = 3.6\,\mathrm{km{\cdot}h^{-1}}$ in a direction opposite to the direction of the field lines. Electric intensity of the field is $500\,\mathrm{V{\cdot}m^{-1}}$. What will be the speed v of the Fykos bird after $100\,\mathrm{m}$ long flight in the electric field? Consider a point-like bird with mass $m = 0.5\,\mathrm{kg}$. *Faleš has been watching birds at a bus stop...*

Because of the charge of the bird, an electric force acts on it. The force is $F = qE$ and it will accelerate the bird because its charge is negative and therefore the force points in the direction opposite to the field.

The force does work (s is the length of trajectory of the bird in the field)

$$W = Fs\,,$$

which can be used in the conservation of energy

$$E_{\mathrm{ki}} = E_{\mathrm{kf}} + W_{\mathrm{e}}\,.$$

Here E_{ki} is the initial and E_{kf} is the final kinetic energy, which are given as follows

$$E_{\mathrm{k}} = \frac{1}{2}mv^2\,.$$

We can substitute and evaluate

$$v^2 = v_0^2 + \frac{2sqE}{m}\,.$$

Therefore

$$v = \sqrt{\frac{2s \cdot 10^{12} \cdot 1.6 \cdot 10^{-19} \cdot E}{m} + v_0^2}\,.$$

Numerically, the result is $v \doteq 1.016\,\mathrm{m{\cdot}s^{-1}}$.

Note that we used only the mass of the bird, because the mass of the electrons is lower by 20 orders of magnitude and can be neglected.

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Problem E.2 ... Náry's hoverboard

How much current would have to flow through a horizontal wire 1 m *long to make the wire levitate above the ground? Magnetic induction is* $5 \cdot 10^{-5}$ T *at the place where the wire is and points to the north in the horizontal plane. The wire has a mass* 50 g. *The direction of the wire is from west to east.*

Faleš and Náry playing with a compass.

The magnetic force has to be equal to the gravitational force and has to point upwards. The expression for magnetic force is

$$F_{\mathrm{m}} = BIl \sin\alpha\,,$$

where the angle α is measured between the vector of magnetic induction and the wire. In our setting, the angle is $\alpha = \pi/2$. We can express the current from the equation

$$BIl = mg\,.$$

That is,

$$I = \frac{mg}{Bl}\,.$$

Using the numbers gives us $I \doteq 9.8\,\mathrm{kA}$.

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Problem E.3 ... astonished Náry

Náry was playing with electrical components and ended up with four peculiar things. He was told by others that they are a resistor with resistance $R = 10\,000\,\Omega$, *capacitor with capacity* $C = 1\,\mu\mathrm{F}$, *inductor with inductance* $L = 10\,\mathrm{H}$ *and alternating current source with voltage* $U = 230\,\mathrm{V}$ *and frequency* $f = 50\,\mathrm{Hz}$. *By some strange coincidence he managed to build a series circuit. Knowing what he is playing with, he immediately computed the absolute value of phase shift between current and voltage in the circuit. What number did he get?*

Faleš and Náry during experimental afternoon.

The phase shift φ can be simply computed using the phase diagram, where on the positive y-axis we put voltage on the inductor, on its negative half the voltage on the capacitor and the voltage on the resistor is on positive x-axis. This gives us

$$\operatorname{tg}\varphi = \frac{U_L - U_C}{U_R} = \frac{I\omega L - \frac{I}{\omega C}}{IR}\,.$$

We can plug in formula for the angular frequency $\omega = 2\pi f$ to get

$$\operatorname{tg}\varphi = \frac{2\pi f L - \frac{1}{2\pi f C}}{R}\,.$$

The phase shift is

$$\varphi = \operatorname{arctg}\frac{2\pi f L - \frac{1}{2\pi f C}}{R}\,.$$

It gives us numerically $\varphi \doteq -0.24^\circ$. The absolute value therefore is 0.24°

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Problem E.4 ... it's freezing

An arc lamp has resistivity $0.2\,\Omega$ *and it is connected to voltage* $U = 60\,\mathrm{V}$. *How much heat does it release during* $1\,\mathrm{min}$? *Give the result in* MJ. *Faleš shivered with cold.*

We can use the Ohm's law to get the current through the lamp. It is

$$I = \frac{U}{R} = 300\,\mathrm{A}\,.$$

The heat, which is being released, is the Joule heat. The formula for it's flow (or power) P is

$$P = UI = \frac{U^2}{R}\,.$$

We need to multiply it by $t = 60\,\mathrm{s}$ to get the production in one minute. The result is $E = Pt = 1.08\,\mathrm{MJ}$.

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Problem X.1 ... Odysea and Prometheus

There are two space-ships, Odysea and Prometheus. Both of them have rest length L_0 *and both are approaching the Earth, but from opposite direction. They meet in a point where Odysea has speed* $0.25c$ *and Prometheus has speed* $0.75c$. *How long do they appear to each other in the moment when they pass each other? Determine the length as a multiple of* L_0.

Faleš has been watching Stargate.

First, we have to compute the combined velocity of both space-ships. They add together relativistically:

$$v'_\mathsf{P} = \frac{v_\mathsf{P} - v_\mathsf{O}}{1 - \frac{v_\mathsf{O} v_\mathsf{P}}{c^2}}\,,$$

Here v_O is velocity of Odysea (its speed with negative sign as they are approaching from opposite directions) and v_P is velocity of Prometheus (its speed with positive sign). If we plug in the numbers we get the total speed $16/19c$. Prometheus is therefore seem from Odysea to be approaching with speed $16/19c$ (and vice versa).

Now we can easily use the Lorentz formula for length contraction:

$$L' = \frac{L_0}{\gamma} = L_0\sqrt{1 - \left(\frac{16}{19}\right)^2} = \frac{\sqrt{105}}{19}L_0 \doteq 0.539L_0\,.$$

The ships see each other as $0.539L_0$ long.

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Problem X.2 ... who's gonna survive

Let us have a beam of neutrons with kinetic energy $T = 10\,\mathrm{keV}$. Mean lifetime of a neutron is $\tau_\mathrm{n} = 925\,\mathrm{s}$ and its rest energy is $m_\mathrm{n}c^2 = 939.6\,\mathrm{MeV}$. What percentage of neutrons in the beam decays during passage through distance $l = 10\,\mathrm{m}$? *Faleš was watching Independence day.*

Kinetic energy is much smaller than the rest mass and therefore we can solve the problem non-relativistically. Time needed by a neutron to travel a distance l is

$$t = \frac{l}{v} = \frac{l}{\sqrt{\frac{2T}{m_\mathrm{n}}}} = \frac{l}{c}\sqrt{\frac{m_\mathrm{n}c^2}{2T}}\,.$$

This time has to be substituted into the decay law. The portion of decayed neutrons is

$$f_\mathrm{D} = \frac{N_0 - N}{N_0} = 1 - \mathrm{e}^{-\frac{t}{\tau_\mathrm{n}}} = 1 - \mathrm{e}^{-\frac{l}{\tau_\mathrm{n}c}\sqrt{\frac{m_\mathrm{n}c^2}{2T}}} \approx \frac{l}{\tau_\mathrm{n}c}\sqrt{\frac{m_\mathrm{n}c^2}{2T}}\,.$$

Where we have used the Taylor expansion of an exponential. Numerically, we get $7.8 \cdot 10^{-7}\,\%$.

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Problem X.3 ... the winter, again

Consider a block of ice with mass $m_\mathrm{i} = 5\,\mathrm{kg}$ and temperature $t_\mathrm{i} = -2\,^\circ\mathrm{C}$. We place $m_\mathrm{Fe} = 3\,\mathrm{kg}$ block of iron heated to $t_\mathrm{Fe} = 1\,333\,^\circ\mathrm{C}$ on the block of ice. Determine the resulting temperature of the system. The specific heat capacity of ice is $c_\mathrm{i} = 2.1\,\mathrm{kJ\cdot kg^{-1}\cdot K^{-1}}$, the heat of fusion of ice is $l_\mathrm{im} = 334\,\mathrm{kJ\cdot kg^{-1}}$, the specific heat capacity of water is $c_\mathrm{w} = 4.18\,\mathrm{kJ\cdot kg^{-1}\cdot K^{-1}}$, the latent heat of evaporation of water is $l_\mathrm{vw} = 2.26\,\mathrm{MJ\cdot kg^{-1}}$ and the specific heat capacity of iron is $c_\mathrm{Fe} = 473\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ *Faleš has seen ice for the first time this year.*

First, the iron block heats the ice to temperature $0\,^\circ\mathrm{C}$ and then the ice starts to melt (if there is enough heat in the iron). Therefore, let us start with comparison of the two heats. The heat needed to change temperature of an object from initial temperature t_i to temperature t_f is

$$Q = mc(t_\mathrm{f} - t_\mathrm{i})\,.$$

Plugging in the numbers we find the two heats are (their moduli) $Q_\mathrm{i} = 21\,\mathrm{kJ}$ and $Q_\mathrm{Fe} \doteq 1.89\,\mathrm{MJ}$.

We can see the ice is heated up and starts to melt. The heat needed to melt the entire block can be computed from equation

$$Q_\mathrm{im} = m_\mathrm{i}l_\mathrm{im}\,,$$

where l_im is heat of fusion of ice and its value is $334\,\mathrm{kJ\cdot kg^{-1}}$. The heat needed to melt the entire block of ice is $Q_\mathrm{im} = 1.67\,\mathrm{MJ}$. Therefore the block really is melted and starts to heat up. The final equation therefore is

$$m_\mathrm{Fe}c_\mathrm{Fe}(t_\mathrm{Fe} - t) = Q_\mathrm{i} + Q_\mathrm{im} + m_\mathrm{i}c_\mathrm{w}t\,,$$

where $c_\mathrm{w} = 4.18\,\mathrm{kJ\cdot kg^{-1}\cdot K^{-1}}$ is specific heat capacity of water. The resulting temperature is

$$t = \frac{m_\mathrm{Fe}c_\mathrm{Fe}t_\mathrm{Fe} - (Q_\mathrm{i} + Q_\mathrm{im})}{m_\mathrm{i}c_\mathrm{w} + m_\mathrm{Fe}c_\mathrm{Fe}} = \frac{m_\mathrm{Fe}c_\mathrm{Fe}t_\mathrm{Fe} - m_\mathrm{i}c_\mathrm{i}\,(t_0 - t_\mathrm{i}) - m_\mathrm{i}l_\mathrm{im}}{m_\mathrm{i}c_\mathrm{w} + m_\mathrm{Fe}c_\mathrm{Fe}}\,.$$

Numerically, this evaluates to $t \doteq 9.0\,^\circ\mathrm{C}$.

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Problem X.4 ... tiler

Are there such plane edge-to-edge tilings where, in every vertex, t equilateral triangles, s squares, and h regular hexagons meet? We want to know how many such tilings exist. Numbers t, s, h are non-negative integers. Sizes of polygons of the same type in one tiling are identical. The arrangement of polygons about the vertex does not matter (different arrangements for fixed s, t, h count as one). *Mirek got a new book.*

For every such tiling

$$\frac{\pi}{3}t + \frac{\pi}{2}s + \frac{2\pi}{3}h = 2\pi\,,$$

must hold, that is

$$2t + 3s + 4h = 12\,.$$

Now we can see that

$$t \leq 6\,, s \leq 4\,, h \leq 3\,.$$

There are 7 possible combinations of (t, s, h): $(0,0,3)$, $(0,4,0)$, $(6,0,0)$, $(1,2,1)$, $(2,0,2)$, $(3,2,0)$, $(4,0,1)$. Of course, we can find diferent arrangements for these triplets (try $(1,2,1)$), but we asked only for the number of triplets, so the answer is 7.

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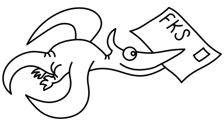

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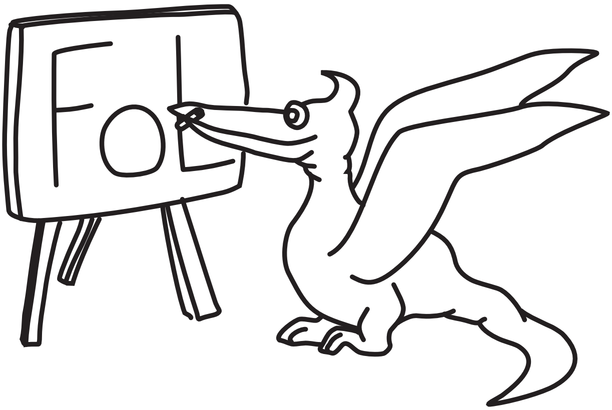
Solutions of 7^{th} Online Physics Brawl



Problem FoL.1 ... tough run

With what velocity should you run at the equator in order to weigh as much as possible (e.g. have the maximum possible weight, not mass) if you can choose the optimal direction?

Matěj's healthy method of gaining weight.

It's best to run against Earth's rotation, i.e. westwards, with the exact velocity that causes centrifugal force to be zero. The velocity should be equal and opposite to the velocity of rotating Earth at that point.

$$v = \omega R = \frac{2\pi R}{T} \doteq 464\,\mathrm{m{\cdot}s^{-1}}\,,$$

where $T = 24\,\mathrm{h}$ and $R = 6{,}380\,\mathrm{km}$.

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Problem FoL.2 ... electrobell

*Mirek's colleague bought a \$1 "small science" apparatus at a flea market. It's a small box with two wires. The ends of these wires are almost connected, leaving a small distance l between them. If we connect the wires to a powerful voltage source, we can create a short electric arc. The sound of this discharge is quite loud, that's why Mirek's colleague uses this device as a lunch bell. Find the largest possible distance (**in micrometres**) of the wires that allows the creation of an electric discharge for a source with peak voltage $325\,\mathrm{V}$. The dielectric strength of air is $D = 3\,\mathrm{MV{\cdot}m^{-1}}$.*

Mirek was looking forward to lunch.

The maximum voltage in the circuit is $U = 325\,\mathrm{V}$, so the largest distance between wires allowing the creation of a discharge is

$$l = \frac{U}{D} \doteq 110\,\mathrm{\mu m}\,.$$

Let us note that the presence of dust, shape of electrodes and other effect could increase this distance. For the purposes of the question, we omitted the Tesla coil inside the real device.

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Problem FoL.3 ... jump

The mass of the average human is $80\,\mathrm{kg}$. How many people would need to gather in one place and jump $1\,\mathrm{m}$ up at the same time in order to shift Earth's center by $0.1\,\mathrm{pm}$?

Matěj was jumping on a trampoline.

The common center of mass of Earth and the people doesn't move. At the moment when all people are $1\,\mathrm{m}$ above Earth's surface, the center of Earth should shift by $0.1\,\mathrm{pm}$.

$$N \cdot 1\,\mathrm{m} \cdot 80\,\mathrm{kg} \doteq 0.1\,\mathrm{pm} \cdot 5.97 \cdot 10^{24}\,\mathrm{kg}\,,$$

where we utilised the approximation $1\,\mathrm{m} - 0.1\,\mathrm{pm} \approx 1\,\mathrm{m}$.

$$N \doteq \frac{0.1\,\mathrm{pm} \cdot 5.97 \cdot 10^{24}\,\mathrm{kg}}{1\,\mathrm{m} \cdot 80\,\mathrm{kg}} \doteq 7.5 \cdot 10^{9}$$

That means we'd need the whole population of Earth.

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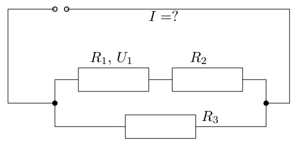
Problem FoL.4 ... there's a current flowing

What is the total current I flowing through the depicted circuit? Resistances in the diagram are $R_1 = 124\,\Omega$, $R_2 = 263\,\Omega$ and $R_3 = 454\,\Omega$. Voltage on the first resistor is $U_1 = 14.8\,\mathrm{V}$.

Enter the result in miliamperes.

Karel wanted you to review electrical circuits.



Let's denote the voltages and currents on each resistor as U_x and I_x. Ohm's law applied on the first resistor gives the current

$$I_1 = \frac{U_1}{R_1} .$$

Charge preservation (or Kirchhoff's first law) tells us that $I_1 = I_2$. This relation can be used to obtain the voltage on the second resistor

$$U_2 = R_2 I_2 = R_2 I_1 = \frac{R_2}{R_1} U_1 .$$

According to Kirchhoff's second law, the total voltage on the upper branch is equal to the voltage on the lower branch. This leads to

$$U = U_3 = U_1 + U_2 = U_1 \left(1 + \frac{R_2}{R_1}\right) .$$

Now we can finally express the total current

$$I = I_1 + I_3 = \frac{U_1}{R_1} + \frac{U_1 \left(1 + \frac{R_2}{R_1}\right)}{R_3} = U_1 \left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{R_2}{R_1 R_3}\right) .$$

Plugging in the numbers gives $I \doteq 221\,\mathrm{mA}$.

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Problem FoL.5 ... a mirror problem

There's a $10\,\mathrm{m}$ wide, $5\,\mathrm{m}$ long rectangular room. In the middle of one of the longer walls, there's a mirror with width $1\,\mathrm{m}$ at eye level. Mikuláš is standing in front of the middle of the mirror at a distance $1\,\mathrm{m}$ from the mirror. What part of the room's area (in %) will he see in the mirror? Neglect the fact that part of this area won't be visible because he sees himself in front of it. Note: when computing the area, consider only the horizontal cross-section of the room at eye level.

Katka was looking out of a window during a lecture.

It's clear that we're mainly interested in the two rays reflected from the mirror's border. Let's denote the distance of Mikuláš from the mirror by v and the half-width of the mirror by z. According to the law of reflection, we know that

$$\frac{v}{z} = \frac{y}{x}\,,$$

where y is the length of the room and x is the perpendicular distance between the point of incidence of the ray (at the mirror) and the ray's intersection with the wall behind Mikuláš. In this case, $x = 2.5\,\mathrm{m}$. It follows from the geometry of the problem that the ray will end up on the back wall of the room and separate the visible and invisible part of the room. After partitioning the visible part and summing up all areas, we find the area S of the visible part

$$S = 2zy + \frac{1}{2}2xy = zy\left(2 + \frac{y}{v}\right)\,,$$

numerically $S = 17.5\,\mathrm{m}^2$. The total area can be computed by simply multiplying its side lengths, numerically $S_\mathrm{p} = 50\,\mathrm{m}^2$. The ratio of visible to total area can be found simply as

$$p = \frac{S}{S_p}\,.$$

We can see that the result is $p = 35\,\%$.

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Problem FoL.6 ... supercooled water

There's a container with supercooled water at temperature $t = -8\,^\circ\mathrm{C}$. Initially there are no condensation nuclei, so it remains in the liquid state. Find the mass percentage of this liquid that will freeze after inserting a condensation nucleus. Neglect the heat capacity of the container. Use the specific heat capacity of water $c = 4{,}180\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ and the latent heat of fusion $l = 334\,\mathrm{kJ{\cdot}kg^{-1}}$.

Remember: This is an easy problem, so keep it simple.

Karel attended a talk by doc. Bochníček on the properties of supercooled water.

Let's break this problem down from the viewpoint of heat balance. The supercooled water will start freezing the moment we insert condensation nucleus in the container and heat will be released during the fusion process. The whole volume of water (liquid or solid) will receive this heat and warm up to $0\,^\circ\mathrm{C}$. The heat balance can be written as $kml = mc\Delta t$, where m is the mass of water (cancels out), k is the fraction of water turned into ice, c is the specific heat capacity fo water and l is the latent heat of fusion of water. The left-hand side stands for the heat released by freezing water. The right-hand side stands for the heat received by the whole volume in order to heat up to $0\,^\circ\mathrm{C}$. Rearrangement of the heat balance equation leads to

$$k = \frac{c\Delta t}{l}\,.$$

For given values we obtain $k = 10\,\%$.

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Problem FoL.7 ... ecological

Imagine that electricity could only be produced by burning wood. Let's assume we have just enough wood to produce the amount of electricity necessary to power a tablet for one hour. How many sheets of paper could be made from this wood instead of burning it? The heating value of wood is $13\,\mathrm{MJ{\cdot}kg^{-1}}$. During production, distribution and storage of electricity, $80\,\%$ of the energy (heat) is lost. The tablet is fueled by a $3.6\,\mathrm{V}$ battery with current consumption of $0.5\,\mathrm{A}$. One sheet of paper weighs $5.0\,\mathrm{g}$ and double of this mass in wood is needed for its production.

Erik is saving the forests.

One sheet of paper of mass m requires $2m$ of wood for its production. Heating value of wood is H. Heat can be transformed to electricity with the efficiency of $\eta = 0.2$. Overall we obtain $2mH\eta$ of electrical energy from one sheet of paper.

The tablet consumes electrical energy UIt, where U is the voltage, I is the current and t is the time of one hour. The number of sheets of paper n necessary to keep the tablet running for one hour is

$$n = \frac{UIt}{2mH\eta} \doteq 0.25\,.$$

Thus we can say that a tablet would consume quarter of a sheet of paper per hour.

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Problem FoL.8 ... headshot

We fire a projectile with mass $m = 0.502\,\mathrm{g}$ from an air gun with mass $M_\mathrm{p} = 5\,\mathrm{kg}$. The projectile impacts a cuboid with mass $M = 182\,\mathrm{g}$, which slides and stops at distance $s = 4.8\,\mathrm{cm}$. The friction coefficient between the cuboid and surface is $f = 0.3$. Determine the muzzle velocity of the projectile.

Assume that the velocity of the projectile at the moment of impact is the same as the muzzle velocity and that the cuboid is far enough to be unaffected by gases emitted from the air gun. The cuboid lies on a horizontal surface.

Adapted by Karel from the article Fyzika jako zážitek ("Physics as an experience").

We can discard the mass of the gun, since we aren't computing the projectile's velocity based on recoil. However, the remaining parameters are relevant. First of all, we'll use the fact that we're dealing with a perfectly inelastic collision of the projectile and the cuboid. That means energy isn't conserved, but momentum is; both objects merge into one and start moving with a common velocity w after the collision. Let's denote the muzzle velocity of the projectile by v. Then we get

$$mv = (m + M)\,w \quad \Rightarrow \quad v = \frac{m + M}{m} w\,.$$

Of course, we don't know the initial velocity of the cuboid+projectile after the collision w. However, we know that due to friction, the cuboid will be moving with acceleration (deceleration) $a = fg$, where g is the acceleration due to gravity, until it stops. The formula for distance traversed during motion with constant acceleration is the well-known $s = at^2/2$; for velocity, it's $w = at$. Expressing the time t, we get

$$s = \frac{1}{2}\frac{w^2}{fg} \quad \Rightarrow \quad w = \sqrt{2fgs}\,.$$

All together, the muzzle velocity is given by

$$v = \frac{m+M}{m}\sqrt{2fgs} \doteq 193\,\mathrm{m{\cdot}s^{-1}}\,.$$

The muzzle velocity of our projectile is therefore $193\,\mathrm{m{\cdot}s^{-1}}$. We could neglect the fact that its mass increases the cuboid's mass in the collision – the result would be the same to three significant figures. We can't neglect the projectile's mass before the collision, though (that would give it zero momentum).

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Problem FoL.9 ... transportation acceleration

So one day Karel was casually riding the Prague metro and saw an information panel about trams. It said that the power of a tram car increased tenfold in the last decades. How many times did the maximum speed increase, assuming their weight didn't change and the resistive force are proportional to the second power of tram's velocity? In other words, we want to know the value of $k = v_1/v_0$, where v_0 is the original maximum speed and v_1 is the current one.

Karel saw a tram praising ad in the Prague metro.

The instantaneous power can be expressed as $P = Fv$, where F is the force exerted by the engine and v is the instantaneous velocity. When tram reaches its maximum speed, v can be viewed as a constant, therefore, the resistive force is also constant. Expression for the resistive force is $F = Cv^2$, where C is some constant. For a non-accelerated motion, the resulting force is zero. Thus the power can now be expressed as $P = Cv^3$. According to the information panel, the ratio of new and original power is $P_1/P_0 = 10$. Now we are ready to find the ratio of new and original maximum velocities

$$\frac{P_1}{P_0} = \frac{Cv_1^3}{Cv_0^3} \quad\Rightarrow\quad k = \frac{v_1}{v_0} = \sqrt[3]{\frac{P_1}{P_0}} = \sqrt[3]{10} \doteq 2.15\,.$$

Trams in Prague can achieve 2.15 times greater velocity.

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Problem FoL.10 ... closer than it seems

*According to ground-based observatories on Earth, the parallax of the star Proxima Centauri is $p = 0.77\,\mathrm{arcsec}$. There's a planet orbiting this star at distance $r = 0.05\,\mathrm{au}$ (assume its orbit is approximately circular). Let's imagine there are intelligent beings living on this planet (let's call them Centaurs) and they measured the parallax of our Sun. The Centaurian definition of the parallax is, of course, based on their home planet's orbit. How large is the parallax they measured? Enter the result in **arcseconds**.*

Mirek was pondering about extraterrestrial physical units.

The parallax of a star is (in a simplified way) defined as the following: let's construct a triangle between Earth, the Sun and the observed object. The parallax is the angle at the observed object–vertex. Since this angle will always be very small, we may use the small angle approximation (often called paraxial). If the orbital radius of the exoplanet corresponds to five hundredths of Earth's orbital radius, the parallax measured by the Centaurs will decrease proportionally, so it will be $p' = 0.77\,\mathrm{arcsec}/20 = 0.0385\,\mathrm{arcsec}$.

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Problem FoL.11 ... soda

How many times would the volume of a soda increase if all dissolved carbon dioxide suddenly turned to gas? A typical soda contains $8\,\mathrm{g{\cdot}dm^{-3}}$ of carbon dioxide. Consider the situation at $25\,^\circ\mathrm{C}$, $101.3\,\mathrm{kPa}$. *Štěpán spilled his drink on himself.*

The amount of carbon dioxide contained in the soda in moles is

$$n = \frac{m}{M}\,,$$

where m is the mass and M is the molar mass. In an ideal gas the amount of substance can be expressed as

$$n = \frac{V}{V_0}\,,$$

where V is the volume of the gas and V_0 is the molar volume at given conditions. The equation of state leeads us to $V_0 = RT/p = 24.5\,\mathrm{dm^3}$. Therefore the volume of gas is

$$V = \frac{V_0 m}{M}\,.$$

For one liter of soda we get $V = 4.45\,\mathrm{dm^3}$ of carbon dioxide gas. The total volume (liquid plus gas) is then $5.45\,\mathrm{dm^3}$, so the volume increased by the factor 5.45. The initial volume of dissolved carbon dioxide was neglected.

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Problem FoL.12 ... don't fall!

The two-wheeled vehicle segway maintains its vertical orientation by accelerating or decelerating. Neglect the mass of the vehicle and consider the driver a point with mass $65\,\mathrm{kg}$ at a distance $\frac{r}{2} = 1\,\mathrm{m}$ from the axis of rotation. What power does the motor need to provide when the segway needs to balance a tilt by $\alpha = 10^\circ$ to the front at speed $v = 10\,\mathrm{km{\cdot}h^{-1}}$? Use $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Michal was wondering about the new road signs.*

Imagine the vehicle as a rod with length r that's tilted from the vertical by the angle α. The system is moving non-inertially (the vehicle makes turns, accelerates, decelerates...), which is why there's an inertial force F_1 acting on its center of mass (at distance $\frac{r}{2}$ from the axis of rotation). The force of gravity F_2 acting on the driver causes torque

$$M_2 = F_2 \frac{r}{2} \sin\alpha = mg\frac{r}{2}\sin\alpha\,.$$

Since the driver remains tilted by the same amount all the time, the net torque on the vehicle must be zero, so (with respect to the axis of rotation)

$$F_1 \cos\alpha \frac{r}{2} = mg\frac{r}{2}\sin\alpha \,.$$

Distance $r/2$ cancels out and we get

$$F_1 = mg\tan\alpha \,.$$

Now, from the point of view of the inertial system of the driver, we can identify the the inertial force as the force exerted by the motor. This force acts in the system the non-inertial system too, but it doesn't cause any torque because it acts at the axis of rotation. And when we know the force of the motor, the power can easily be computed using the well-known formula

$$P = F_1 v = mvg\tan\alpha \,.$$

Plugging in the numerical values, we get $P \doteq 312\,\mathrm{W}$.

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Problem FoL.13 ... shoot it down!

What's the probability (in percent) that if we shine a laser vertically upwards for a short time, the ray hits a passenger airplane? The average horizontal cross-section of an airplane is $S = 300\,\mathrm{m}^2$ and at each moment, there are approximately ten thousand airplanes in the air. Assume that airplanes are distributed homogeneously across the sky and fly at heights much smaller than the radius of Earth. *Matěj read about a terror attack.*

We may compute the probability as the ratio of total cross-section of all planes and Earth's surface area.

$$p = \frac{10{,}000S}{4\pi R^2} = 5.9\cdot 10^{-9}\,.$$

The probability that the laser hits an airplane is $5.9\cdot 10^{-7}\,\%$.

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Problem FoL.14 ... Young's thermal stress

A copper rod of length $l = 12.3\,\mathrm{cm}$ and cross-sectional area $S = 1.02\,\mathrm{cm}^2$ is fixed so that its length cannot thermally increase. The linear coefficient of thermal expansion for copper is $\alpha = 1.70\cdot 10^{-5}\,\mathrm{K}^{-1}$ and Young's elastic modulus is $E = 117\,\mathrm{GPa}$. Find the force exerted by the rod on the holder due to thermal expansion. Give the result for $\Delta T = 15.0\,\mathrm{K}$ and only for one end of the rod. *Karel combined formulas.*

The idea behind this is problem is simple. The rod's length should increase, but it cannot, so it will deform. The deformation is due to force exerted on the rod by the holder and according to Newton's 3rd law, the rod exerts force of the same magnitude on the holder. This magnitude F can be expressed as $F = S\sigma$, where σ is the stress in the rod and S is the area of the contact surface, i. e., the cross-sectional area of the rod; radial expansion is negligible. The defining relation of Young modulus is $F = SE\varepsilon$, where ε is the strain. Assumption of linear dependence between length and temperature allows us to express

$$\varepsilon \approx \frac{\Delta l}{l} = \frac{(1 + \alpha\Delta T)l - l}{l} = \alpha\Delta T\,,$$

where Δl is the absolute elongation. Combining all formulas we get

$$F = SE\alpha\Delta T\,.$$

The numerical result is $F = 3.04\,\mathrm{kN}$.

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Problem FoL.15 ... rubber bands

Consider two rubber bands (here, a rubber band is a single strip of rubber) with equal stiffness k and equal rest length. One of the bands breaks if acted upon by a force greater than F_1. The other band breaks if acted upon by a force greater than cF_1, where $c > 1$ is a constant. We take a weight and hang it from both rubber bands in parallel so that they don't break. Then, we slowly and continuously increase its weight (so that it does not oscillate) until the first band breaks. What's the minimal value of the constant c for which the second band doesn't break afterwards? *Michal was shooting rubber bands at people.*

According to the problem statement, the first rubber band breaks when there's a force F_1 acting on it. It follows that the weight must be acting with total force due to gravity $2F_1$ at the moment when the band breaks. At this point, the first band breaks and the weight is hanging only on the second band, which isn't in its equilibrium position.

We can view the resulting situation as a harmonic oscillator, which is initially in the position with maximum upwards displacement. The equilibrium position of this oscillator is a certain distance Δh below its current position. As is well known, the maximum downwards displacement of this weight must be located Δh below the equilibrium position. That's the position the weight will try to get to following the breaking of the first band; afterwards, it will oscillate between the two maximum positions.

Since the force acting upon the rubber band depends only on its displacement from the rest position, the force acting upon it will be maximum when the weight is in the position with maximum downwards displacement. In addition, we know that when the first band broke, the force acting upon the second band is F_1 and in the equilibrium position, the force acting upon the second band is $2F_1$. We can simply conclude that when the displacement is maximum downwards, the force is $3F_1$. Due to the previous reasoning, we know this is the maximum force which will act upon the second band during the oscillations.

The second rubber band has to be able to endure forces greater than $3F_1$. We can see that the constant c must satisfy $c > 3$.

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Problem FoL.16 ... bungee jumping

There's a bridge of height $h = 50\,\mathrm{m}$ and a bungee jumping rope with rest length $l = 10\,\mathrm{m}$ and stiffness $k = 80\,\mathrm{N{\cdot}m^{-1}}$. A man with mass m attaches himself with this rope to the bridge and jumps off (consider his initial velocity to be zero). What's the largest possible mass of this man such that the rope will stop him before hitting the surface below?

Michal is afraid to go bungee jumping.

First of all, let's derive a formula for the depth at which the rope stops the jumper. At this depth, the potential energy of the stretched rope will be equal to the potential energy of the jumper. Therefore, let's assume the rope stops the jumper at depth v (measured downwards from the top of the bridge). Then, we've got

$$\frac{1}{2}k(v - 10\,\mathrm{m})^2 = mgv\,,$$

where k is the stiffness of the rope, g the acceleration due to gravity and m is the mass of the jumper. The term $(v - 10\,\mathrm{m})$ at the left hand side of the equation has this form because the rest length of the rope is $10\,\mathrm{m}$.

We have two unknown quantities in this equation: the depth v, at which the jumper stops, and the mass of the jumper m. We can determine the first of these variables and determine the other one from this equation. Since we want to know the critical mass of the jumper, let's consider the case when the jumper is stopped by the rope exactly at ground level, that is, at depth $v = 50\,\mathrm{m}$. Now we can plug all the values to our equation and express the critical mass of the jumper as

$$m = \frac{\frac{1}{2}k(v - 10\,\mathrm{m})^2}{gv} \doteq 130.48\,\mathrm{kg}\,.$$

It's trivial that a jumper with larger mass won't be stopped by the rope in time.

For the rope to stop the jumper before hitting the ground, the mass of the jumper cannot be greater than $m = 130.48\,\mathrm{kg}$.

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Problem FoL.17 ... fidget spinner

A fidget spinner rotates with angular frequency $\omega = 12.34\,\mathrm{s^{-1}}$. Inner and outer radii of the ball bearing inside a fidget spinner are in ratio $k = 0.432$. What is the period of rotation of one ball inside the bearing around the center of the toy? The inner part of the bearing is static. Assume there is no slipping.

Matěj bought this autistic toy.

We know the ratio $k = r/R$ of the inner radius r and the outer radius R. The instantaneous velocity of a point on the contact between a bearing ball and the outer part of the bearing is $v = \omega/R$. A point on the contact of the ball and the inner part of the bearing has zero instanteneous velocity, because the inner part is static. The velocity of the center of the ball is given by the average of these velocities, i. e. $v/2$. The distance between the center of the ball and the center of the bearing is $(r + R)/2$. The velocity of the center of the ball is therefore

$$\omega_k = \frac{v}{r + R} = \frac{\omega R}{r + R}\,.$$

Now we can express the period

$$T = \frac{2\pi}{\omega_k} = \frac{2\pi\,(r+R)}{\omega R} = \frac{2\pi}{\omega}\,(k+1)\ .$$

The numerical result is $T \doteq 0.729\,\mathrm{s}$.

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Problem FoL.18 ... insect in amber

Amber (with refractive index $n = 1.55$) is a transparent yellow-brown fossil resin. A small beetle, trapped and preserved within the stone, appears to be $2.25\,\mathrm{cm}$ below the surface when viewed directly from above. We are looking at the stone from a large distance. How far beneath the surface (in centimetres) is the beetle actually located?

Karel adapted a problem from Cutnell and Johnson: Physics 9e.

The beetle is at depth a below the surface of amber, but we see it as if the depth was b. Assume that our eyes are at the distance y above the surface. Think of a line perpendicular to the surface, going through the beetle. This line also goes exactly between our eyes since we are looking directly from above. We will denote the distance of each eye from the perpendicular line as x.

A light ray propagating from the beetle through the amber hits the surface at distance d, refracts and then goes straight to one of our eyes. Let's denote the angle of incidence and angle of refraction as α and β, respectively. Now we can write down a system of equations

$$\tan\alpha = \frac{d}{a}\,,$$
$$\tan\beta = \frac{x-d}{y} = \frac{d}{b}\,,$$

where the last expression was obtained by similarity of triangles, because we see the beetle in the direction of the refracted ray. By extending this ray we get a point of intersection with the perpendicular line and this point is b below the surface.

Snell's law leads to

$$\frac{\sin\alpha}{\sin\beta} = \frac{n_0}{n}\,,$$

where n_0 is the refractive index of air.

The accommodation range of eyes of an average adult is bounded from below by $\sim 15\,\mathrm{cm}$, therefore we can safely assume $x \ll y$. This allows us to use approximations $\tan\alpha \approx \sin\alpha \approx \alpha$. Thus we can plug the expressions for $\tan\alpha$ and $\tan\beta$ into Snell's law and obtain

$$\frac{\frac{d}{b}}{\frac{d}{a}} = \frac{n_0}{n}\,.$$

Rearranging for a gives

$$a = b\frac{n}{n_0}\,,$$

and for the given values we get $a \doteq 3.49\,\mathrm{cm}$.

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Problem FoL.19 ... swung rod

There's a rigid rod with length $l = 1.2\,\mathrm{m}$ and zero mass, which is attached at one end in such a way that it can rotate around this fixed end. There are three small spheres with equal mass attached to this rod (you may consider them point masses). One of the spheres is at the free end of the rod and the other two at 1/3 and 2/3 of its length. The rod is held horizontally at first. Then, we release it. What will be the velocity of its free end when it passes through the equilibrium position (the bottommost point)? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Karel was teaching mechanics.

The moment of inertia of a point mass m with respect to an axis at distance d is $J = md^2$. If we denote the masses of the spheres as m, the moment of inertia of the whole system with respect to the fixed end of the rod is

$$J = m\left(\frac{l}{3}\right)^2 + m\left(\frac{2}{3}\right)^2 + ml^2 = \frac{14}{9}ml^2\,.$$

When the rod moves from the horizontal to vertical position, the potential energy that's released is

$$E = mg\frac{l}{3} + mg\frac{2}{3} + mgl = 2mgl\,.$$

It follows from the law of energy conservation that all this energy is converted to kinetic energy of the rod, which satisfies

$$E_\mathrm{k} = \frac{1}{2}J\omega^2 = \frac{1}{2}J\frac{v^2}{l^2}\,.$$

Substituting for E_k and J, we obtain the equation

$$2mgl = \frac{1}{2}\frac{14}{9}ml^2\frac{v^2}{l^2}\,,$$

from which we can easily express

$$v = \sqrt{\frac{18}{7}gl} \doteq 5.50\,\mathrm{m{\cdot}s^{-1}}\,.$$

The end of the rod will move with velocity $5.50\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.20 ... deviated

There's a pendulum consisting of a thin, rigid rod and a heavy weight attached to the end of the rod. The rod is then attached (by its other end) to another, horizontal rod. So, this second rod represents the axis of rotation of the pendulum. Now we rotate the second rod so that it is inclined at angle $\varphi = 30^\circ$ with respect to the horizontal plane (and it still represents the axis of rotation of the pendulum). Find the period of small oscillations T' of the pendulum and compare it with the period T of a pendulum with a horizontal axis of rotation. Give the ratio T'/T as the result. *Mirek was watching the new series Genius – Einstein.*

Let's denote the gravitational acceleration by g. The movement of the deviated pendulum is constrained to a plane inclined at angle φ w.r.t. the horizontal plane. Projection of the gravitational acceleration on this plane is $g_{||} = g\cos\varphi$. The perpendicular component is $g_{\perp} = g\sin\varphi$ and is balanced out by forces inside the pivot (we assume zero friction, of course).

The period of small oscillations of a pendulum is related to g by

$$T \sim g^{-1/2}\,,$$

and for the deviated pendulum

$$T' \sim g_{||}^{-1/2}\,.$$

The ratio of these periods is

$$\frac{T'}{T} = \left(\frac{g\cos\varphi}{g}\right)^{-1/2} = \sqrt{\frac{1}{\cos\varphi}}\,.$$

Plugging in the numbers we get 1.075.

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Problem FoL.21 ... slinger

We've got a slingshot made by attaching two ends of a massless rubber band to two points $l_0 = 15\,\mathrm{cm}$ apart. Rest length of the rubber band is equal to this distance. We place a pebble with mass $m = 5\,\mathrm{g}$ in the middle of the rubber band and stretch it horizontally, forming the legs of an isosceles triangle with height $h_0 = 8\,\mathrm{cm}$. Then, we release the pebble. What's the maximum velocity the pebble will reach? The rubber band (as a whole) has stiffness $k = 50\,\mathrm{kg{\cdot}s^{-2}}$.

Mirek was remembering his childhood toy.

Before the rubber band is stretched, the potential energy $E_{\mathrm{p}0}$ is zero (even if the band was stretched between the attachment points, we could still take it to be zero). After stretching, its length increases from l_0 to

$$l = 2\sqrt{h_0^2 + (l_0/2)^2}\,.$$

The potential energy after stretching is

$$E_p = \frac{1}{2}k(l - l_0)^2 = \frac{1}{2}k(2\sqrt{h_0^2 + (l_0/2)^2} - l_0)^2$$

and when the pebble is launched, that's fully converted to kinetic energy

$$E_k = \frac{1}{2}mv^2 = E_p\,.$$

The pebble's launch velocity can now be expressed as

$$v = \sqrt{\frac{k}{m}}(2\sqrt{h_0^2 + (l_0/2)^2} - l_0)\,;$$

after plugging in the numbers, $v \doteq 6.9\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.22 ... energies of rotation

There are two identical homogeneous cylinders rotating with the same angular velocity ω. One cylinder, let's denote it A, is rotating around its main axis. The other cylinder B is rotating around a parallel axis with distance $4R/5$ from the center of the cylinder, where R is the radius of the cylinder. What's the ratio of rotational kinetic energies of the cylinders? We're interested in E_B/E_A, where E_A and E_B are rotational kinetic energies of cylinders A and B, respectively.

Karel and Lukáš were riding on carousel.

The rotational kinetic energy of a rigid body with moment of inertia J rotating with angular velocity ω is

$$E_\mathrm{k} = \frac{1}{2}J\omega^2\,.$$

The moment of inertia of a cylinder with respect to its main axis (axis of symmetry) is

$$J = \frac{1}{2}mR^2\,.$$

The kinetic energy of cylinder A can be computed easily as

$$E_A = \frac{1}{4}mR^2\omega^2\,.$$

For the moment of inertia with respect to the axis displaced by d, we can utilise Steiner's (parallel axis) theorem $J = J_0 + md^2$. Substituting in the formula for kinetic energy, we get

$$E_B = \frac{1}{2}\left(\frac{1}{2}mR^2 + m\left(\frac{4}{5}R\right)^2\right)\omega^2 = \frac{57}{100}mR^2\omega^2\,.$$

Therefore, the result is

$$\frac{E_B}{E_A} = \frac{\frac{57}{100}mR^2\omega^2}{\frac{1}{4}mR^2\omega^2} = \frac{57}{25} = 2.28\,.$$

Ratio of rotational kinetic energies of the cylinders is 2.28.

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Problem FoL.23 ... the Ripper

Consider a barbell hanging horizontally in the air. For the purposes of this problem, we can imagine the barbell as a long thin rod with a small disk on each end. Let's charge one disk with charge $Q = 100\,\mu\mathrm{C}$, the other disk with charge $-Q$; the rod is perfectly insulating and no charge can be transfered to the rod. Then, let's activate a homogeneous electric field with intensity $E = 1\,\mathrm{MV{\cdot}m^{-1}}$ parallel to the rod in the direction from negative to positive charge. Determine the stress (in units Pa, with positive sign) in the middle of the rod caused by electrostatic forces. The diameter of the rod is $d = 2\,\mathrm{cm}$, the length of the rod $l = 1\,\mathrm{m}$. Neglect polarisation of dielectrics.

Can't get ripped by lifting weights? Rip the weights!

Let's utilise the superposition of electric fields. The charges, which can be treated as point charges due to small sizes of the disks compared to the whole barbell, attract each other with force

$$F_1 = \frac{kQ^2}{l^2}\,.$$

The rod is therefore compressed with force F_1, which corresponds to pressure

$$p_1 = \frac{F_1}{\frac{\pi d^2}{4}} = \frac{4kQ^2}{\pi d^2 l^2}\,.$$

At the same time, there's an external electric field acting on the charges. This field repels the disks from each other with force

$$F_2 = EQ\,.$$

The stress caused by this is

$$p_2 = \frac{F_2}{\frac{\pi d^2}{4}} = \frac{4EQ}{\pi d^2}\,.$$

Subtracting the stresses, we get

$$p_2 - p_1 = \frac{4Q}{\pi d^2}\left(E - \frac{k\ Q}{l^2}\right) \doteq 32{,}000\,\mathrm{Pa}\,,$$

which is the total stress in the rod. We can see that the rod wouldn't be torn apart, not even if it was made of soft plastic. Increasing the intensity of the electric field would just result in a spark discharge between the disks.

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Problem FoL.24 ... inhomogeneous

Consider a method of launching rockets by imparting all the necessary momentum at the moment of launch. A rocket with mass $m = 10\,\mathrm{t}$ is launched directly upwards at its escape velocity. How high above the Earth's surface will the rocket be when its velocity drops to half of the initial velocity? Express this result as a multiple of Earth's radius R (i.e. in units R). Neglect the effects of Earth's atmosphere and rotation.

Kuba meditating on the physics of balistic missiles.

The escape velocity is just enough for the rocket to stop at infinity. The law of energy conservation holds during the whole flight, so we can take the energy of the rocket at distance r from the center of Earth to be equal to the energy at infinity. That gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = 0\,,$$

where G is the gravitational constant and M is the mass of Earth. Therefore, we get

$$\frac{1}{2}mv^2 = \frac{GMm}{r}\,.$$

Taking the ratio of two such equations, we can see that

$$\left(\frac{v}{v_0}\right)^2 = \frac{R}{r}\,,$$

where $R = 6{,}378\,\mathrm{km}$ is the Earth's radius and v_0 is the initial, escape velocity. Now we can easily express the rocket's height at $v = v_0/2$ as

$$h = r - R = R\left(\frac{v_0}{v}\right)^2 - R = 3R\,.$$

The rocket's velocity drops to half at height $3R$.

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Problem FoL.25 ... Jáchym-style

Consider an isotropic point source of β radiation with activity $A = 3.4567\,\mathrm{MBq}$, located in a pool filled with a substance which dampens the radiation according to Lambert-Beer law $N(r) = N_0 \exp(-\mu r)$, where the absorption coefficient is $\mu = 1.1198 \cdot 10^{-1}\,\mathrm{m}^{-1}$. To what distance r from the source should we place a Geiger-Müller counter with detection window area $S = 2.7183 \cdot 10^{-6}\,\mathrm{m}^2$, if we want it to detect $N = 10$ particles per second on average? Assume that the detector registers all particles which hit the detection window.

Lukáš didn't want to use a computer just to download problems.

Let's at first neglect the substance dampening the radiation. The source is radiating isotropically into a spherical shell with surface area $4\pi r^2$, but we're only detecting the part of that radiation incident on a small surface with area S. The number of detected particles is

$$N = \frac{AS}{4\pi r^2}\,.$$

However, the radiation is dampened, so we need to multiply this formula by a Lambert factor. The resulting formula is

$$N = \frac{AS}{4\pi r^2} \exp\left(-\mu r\right)\,.$$

We've got a non-linear equation for r, which cannot be solved analytically. However, we can solve it either numerically or graphically. The graphical solution consists just of drawing a graph of the function $N(r)$ and reading out the value of r for the given N.

For a numerical solution, we can use a plethora of methods; let's use the interval halving method (bisection method). First, we choose a sufficiently large R so that the root of the equation

$$\frac{AS}{4\pi r^2} \exp\left(-\mu r\right) - N = f(r) = 0$$

which we're looking for would lie in the interval $(0, R)$. We can see that it's true e.g. if $f(0)$ and $f(R)$ have different signs. Then, we take the two halves of the interval. From these two halves, we pick one the root should definitely lie in and apply the same procedure again to that smaller interval until reaching the required precision.

The result is that the source should be at distance $r \doteq 0.269{,}35\,\mathrm{m}$.

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Problem FoL.26 ... drop the gallium into hot water

We've got $m_{\mathrm{Ga}} = 24\,\mathrm{g}$ of gallium and we'd like to perform an interesting experiment with it. We decide to prepare hot water with temperature $t_0 = 93\,^\circ\mathrm{C}$ and volume $V = 250\,\mathrm{ml}$. We place the gallium into an imperfect calorimeter with heat capacity $C = 95\,\mathrm{J{\cdot}K^{-1}}$ and then pour the hot water on it. What will the temperature (in degrees Celsius) of the calorimeter, gallium and water be after reaching thermodynamic equilibrium? Consider the calorimeter with water and *gallium an isolated, closed system. The initial temperatures of the gallium and the calorimeter were $t_1 = 22\,^\circ\mathrm{C}$. The latent heat of fusion of gallium is $l = 5.59\,\mathrm{kJ{\cdot}kg^{-1}}$, specific heat capacity as a solid $c_1 = 370\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ and as a liquid $c_2 = 400\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. The specific heat capacity of water is $c = 4{,}180\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$.*

Karel was thinking about the price of gallium, so he at least set this problem.

The problems seems to be clear at first. We're balancing the heat absorbed and given out. The only complication is caused by the fact that gallium melts at $t_t = 29.8\,^\circ\mathrm{C}$. Therefore, let's first determine by how much (let's denote it by ΔT) the water cools down when it heats up the gallium by $\Delta t_s = t_t - t_1$ and melts it. We can write the heat balance for this as

$$cV\varrho\Delta T = (C + c_1 m_{\mathrm{Ga}})\Delta t_s + m_{\mathrm{Ga}} l\,,$$

where ϱ is water density, which we'll consider to be $1\,\mathrm{g{\cdot}cm^{-3}}$. We find out that melting the gallium at the given initial temperature only cools down the water by very little. That means we'll have thermodynamic equilibrium between the calorimeter, water and gallium at a temperature higher than the melting point of gallium. We can now write the overall heat balance, with t denoting the final temperature.

$$(C + c_1 m_{\mathrm{Ga}})\Delta t_s + m_{\mathrm{Ga}} l + (C + c_2 m_{\mathrm{Ga}})(t - t_t) = (t_0 - t)cV\varrho$$

If we substitute Δt_s and express t, we obtain the equation

$$t = \frac{t_0 cV\varrho + t_t(C + c_2 m_{\mathrm{Ga}}) - m_{\mathrm{Ga}} l - (C + c_1 m_{\mathrm{Ga}})(t_t - t_1)}{(C + c_2 m_{\mathrm{Ga}}) + cV\varrho}\,.$$

After plugging in the numerical values, we get $t \doteq 86.4\,^\circ\mathrm{C}$.

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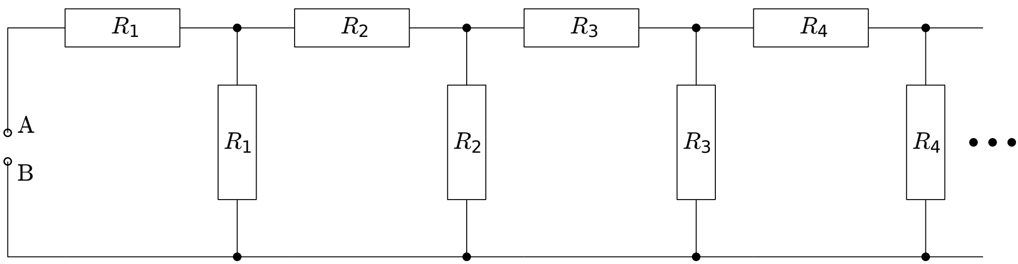
Problem FoL.27 ... another infinite circuit?



What's the resistance between points A and B of an infinite resistor network in the figure? The resistors' resistances are $R_i = 2^{i-1}R$ – each pair of resistors has the same resistance, which is twice as large as that of the previous pair. Compute the numerical result for $R = 3.002\,\Omega$.

Karel was thinking about variations of a standard problem.

We want to find the total resistance of the whole network; let's denote it by R_∞. One possibility is to compute partial resistances one by one and watch what number they converge to. That takes a lot of work, so it's much better to use a trick. Let's try to find the resistance of the whole network in some other way and get a quadratic equation, which can be solved to find R_∞.

In this case, let's imagine disconnecting the two resistors with resistance $R_1 = R$. How does the resulting circuit look? It's very similar to the previous circuit, but all resistors in it have doubled resistances. That's exactly what we need. Based on this idea, we can write the equation

$$R_\infty = R + \frac{2RR_\infty}{R + 2R_\infty}\,.$$

Now we only need to solve it

$$2R_\infty^2 - 3RR_\infty - R^2 = 0\,, \quad \Rightarrow \quad R_\infty = \frac{3 \pm \sqrt{17}}{4} R\,.$$

We need to think about which solution of the quadratic equation is the correct one. Since one solution is negative, we can easily pick the only positive solution. The total resistance is $R_\infty \doteq 5.346\,\Omega$.

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Problem FoL.28 ... medal

The organizers of FYKOS decided to award the top three participants with medals. The FYKOS medal is a flat cylinder consisting of three layers. The first layer is made of gold, the second layer is made of silver and the third one is copper. We know that the electrical resistance between the lower and upper base of the medal is the same as if all three layers were made of copper. We also know that the heat capacity of the medal is the same as if all three layers were made of gold. Find out the mass ratio of the FYKOS medal and another medal of identicial proportions that is made of silver only. Heat capacities, densities and resistivities of pure metals can be found on the Internet (or in printed engineering tables). *Jáchym thinks that a diploma is not enough.*

The height of the medal is $h = h_{\mathrm{Au}} + h_{\mathrm{Ag}} + h_{\mathrm{Cu}}$ where h_{Au}, h_{Ag} and h_{Cu} are the heights of the gold, silver and copper layers, respectively. We will make use of the relation for electrical resistance

$$R = \frac{l}{S}\zeta_{\mathrm{X}}\,,$$

denoting the resistivity as ζ_{X} to avoid confusing with mass density ϱ. The electrical resistance of the whole medal is given by the sum of respective resistances, which leads us to

$$\frac{h_{\mathrm{Au}}}{S}\zeta_{\mathrm{Au}} + \frac{h_{\mathrm{Ag}}}{S}\zeta_{\mathrm{Ag}} + \frac{h_{\mathrm{Cu}}}{S}\zeta_{\mathrm{Cu}} = \frac{h}{S}\zeta_{\mathrm{Cu}}\,,$$

where the RHS represents a copper medal of identical size. By substituting for h and multiplying by the cross-section S we get

$$h_{\mathrm{Au}}\zeta_{\mathrm{Au}} + h_{\mathrm{Ag}}\zeta_{\mathrm{Ag}} + h_{\mathrm{Cu}}\zeta_{\mathrm{Cu}} = h_{\mathrm{Au}}\zeta_{\mathrm{Cu}} + h_{\mathrm{Ag}}\zeta_{\mathrm{Cu}} + h_{\mathrm{Cu}}\zeta_{\mathrm{Cu}}\,,$$

$$h_{\mathrm{Au}} = h_{\mathrm{Ag}}\frac{\zeta_{\mathrm{Cu}} - \zeta_{\mathrm{Ag}}}{\zeta_{\mathrm{Au}} - \zeta_{\mathrm{Cu}}} = k_1 h_{\mathrm{Ag}}\,, \tag{1}$$

where we introduced the ratio k_1 of the heights of gold and silver layer.

The heat capacity of a body made of material X can be expressed as

$$C = mc_\mathrm{X} = V\varrho_\mathrm{X}c_\mathrm{X}\,.$$

Total heat capacity of the medal is again obtained as a sum of respective heat capacities

$$h_\mathrm{Au}S\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Ag}S\varrho_\mathrm{Ag}c_\mathrm{Ag} + h_\mathrm{Cu}S\varrho_\mathrm{Cu}c_\mathrm{Cu} = hS\varrho_\mathrm{Au}c_\mathrm{Au}\,.$$

By substituting for h and dividing by the cross-section S we get

$$h_\mathrm{Au}\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Ag}c_\mathrm{Ag} + h_\mathrm{Cu}\varrho_\mathrm{Cu}c_\mathrm{Cu} = h_\mathrm{Au}\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Cu}\varrho_\mathrm{Au}c_\mathrm{Au}\,,$$

$$h_\mathrm{Cu} = h_\mathrm{Ag}\frac{\varrho_\mathrm{Au}c_\mathrm{Au} - \varrho_\mathrm{Ag}c_\mathrm{Ag}}{\varrho_\mathrm{Cu}c_\mathrm{Cu} - \varrho_\mathrm{Au}c_\mathrm{Au}} = k_2h_\mathrm{Ag}\,, \tag{2}$$

where we introduced the ratio k_2 of the heights of copper and silver layer.

The mass ratio of the FYKOS medal and a silver medal is

$$k = \frac{h_\mathrm{Au}S\varrho_\mathrm{Au} + h_\mathrm{Ag}S\varrho_\mathrm{Ag} + h_\mathrm{Cu}S\varrho_\mathrm{Cu}}{hS\varrho_\mathrm{Ag}} = \frac{h_\mathrm{Au}\varrho_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Cu}\varrho_\mathrm{Cu}}{h_\mathrm{Au}\varrho_\mathrm{Ag} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Cu}\varrho_\mathrm{Ag}}\,.$$

Substituting for h_Au and h_Cu from equations (1) and (2) we establish the result

$$k = \frac{h_\mathrm{Ag}k_1\varrho_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Ag}k_2\varrho_\mathrm{Cu}}{h_\mathrm{Ag}k_1\varrho_\mathrm{Ag} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Ag}k_2\varrho_\mathrm{Ag}} = \frac{1 + \frac{1}{\varrho_\mathrm{Ag}}\left(k_1\varrho_\mathrm{Au} + k_2\varrho_\mathrm{Cu}\right)}{1 + k_1 + k_2}\,.$$

For the given values we get $k \doteq 1.1$.

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Problem FoL.29 ... Dyson sphere under construction

A Dyson sphere is a hypothetical construction surrounding a star, built by an advanced civilisation in order to utilise all energy coming from their star. It should be a relatively thin shell with radius comparable to the distance of planets from the star. What would be the equilibrium temperature T_S of this shell compared to the equilibrium temperature T_P of a planet orbiting the same star on a circular orbit with distance equal to the radius of the Dyson sphere (in a system without the Dyson sphere)? Assume that the planet and the Dyson sphere are black bodies and all bodies in this problem radiate isotropically. Neglect the cosmic background radiation and the radiation of other space objects. As the result, compute the ratio $k = T_\mathrm{S}/T_\mathrm{P}$.

Karel was thinking about radiative heat transfer.

A Dyson sphere absorbs all solar radiation and radiates it in both directions (inwards and outwards). However, the sphere has to absorb everything it radiates inwards anyway (radiation which reaches the sun is negligible). In order to reach equilibrium, it has to radiate outwards the same power as the power radiated by the sun (let's denote it by P_S).

$$P_\mathrm{S} = 4\pi R^2\sigma T_\mathrm{S}^4\,,$$

$$T_\mathrm{S} = \sqrt[4]{\frac{P_\mathrm{S}}{4\pi R^2\sigma}}\,,$$

where R is the distance from the sun and $4\pi R^2$ is the surface area of the sphere. The sun radiates isotropically, so the radiation reaching the planet is

$$P = P_\mathrm{S} \frac{\pi r^2}{4\pi R^2}\,,$$

where πr^2 is the area of the planet's cross-section. The planet also has to radiate the same power from its surface

$$P = 4\pi r^2 \sigma T_\mathrm{P}^4\,,$$

$$T_\mathrm{P} = \sqrt[4]{\frac{P}{4\pi r^2\sigma}} = \sqrt[4]{\frac{P_\mathrm{S}\frac{\pi r^2}{4\pi R^2}}{4\pi r^2\sigma}} = \sqrt[4]{\frac{P_\mathrm{S}}{16\pi R^2\sigma}}\,.$$

Now we can find the ratio

$$k = \frac{T_\mathrm{S}}{T_\mathrm{P}} = \sqrt[4]{4} = \sqrt{2}\,.$$

The equilibrium temperature of a Dyson sphere is $\sqrt{2}$ times the temperature of the planet.

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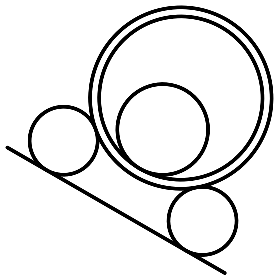
Problem FoL.30 ... too many cylinders

On an inclined plane with inclination angle 35°*, there's a system of three full cylinders and one hollow cylinder. Two smaller cylinders have radius* $r_1 = 0.1\,\mathrm{m}$ *and moment of inertia* $J_1 = 2\,\mathrm{kg{\cdot}m^2}$*, the middle cylinder has* $r_2 = 0.15\,\mathrm{m}$ *and* $J_2 = 10\,\mathrm{kg{\cdot}m^2}$*, and the hollow cylinder has* $r_3 = 0.3\,\mathrm{m}$*,* $d = 0.02\,\mathrm{m}$ *and* $J_3 = 20\,\mathrm{kg{\cdot}m^2}$*, where* r_3 *is the outer radius and* d *is the thickness of its walls. All cylinders are homogeneous. A rigid construction of negligible mass keeps the cylinders in the same relative positions, but allows them to rotate. Assume there is no slipping anywhere. What will be the distance travelled by this system during the initial* $15\,\mathrm{s}$ *after being released from rest?* *Because one cylinder is too mainstream.*



Let's denote the angular velocities of both small cylinders by ω_1. The whole system then moves with velocity $v = \omega_1 r_1$. The angular velocity of the hollow cylinder satisfies

$$\omega_3 = \omega_1 \frac{r_1}{r_3}\,.$$

For the angular velocity of the middle cylinder, we get

$$\omega_2 = \omega_3 \frac{r_3 - d}{r_2} = \omega_1 \frac{r_1}{r_3}\frac{r_3 - d}{r_2}\,.$$

We can determine the masses of individual bodies from the formulae for moment of inertia

$$
\begin{aligned}
m_1 &= \frac{2J_1}{r_1^2}\,,\\
m_2 &= \frac{2J_2}{r_2^2}\,,\\
m_3 &= \frac{2J_3}{\left(r_3^2 + (r_3 - d)^2\right)}\,.
\end{aligned}
$$

The kinetic energy of one cylinder is $E_\mathrm{k} = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2$. If we substitute for the unknowns of each cylinder from the formulae above, we find that the total kinetic energy of the system satisfies

$$E_\mathrm{k} = \frac{3J_1}{r_1^2}v^2 + \frac{J_2}{r_2^2}v^2 + \frac{(r_3-d)^2\,J_2}{2r_2^2r_3^2}v^2 + \frac{J_3}{\left(r_3^2 + (r_3-d)^2\right)}v^2 + \frac{J_3}{2r_3^2}v^2 = kv^2\,.$$

Initially, the system is at zero height with zero potential energy. When it travels distance x, it reaches height $h = -x\sin\alpha$, where α is the inclination angle of the plane. Its potential energy will be $E_\mathrm{p} = mgh = -mgx\sin\alpha$, where $m = 2m_1 + m_2 + m_3$. The total energy of the system is constant, so

$$
\begin{aligned}
E_\mathrm{k} + E_\mathrm{p} = kv^2 - mgx\sin\alpha &= 0\,,\\
v^2 &= \frac{mg\sin\alpha}{k}x\,.
\end{aligned}
$$

The acceleration of the system is constant, so we can use the formulae $v = at$ and $x = \frac{1}{2}at^2$. Then, we have

$$a = \frac{mg\sin\alpha}{2k}\,.$$

Now we only need to substitute for the acceleration in the equation for x

$$x = \frac{mg\sin\alpha}{4k}t^2\,,$$

so $x \doteq 415\,\mathrm{m}$.

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Problem FoL.31 ... do I hear that?

They say Chuck Norris can go get his beer so fast he meets himself. However, how would that work if Chuck couldn't break the laws of physics? According to the theory of relativity, he can't move faster than light, so he can't see the photons he sent out (in vacuum, without reflection or refraction). However, could he hear himself? What's the minimum possible velocity with which he has to run if he wants to hear (faster, in reverse) what he said on the way there when running back? Chuck speaks with frequency $f = 200\,\mathrm{Hz}$ and the human ear (Chuck's ear too) can hear frequencies in the range $20\,\mathrm{Hz}$ through $20\,\mathrm{kHz}$. The speed of sound is $c = 340\,\mathrm{m{\cdot}s^{-1}}$.

Matěj imagined what it'd be like if Chuck Norris was subject to the laws of physics.

In order to hear himself, he has to move faster than the sound waves he's emitting. Therefore, Chuck's velocity v has to be higher than c. When he runs back with velocity v, he'll meet the sound waves in reverse order, so he'll hear what he said backwards. Due to the Doppler effect, the frequency of sound when the source and receiver move towards each other with velocities equal to v satisfies

$$\frac{f'}{f} = \frac{c+v}{c-v}\,.$$

The condition required for him to hear himself is

$$20\,\mathrm{Hz} \leq -f' \leq 20{,}000\,\mathrm{Hz}\,,$$
$$0.1 \leq -\frac{f'}{f} \leq 100\,.$$

It also follows from the Doppler law that when the source and receiver move towards each other with equal velocities, the absolute value of the incoming frequency can't be smaller than that of the original frequency. Therefore, we're only interested in the second condition, which says that the ratio of frequencies has to be smaller than 100. The negative sign of the frequency means Chuck hears himself in reverse. The resulting frequency has to be negative too. From the previous equation, we can express the velocity v as a function of the change in frequency

$$v = \frac{\frac{f'}{f} - 1}{\frac{f'}{f} + 1} c\,.$$

Substituting $f'/f = -100$, we obtain a condition for the velocity with which Chuck can move

$$v \geq \frac{101}{99} c = 346.9\,\mathrm{m{\cdot}s^{-1}}\,.$$

For any higher velocity, he'll hear himself clearly and the frequency he hears will approach the emitted (negative) frequency.

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Problem FoL.32 ... we'll be there in no time

*In a galaxy far far away, a gigantic spaceship that can move with velocity $v = 0.002c$ was built. After reaching this velocity, the spaceship set course for Earth. What will be the error made by an observer on Earth when estimating its distance, if he believes the spaceship to be a star and measures its redshift as $z = 0.005$? Assume the Hubble constant is $H = 70\,\mathrm{km{\cdot}s^{-1}{\cdot}MPc^{-1}}$. Enter the result in **megaparsecs**.* *Mirek was afraid of space invasion.*

The redshift and spaceship velocity are small enough, so we can use the linear approximation of Doppler effect

$$v_\mathrm{r} = zc\,,$$

where v_r is the velocity with which an object is moving away from us. According to Hubble's law,

$$v_\mathrm{r} = HD\,,$$

where D is the object's distance. According to an observer on Earth, the spaceship's distance is

$$D = \frac{zc}{H}\,,$$

but its real distance is

$$D' = \left(z + \frac{v}{c}\right)\frac{c}{H}\,.$$

The observer's error is

$$|D' - D| = \frac{v}{H} \doteq 8.6\,\mathrm{MPc}\,.$$

The spaceship will never reach us, of course – this holds for any object for which we observe redshift.

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Problem FoL.33 ... boring laboratory routine

We are sitting in a lab, measuring the spectral lines of hydrogen. One value of wavelength in the series of measurements is extremely low, $\lambda = 91.184\,\mathrm{nm}$*. Assuming the Bohr model of hydrogen is exact, what would be the distance of the electron whose transition caused this exceptional emission? We also assume that the electron was in a bound state, no matter how large the initial distance was. In your calculations, use the following constants: ionization energy of hydrogen* $E_0 = 13.598{,}4\,\mathrm{eV}$*, Planck constant* $h = 6.626{,}07\cdot 10^{-34}\,\mathrm{J{\cdot}s}$*, speed of light* $c = 2.997{,}92\cdot 10^{8}\,\mathrm{m{\cdot}s^{-1}}$ *and elementary charge* $e = 1.602{,}18\cdot 10^{-19}\,\mathrm{C}$*.*

Hint *Use all the qunatities to the full given precision!* *Mirek sees atoms.*

Using the Bohr model we can derive an expression for emitted wavelength

$$\frac{1}{\lambda} = \frac{E_0}{hc}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\,,$$

where n_2 is an integer describing the initial energy level of the electron and n_1 is the final level. E_0 is the ionization energy (expressed in J). Basic knowledge of hydrogen series tells us that our observation fits into the Lyman series, i. e. $n_1 = 1$ (this can be confirmed by a short computation).

Electron-proton distance in the Bohr model is given by

$$r_n = r_0 n^2\,,$$

where $r_0 \doteq 52.9\,\mathrm{pm}$ is the so called Bohr radius. Expressing n_2 from the first equation (there's no point in rounding to the nearest integer due to very limited precision) and substituing to the formula for radius, we get

$$r = \frac{r_0}{1 - \frac{hc}{\lambda E_0}} \doteq 0.55\,\mathrm{\mu m}\,.$$

We can concluded that before emission, the radius of the hydrogen atom was comparable to the size of an average bacterium.

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Problem FoL.34 ... loud music

At a disco, there's loud music blaring from one loudspeaker with power 200 W*. Matěj doesn't like that music though, so he decides to play his own on his phone via its speaker with power* 4 W*. How many people will hear Matěj's track louder than the disco? Matěj is standing* 10 m *from the loudspeaker and the number density of people is* $2\,\mathrm{m^{-2}}$*.*

It's common to think about such things at a disco, right?

People will hear better the song whose intensity at that point is higher. Let's denote the power of the loudspeaker by P_D, the power of the phone speaker by P_M. We'll use the fact that intensity of sound I is actually power incident upon a unit area

$$I = \frac{P}{4\pi r^2}\,,$$

where r is the distance from a source with power P. Let's set up cartesian coordinates: the loudspeaker is located at the origin and the phone at the coordinates $(l, 0)$, where $l = 10\,\mathrm{m}$. The individual intensities depend on the position of the receiver,

$$I_\mathrm{D} = \frac{P_\mathrm{D}}{4\pi(x^2+y^2)}\,,$$
$$I_\mathrm{M} = \frac{P_\mathrm{M}}{4\pi[(x-l)^2+y^2]}\,.$$

Let's find the curve describing the border of the area where the phone has higher intensity

$$I_\mathrm{D} = I_\mathrm{M}\,,$$
$$\frac{P_\mathrm{D}}{4\pi\left(x^2+y^2\right)} = \frac{P_\mathrm{M}}{4\pi\left((x-l)^2+y^2\right)}\,,$$
$$P_\mathrm{D}(x-l)^2 + P_\mathrm{D}y^2 = P_\mathrm{M}x^2 + P_\mathrm{M}y^2\,,$$
$$x^2(P_\mathrm{D}-P_\mathrm{M}) + y^2(P_\mathrm{D}-P_\mathrm{M}) - 2xlP_\mathrm{D} + P_\mathrm{D}l^2 = 0\,,$$
$$x^2 - x\frac{2lP_\mathrm{D}}{P_\mathrm{D}-P_\mathrm{M}} + y^2 + \frac{P_\mathrm{D}l^2}{P_\mathrm{D}-P_\mathrm{M}} = 0\,,$$
$$\left(x - \frac{2lP_\mathrm{D}}{P_\mathrm{D}-P_\mathrm{M}}\right)^2 + y^2 = \frac{P_\mathrm{D}^2l^2}{(P_\mathrm{D}-P_\mathrm{M})^2} - \frac{P_\mathrm{D}l^2}{P_\mathrm{D}-P_\mathrm{M}}\,,$$
$$\left(x - \frac{2lP_\mathrm{D}}{P_\mathrm{D}-P_\mathrm{M}}\right)^2 + y^2 = \frac{P_\mathrm{D}P_\mathrm{L}l^2}{(P_\mathrm{D}-P_\mathrm{M})^2}\,.$$

We managed to convert this equation to the form describing a circle with center away from the origin. This circle is known as the circle of Apollonius (it's defined by a fixed ratio of distances to 2 points) and its radius is

$$r^2 = \frac{P_\mathrm{D}P_\mathrm{L}l^2}{(P_\mathrm{D}-P_\mathrm{M})^2}\,.$$

The area of this circle is

$$S = \frac{\pi P_\mathrm{D}P_\mathrm{L}l^2}{(P_\mathrm{D}-P_\mathrm{M})^2} \doteq 6.54\,\mathrm{m^{-2}}\,.$$

The number of people hearing the phone with higher intensity is $S\cdot 2\,\mathrm{m^{-2}} \doteq 13$ people.

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Problem FoL.35 ... football player

What's the lowest velocity which a footballer has to impart to a ball located at the border of the penalty area in order to hit the crossbar? The distance to the goal is $d = 16.5\,\mathrm{m}$ and the crossbar is at height $h = 2.44\,\mathrm{m}$. Neglect air resistance and the dimensions of the ball and the crossbar.

Kuba wanted to cheat.

The minimum necessary velocity corresponds to minimum distance to the goal, so we should have the footballer stand straight in front of the goal and kick perpendicularly to the goal line. Now we're dealing with a 2D problem.

The ball follows a standard trajectory in a homogeneous gravity field, which is described by the equations

$$x = vt\cos\alpha\,,$$
$$y = vt\sin\alpha - \frac{1}{2}gt^2\,,$$

where v is the initial velocity of the ball and α is its elevation angle. The crossbar is hit at time t, so

$$d = vt\cos\alpha\,,$$
$$h = vt\sin\alpha - \frac{1}{2}gt^2\,.$$

This gives us two solutions (v, t), but only one of them has positive t. We get

$$v = \frac{d}{\cos\alpha}\sqrt{\frac{g}{2(d\tan\alpha - h)}} = d\sqrt{\frac{g}{d\sin(2\alpha) - h\cos(2\alpha) - h}}\,,$$

where we used the formulas for sine and cosine of a double angle.

Now, we've got velocity as a function of elevation angle only. The minimum of velocity occurs when the denominator under the square root is maximised, which means its derivative with respect to α must vanish. We get

$$\frac{\partial}{\partial\alpha}\left(d\sin(2\alpha) - h\cos(2\alpha) - h\right) = 2d\cos(2\alpha) + 2h\sin(2\alpha) = 0\,,$$
$$\tan(2\alpha) = -\frac{d}{h}\,.$$

We obtained a single extremum, which has to be the minimum, because $v(\alpha)$ is continuous and for angles 90° and $\arctan(h/d)$, we've got $v = +\infty$.

Since $\tan(2\alpha) < 0$ and $\alpha \in (0, 90^\circ)$, we also need $2\alpha \in (90^\circ, 180^\circ)$, when $\sin(2\alpha) > 0$ and $\cos(2\alpha) < 0$. Using the relations between goniometric functions, we can now express

$$\cos(2\alpha) = -\frac{1}{1 + \tan^2(2\alpha)} = -\frac{h}{\sqrt{d^2 + h^2}}\,,$$
$$\sin(2\alpha) = \sqrt{1 - \cos^2(2\alpha)} = \frac{d}{\sqrt{d^2 + h^2}}\,.$$

The last step is to substitute this back to the expression for velocity, which gives us the final expression for minimal velocity of the ball

$$v = \sqrt{g}\sqrt{\sqrt{d^2+h^2}+h} \doteq 13.7\,\mathrm{m\cdot s^{-1}}\,.$$

Therefore $v \doteq 13.7\,\mathrm{m\cdot s^{-1}}$.

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Problem FoL.36 ... water over gold

Lord Waterboard wanted to have his waterproof castle filled with water. The drawbridge of the castle is rectangular (height $h = 3\,\mathrm{m}$ and width $s = 2\,\mathrm{m}$) and rotates around its bottom side. The drawbridge is drawn up and locked at the top; the lock will hold up against forces up to $F = 50\,\mathrm{kN}$ (the hinges at the bottom can withstand any force). What will the water level in the castle be at the moment when the lock breaks and the water flows out and onto the poor villagers? *Matěj was thirsty.*

Since the drawbridge acts as a lever, the lock is exposed to a force that differs from the outward force of the water pressure. For this reason, we need to calculate the force through its torque.

The maximum allowed torque at the drawbridge is$M = hF$. Let's denote the water level by H. The pressure at height x is $p(x) = (H - x)\varrho g$. There are two possible situations we can deal with depending on whether H is larger or smaller than h. In this specific case, $H > h$ (see the result below). The torque with which water pushes against the drawbridge can be computed by integrating

$$M = s\int_0^h xp(x)\,\mathrm{d}x = s\varrho g\left(\frac{1}{2}Hh^2 - \frac{1}{3}h^3\right)\,.$$

We obtain

$$hF = s\varrho g\left(\frac{1}{2}Hh^2 - \frac{1}{3}h^3\right)\,,$$

$$H = \frac{2s\varrho gh^3 + 6hF}{3s\varrho gh^2} \doteq 3.7\,\mathrm{m}\,,$$

which satisfies the condition $H > h$. If we tried to solve the other case (when the water level is below the top of the drawbridge), we'd get $H \doteq 3.6\,\mathrm{m}$, which contradicts the initial assumption.

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Problem FoL.37 ... unbent, unbowed, unsunken

On the flat water surface of a pond floats a hollow half-sphere with diameter $d = 30\,\mathrm{cm}$ and mass $m = 0.2\,\mathrm{kg}$. By a vertical impulse the sphere is forced into oscillation. What is the maximum kinetic energy of the impulse that won't submerge the half-sphere? Neglect the resistive forces of water. *Xellos placed a spoon on his tea.*

Oscillations are irrelevant, all we need to know is that the half-sphere will submerge when the surrounding water reaches its edge. This means that the maximum kinetic energy is equal to work done on the half-sphere that is necessary to move it from the equilibrium position to the critical position.

The resulting force exerted on the sphere is given as the difference between gravity and buoyancy

$$F(x) = \varrho\pi \left(Rx^2 - \frac{x^3}{3} \right) g - mg\,,$$

where x is the vertical size of the submerged part; we used the formula $V = \pi(Rx^2 - x^3/3)$ for the volume of a spherical cap. By integration we get

$$W = \varrho\pi(\frac{Rx^3}{3} - \frac{x^4}{12})g - mgx\,.$$

We need to integrate from equilibrium $F = 0$ to $x = R$. At the equilibrium, the balance equation is

$$Rx^2 - \frac{x^3}{3} = \frac{m}{\varrho\pi}\,,$$

which is a cubic equation with no obvious solution – numerically we get $x_0 \doteq 2.11\,\mathrm{cm}$. The maximum kinetic energy is then equal to $W(R) - W(x_0) \doteq 3.63\,\mathrm{J}$.

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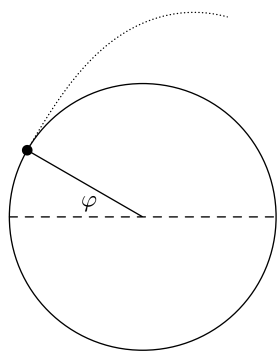
Problem FoL.38 ... in the rain

You're riding a bike, it's raining and there's a puddle in front of you. You don't have any mudguards, so you slow down to $3\,\mathrm{m\cdot s^{-1}}$*. What will be the maximum height (in* cm *above ground) to which the water will splash when flying off a tyre with radius* $35\,\mathrm{cm}$*? Neglect all resistive forces.* *Štěpán rode a bike in the rain.*



Due to centrifugal force, water flies off all points on the tyre. Consider a point at angle φ as shown in the figure. That is, water flies vertically upwards from the point at $\varphi = 0$ and horizontally from the point at $\varphi = \pi/2$, which is the highest point on the tyre.

The vertical component of velocity of water flying off from the point at angle φ is $v_y = v\cos\varphi$. The height of this point above the ground is $y_0 = r\sin\varphi + r$ – we shouldn't forget that the center of the wheel is at height r.

Using the well-known formula for motion under influence of gravity, we can obtain the maximum height reached from this point at angle φ

$$h(\varphi) = y_0 + \frac{v_y^2}{2g} = r\sin\varphi + r + \frac{v^2\cos^2\varphi}{2g}\,.$$

We're looking for the maximum height which the water can reach. The solution will clearly correspond to $\varphi \in \langle 0, \pi/2 \rangle$, since from any other point, water either flies off downwards or doesn't reach as high as from some other point.

The first derivative of h

$$\frac{\mathrm{d}h}{\mathrm{d}\varphi} = \cos\varphi \left(r - \frac{v^2}{g} \sin\varphi \right) .$$

An extremum can occur where the derivative is zero, that is, at points

$$\varphi_1 = \frac{\pi}{2} ,$$
$$\varphi_2 = \arcsin\left(\frac{gr}{v^2}\right) .$$

We can check that φ_1 is a minimum, while φ_2 is a maximum. The maximum height above the ground to which water splashes is

$$h(\varphi_2) = \frac{\left(gr + v^2\right)^2}{2gv^2} ,$$

so $h(\varphi_2) \doteq 87.5\,\mathrm{cm}$.

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Problem FoL.39 ... fire up the synapses

The magnetic field of Earth suddenly vanished and we need to replace it somehow. We know that the original field was approximately like that of a dipole and its magnitude at the equator is approximately given by the formula $|B| = B_0/R^3$, where $B_0 = 3.1 \cdot 10^{-5}\,\mathrm{T}$ and R is the distance from the center of Earth as a multiple of Earth's radius. We want to replace the field using a small coil (solenoid) inserted into the center of Earth. The coil has $N = 10^6$ loops and its radius is $\varrho = 1\,\mathrm{m}$. What current should flow through the coil if the new magnetic field is to be equal to the initial field? Assume that the permeability of all materials is the same as that of vacuum. *Mirek cannot count to four.*

The solenoid with current passing through it generates a magnetic field that's, at sufficient distances from the solenoid, the same as the field of a dipole. The dipole moment of one loop is given by

$$m = IS ,$$

where I is the current passing through it and S is the area enclosed by the loop. For our coil with N loops, the magnetic moment M can be expressed as

$$M = NI\pi\varrho^2 .$$

The field generated by a dipole moment $\boldsymbol{M}$ is described by the formula

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \left(\frac{3\boldsymbol{r}(\boldsymbol{M}\cdot\boldsymbol{r})}{|\boldsymbol{r}|^5} - \frac{\boldsymbol{M}}{|\boldsymbol{r}|^3} \right) ,$$

where μ_0 is the permeability of vacuum and $\boldsymbol{r}$ is the position vector pointing from the center of the dipole. The first term inside the brackets is zero on the equator ($\boldsymbol{M}$ is oriented north-south), so the magnitude of the coil's field at the equator is

$$|B| = \frac{\mu_0}{4\pi} \frac{M}{r^3} = \frac{\mu_0}{4\pi} \frac{NI\pi\varrho^2}{r^3} .$$

By comparing it with Earth's magnetic field, we get

$$\frac{B_0 R_\mathrm{E}^3}{r^3} = \frac{\mu_0}{4\pi}\frac{NI\pi\varrho^2}{r^3}\,,$$

and from this, the current can be expressed as

$$I = \frac{4B_0 R_\mathrm{E}^3}{\mu_0 N\varrho^2}\,.$$

After plugging in all numbers, we get $I \doteq 2.6 \cdot 10^{16}\,\mathrm{A}$. We can safely claim that even if the coil didn't melt in Earth's core by some miracle, it would definitely melt because of the current flowing through it (melt is a serious understatement).

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Problem FoL.40 ... efficient ride

An electric car is driving on a level road. Its effective cross-section is $S = 2\,\mathrm{m}^2$ and its drag coefficient is $C = 0.2$. Its motor has efficiency $\eta = 40\,\%$. What's the optimal constant velocity (which maximises the car's range), if the power consumed inside the car (by air conditioning, radio,...) is a constant $P_0 = 400\,\mathrm{W}$? The density of air is $\varrho = 1.29\,\mathrm{kg{\cdot}m^{-3}}$.

Matěj dreams of driving a Tesla.

Using Newton's formula for air drag

$$F_\mathrm{o} = \frac{1}{2}CS\varrho v^2\,.$$

The power spent when driving with velocity v is

$$P_v = F_\mathrm{o} v = \frac{1}{2}CS\varrho v^3\,.$$

The total power consumption of the car is the sum of this power (divided by efficiency) and the power spent on appliances inside the car. That means

$$P = \frac{1}{\eta}P_v + P_0\,.$$

If the car's battery capacity is E, it can keep going for time $t = \frac{E}{P}$ and its range is

$$s = vt = \frac{Ev}{P} = \frac{Ev}{\frac{1}{2\eta}CS\varrho v^3 + P_0}\,.$$

The first derivative of the distance with respect to velocity vanishes at the maximum,

$$\frac{\mathrm{d}s}{\mathrm{d}v} = \frac{E\frac{1}{2\eta}CS\varrho v^3 + EP_0 - Ev\frac{3}{2\eta}CS\varrho v^2}{\left(\frac{1}{2\eta}CS\varrho v^3 + P_0\right)^2} = 0\,,$$

$$\frac{1}{2\eta}CS\varrho v^3 + P_0 - \frac{3}{2\eta}CS\varrho v^3 = 0\,,$$

$$v = \sqrt[3]{\frac{P_0\eta}{CS\varrho}}\,.$$

Numerically, $v \doteq 6.77\,\mathrm{m\cdot s^{-1}}$.

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Problem FoL.41 ... hot day

During the spring equinox a little boat is swimming in the equatorial seas. The boat has the shape of a rectangle with surface area $5\,\mathrm{m^2}$. The Sun shines on the surface of Earth with intensity $1.3\,\mathrm{kW\cdot m^{-2}}$. How much solar energy in MJ *will the boat receive from sunrise to sunset?* *Štěpán gazed on hot metal roofs.*

The first step is to realize that the angle of incidence of sun rays is changing throughout the day. Right after the sunrise and right before the sunset, the incident power will be minimal, while at noon it will reach the maximum value. Let us assume that the speed of the boat is negligible in comparison to Earth's rotation.

For an angle of incidence α the power can be expressed as $P(\alpha) = SI\sin(\alpha)$, where S is the surface area of our boat and I is the intensity of solar radiation. The angle α changes linearly with time and the dependence can be written as $\alpha(t) = \pi t/T$, where $T = 12\,\mathrm{h}$ is the length of one day. The total energy received by the boat is given by the integral

$$E = \int_0^T P(\alpha)\,\mathrm{d}t = SI\int_0^T \sin\left(\pi\frac{t}{T}\right)\,\mathrm{d}t = \frac{2SIT}{\pi}\,.$$

The boat will receive $178.8\,\mathrm{MJ}$ in the form of solar energy, which is equal to $64\,\%$ of energy received during twelve hours of perpendicular irradiation.

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Problem FoL.42 ... all the way up, please

What's the minimum length (in kilometres) of a space elevator built at the equator, so that it doesn't collapse under gravity? Consider the elevator to be just a few straight ropes leading to space. The elevator is homogeneous and doesn't contain any extra weight at the end.

We're not interested in the trivial solution of zero length. *Štěpán forgot how to use a lift.*

The minimum length corresponds to the case in which the whole rope is vertical.

In order to simplify the solution, let's define linear density of the elevator σ.

Now, let's focus on a small segment of the elevator with length $\mathrm{d}r$ and distance r from the center of Earth. The mass of this segment is $\mathrm{d}m = \sigma\mathrm{d}r$. This small segment is attracted to Earth by gravitational force $G\frac{M\mathrm{d}m}{r^2}$, where G is the gravitational constant and M is the mass of Earth. At the same time, this segment is repelled from Earth by centrifugal force $-\omega^2 r\mathrm{d}m$, where ω is the angular velocity of Earth's rotation. These forces act in opposite directions, so we need to change the signs of one of them.

Summing up these two (counteracting) forces, we get the total force acting on a small segment of the elevator

$$\mathrm{d}F = \sigma\left(G\frac{M}{r^2} - \omega^2 r\right)\mathrm{d}r\,.$$

At small distances, the gravitation is larger, so $\mathrm{d}F > 0$. At some height, the so-called geostationary orbit, $\mathrm{d}F = 0$. For all higher segments, the centrifugal force will be stronger and $\mathrm{d}F < 0$. We're interested in the length of the elevator, measured from the surface of Earth with radius R up to some height h above the surface, which leads to the net force summed over all segments being zero. That means the gravitational forces at small heights and centrifugal forces at large heights cancel out and the elevator will neither fall down nor be ripped off at the base. That means

$$\int_R^{R+h} \mathrm{d}F = 0\,,$$

$$\sigma\left(\frac{hGM}{R(R+h)} - \frac{1}{2}h\omega^2(2R+h)\right) = 0\,.$$

In this last equation, the only unknown is h. After some manipulation, we obtain a quadratic equation with a suitable solution

$$h = \sqrt{\frac{2GM}{R\omega^2} + \frac{R^2}{4}} - \frac{3R}{2}\,,$$

so $h \doteq 144{,}000\,\mathrm{km}$.

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Problem FoL.43 ... shower

What's the maximum useful running time for a shower fed from a cylindrical boiler with cross-section $S = 0.8\,\mathrm{m}^2$ and height $H = 1.5\,\mathrm{m}$? The showerhead is connected directly to the bottom of the boiler, the area through which water flows out of the showerhead is $s = 0.8\,\mathrm{cm}^2$. Consider the minimum volumetric flow rate for which showering is possible to be $Q_0 = 2\,\mathrm{dl{\cdot}s^{-1}}$. Assume that there's no water flowing into the boiler and all water flowing into the showerhead comes from the boiler.

Enter the result in minutes. *There wasn't enough water for Kuba!*

The flow of water must satisfy Bernoulli's equation

$$Q = Sv = su\,,$$

$$\frac{1}{2}\varrho v^2 + h\varrho g = \frac{1}{2}\varrho u^2\,,$$

where v is the speed with which the water level in the boiler drops, u is the speed with which water flows out of the showerhead, ϱ is the density of water, h the height of the water level in the boiler and g is the acceleration due to gravity. Since $S \gg s$, it follows from the continuity equation that $v \ll u$, so we may neglect the term with v in the Bernoulli equation.

From this, we can express v as

$$v = \frac{s}{S}\sqrt{2gh}\,.$$

Now, we can compute the minimum allowed height of water in the boiler

$$Q_0 = Sv_0 = s\sqrt{2gh_0}\,,$$

which gives us

$$h_0 = \frac{Q_0^2}{2gs^2} \doteq 32\,\mathrm{cm}\,.$$

Now we have to solve the differential equation

$$-\frac{\mathrm{d}h}{\mathrm{d}t} = v = \frac{s}{S}\sqrt{2gh}\,,$$

which can be done as

$$\frac{s}{S}\sqrt{2g}\int_0^t \mathrm{d}t = -\int_H^{h_0} \frac{\mathrm{d}h}{\sqrt{h}} = 2\left(\sqrt{H} - \sqrt{h_0}\right)\,,$$

$$t = \frac{S}{s}\sqrt{\frac{2}{g}}\left(\sqrt{H} - \sqrt{h_0}\right) = 50\,\mathrm{min}\,.$$

We can keep showering for 50 minutes using only water from the boiler. In reality, water from the boiler would be mixed with an approximately equal amount of tap water, which would give us an hour and half (provided a suitable technical execution).

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Problem FoL.44 ... space race

*Two spacehips, USS Annie and USS Bonnie, are racing. Both are approaching the finishing line with their maximum speeds $v_\mathrm{A} = c/4$ and $v_\mathrm{B} = c/2$. An observer at rest at the finishing line at time 0 observes Bonnie at distance $d_\mathrm{B} = 250\,\mathrm{km}$ and Annie at distance $d_\mathrm{A} = 100\,\mathrm{km}$. Determine the absolute difference (positive number) between the times when the spaceships reach the finishing line according to this observer in **milliseconds**.*

According to Mirek, placings in a race are relative.

At the moment when we see Annie at distance d_A, the spaceship is actually closer, because the light signal reached us with a delay. Let's denote the travel time of the signal by t and the real distance of Annie from the goal by d'_A. The signal travelled for

$$t = \frac{d_\mathrm{A}}{c}\,,$$

so the real distance is

$$d'_\mathrm{A} = d_\mathrm{A} - v_\mathrm{A}t = d_\mathrm{A}\left(1 - \frac{v_\mathrm{A}}{c}\right)\,.$$

We can similarly compute the real distance of Bonnie

$$d'_\mathrm{B} = d_\mathrm{B} - v_\mathrm{B}t = d_\mathrm{B}\left(1 - \frac{v_\mathrm{B}}{c}\right)\,.$$

The difference between finish times of the spaceships is

$$|t_\mathrm{A} - t_\mathrm{B}| = \left|\frac{d'_\mathrm{A}}{v_\mathrm{A}} - \frac{d'_\mathrm{B}}{v_\mathrm{B}}\right| = \left|\frac{d_\mathrm{A}}{v_\mathrm{A}}\left(1 - \frac{v_\mathrm{A}}{c}\right) - \frac{d_\mathrm{B}}{v_\mathrm{B}}\left(1 - \frac{v_\mathrm{B}}{c}\right)\right|\,.$$

Numerically, we get $|t_{\mathrm{A}} - t_{\mathrm{B}}| \doteq 0.167\,\mathrm{ms}$. The first ship to reach the goal is USS Bonnie.

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Problem FoL.45 ... gray plates

There are two parallel infinite planes in vacuum. The first plane has fixed temperature $T_1 = 200\,\mathrm{K}$ and reflectivity $R_1 = 1/2$ (the ratio of reflected to incident light intensity). The second plane has fixed temperature $T_2 = 300\,\mathrm{K}$ and reflectivity $R_2 = 1/3$. What will be the temperature of a third plane with reflectivity $R = 1/3$ placed in parallel between them?

Assume that all planes radiate as black bodies (have emissivity 1), they just partially reflect incident light. The transmissivities of all bodies are zero.

Kuba wanted to quantify the cooling effect of reflection

A radiated ray will keep being partially reflected between planes and partially absorbed. Therefore, part of the energy radiated by each plane returns back to it. We must compute what part of radiated heat is actually transferred to other planes.

When plane R radiates a ray with intensity I in the direction of plane R_1, the ray reflected from that plane has intensity $R_1 I$, which means that plane R_1 absorbed $(1-R_1)I$. The reflected ray is reflected a second time from plane R, which leads us to the original situation, just with intensity I replaced by $R_1 R I$. This way, we could keep going to get a geometric series for intensity absorbed at plane R_1 in the form

$$I_1 = \sum_{k=0}^{\infty} I (RR_1)^k (1 - R_1) = I \frac{1 - R_1}{1 - RR_1}\,.$$

The complement of this intensity $I - I_1$ is equal to the intensity absorbed by plane R.

Now we've got full information about energy transfer between planes. Replacing R_1 by R_2, we obtain the formula for intensity transferred from plane R to plane R_2 and replacing R_i by R, we obtain the formula for intensity transferred from plane R_i to plane R.

In stationary state, the total radiated energy is equal to total incident energy. This holds for radiated and incident power and therefore for intensities of exchanged radiation. After multiplying terms from Stefan-Boltzmann law by coefficients given by multiple reflection between planes, we may write the law of energy conservation in the form

$$\sigma T^4 \left(\frac{1 - R_1}{1 - RR_1} + \frac{1 - R_2}{1 - RR_2}\right) = \sigma T_1^4 \frac{1 - R}{1 - RR_1} + \sigma T_2^4 \frac{1 - R}{1 - RR_2}\,.$$

After plugging in all reflectivities, the solution is

$$T = \sqrt[4]{\frac{16 T_1^4 + 15 T_2^4}{27}} \doteq 272\,\mathrm{K}\,.$$

This result says that if $T_1 = T_2$, the resulting T would be higher. That's caused by asymmetry between planes R and R_1, where both the energy radiated by plane R_1 and that radiated by plane R flow mostly towards plane R. It's all counter to physical intuition and says that the emissivity of a grey body is actually less than 1.

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Problem FoL.46 ... do not drown!

There's a river of width $d = 20\,\mathrm{m}$ with a parabolic velocity profile, meaning that at the distance y from either bank, the speed of the current is $v(y) = 4v_0 \frac{y}{d}\left(1 - \frac{y}{d}\right)$, where $v_0 = 1\,\mathrm{m{\cdot}s^{-1}}$ is the speed in the middle. The vertical velocity profile is constant. How long would it take us to swim across the river if we stay on a trajectory perpendicular to the bank? In the absence of any current our swimming speed would be $w = 2v_0$. *Kuba was carried away by the current.*

In order to stay on the given trajectory, the swimmer must compensate for the current speed. Let us denote the angle between the vector of the swimmer's velocity and the perpendicular trajectory as α. This angle appears in the formula for the x component of swimmer's velocity

$$w \sin\alpha = v(y) = 4v_0 \frac{y}{d}\left(1 - \frac{y}{d}\right) .$$

The y component can be obtained from the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = w\cos\alpha = w\sqrt{1 - \left[\frac{4v_0}{w}\frac{y}{d}\left(1 - \frac{y}{d}\right)\right]^2} .$$

Here we used $\cos\alpha = \sqrt{1 - \sin^2\alpha}$.

The solution of this differential equation will lead us to the expression for travel time τ. Plugging in the value $w = 2v_0$ and substituing $\psi = y/d$, we get

$$\tau = \frac{1}{w}\int_0^d \frac{\mathrm{d}y}{\sqrt{1 - \left[\frac{4v_0}{w}\frac{y}{d}\left(1 - \frac{y}{d}\right)\right]^2}} = \frac{d}{w}\int_0^1 \frac{\mathrm{d}\psi}{\sqrt{1 - 4\psi^2\left(1 - \psi\right)^2}} \doteq 1.078\frac{d}{w} \doteq 10.8\,\mathrm{s} .$$

The elliptical integral has to be solved numerically. We can now see that the resulting time is only by 0.8 s longer than it would be if the river was not flowing.

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Problem FoL.47 ... pressure cooking

A pressure cooker is a closed pot with walls of thickness $t = 5\,\mathrm{mm}$, thermal conductivity $\lambda = 9\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$, volume $V = 3\,\mathrm{l}$ and surface area $S = 4\,\mathrm{dm^2}$. There is a single hole in its walls with diameter $d = 4\,\mathrm{mm}$. We start heating the pot with power $P = 7\,\mathrm{kW}$, pour $V_\mathrm{v} = 1\,\mathrm{l}$ of water inside, close the lid and bring it boil. The room temperature is $T_\mathrm{i} = 20\,^\circ\mathrm{C}$. Neglect the dependence of boiling point on pressure. What will be the pressure increase (with respect to atmospheric pressure) inside the pot? *Xellos knows how to cook water.*

Let's assume that the system of pot and steam is in thermodynamic equilibrium, so the power P is spent only on heat loss through the walls of the pot and evaporation of water. If the temperature of the pot and water is equal to the boiling point $T = 100\,^\circ\mathrm{C}$, the power lost through the walls of the pot is

$$P_\mathrm{e} = \frac{\lambda S}{t}(T - T_\mathrm{i}) ,$$

so the water evaporates at a rate

$$\frac{\mathrm{d}m}{\mathrm{d}\tau} = \frac{1}{l}\left(P - \frac{\lambda S}{t}(T - T_\mathrm{i})\right) ,$$

where $l \doteq 2.26\,\mathrm{MJ{\cdot}kg^{-1}}$ is the latent heat of vaporisation of water. Since the volume of steam is much larger than volume of water with equal mass, we need an approximately equal mass of water to escape from the pot through the hole; this gives us the velocity of steam escaping through the hole

$$v = \frac{\mathrm{d}m}{\mathrm{d}\tau}\frac{1}{\varrho}\frac{4}{\pi d^2} .$$

We can tie this velocity to an increase in pressure based on Bernoulli equation

$$\Delta p = \frac{1}{2}\varrho v^2 .$$

The density of steam ϱ at temperature T is given by the ideal gas equation of state as

$$\varrho = \frac{Mp}{RT} ,$$

where $M = 18\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of water and p is the pressure (we may assume the increase in pressure is sufficiently small, so p is approximately equal to atmospheric pressure). All together, we get

$$\Delta p = \frac{RT}{Mp}\frac{8}{\pi^2 d^4 l^2}\left(P - \frac{\lambda S}{t}(T - T_\mathrm{i})\right)^2 \doteq 1.6\,\mathrm{kPa} .$$

The assumption $T = 100\,^\circ\mathrm{C}$ isn't completely correct, since the boiling point depends on pressure. However, we can see that we really have $\Delta p \ll p$, so the real boiling point of water in the pot is close to the standard one and the result is approximately correct.

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Problem FoL.48 ... thirteen barrels

Consider a hermetically sealed container with known volume containing a gas with known pressure. We also have a barrel with volume equal to one thirteenth of the volume of the container. First, we connect the barrel to a source of gas (which fills it with gas with fixed pressure) and wait for the pressure in the barrel to equalise. Then, we disconnect the barrel from the source, connect it to the container and wait for the pressures to equalise. Finally, we disconnect the barrel from the container. What must the ratio of pressure of gas in the source to initial pressure in the container be if we want to repeat this process exactly 13 times in order to increase the pressure in the container to exactly 13 times its initial value? Assume that the ambient temperature is constant and that both the barrel and the container are in thermal contact with their surroundings. *Jáchyma got an idea on Friday the 13th.*

After connecting a barrel with volume V_b to the source, the pressure in the barrel will become equal to the pressure of the source p_z. In the container with volume V_n, we have gas with pressure

p_0. After connecting the barrel to the container, the pressure in both vessels will become p_1. Since everything happens at constant temperature, it follows from the equation of state that

$$p_1 \left(V_\mathrm{b} + V_\mathrm{n}\right) = p_\mathrm{z} V_\mathrm{b} + p_0 V_\mathrm{n} \,.$$

We can write this in the form $p_1 = k\, p_0 + (1-k)\, p_\mathrm{z}$, where we used the substitution

$$k = \frac{V_\mathrm{n}}{V_\mathrm{n} + V_\mathrm{b}} \,. \tag{3}$$

We found an equation describing the change in pressure in the container after connecting and disconnecting the barrel once. If the whole process happens n times, the pressure in the container will be

$$p_\mathrm{n} = k^n p_0 + (1-k)\, p_\mathrm{z} \sum_{i=1}^{n} k^{i-1} = k^n p_0 + (1-k)\, p_\mathrm{z} \frac{k^n - 1}{k-1} = k^n p_0 + (1-k^n)\, p_\mathrm{z} \,.$$

The ratio of pressure p_z to p_0 then satisfies

$$\frac{p_\mathrm{z}}{p_0} = \frac{\frac{p_\mathrm{n}}{p_0} - k^n}{1 - k^n} \,.$$

If we substitute 13 for n, the expression in (3) for k, and use conditions from the problem statement for V_n and p_{13}, that is, $V_\mathrm{n} = 13 V_\mathrm{b}$ and $p_{13} = 13 p_0$, we can compute the result in the form

$$\frac{p_\mathrm{z}}{p_0} = \frac{13 - \left(\frac{13}{14}\right)^{13}}{1 - \left(\frac{13}{14}\right)^{13}} \,.$$

Numerically, $p_\mathrm{z}/p_0 \doteq 20.40$.

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Problem FoL.49 ... space cowboy

An astronaut with a space gun is floating freely in empty space. The gun is a futuristic weapon, that, when fired, imparts the projectile with kinetic energy $E_\mathrm{k} = 1 \cdot 10^{17}\,\mathrm{J}$ (measured in the reference frame in which the center of mass of the gun and projectile is at rest). Find the velocity with which the projectile will move away from the astronaut (in the reference frame of the astronaut). The mass of the astronaut with the gun is $m_1 = 100\,\mathrm{kg}$, the mass of the projectile is $m_2 = 10\,\mathrm{kg}$. Neglect any decrease in mass due to burning of explosive charges and rotation of objects. ***Enter the result as a multiple of the speed of light (in units of*** *c).*

Mirek thinks that guns are operational in vacuum.

Let's observe the event from the reference frame of the center of mass, in which the astronaut is initially at rest. The kinetic energy of the fired projectile is

$$E_\mathrm{k} = \sqrt{m_2^2 c^4 + p_2^2 c^2} - m_2 c^2 \,.$$

Next, let's express the momentum of the projectile

$$p_2 = \frac{\sqrt{E_\mathrm{k}^2 + 2 E_\mathrm{k} m_2 c^2}}{c} \,.$$

In this problem, energy is not conserved, but momentum is, so $\boldsymbol{p}_1 = -\boldsymbol{p}_2$ holds. The momentum can also be expressed in the form $\boldsymbol{p}_i = \gamma_i m_i \boldsymbol{v}_i$, where

$$\gamma = \frac{1}{\sqrt{1-\frac{v_i^2}{c^2}}}\,.$$

We get an equation for the velocity of the projectile

$$\frac{\sqrt{E_\mathrm{k}^2 + 2E_\mathrm{k}m_2c^2}}{c} = \frac{1}{\sqrt{1-\frac{v_2^2}{c^2}}}m_2v_2\,. \tag{4}$$

We can express the velocity of the projectile as

$$v_2 = c\frac{\sqrt{\varepsilon^2+2\varepsilon}}{1+\varepsilon}\,,$$

where $\varepsilon = E_\mathrm{k}/(m_2c^2)$ is the ratio of kinetic energy to rest energy of the projectile. To compute the velocity of the astronaut, we'll use equation (4) too, we only need to change the indices on the right hand side from 2 to 1 (and watch out for the sign – the astronaut and the projectile are moving in opposite directions). We find the expression

$$v_1 = c\sqrt{\frac{\varepsilon^2+2\varepsilon}{\mu^2+2\varepsilon+\varepsilon^2}}\,,$$

where $\mu = m_1/m_2$.

Now we just need to combine the velocities correctly. Our center-of-mass observer is moving with velocity v_1 with respect to the astronaut and the projectile is moving with velocity v_2 with respect to the observer. From the astronaut's point of view, using the relativistic velocity addition formula, the projectile will move with velocity

$$u_2 = \frac{v_2+v_1}{1+\frac{v_1v_2}{c^2}}\,.$$

After plugging it all in, the result is numerically $u_2 \doteq 0.4743c$.

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Problem FoL.50 ... pressure cooking reloaded

A pressure cooker is a closed pot with walls of thickness $t = 5\,\mathrm{mm}$, thermal conductivity $\lambda = 9\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$, volume $V = 3\,\mathrm{l}$ and surface area $S = 4\,\mathrm{dm}^2$. There is a single hole in its walls with diameter $d = 4\,\mathrm{mm}$. We start heating the pot with power $P = 7\,\mathrm{kW}$, pour $V_\mathrm{v} = 1\,\mathrm{l}$ of water inside close the lid and bring it to boil. The room temperature is $T_\mathrm{i} = 20\,{}^\circ\mathrm{C}$. Compute the increase in the boiling point of water (in ${}^\circ\mathrm{C}$) with respect to the boiling point at standard conditions (in an open pot), which is $100\,{}^\circ\mathrm{C}$. *Xellos knows how to cook water.*

Let's use the result of the previous version of this problem: the pressure inside the pot is given by the formula

$$p = p_\mathrm{a} + \frac{RT}{Mp}\frac{8}{\pi^2d^4l^2}\left(P - \frac{\lambda S}{t}(T-T_\mathrm{i})\right)^2\,.$$

We still know neither the pressure nor the temperature in the pot. The pressure depends on the boiling point according to Clausius-Clapeyron equation

$$p = p_\mathrm{a} \exp\left(-\frac{L}{R}\left(\frac{1}{T} - \frac{1}{T_\mathrm{v}}\right)\right) ,$$

where $T_\mathrm{v} = 100\,^\circ\mathrm{C}$ is the boiling point at atmospheric pressure and $L = lM$ is the molar latent heat of evaporation of water. We could combine these equations, but we'd get a nasty equation for T which we could only solve numerically. We can, however, use the fact that the pressure and boiling point won't change much inside the pot. Let's denote $\Delta T = T - T_\mathrm{v}$ and expand

$$\begin{aligned} \frac{p}{p_\mathrm{a}} &\approx 1 - \frac{L}{R}\left(\frac{1}{T} - \frac{1}{T_\mathrm{v}}\right) \approx 1 + \frac{L}{R}\frac{\Delta T}{T_\mathrm{v}^2} , \\ \frac{lM}{R}\frac{\Delta T}{T_\mathrm{v}^2} &\approx \frac{8RT_\mathrm{v}}{\pi^2 d^4 l^2 M p_\mathrm{a}}\left(P - \frac{\lambda S}{t}(T_\mathrm{v} - T_\mathrm{i})\right)^2 , \\ \Delta T &\approx \frac{8R^2T_\mathrm{v}^3}{\pi^2 d^4 l^3 M^2 p_\mathrm{a}^2}\left(P - \frac{\lambda S}{t}(T_\mathrm{v} - T_\mathrm{i})\right)^2 . \end{aligned}$$

We utilised the approximations $T \approx T_\mathrm{v}$, $p \approx p_\mathrm{a}$. We get $\Delta T \doteq 0.46\,^\circ\mathrm{C}$ – it is clear that the assumption $T \approx T_\mathrm{v}$ holds. Since the change in the boiling point is sufficiently small, we'd reach a very similar result ($0.45\,^\circ\mathrm{C}$) by directly using the pressure computed in the previous version of this problem and the Clausius-Clapeyron equation.

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Problem FoL.51 ... balls of steel

There are two conducting balls with radii $R_1 = 0.1\,\mathrm{m}$, $R_2 = 0.2\,\mathrm{m}$ inside a homogeneous electric field $\boldsymbol{E} = E\boldsymbol{e}_z$, where $E = 50\,\mathrm{kV{\cdot}m^{-1}}$. The spherical coordinates of the center of the second ball with respect to the center of the first one are (r, ϑ, φ), where the vector corresponding to $\vartheta = 0$ lies in the direction of cartesian axis z and $0 \leq \varphi < 360^\circ$. Find the force between those balls for $R_{1,2} \ll r = 5\,\mathrm{m}$, $\vartheta = 30^\circ$, $\varphi = 50^\circ$. *Xellos wanted to make a problem that has balls.*

Let us begin with only one ball R in a homogeneous field. The ball is a conductor, so there will be induced charge on its surface distributed in such a way that the resulting electric potential on the ball will be constant. In our case this means that there is a linear dependence between the potential of the induced charge and the coordinate z.

This condition is fulfilled by the dipole potential. Two charges $\pm q$ placed along the z axis at a mutual distance $d \ll R$, symetrically with respect to the center of the ball, create a dipole potential

$$V_\mathrm{i}(\boldsymbol{r}) \approx \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{\sqrt{r^2 + zd}} + \frac{1}{\sqrt{r^2 - zd}}\right) \approx \frac{qzd}{4\pi\varepsilon_0 r^3} .$$

On the surface of the ball, we have $r = R$; the potential of the ambient field is $V_\mathrm{e} = -Ez$ (plus a constant), so we need to create a dipole moment

$$p = qd = 4\pi\varepsilon_0 R^3 E .$$

With two balls, we would have to account for the mutual induction between those two balls. In our case, $R_{1,2} \ll r$, thus we can introduce an approximation that neglects this effect.

Now we have to find the force between two (induced) dipoles. We know that the resulting force will be independent of φ due to symmetry. The dipole moments are

$$\boldsymbol{p}_{1,2} = q_{1,2} d_{1,2} \boldsymbol{e}_z = 4\pi\varepsilon_0 R_{1,2}^3 E\,.$$

The formula for the force between two dipoles is well known:[1]

$$\boldsymbol{F} = \frac{3}{4\pi\varepsilon_0 r^5}\left((\boldsymbol{p}_1\cdot\boldsymbol{r})\boldsymbol{p}_2 + (\boldsymbol{p}_2\cdot\boldsymbol{r})\boldsymbol{p}_1 + (\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{r} - 5\frac{(\boldsymbol{p}_1\cdot\boldsymbol{r})(\boldsymbol{p}_2\cdot\boldsymbol{r})}{r^2}\boldsymbol{r}\right)\,.$$

Since $\boldsymbol{p}_1, \boldsymbol{p}_2$ point in the direction of z and $z = r\cos\vartheta$, we obtain

$$\boldsymbol{F} = \frac{3p_1p_2}{4\pi\varepsilon_0 r^5}\left(2r\cos\vartheta\,\boldsymbol{e}_z + (1-5\cos^2\vartheta)\boldsymbol{r}\right)\,.$$

The expression in parentheses has a perpendicular component $(1-5\cos^2\vartheta)r\sin\vartheta$ and parallel component $(3-5\cos^2\vartheta)r\cos\vartheta$. Its absolute value then is, by the Pythagorean theorem,

$$r\sqrt{1+5\cos^4\vartheta - 2\cos^2\vartheta} = r\sqrt{\sin^4\vartheta + 4\cos^4\vartheta}$$

and the resulting force is

$$F = \frac{3p_1p_2}{4\pi\varepsilon_0 r^4}\sqrt{\sin^4\vartheta + 4\cos^4\vartheta} = \frac{12\pi\varepsilon_0 R_1^3 R_2^3 E^2}{r^4}\sqrt{\sin^4\vartheta + 4\cos^4\vartheta} \doteq 1.62\cdot 10^{-8}\,\mathrm{N}\,.$$

We can easily show that the dipole field decreases with distance so fast that the mutual induction is negligible. The dipole field of the second ball at the distance r is weaker than the homogeneous field by approximately $(R_2/r)^3 < 1\cdot 10^{-4}$, a similar evaluation can be done for the first ball.

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Problem M.1 ... tic-toc

How many times greater is the distance travelled by the tip of the second hand on a clock face than the distance travelled by the tip of the minute hand, during one day (exactly 24 hours)? The length of the second hand is 105 mm*, of the minute hand* 100 mm*.*

Kiki was killing some time.

The travelled distance s is given by $s = r\varphi$, where r is the radius and φ is the angle (in radians). The full angle is 2π. Let us see how many full rotations does each hand make during one day. The second hand makes one rotation per minute, making it 1,440 rotations per day. The minute hand makes 24 rotations per day. Since we are interested in the ratio of distances, factor 2π cancels out and the result is

$$x = \frac{1{,}440 r_1}{24 r_2}\,.$$

For given values we get $x = 63$. The second hand will travel 63 times greater distance than the minute hand.

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[1] The derivation is not hard, but it is onerous. For two magnetic dipoles, the formula can be found e. g. on Wikipedia https://en.wikipedia.org/wiki/Magnetic_dipole#Forces_between_two_magnetic_dipoles. To express the force between electric dipoles, we just need to make substitutions $\boldsymbol{m} \leftrightarrow \boldsymbol{p}$ and $\mu_0 \leftrightarrow 1/\varepsilon_0$.

Problem M.2 ... far from school

Luboš and Náry leave the student dormitory at the same time and both head (separately) to school. The distance between the school and the dormitory is $l = 350\,\mathrm{m}$. Luboš has got a swift pace, $v_\mathrm{L} = 2\,\mathrm{m{\cdot}s^{-1}}$, but he is also very forgetful. During his walk to school he had to return three times to the dormitory – the turning points were at distances $100\,\mathrm{m}$, $200\,\mathrm{m}$ and $300\,\mathrm{m}$ from the dormitory. Each time he spent $t_\mathrm{L} = 1\,\mathrm{min}$ inside the dormitory. Náry walks at the same speed as Luboš, $v_\mathrm{N} = v_\mathrm{L}$, but he doesn't turn back to get the things he forgot. However, he has another bad habit – stopping and talking to passersby. Let's assume he bumps into someone every $50\,\mathrm{m}$ (including the dormitory entrance and school entrance) and talks to him for $t_\mathrm{N} = 1.5\,\mathrm{min}$. What will be the difference of Luboš's and Náry's travel times? If Náry arrives first, give the result as a negative number. *Mirek counted how many times Luboš returned.*

Luboš returns three times, every time walking the same distance ($100\,\mathrm{m}$, $200\,\mathrm{m}$ and $300\,\mathrm{m}$) twice, and then he walks straight from dormitory to school ($350\,\mathrm{m}$), so the total travel time is

$$T_\mathrm{L} = 3t_\mathrm{L} + \frac{2}{v_\mathrm{L}}\left(100\,\mathrm{m} + 200\,\mathrm{m} + 300\,\mathrm{m}\right) + \frac{350\,\mathrm{m}}{v_\mathrm{L}} \doteq 955\,\mathrm{s}\,.$$

Náry stops eight times, but walks without returning, so his travel time is

$$T_\mathrm{N} = 8t_\mathrm{N} + \frac{350\,\mathrm{m}}{v_\mathrm{N}} \doteq 895\,\mathrm{s}\,.$$

The difference then is $T_\mathrm{N} - T_\mathrm{L} \doteq -60\,\mathrm{s}$, meaning Luboš will arrive one minute after Náry.

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Problem M.3 ... not falling well

A cylindrical glass with radius of the base $R = 5\,\mathrm{cm}$ is filled with beer so that the surface of the liquid reaches $h = 4\,\mathrm{mm}$ above the edge (measured at the center of the surface). This is due to the strong surface tension of $\sigma = 72\,\mathrm{mN{\cdot}m^{-1}}$. A fly just drowned in the beer and floats at the distance $r = 2\,\mathrm{mm}$ from the center of the surface. How fast will the fly be moving when it reaches the edge of the glass? Assume it was initially at rest, the slope of the surface close to the center is small (but non-zero) and all resistive forces are negligible.

Xellos was drinking a beer.

This is an easy problem relying on the law of energy conservation – velocity is independent of the trajectory and is related to the height difference Δh through the formula

$$v = \sqrt{2g\Delta h}\,.$$

Since we assumed the surface is almost level near the center, the fly must have been initially in the height h above the edge of the glass. Right before reaching the edge of the glass, its height will be 0, therefore $\Delta h = h$ and $v \approx \sqrt{2gh} = 0.28\,\mathrm{m{\cdot}s^{-1}}$.

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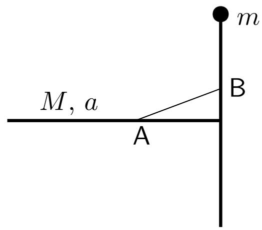
Problem M.4 ... don't fall at all



In the figure there is a construction made of two rods as seen from above. The length and mass of each rod are a and M. A mass $m = M/4$ is attached to the end of one of the rods. We have welded another homogeneous rod to the construction (the thin line in the figure) so that the center of mass of the new rod coincides with the center of mass of the construction. Find the length of the third rod (distance between A and B in the figure) and give the result as a multiple of a.

Mirek was unhappy because he couldn't touch the center of mass.

First, let us find the coordinates of the center of mass T of the construction

$$x_{\mathrm{T}} = \frac{M\frac{a}{2} + Ma + \frac{M}{4}a}{\frac{9}{4}M} = \frac{7}{9}a\,,$$

$$y_{\mathrm{T}} = \frac{\frac{M}{4}\frac{a}{2}}{\frac{9}{4}M} = \frac{1}{18}a\,,$$

where the origin was put at the left end of the left-right pointing rod. The new rod must be welded on the construction in such a way that the distance between A and the center of mass is equal to the distance between B and the center of mass. So

$$|AB| = \sqrt{(2(a - x_{\mathrm{T}}))^2 + (2y_{\mathrm{T}})^2} = \frac{\sqrt{17}}{9}a\,.$$

In multiples of a, the length of the new rod is approximately 0.46.

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Problem E.1 ... breaking a capacitor

We've got a capacitor formed by two separated conducting plates. We charge it and disconnect it from the voltage source. How many times larger will the charge we can get from the capacitor be, if we break it into four identical pieces and place them on top of each other without rotating them in any way? *Štěpán was breaking chocolate into pieces.*

Let's denote the capacity of the original capacitor by C_0 and its voltage by U. If we split it into 4 identical pieces, we get capacitors with quarter capacities (since their surfaces decrease to a quarter of the original) and the same voltage U.

If we combine them in series, the total capacity will be $C_0/16$ (you can verify this yourselves) and the voltage will be 4 times larger.

We can compute the total charge as

$$Q = 4U\frac{1}{16}C_0 = \frac{1}{4}UC_0 = \frac{1}{4}Q_0\,.$$

The charge will be four times smaller.

We can reach the same result by working with capacitor energy, which must be conserved.

Note that the problem can be solved in an even simpler way. We only need to notice that the pieces of plates will have only a quarter of the original charge, so the total charge we can obtain will also be just a quarter of the original.

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Problem E.2 ... ideal voltage

Consider an ideal voltage source. First, we connect to it in parallel three identical resistors and find out that the total power consumed by the resistors is P. What will be the consumed power if we connect the resistors to the source in series? Compute the result as a multiple of P. *Karel was teaching about electric voltage.*

When the resistors with identical resistances R are connected in parallel, their total resistance is equal to $R/3$. When they are connected in series, it's $3R$ (you can verify that yourselves). In order to compute the power P of resistors connected in parallel, we'll use the well-known formula for power consumed by a resistor and Ohm's law. We get

$$P = UI = \frac{U^2}{R_c} = \frac{3U^2}{R}\,,$$

where I is the total current flowing through the circuit, U is the voltage of the source and R_c is the total resistance of the circuit. The power P' of resistors connected in series is

$$P' = UI = \frac{U^2}{R_c} = \frac{U^2}{3R}\,.$$

Simple division now yields $P' = \frac{1}{9}P$.

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Problem E.3 ... ideal current

Consider an ideal current source. First, we connect to it three identical resistors in parallel and find that the total power consumed by the resistors is P. What will the consumed power be if we connect the resistors to the source in series? Compute the result as a multiple of P.
Karel was teaching about electric current.

When the resistors with identical resistances R are connected in parallel, their total resistance is equal to $R/3$. When they are connected in series, it's $3R$ (you can verify that yourselves). In order to compute the power P of resistors connected in parallel, we'll use the well-known formula for power consumed by a resistor and Ohm's law. We get

$$P = UI = R_c I^2 = \frac{1}{3} R I^2\,,$$

where I is the total current flowing through the circuit, U is the voltage of the source and R_c is the total resistance of the circuit. The power P' of resistors connected in series is

$$P' = UI = R_c I^2 = 3RI^2\,.$$

Simple division now yields $P' = 9P$.

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Problem E.4 ... loaded triangle

We built a triangle from three resistors with equal resistances and connected voltage sources parallel to two of the resistors in such a way that their negative poles are connected to the same vertex. How many times larger will the current flowing through the source with voltage $7U$ be compared to the current flowing through the other source with voltage $5U$?

Kuba forced Štěpán to make some problems instead of programming.

First, let's perform the triangle-star transformation. We only need to realise that all resistances are identical, so the resistors in the star will also be identical; let's denote it by R.

We can now apply Kirchhoff's laws. Let's denote the current flowing through the source with smaller voltage by I_1, the current flowing through the larger source by I_2 and the current through the last resistor by I_3. We assume that the currents flow through the sources in the standard direction (we aren't charging them). Then, we get the equation $I_1 + I_2 = I_3$. (We could get slightly different equations if we defined the currents in different directions, but the final result will always be the same.)

For the first loop, we get $I_1R + I_3R = 5U$ and for the second one, $I_2R + I_3R = 7U$. Solving these three equations with three unknown variables, we find that $I_2 = 3U/R$ and $I_1 = U/R$, so the current flowing through the larger source is 3 times larger.

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Problem X.1 ... a nice cup of tea

Somebody mixed the polonium-210 isotope (half-life 138 days) into Mikuláš's tea. Luckily, Mikuláš noticed that. However, he really wants some tea, so he measured the amount of radionuclide in his tea and calculated when will the radioactivity decrease to a safe-to-drink value. After this time of 342 days he drank his tea with no side effects. A few years later, somebody mixed twice the original amount of polonium-210 into his tea. How long does Mikuláš have to wait this time?

Guys, I really am not a retired Russian agent...

All we need is to recall the definition of half-life. Then it is clear that if we wait 138 days, the new problem will reduce itself to the old one (i. e. the amount of the radionuclide will decrease to the original value). Therefore, the waiting time will be $138 + 342 = 480$ days.

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Problem X.2 ... cheesy

Vítek found a shop with suspiciously cheap cheese. On Monday at 14:15, he bought 250 g of cheese there. On Tuesday at 17:21, he bought another 200 g and on Wednesday at 19:45, yet another 160 g. He always stored the cheese he bought carefully in the fridge. Unfortunately, he hadn't noticed that the cheese contained the radioactive nuclide ^{24}Na with half-life 15 hrs.

The mass fraction of this substance in the cheese at the moment when the cheese is bought is $w = 0.0013$. What will be the mass (in grams) of non-decayed ^{24}Na in the cheese at the moment when Vítek wants to eat it – on Thursday at 9:11? Jáchym was skipping through sale flyers.

Radioactive decay follows the equation

$$m' = m\mathrm{e}^{-\frac{t}{T}\ln 2}\,,$$

where m is the initial mass of the isotope and m' is its mass at time t. The last variable in the equation is half-life T.

Let's denote the time period between the first sale and eating the cheese by t_1 and the mass of bought cheese by M_1. The initial mass of the isotope in cheese is $m_1 = M_1 w$. The mass of the isotope after time t_1 then is

$$m_1' = m_1\mathrm{e}^{-\frac{t_1}{T}\ln 2} = M_1 w\mathrm{e}^{-\frac{t_1}{T}\ln 2}\,.$$

Similarly, we can compute the masses of the isotope in the other two pieces of cheese, which we'll denote by m_2' and m_3'. The resulting mass is the sum of masses in individual pieces, which is

$$m_1' + m_2' + m_3' \doteq 0.168\,\mathrm{g}\,.$$

The mass of the non-decayed sodium isotope is 168 mg.

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Problem X.3 ... yoghurt

Jáchym eats half a kilogram of yoghurt for dinner every day. The yoghurt contains a mass fraction $w = 10^{-4}$ of a certain radioactive isotope with half-life 26 days. How many grams of this isotope will be present in Jáchym's body every day before dinner, if he's been eating this way for a long time? Assume that the isotope doesn't leave his body in any way other than through radioactive decay. Jáchym, eating a yoghurt.

Right before dinner, Jáchym contains mass m_0 of this isotope. The yoghurt contains mass $\Delta m = w m_\mathrm{j}$ of the isotope, where m_j is the mass of the yoghurt. After Jáchym eats the yoghurt, the mass of the isotope he contains increases to $m_1 = m_0 + \Delta m$. During the following 24 hours, this mass gradually decreases to the original value m_0. We can use the formula for radioactive decay

$$m_0 = m_1\mathrm{e}^{-\lambda t}\,.$$

Substituting for m_1, we get

$$m_0 = (m_0 + \Delta m)\,\mathrm{e}^{-\lambda t}\,.$$

Now, we can express

$$m_0 = \Delta m\frac{\mathrm{e}^{-\lambda t}}{1 - \mathrm{e}^{-\lambda t}}\,.$$

The time period t has length 1 day and the radioactive decay constant λ satisfies

$$\lambda = \frac{\ln 2}{t_{\frac{1}{2}}}\,,$$

where $t_{\frac{1}{2}}$ is the half-life. After plugging in all the numbers, we get $m_0 \doteq 1.85\,\mathrm{g}$.

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Problem X.4 ... beer

The average volume ratio of air to beer in beer foam is 6.87*. The half-life of an average bubble is* 73 s*. If, initially, the beer reaches the height* 13 cm *above the bottom of the glass and the foam reaches height* 16.5 cm*, what height (in centimetres) will the foam reach after two minutes?*

Made up by Jáchym while watching a Czech cult film.

The beer reaches the height h_1 and the foam h_2. The initial height of foam is $h_0 = h_2 - h_1$. The decay of bubbles will proceed in the same way as radioactive decay – the height of foam at time t is

$$h(t) = h_0 \mathrm{e}^{-\frac{t}{T}\ln 2}\,,$$

where T denotes the half-life. The beer and air released by this decay will have height h_p and h_v respectively. We obtain the equation

$$\begin{aligned} h_\mathrm{p} + h_\mathrm{v} &= h_0 - h\,,\\ 1 + \frac{h_\mathrm{v}}{h_\mathrm{p}} &= \frac{h_0 - h}{h_\mathrm{p}}\,,\\ h_\mathrm{p} &= \frac{h_0 - h}{k+1}\,, \end{aligned}$$

where $k = h_\mathrm{v}/h_\mathrm{p}$ is the given ratio of air to beer. The height which the foam will reach in the glass can be computed as

$$H = h_1 + h_\mathrm{p} + h = h_1 + (h_2 - h_1)\,\frac{k\mathrm{e}^{-\frac{t}{T}\ln 2} + 1}{k+1} \doteq 14.42\,\mathrm{cm}\,.$$

The foam will reach 14.42 cm from the bottom of the glass.

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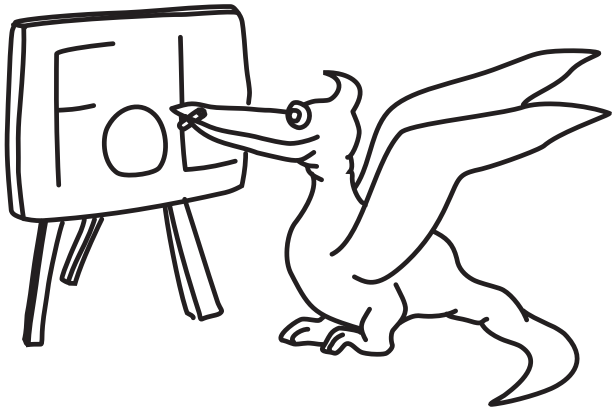
Solutions of 8^{th} Online Physics Brawl



Problem FoL.1 ... underground 3 points

Matěj, with mass $m = 70\,\mathrm{kg}$, entered an underground train car and spotted his friend Jáchym at the other side of the train car, at distance $s = 20\,\mathrm{m}$. Matěj started walking towards Jáchym, in the direction of motion of the train. The train was moving with constant acceleration $a = 1\,\mathrm{m{\cdot}s^{-2}}$. How much work must Matěj do to reach Jáchym?

Matěj loves travelling by underground.

In a train car traveling with a constant velocity, the mechanical work done by Matěj would be zero, since his initial and final position are at the same height and there is no opposing force. While accelerating, the underground becomes a non-intertial reference frame. Matěj is heading in the direction of train car's acceleration, so he must do some work to actually move. Newton's second law says in this case that the force exerted by Matěj in the direction of his motion is (assuming he moves with a constant velocity)

$$F = ma\,.$$

This force is kept constant along the path of length s, so the work done by Matěj is

$$W = Fs = mas = 1{,}400\,\mathrm{J}\,.$$

This is equivalent to the energy contained in about 10 milligrams of sugar (approx. 50 grains). It seems Matěj won't strain too much.

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Problem FoL.2 ... passing trains 3 points

Danka was watching a train with length $l_1 = 120\,\mathrm{m}$, which was passing her with velocity $v_1 = 60\,\mathrm{km{\cdot}h^{-1}}$. She also saw a train of unknown length l_2 approaching from the opposite direction with velocity $v_2 = 80\,\mathrm{km{\cdot}h^{-1}}$. In order to find l_2, Danka measured the time the trains spent passing each other, i.e. the interval between the times when the fronts of the trains met and when the backs of the trains met. She measured $t = 9\,\mathrm{s}$. How long was the second train?

Danka was waiting for a train.

This problem is best solved in the reference frame of the first train. Then the second train approaches the first (stationary) train with velocity $v_1 + v_2$. In order for the trains to pass each other, the second train must travel the distance equal to the length of the first train (so that the front of the second train reaches the back of the first train) plus the length of the second train (so that the backs of both trains are next to each other). Let's denote the total travel time t and the distance covered

$$s = l_1 + l_2\,.$$

It follows

$$l_1 + l_2 = (v_1 + v_2)t\,.$$

Now we express the length l_2 as

$$l_2 = (v_1 + v_2)t - l_1 = 230\,\mathrm{m}\,.$$

The length of the second train is $230\,\mathrm{m}$.

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Problem FoL.3 ... let's play a game

3 points

You are playing a game which consists of separate rounds. When you lose a round, you lose a point. When you win a round, you gain a point, and if you have also won the previous two rounds, you gain another point as a bonus. If the chance of winning a round is 50 % (draw is not possible), what is the long-term average (expected value) of the number of points gained per round? *Jáchym plays only ranked matches in Hearthstone.*

Let's assume a round of the game just started. There's a 50 % chance we will lose this round and thus lose a point, and there's a 50 % chance we will win and gain a point. Independently of those probabilities we have won both previous rounds with probability 25 %. The probability to gain a bonus point is therefore 12.5 %. Putting all this together we get

$$p = 0.5 \cdot (-1) + 0.5 \cdot 1 + 0.125 \cdot 1 = 0.125\,.$$

The expected point gain per round is 0.125.

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Problem FoL.4 ... Keep pulling or I'll whip you!

3 points

When building the pyramids, the transport of giant stone blocks was done by exploitation of the good old slave labor. The blocks are transported on an inclined plane with slope $\alpha = 15^\circ$. Whipmaster Jáchym found out that to pull a block up the inclined plane, the full force of 15 slaves is needed. However, when transporting the same block down, he only needs five slaves pulling with full power. Determine the friction coefficient between the block and the inclined plane. Each slave is equally strong and pulls a rope parallel to the inclined plane with constant force. *Matěj was interested in pulling of blocks.*

Let us denote the weight of a block by m and the friction coefficient by f. The normal force pushing the block to the plane is $mg\cos\alpha$. Then the friction force is given by $fmg\cos\alpha$. When the slaves pull a block downward, they are help by the tangential component of the gravitational force $mg\sin\alpha$. When they pull the same block upward, they have to compensate for the same force component. The maximum force exerted by one slave will be denoted F in the following. We get two balance equations (one for pulling upward, one for pulling downward)

$$15F = fmg\cos\alpha + mg\sin\alpha\,,$$
$$5F = fmg\cos\alpha - mg\sin\alpha\,.$$

Solving for the friction coefficient we obtain

$$fmg\cos\alpha + mg\sin\alpha = 3fmg\cos\alpha - 3mg\sin\alpha\,,$$
$$4\sin\alpha = 2f\cos\alpha\,,$$
$$f = 2\tan\alpha\,.$$

For given values we get $f \doteq 0.536$.

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Problem FoL.5 ... Theophist temperature 3 points

In hell, temperature is measured on the Theophist scale. The conversion formula between Theophist and Celsius scales is linear. The temperature $100\,^\circ\mathrm{T}$ *(degrees Theophist) is the boiling temperature of sulfur* $445\,^\circ\mathrm{C}$*. Once it was so cold in hell that paraffin wax solidified. At that time, the hell's meteorological station measured* $-62\,^\circ\mathrm{T}$*, which corresponds to* $40\,^\circ\mathrm{C}$*. Determine the boiling temperature of mercury* ($357\,^\circ\mathrm{C}$) *in hell's units* $^\circ\mathrm{T}$*.*

You are going to get into hot mercury about that.

Let us define the conversion formula $y(^\circ\mathrm{T}) = kx(^\circ\mathrm{C}) + q$ for the two temperature scales, where x is the temperature given in degrees of Celsius, y is the temperature given in degrees of Theophist and q and k are some coefficients. Then by solving the set of equations

$$100 = 445k + q\,,$$
$$-62 = 40k + q$$

we obtain $k \doteq 0.4$ a $q \doteq -78$. We substitute for the coefficients in the conversion formula $y(^\circ\mathrm{T}) = 0.4x(^\circ\mathrm{C}) - 78$. Plugging in the value for the boiling temperature of mercury in Celsius we get $65\,^\circ\mathrm{T}$.

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Problem FoL.6 ... stairs 3 points

Danka steps on the top stair of an escalator. When she is standing still, the escalator carries her to the bottom in $t_1 = 2.0\,\mathrm{min}$*. If the escalator was not moving and Danka walked down it, it would take her* $t_2 = 2.5\,\mathrm{min}$ *to get to the bottom. How long will it take her to get from the top to the bottom if she starts walking down the moving escalator in the middle of the path? Danka walks with constant velocity.*

Danka stopped to think in an underground.

Denote v_1 the velocity of the escalator, v_2 the velocity of Danka walking and t_3 the time in question. If we further denote l the length of the escalator, then we can write for the given velocities

$$v_1 = \frac{l}{t_1}\,,$$
$$v_2 = \frac{l}{t_2}\,.$$

For half of the path Danka is standing still, so she covers this segment in $t_1/2$. Her velocity on the other half of the escalator is given by the sum of v_1 and v_2. The time of travel, t_3, is then expressed through

$$t_3 = \frac{t_1}{2} + \frac{\frac{l}{2}}{v_1 + v_2}\,,$$
$$t_3 = \frac{t_1}{2} + \frac{1}{2}\frac{l}{\frac{l}{t_1} + \frac{l}{t_2}}\,,$$
$$t_3 = \frac{t_1\left(t_1 + 2t_2\right)}{2\left(t_1 + t_2\right)} \doteq 1.6\,\mathrm{min}\,.$$

The travel on escalator will take Danka 1.6 min.

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Problem FoL.7 ... hockey-football problem 4 points

There is a radioactive substance with N_0 particles. Let's denote the number of particles that decay during the second $1/3$ of the first half-life by ΔN. Find the ratio $\Delta N/N_0$.

Waiting for the third third of Half-Life.

The number of undecayed particles after the first third of a half-time is

$$N_1 = N_0 \left(\frac{1}{2}\right)^{\frac{1}{3}} \,,$$

after the second third

$$N_2 = N_0 \left(\frac{1}{2}\right)^{\frac{2}{3}} \,.$$

In result,

$$\frac{N_1 - N_2}{N_0} = \left(\frac{1}{2}\right)^{\frac{1}{3}} - \left(\frac{1}{2}\right)^{\frac{2}{3}} \doteq 0.1637$$

is the portion of particles that decayed during the second third of a half-time.

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Problem FoL.8 ... resistive block 3 points

We have $l = 11.4\,\mathrm{m}$ of wire with circular cross-section with diameter $d = 0.61\,\mathrm{mm}$. The resistance of the wire is $R = 19.6\,\Omega$. What would be the resistance R_a of a cube formed by smelting this wire, if it was measured under direct current, between two perfectly conductive plates touching two opposite sides of the cube?

Karel was thinking up simple physics problems.

For resistance of a wire with circular cross-section with length l and cross-section $S = \pi d^2/4$, the following formula can be applied:

$$R = \varrho \frac{l}{S} = \varrho \frac{4l}{\pi d^2} \,,$$

where ϱ is material resistivity of which the wire is made out, and which we want to obtain from the relation

$$\varrho = \frac{\pi d^2 R}{4l} \,.$$

We can work out the resistance of a cube as

$$R_a = \varrho \frac{l_a}{S_a} = \frac{\pi d^2 R}{4l} \frac{a}{a^2} = \frac{\pi d^2 R}{4la} \,,$$

where a is the length of the edge of the cube, $l_a = a$ and $S_a = a^2$. Now we have to work out the length of the cube's edge. This can be determined from the conserved mass and volume

$$V_a = V \quad \Rightarrow \quad a^3 = lS \quad \Rightarrow \quad a = \sqrt[3]{\frac{\pi d^2 l}{4}} \,.$$

We substitute into the resistance formula and get

$$R_a = \frac{\pi d^2 R}{4l} \sqrt[3]{\frac{4}{\pi d^2 l}} = \left(\frac{\sqrt{\pi} d}{2l}\right)^{\frac{4}{3}} R \doteq 3.36 \cdot 10^{-5}\,\Omega\,.$$

The resistance of the cube with contacts on opposite sides would be only $3.36 \cdot 10^{-5}\,\Omega$. As a side note, the length of the cube is around 1.5 cm.

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Problem FoL.9 . . . queen's justice — 3 points

Poor serfs are bringing eggs to the queen and receive gold coins in exchange. Each egg is worth ten gold coins. However, if a serf brings eggs worth more than 6,700 *gold coins in total, the queen will pay* 1,600 *gold coins less for them to make sure landowners won't get as rich as Her Majesty. If they bring eggs worth from* 5,000 *to* 6,700 *gold coins (inclusive), the pay is reduced only by "only"* 1,000 *gold coins.*

A farmer is breeding hens that lay from 0 to 1,000 *eggs per week (uniformly randomly distributed), which he brings to the queen each Sunday. What is the average (expected) amount of gold coins he earns per week?* *Matěj and Jáchym are obedient servants of Her Majesty.*

Ignoring the money deduction the farmer would earn on average 5,000 gold coins each week (the expected value of the uniform distribution). In 50 % of all cases there will be no deduction because the farmer's weekly earnings will be less than 5,000 (actually the probability is a little bit lower since there are 1,001 possible cases and only in 500 of them there is no deduction, but we can neglect this). Then there is a 17% probability of facing a 1,000 tax and 33% probability of paying 1,600 in tax. The resulting average is

$$5{,}000 - 0.17 \cdot 1{,}000 - 0.33 \cdot 1{,}600 = 4{,}302$$

gold coins.

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Problem FoL.10 . . . scuba diver — 4 points

A scuba diver is $h = 10\,\mathrm{m}$ under the water surface. How large is the area of the surface on which he can see the sky? The refractive index of water is 1.33 *and the refractive index of air is* 1.00. *Šimon is reminiscing about summer.*

The refraction of light passing through a boundary between two media is described by the well-known Snell's law

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2\,,$$

where in this case n_1 denotes the refractive index of water, n_2 is the refractive index of air and α_1 stands for the angle of refraction. The maximum value of the angle of incidence α_2 is 90°. For this value we can find the critical angle of refraction

$$\sin \alpha_\mathrm{m} = \frac{1}{n_1}\,.$$

Invoking the symmetry properties of the given problem, we conclude that the surface area, where the sky reflection is visible, is a circle. The radius of this circle r can be expressed with the help of the tangent function

$$\tan \alpha_{\mathrm{m}} = \frac{r}{h} \,.$$

Then the area of the circle is given by

$$S = \pi r^2 = \pi h^2 \tan^2 \alpha_{\mathrm{m}} = \pi h^2 \frac{\sin^2 \alpha_{\mathrm{m}}}{\cos^2 \alpha_{\mathrm{m}}} = \pi h^2 \frac{\sin^2 \alpha_{\mathrm{m}}}{1 - \sin^2 \alpha_{\mathrm{m}}} \,,$$

where we expressed radius r in terms of depth of submersion of the scuba diver h and angle α_{m}. In the last two steps we also used trigonometric formulas to substitute tan with the sine function. The final result is

$$S = \pi h^2 \frac{1/n_1^2}{1 - 1/n_1^2} = \pi h^2 \frac{1}{n_1^2 - 1} \,,$$

which is approximately $409\,\mathrm{m}^2$.

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Problem FoL.11 ... fifty 3 points

Vašek calibrated his tachometer so that it would show the exact velocity of his car based on the rotation speed of the wheels. However, his tyres are worn and their radius is $35.7\,\mathrm{cm}$, so he replaces them by new ones with $4\,\mathrm{mm}$ larger radius, but forgets to recalibrate the tachometer. How much (in $\mathrm{km{\cdot}h^{-1}}$) does his real velocity differ from the velocity displayed on the tachometer, which is $50\,\mathrm{km{\cdot}h^{-1}}$? *Vašek follows traffic laws.*

Car's velocity is measured via rotational speed of the wheels, so the velocity displayed on the tachometer is calculated from angular frequency ω of the wheels. The velocity v on the tachometer is equal to the velocity of the car with worn tyres of a radius R. After Vašek changes tyres from the old ones to the new with radius $R' = R + \Delta R$, he drives with a velocity $v' = \omega R'$, whereas (because angular frequency ω remains the same) tachometer still shows velocity $v = \omega R$. We want to find the difference Δv of these velocities,

$$\Delta v = v' - v = \omega \left(R' - R\right) = v \frac{\Delta R}{R} \,.$$

In conclusion, Vašek breaks the speed limit by $0.56\,\mathrm{km{\cdot}h^{-1}}$.

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Problem FoL.12 ... Watch out for the turn! 3 points

A car with mass $m = 1.50\,\mathrm{t}$ is taking a turn with radius $R = 30.0\,\mathrm{m}$. The static friction coefficient between the tyres and road surface is $f = 0.55$. With what maximum velocity may the car take the turn without skidding? *Danka took a turn too fast.*

Centrifugal force

$$F_\mathrm{c} = m\frac{v^2}{R}$$

acts in the outward direction on the car taking the turn. A static friction force acts in the opposite direction and compensates F_c completely. However, the strength of static friction is limited by

$$F_\mathrm{t,max} = fmg\,.$$

In order to avoid a car slide, the centrifugal force must be less than or equal to the maximum friction force

$$F_\mathrm{c} < F_\mathrm{t,max}\,,$$

$$m\frac{v^2}{R} < fmg\,.$$

The condition on allowed values of velocity follows

$$v < \sqrt{fgR} \doteq 45.8\,\mathrm{km\cdot h^{-1}}\,.$$

The car can take the turn with a maximum velocity of $45.8\,\mathrm{km\cdot h^{-1}}$.

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Problem FoL.13 ... Planck linear density 5 points

Planck units are physical quantities formed as suitable combinations of speed of light $c = 3.00\cdot 10^8\,\mathrm{m\cdot s^{-1}}$, *reduced Planck constant* $\hbar = 1.05\cdot 10^{-34}\,\mathrm{kg\cdot m^2\cdot s^{-1}}$ *and the gravitational constant* $G = 6.67\cdot 10^{-11}\,\mathrm{kg^{-1}\cdot m^3\cdot s^{-2}}$. *That means we may write* $A = c^\alpha \cdot \hbar^\beta \cdot G^\gamma$, *obtaining all sorts of physical quantities. Consider such an expression for linear density, which is usually denoted by* λ *and has units* $\mathrm{kg\cdot m^{-1}}$. *Compute the sum of exponents* $\alpha + \beta + \gamma$.

Karel likes to think about Planck's units.

Problems of this kind are usually solved through dimensional analysis, so let's try that. The assumption is

$$\lambda = c^\alpha \cdot \hbar^\beta \cdot G^\gamma\,.$$

Plugging in the physical units, this amounts to

$$\mathrm{kg\cdot m^{-1}} = \mathrm{m}^\alpha\cdot\mathrm{s}^{-\alpha}\cdot\mathrm{kg}^\beta\cdot\mathrm{m}^{2\beta}\cdot\mathrm{s}^{-\beta}\cdot\mathrm{kg}^{-\gamma}\cdot\mathrm{m}^{3\gamma}\cdot\mathrm{s}^{-2\gamma}\,.$$

For each unit (kg, m and s) we get a linear equation, resulting in a set

$$\begin{aligned} 1 &= 0\alpha + \beta - \gamma\,,\\ -1 &= \alpha + 2\beta + 3\gamma\,,\\ 0 &= -\alpha - \beta - 2\gamma\,. \end{aligned}$$

We can easily solve this and obtain $\alpha = 2$, $\beta = 0$ and $\gamma = -1$, and so

$$\lambda = \frac{c^2}{G} \doteq 1.35\cdot 10^{27}\,\mathrm{kg\cdot m^{-1}}\,.$$

An alternative approach starts with the definitions of Planck mass and Planck length, which can be then divided to immediately obtain

$$\lambda = \frac{m}{l} = \sqrt{\frac{c\hbar}{G}}\sqrt{\frac{c^3}{\hbar G}} = \frac{c^2}{G}\,.$$

The task requires us to sum the exponents we found, so the final result is 1.

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Problem FoL.14 ... drop velocity

3 points

Compute the velocity of a raindrop (a sphere with radius 3 mm) immediately before hitting the ground. Assume the flow of air around a raindrop is turbulent. The density of air is $\varrho = 1.25\,\mathrm{kg{\cdot}m^{-3}}$ and the drag coefficient of a spherical raindrop is $C = 0.5$.

Katarína likes walking in the rain.

When a rain droplet falls to the ground, gravitational force and drag force affect it. The gravitational force is given as $F_G = mg$. The drag force is (considering turbulent flow) $F_\mathrm{D} = \frac{1}{2}CS\varrho v^2$ and it depends on the drag coefficient C, which is a constant, the cross-sectional area of falling object S, the density of the fluid ϱ, and the velocity of the falling object v. During the fall, the object's velocity changes, so the drag force affecting it changes as well. When the drag force becomes equal to the gravitational force effecting the droplet, there is no resultant force effecting the object and thus the object continues to fall with its terminal velocity. We need to find this velocity from the equation

$$F_\mathrm{D} = F_G \qquad \Rightarrow \qquad \frac{1}{2}CS\varrho v^2 = mg\,.$$

Drag coefficient of a sphere is a constant with known value $C = 0.50$. Also, we know the cross-sectional area of the droplet, calculated from known variables as $S = \pi r^2$, where r is radius of the droplet. The density of the surroundings is the density of air since the droplet is surrounded by air. We want to find the velocity of the droplet, so v is our unknown. We can work out mass of the droplet from its density and volume:

$$m = \varrho_\mathrm{water} V_\mathrm{droplet} = \varrho_\mathrm{water}\frac{4}{3}\pi r^3\,.$$

After rearranging the first equation, we get the desired formula for the velocity of the droplet

$$v = \sqrt{\frac{\varrho_\mathrm{water}\frac{4}{3}2rg}{C\varrho_\mathrm{air}}} \doteq 11.2\,\mathrm{m{\cdot}s^{-1}}\,.$$

The velocity of the droplet when it hits the ground is $11.2\,\mathrm{m{\cdot}s^{-1}}$. We accentuate that this solution works only when there is enough time for gravitational and drag forces to reach an equilibrium.

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Problem FoL.15 ... giga-cave 5 points

Consider a perfectly spherical planet with radius R. The planet is homogeneous, except for a spherical cave with radius $R/8$ touching the surface of the planet. What is the ratio of gravitational acceleration close to the surface at the point where the cave touches the surface to gravitational acceleration at the opposite point? *Karel likes spherical planets.*

Gravitational interaction of individual physical objects is additive. Therefore, we may model our planet as the difference of a full homogeneous sphere and a small full sphere located in the cave. Let ϱ be the density of the material of the planet. Then the mass of the full sphere equals

$$M = \frac{4}{3}\pi R^3 \varrho\,,$$

whereas for the smaller full sphere we have

$$m = \frac{4}{3}\pi \left(\frac{R}{8}\right)^3 \varrho.$$

For gravitational acceleration on a surface of the planet directly above the cavity we obtain

$$g_1 = \frac{GM}{R^2} - \frac{Gm}{\left(\frac{R}{8}\right)^2} = \frac{7}{6}\pi RG\varrho\,,$$

and on the opposite side

$$g_2 = \frac{GM}{R^2} - \frac{Gm}{\left(\frac{15R}{8}\right)^2} = \frac{1{,}799}{1{,}350}\pi RG\varrho\,.$$

Final ratio of these accelerations is

$$\frac{g_1}{g_2} = \frac{225}{257}\,,$$

which roughly equals 0.875.

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Problem FoL.16 ... compound bow or semiautomatic pistol? 5 points

We fire an arrow from a compound bow PSE X-Force Omen and a bullet from a semi-automatic pistol Colt 1903 .32 directly against each other. If these two projectiles collided in a perfectly inelastic collision and merged together, what would be the velocity (including the direction) of the resulting projectile? The arrow has mass $m_1 = 350\,\mathrm{grain}$ (as you surely know, $1\,\mathrm{grain} = 6.479\cdot 10^{-5}\,\mathrm{kg}$) and velocity $v_1 = 366\,\mathrm{ft{\cdot}s^{-1}}$ (feet per second, $1\,\mathrm{ft{\cdot}s^{-1}} = 0.3048\,\mathrm{m{\cdot}s^{-1}}$). The bullet, 7.65 mm Browning Short, has mass $m_2 = 65\,\mathrm{grain}$ and velocity $v_2 = 925\,\mathrm{ft{\cdot}s^{-1}}$. Assume non-rotating point projectiles. The answer should be positive if the resulting projectile flies in the direction of the arrow or negative if it flies in the direction of the bullet.

Karel was thinking about ranged weapons.

Total mechanical energy is not conserved in case of perfectly inelastic collision. On the other hand we can use conservation of linear momentum p and fact, that after the collision projectiles move together with a common velocity w. As a positive direction we choose direction of the arrow as mentioned in the task.

$$p = m_1v_1 - m_2v_2 = (m_1 + m_2)w\,.$$

We can express the velocity as

$$w = \frac{m_1v_1 - m_2v_2}{m_1 + m_2} \doteq 164\,\mathrm{ft{\cdot}s^{-1}} \doteq 50\,\mathrm{m{\cdot}s^{-1}}\,.$$

As we may see, it is better to stand on the side of the bowman. Arrow "pushes" the bullet and they continue with relatively high velocity (roughly 45 % of original velocity of the arrow) towards gunman.

Of course, the in-flight collision of arrow and bullet is not very realistic. Even if they collided, they would probably just change their directions or fracture. To conclude, we should mention that the original velocities in common units are $v_1 = 112\,\mathrm{m{\cdot}s^{-1}}$ and $v_2 = 282\,\mathrm{m{\cdot}s^{-1}}$ and kinetic energies of projectiles are $E_1 = 141\,\mathrm{J}$ and $E_2 = 167\,\mathrm{J}$.

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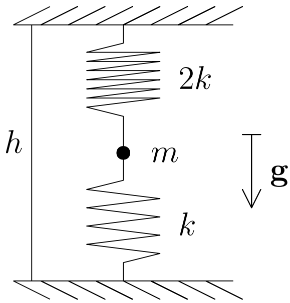
Problem FoL.17 ... twice as stiff spring 3 points



There are two springs with negligible masses and one point mass in the configuration depicted in the figure. The stiffness of the lower spring is $k = 5.0\,\mathrm{kg{\cdot}s^{-2}}$ and the stiffness of the upper spring is $2k$. The rest length of each spring is $l_0 = h/4 = 10\,\mathrm{cm}$ and they are both stretched to lengths $2l_0$. What should be the mass of the point mass if we want the whole configuration to be in equilibrium? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Karel varied Paťo's problem from Slovak Physics Olympiad.

We can divide the upper spring with stiffness $2k$ into two springs with the same rest length and with stiffness k. Force from one of them cancels force from the lower spring and so only one spring with stiffness k and a point of mass is left. Elongation of upper spring is $\Delta l = l_0$, so we have an equation

$$mg = \Delta l k = l_0 k\,,$$

from which we get

$$m = \frac{l_0 k}{g} \doteq 51\,\mathrm{g}\,.$$

The mass is 51 g.

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Problem FoL.18 ... a warm-up hurricane

4 points

The main driving force of a cyclone is the ocean, or more specifically, the heat lost by water evaporating from the surface of the ocean. Steam particles are quickly lifted up to the upper part of the cyclone, where water condensates due to lower temperature, pressure and presence of condensation nuclei. In the last part of the cycle, it falls down to the ocean surface as rain (this process is also very fast). Therefore, the cyclone works between two temperatures – the ocean temperature $T_1 = 280\,\mathrm{K}$ and the cloud temperature $T_2 = -26\,^\circ\mathrm{C}$. It turns out that this process can be described as a thermodynamic cycle with the ideal gas. Using the given information, compute the efficiency of a cyclone as a thermodynamic heat engine with water as the working substance.

Vítek watched news.

The task tells us the cycle consists of four processes: water evaporation, rising of the steam, phase transition in the clouds and precipitation. Processes 2 and 4, i. e. rising and descent, are both very fast. Therefore we will neglect any heat exchange and consider these two processes as adiabatic ($\Delta Q = 0$). We've been also told that the engine works between two constant temperatures. This implies that the other two processes, 1 and 3, are isothermal. A thermodynamic cycle consisting of two isothermal and two adiabatic processes is called the Carnot cycle. And for the Carnot cycle the efficiency is given by

$$\eta = 1 - \frac{T_2}{T_1} \doteq 0.12\,.$$

The efficiency is 0.12.

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Problem FoL.19 ... rain

3 points

It started raining and the Fykos bird noticed that the density of raindrops (hits per unit area) on the western side of the faculty building is twice as large as on the southern side. Determine the direction of wind – specifically, the angle between this direction and the northward direction, in degrees. Assume the sides of the building are exactly facing the four cardinal directions.

Matěj likes to sit inside college during the rain.

Let us denote the density of rain droplets on the southern side by σ. Density on the western side is then 2σ. We want to find the direction along which the densities become seemingly identical.

Assume that wind blowing in one of the cardinal directions results in density σ_0 on the respective side of the building. An inclination of α decreases this density to $\sigma_0 \cos\alpha$ since the projected area of the wall is multiplied by $\cos\alpha$ and therefore receives ($\cos\alpha$)-times as much raindrops.

If the angle α is measured from the northern direction, the density of droplets will be

$$\sigma_0 \cos\alpha$$

on the southern wall and

$$\sigma_0 \cos(90^\circ - \alpha) = \sigma_0 \sin\alpha$$

on the western wall. Thus we obtain a set of linear equations

$$\sigma = \sigma_0 \cos\alpha\,,$$
$$2\sigma = \sigma_0 \sin\alpha\,.$$

solved by

$$\alpha = \arctan 2 \doteq 63.43^\circ\,.$$

As one would expect, the wind blows from the east with a slight inclination to the north.

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Problem FoL.20 ... throwing balls 4 points

Mišo and Dano are standing next to each other. At the same time, each of them throws a tennis ball with the same velocity $v = 15\,\mathrm{m\cdot s^{-1}}$. Dano throws his ball under an elevation angle $\alpha_1 = 45^\circ$ and Mišo throws his under an elevation angle $\alpha_2 = 35^\circ$. How much farther will the winning ball go? Assume that the balls fly next to each other in parallel planes and start from zero height. Neglect air resistance. *Danka remembered physical education lessons.*

First we decompose the velocity of the tennis ball into perpendicular components

$$v_x = v_0 \cos\alpha\,,$$
$$v_y = v_0 \sin\alpha\,.$$

Choosing the coordinate system so that the initial postion of a ball lies in its origin (initial elevation is zero), we can express the time dependence of a ball's coordinates as

$$x = v_x t\,,$$
$$y = v_y t - \frac{1}{2} g t^2\,.$$

Now we want to find the maximum distance d. For time of flight t_d we have

$$x_d = d = v_x t_d\,,$$
$$y_d = 0 = v_y t_d - \frac{1}{2} g t_d^2\,.$$

We solve the first equation for t_d,

$$t_d = \frac{d}{v_x}$$

and substitute into the second equation. After few simple manipulations we obtain

$$d = \frac{2 v_x v_y}{g}\,,$$
$$d = \frac{2 v^2 \sin\alpha\cos\alpha}{g}\,,$$
$$d = \frac{v^2}{g} \sin 2\alpha\,.$$

To compute the difference between ranges d_1 and d_2 for each ball, we just subtract d_2 from d_1 because we know that without air drag the distance d_1 (45° elevation angle) must be larger.

$$\Delta d = d_1 - d_2 = \frac{v^2}{g} \left(\sin 2\alpha_1 - \sin 2\alpha_2\right) \doteq 1.38\,\mathrm{m}\,.$$

The winning ball will go 1.38 m farther than the other ball. However, in real conditions the ball thrown by Mišo might go further because air drag with v^2 dependence decreases the optimal elevation angle.

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Problem FoL.21 ... shadow

4 points

What is the smallest (absolute) latitude such that when you are standing on a horizontal surface at this latitude, your shadow (created by the Sun) will never be shorter than you?

Matěj is hiding in the shadows of the others.

Shadows cast by the sun are the shortest at high noon (Solar noon). First we solve the task for equinox and then we add the effect of inclination of the Earth's rotation axis. The angle between the vertical and the direction of the sun rays is φ, where φ denotes the geographic latitude. Using simple trigonometry we obtain the length of the shadow

$$s = h \tan\varphi\,,$$

where h is the height of the object (i. e. us). We are interested only in the critical case of $s = h$. Then we have

$$\tan\varphi = 1\,,$$
$$\varphi = 45^\circ\,.$$

Now we include inclination of the rotation axis $\Phi = 23.4^\circ$. In the worst case scenario, that is during solstice, the angle between sun rays and the vertical is going to be $\varphi - \Phi$. So

$$\varphi - \Phi = 45^\circ\,,$$
$$\varphi = 45^\circ + 23.4^\circ = 68.4^\circ\,.$$

This means that at latitudes higher than this value, our shadow can never be shorter than us. The result accidentally almost coincides with the latitude of the polar circle $\varphi \approx 66.5^\circ$.

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Problem FoL.22 ... elevator acceleration

4 points

An elevator should travel from the ground level of a building to the top at height 60 m*. During the acceleration phase, its acceleration must not exceed* $4\,\mathrm{m\cdot s^{-2}}$*; when it decelerates, it must not exceed* $6\,\mathrm{m\cdot s^{-2}}$ *(in absolute value). What is the minimum time the elevator needs to reach the top of the building?*

Are there any limits to the optimization of the elevator usage?

The fastest operation regime is achieved when the elevator accelerates as much as possible for time t_1 and then immediately starts decelerating. Let's denote the deceleration time as t_2. Initially the elevator is still and it again comes to a halt at the end of the travel. Therefore

$$v = a_1 t_1 = a_2 t_2\,.$$

We can rewrite this equation to relate the two times,

$$t_2 = \frac{a_1}{a_2} t_1\,.$$

The height (distance travelled under constant acceleration) is given by the well known formula

$$h = \frac{1}{2} a_1 t_1^2 + \frac{1}{2} a_2 t_2^2 = \frac{a_1 (a_1 + a_2)}{2a_2} t_1^2\,.$$

This leads to

$$t_1 = \sqrt{\frac{2ha_2}{a_1(a_1 + a_2)}}\,,$$

and similarly,

$$t_2 = \sqrt{\frac{2ha_1}{a_2(a_1 + a_2)}}\,.$$

The total time is then given by the sum

$$t = t_1 + t_2 = \sqrt{\frac{2h(a_1 + a_2)}{a_1 a_2}} \doteq 7.07\,\mathrm{s}\,.$$

The minimum time of travel is 7.07 s.

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Problem FoL.23 ... Kevin the Cube 5 points

A large homogeneous cube with side length $a = 15\,\mathrm{m}$ decided to roll around one of its edges on a horizontal surface in such a way that the energy used for this would be the minimum possible. Unfortunately, at distance a from this egde, there is a tomato that is soon to be squashed. With what velocity will an edge of the cube hit the tomato? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Matěj is fascinated by rolling balls.*

During the rolling motion the center of mass of the cube will reach its maximum height of $\frac{\sqrt{2}}{2}a$. After reaching the maximum, the following ("falling") motion will decrease the potential energy of the cube by

$$\Delta E = \frac{1}{2}\left(\sqrt{2} - 1\right) amg\,,$$

where m is the mass of Kevin the Cube. According to the Steiner theorem, the moment of inertia of a cube rolling around its edge is

$$J = \frac{1}{6} ma^2 + \frac{1}{2} ma^2 = \frac{2}{3} ma^2\,.$$

Now from the kinetic energy formula $E_\mathrm{k} = \frac{1}{2}J\omega^2$ we can easily obtain

$$v = a\omega = a\sqrt{\frac{2\Delta E}{J}} = a\sqrt{\frac{\left(\sqrt{2}-1\right)amg}{\frac{2}{3}ma^2}} = \sqrt{\frac{3(\sqrt{2}-1)}{2}ag} \doteq 9.562\,\mathrm{m{\cdot}s^{-1}}\,.$$

The edge will smash the tomato with velocity $9.56\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.24 ... chained

5 points

A chain with mass $6.0\,\mathrm{kg}$ *and length* $11\,\mathrm{m}$ *is hanging down from one end at height* $15\,\mathrm{m}$ *above the ground. Find the maximum kinetic energy of the chain after it is released. Consider that the chain hits the ground perfectly inelastically.*

Josef Jírů wanted to enrich our participants with a problem with a chain.

Let's denote the mass of the chain as m, its length as l, the height from the upper end above the ground as h and x is the distance of the chain link, that has just hit the ground, from the chain's upper end. If the entire chain is moving, its kinetic energy E_k is increasing. At the instant of impact, the kinetic energy equals $E_\mathrm{k} = mg(h-l) = 235\,\mathrm{J}$. Later, the kinetic energy of the chain for $x \in \langle 0, l\rangle$ is given by

$$E_\mathrm{k}(x) = \frac{1}{2}\frac{x}{l}m2g(h-x) = \frac{mg}{l}\left(-x^2 + hx\right)\,.$$

The expression in the brackets is a quadratic function with zeroes 0 and h, the function attains its maximum for $x = h/2$. For the maximum energy, one obtains

$$E_\mathrm{kmax} = E_\mathrm{k}(h) = \frac{mgh^2}{4l} \doteq 301\,\mathrm{J}\,,$$

which is, indeed, a higher value than $235\,\mathrm{J}$.

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Problem FoL.25 ... cork problem

4 points

Vítek was investigating some fluid properties using a water surface with surface tension $\sigma_1 = 70\,\mathrm{mN{\cdot}m^{-1}}$. *For this purpose, he made a thin cork board with mass* $m = 100\,\mathrm{g}$ *and side lengths* $b = 5\,\mathrm{cm}$ *and* $c = 7\,\mathrm{cm}$, *which was – to his surprise – floating on the surface. He decided to get the board moving, so he added a bit of detergent to one of its shorter sides. The surface tension of the resulting soap (detergent) solution is* $\sigma_2 = 40\,\mathrm{mN{\cdot}m^{-1}}$. *Help Vítek find the acceleration of the cork board. For simplicity, assume that water resistance is negligible and that the contact angles are always* 90°.

Vítek likes to accelerate cork with some detergent.

We dump the detergent to the shorter edge of the board with a length of b. The formed soapsuds have different surface tension σ_2 and therefore the board is no longer in the equilibrium. The forces acting on sides with a length of c will cancel out. The net force is given by

$$F_1 - F_2 = b(\sigma_1 - \sigma_2) = ma \qquad \Rightarrow \qquad a = \frac{b(\sigma_1 - \sigma_2)}{m}\,.$$

If we evaluate this equation, we obtain $a = 1.5 \cdot 10^{-2}\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.26 ... olive one — 6 points

An ideal, seedless olive may be modelled by a hollow sphere with inner and outer radii $r_1 = 3.00\,\mathrm{mm}$ and $r_2 = 10.0\,\mathrm{mm}$, respectively. If the density of an ideal olive is $\varrho_\mathrm{o} = 816\,\mathrm{kg{\cdot}m^{-3}}$, how deep will it submerge in brine with density $\varrho_\mathrm{b} = 1{,}211\,\mathrm{kg{\cdot}m^{-3}}$? Assume that an ideal olive is perfectly tasty and that brine fills the hole created by the missing seed.

Jáchym is attracted by olives.

Let $V(x)$ denote the volume from the bottom of the olive up to a distance x. The volume of the whole olive is given by

$$V(2r_2) = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\,.$$

Archimedes' principle tells us that the volume $V(h)$ submerged in the brine may be written as

$$V(h) = \frac{\varrho_\mathrm{o}}{\varrho_\mathrm{b}} V(2r_2)\,.$$

Let's divide $V(x)$ into three separate functions on three disjoint intervals. For $x \in \langle 0, r_2 - r_1\rangle$, we may write

$$V(x) = \int_0^x S(y)\mathrm{d}y\,,$$

where the cross-sectional area at height y above the bottom of the olive is given by

$$S(y) = \pi r^2(y) = \pi\left(r_2^2 - (r_2 - y)^2\right) = \pi\left(2r_2 y - y^2\right)\,.$$

Substituting for the integrand, we get the well-known formula for the volume of a spherical cap

$$V(x) = \pi\int_0^x \left(2r_2 y - y^2\right)\mathrm{d}y = \pi\left[r_2 y^2 - \frac{y^3}{3}\right]_0^x = \pi\left(r_2 x^2 - \frac{x^3}{3}\right)\,.$$

Obviously, $V(x) < V(h)$ for all $x \in \langle 0, r_2 - r_1\rangle$, since $V(h) > V(2r_2)/2 = V(r_2)$. Using symmetry, we could calculate $V(x)$ for $x \in \langle r_2 + r_1, 2r_2\rangle$, but it can be shown that these volumes would be too large. Therefore, we need to find a formula for $V(x)$ with x in the middle interval $(r_2 - r_1, r_2 + r_1)$. Again, this is not particularly difficult, we can just subtract the volume of a smaller spherical cap from the volume $V(x)$ obtained in the same way as for the first interval, that is

$$\begin{aligned} V(x) &= \pi\left(r_2 x^2 - \frac{x^3}{3}\right) - \pi\left(r_1\left(x - r_2 + r_1\right)^2 - \frac{\left(x - r_2 + r_1\right)^3}{3}\right) \\ &= \pi x\left(r_2^2 - r_1^2\right) - \pi\left(r_2 - r_1\right)^2\frac{r_2 + 2r_1}{3}\,. \end{aligned}$$

Finally, we calculate the height

$$h = \frac{3V(h) + \pi\left(r_2 + 2r_1\right)\left(r_2 - r_1\right)^2}{3\pi\left(r_2^2 - r_1^2\right)} = \frac{4\frac{\varrho_\mathrm{o}}{\varrho_\mathrm{b}}\left(r_2^3 - r_1^3\right) + \left(r_2 + 2r_1\right)\left(r_2 - r_1\right)^2}{3\left(r_2^2 - r_1^2\right)} \doteq 12.5\,\mathrm{mm}\,.$$

A perfect olive will submerge $h \doteq 12.5\,\mathrm{mm}$ deep in the brine.

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Problem FoL.27 ... Paťo is doing sports 5 points

Legend says Paťo finished a twelve-minute run in six minutes. Along the track, there are milestones at equal distances $\Delta l = 5.000\,000\,000\,\mathrm{km}$; the first milestone is Δl from the starting point. Find the number of milestones Paťo passed if his watch showed that he ran for six minutes when he finished the run. The speed of light is $299{,}792{,}458\,\mathrm{m{\cdot}s^{-1}}$.

Every evening, Lukáš looks under his bed to make sure Paťo isn't there.

The difference in time readings performed by Paťo and by a referee is due to time dilation. If we denote Paťo's proper time by $\Delta\tau = 6\,\mathrm{min}$ and the referee's time by $\Delta t = 12\,\mathrm{min}$, the relation for the time dilation reads

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\,.$$

From this formula we work out the velocity

$$v = c\sqrt{1-\left(\frac{\Delta\tau}{\Delta t}\right)^2}\,,$$

so Paťo traveled distance (observed from the reference frame of milestones)

$$l = v\Delta t = c\Delta t\sqrt{1-\left(\frac{\Delta\tau}{\Delta t}\right)^2} \doteq 186{,}932{,}077\,\mathrm{km}\,.$$

This amounts to 37,386,415 passed milestones (the ratio $\Delta l/l$ was rounded down to an integer value).

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Problem FoL.28 ... energetic pendulum 5 points

A mathematical pendulum is a small ball with mass $m = 5\,\mathrm{g}$ hanging from a (massless) rope with length $l = 2\,\mathrm{m}$. We displace the ball from its equilibrium position by an angle $\alpha = 4^\circ 10'$ and let it oscillate. After completing one swing (half-oscillation), the ball retains $99\,\%$ of the mechanical energy it had before that swing. After how many full swings (half-oscillations) will the velocity of the ball when passing through the equilibrium position decrease under $v_h = 0.2\,\mathrm{m{\cdot}s^{-1}}$? Assume the pendulum loses its energy symmetrically with respect to the equilibrium position.

Daniela closely watched the "renovation" of student cafeteria in Troja.

Let's set ground level of potential energy to the level of ball's equilibrium position. After displacing the pendulum by an angle α, the ball obtains the initial potential energy

$$E_{p_0} = mgh_0\,,$$

while h_0 satisfies

$$h_0 = l\left(1 - \cos\alpha\right) \,.$$

During the oscillations, potential energy changes to kinetic and vice versa, while part of the energy dissipates. Let's denote $\eta = 0.99$. After one swing, the potential energy of the ball equals

$$E_{p_1} = \eta E_{p_0} \,.$$

Let's denote n number of swings, after which the condition from assignment is satisfied. Then after n swings, the potential energy in an amplitude equals

$$E_{p_n} = \eta^n E_{p_0} \,.$$

When the ball passes the equilibrium position for the first time, it has zero potential energy and it's kinetic energy equals

$$E_{k_1} = \frac{1}{2} m v_1^2 \,,$$

while the equation

$$E_{k_1} > E_{p_1}$$

is satisfied. During n-th passing through the equilibrium position it therefore holds

$$\begin{aligned} E_{k_n} &> E_{p_n} \,, \\ \frac{1}{2} m v_h^2 &> \eta^n m g h_0 \,, \\ n &> \log_\eta \frac{v_h^2}{2gl\left(1 - \cos\alpha\right)} \,, \\ n &> 94.8 \,. \end{aligned}$$

So the pendulum must perform 95 full swings.

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Problem FoL.29 ... Einstein with Cherenkov 6 points

A photon collides with a gold surface submerged in a medium with refractive index $n = 1.804{,}17$ *and ejects an electron. This electron is so fast that in this medium, it is moving faster than the speed of light in this medium. What is the maximum possible wavelength of the photon? The work function of gold is* $W_0 = 5.45000\,\mathrm{eV}$*. The speed of light in vacuum is* $c = 2.997{,}925 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$*, Planck constant* $h = 6.626{,}07 \cdot 10^{-34}\,\mathrm{J{\cdot}s}$*, electron rest mass* $m_0 = 9.109{,}38 \cdot 10^{-31}\,\mathrm{kg}$ *and elementary charge* $e = 1.60218 \cdot 10^{-19}\,\mathrm{C}$.

Karel was looking into a nuclear reactor.

A photon with frequency f and wavelength λ traveling in medium with refractive index n has energy

$$E = hf = \frac{hc}{n\lambda} \,. \tag{1}$$

The resulting energy of the ejected electron is calculated as

$$W = E - W_0 \,.$$

Then the velocity of the electron must be at least

$$v = \frac{c}{n}$$

and this velocity is used for computation of relativistic kinetic energy

$$W = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = m_0 c^2 \left(\frac{n}{\sqrt{n^2 - 1}} - 1 \right) .$$

Now we use (1) and easily obtain the maximum wavelength of the electron

$$\lambda = \frac{hc}{nE} = \frac{hc}{n\left(W_0 + W\right)} = \frac{hc}{n\left(W_0 + m_0 c^2 \left(\frac{n}{\sqrt{n^2-1}} - 1\right)\right)} \doteq 6.675{,}82 \cdot 10^{-12}\,\mathrm{m} = 6.675{,}82\,\mathrm{pm}\,.$$

The maximum wavelength is 6.675,82 pm.

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Problem FoL.30 ... two-state problem — 5 points

The Fykos bird started learning statistical physics, but he cannot solve the following introductory problem. Consider N particles in a closed box. Let's split them into two groups – particles in the right half of the box are in state 0 and those in the left half are in state 1. We do not care about which individual particles have which state, only that N_0 have state 0 and N_1 have state 1 ($N_1 + N_0 = N$). Microstates of the system are described by a number $m = N_1 - N_0$, which describes the distribution of particles between the two states. Let's denote the number of ways to split the particles into the two states by $t(m)$ (e.g. for $N = m = 10$, $t = 1$). Note that the exact analytical formula may be simplified to

$$t(m) = \sqrt{\frac{2}{\pi N}}2^N \exp\left(-\frac{m^2}{2N}\right)$$

using Stirling approximation. Evaluate both formulas (exact and approximate) for $N = 20$ and $m = 8$ and find the relative error (in percentage) of the approximate value with respect to the exact value, i.e. by how many percent the approximate value differs from the exact one.

Vítek is in another state.

To solve this task we must realize that the distribution described above is binomial – the problem is the same as e. g. "find the probability of getting 8 heads in 20 coin tosses". Our job is to calculate the binomial coefficient for the event N_1 within N possible states (or even N_1, since the probabilities for 0 and 1 are the same, that is 1/2). From the definition of binomial coefficient,

$$t = \binom{N}{N_1} = \frac{N!}{(N - N_1)!N_1!}\,,$$

which is the number of ways of choosing N_1 particles from the set of N particles. The expression $N - N_1$ is exactly N_0. Substituting $N_0 = (N-m)/2$ and $N_1 = (N+m)/2$ we get

$$t = \frac{N!}{\left(\frac{N-m}{2}\right)!\left(\frac{N+m}{2}\right)!}\,.$$

Now we plug in the number into this formula and also into the approximate one. The resulting difference is

$$q = 1 - \frac{37{,}771}{38{,}760}\,,$$

which can be also written as 2.55 %.

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Problem FoL.31 ... spacey

5 points

In a lab on Earth, we measure the radiation spectrum of a star that is moving away from Earth with velocity $v = 1.70\cdot 10^7\,\mathrm{m\cdot s^{-1}}$. The maximum intensity in this spectrum is at wavelength $\lambda_0 = 700\,\mathrm{nm}$. Determine the surface temperature of the star. Assume that the star is an ideal black body. The speed of light is $c = 3.00\cdot 10^8\,\mathrm{m\cdot s^{-1}}$.

Vítek was reading Fundamental Astronomy.

Given that the star is receding the Doppler redshift must apply. The relation between emitted and observed wavelength reads $\lambda_0 = \lambda_\mathrm{e}(1+z)$, where z is the redshift factor. According to Wien's law a black body with a temperature T has radiation peak at wavelength λ_m (in the reference frame of the star) given by

$$\lambda_\mathrm{m} = \frac{b}{T}\,,$$

where $b = 2.898\,\mathrm{mm\cdot K}$ is Wien's constant and λ_m is identified as λ_e.

Redshift can be expressed in the terms of receding velocity as

$$z = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}} - 1\,,$$

where c is the speed of light. Combining all formulas we determine the temperature as

$$T = \frac{b}{\lambda_0}\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{\frac{1}{2}}\,.$$

For given numerical values we arrive at $T \doteq 4{,}380\,\mathrm{K}$.

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Problem FoL.32 ... one-third train 3 points

A train was moving with velocity $96\,\mathrm{km\cdot h^{-1}}$. It engaged its brakes for the first time at time $99\,\mathrm{s}$ before coming to a halt. What was the distance at which it engaged the brakes? During the first $1/3$ of this distance, the train was decelerating with constant acceleration and decreased its velocity to $1/2$. During the second $1/3$ of the same distance, it was moving with constant velocity, and during the last $1/3$, it was again decelerating with constant acceleration until it stopped. *Josef Jírů wanted to enrich our participants with a problem with trains.*

Let v_0 denote the speed of the train before engaging the brakes and s_1 is $1/3$ of the distance mentioned above. Different times, that the train spent in each $1/3$ of the distance, are in a ratio of

$$t_1 : t_2 : t_3 = \frac{s_1}{\frac{3}{4}v_0} : \frac{s_1}{\frac{1}{2}v_0} : \frac{s_1}{\frac{1}{4}v_0}\,,$$

where denominators of right hand side of equation above equal average speed of the train in given $1/3$ of the distance (we assume $v_{\mathrm{avg}} = s_1/t_{1,2,3}$). Also, right hand side may be written as

$$\frac{s_1}{\frac{3}{4}v_0} : \frac{s_1}{\frac{1}{2}v_0} : \frac{s_1}{\frac{1}{4}v_0} = \frac{4}{3} : 2 : 4 = 2 : 3 : 6\,.$$

E.g. time, that the train spent in second third of the distance, equals $t_2 = 3/11 \cdot 99\,\mathrm{s} = 27\,\mathrm{s}$, which means that it covered distance $s_1 = v_0 t_2/2 = 360\,\mathrm{m}$. Therefore the total distance is three times s_1, which means $1{,}080\,\mathrm{m}$.

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Problem FoL.33 ... marble on a spiral 5 points

There is a spiral defined by parametric equations $x = \sin\varphi$, $y = -\cos\varphi$, $z = \varphi/2\pi$, placed in a homogeneous gravitational field $g = 9.81\,\mathrm{m\cdot s^{-2}}$ in the direction of the negative z-axis. We place a marble on the spiral. The marble does not rotate and moves on the spiral without friction. What will the negative z-component of its acceleration be?

Karel likes to play with marbles.

In the xy plane, the spiral looks like a unit circle. We are interested only in z compound of the acceleration. Because the force, causing the marble to follow a curved path, is perpendicular to the direction of motion and therefore does no mechanical work, we can unwrap the whole spiral into a straight line. By doing this, we transform the issue into the problem of an inclined plane. Let α denote the angle between the inclined and horizontal plane. With every circle travelled the marble changes its height by 1. Therefore we obtain

$$\tan\alpha = \frac{1}{2\pi}\,,$$

$$\alpha = \arctan\frac{1}{2\pi} \doteq 0.16\,.$$

The compound of gravitational acceleration, affecting the marble in the direction of its movement, will be $g\sin\alpha$. However, we are interested in the marble's acceleration in the direction of z axis. To calculate this, we multiply the previous result once more by $\sin\alpha$. We obtain

$$a = g\sin^2\alpha = g\sin^2\arctan\frac{1}{2\pi} = \frac{g}{1+(2\pi)^2}\,,$$

what is approximately $0.242\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.34 ... olive two 6 points

An ideal olive is a perfect sphere containing a spherical seed with radius $r_1 = 3\,\mathrm{mm}$. The density of an ideal olive is equal to the density of brine $\varrho_\mathrm{n} = 1{,}211\,\mathrm{kg{\cdot}m^{-3}}$ so that it would freely float in a can. Consider an open cylindrical can of olives in brine with base radius $R = 31\,\mathrm{mm}$. If we remove the seed from an olive, the rest of the olive floats up to the surface of the brine and this surface decreases by $\Delta h = 4\cdot 10^{-5}\,\mathrm{m}$ in the process. What is the density of the seed? Assume that the brine fills the hole created by removing the seed, too.

Jáchym cannot help himself and still thinks just about olives.

Let r_2 denote radius of the olive. Total density of the perfect olive is ϱ_n. If we mark density of the seed as ϱ_p and of the rest of the olive ϱ_o, we obtain equation

$$\frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\varrho_\mathrm{o} + \frac{4}{3}\pi r_1^3\varrho_\mathrm{p} = \frac{4}{3}\pi r_2^3\varrho_\mathrm{n}\,,$$

from which we can express

$$\varrho_\mathrm{p} = \frac{r_2^3\varrho_\mathrm{n} - \left(r_2^3 - r_1^3\right)\varrho_\mathrm{o}}{r_1^3}\,.$$

When the rest of the olive will be in rest on the surface, we will mark the volume above and under surface as V_1 and V_2, respectively. Then

$$V_1 + V_2 = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\,,$$
$$V_2\varrho_\mathrm{n} = \left(V_1 + V_2\right)\varrho_\mathrm{o}\,,$$

from where we can express

$$V_1 = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\left(1 - \frac{\varrho_\mathrm{o}}{\varrho_\mathrm{n}}\right)\,.$$

Total volume under that surface decreased by volume of the seed and also by volume V_1. This volume is nercessarily equal to the volume, that decreased in the can. That one we can express as $\pi R^2\Delta h$. From this equality we find out that

$$\pi R^2\Delta h = V_1 + \frac{4}{3}\pi r_1^3 = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\left(1 - \frac{\varrho_\mathrm{o}}{\varrho_\mathrm{n}}\right) + \frac{4}{3}\pi r_1^3\,,$$

from which we express

$$\varrho_\mathrm{o} = \left(1 - \frac{3R^2\Delta h - 4r_1^3}{4\left(r_2^3 - r_1^3\right)}\right)\varrho_\mathrm{n}\,.$$

Adding into one of equations found above we obtain the result

$$\varrho_\mathrm{p} = \frac{3R^2\Delta h}{4r_1^3}\varrho_\mathrm{n} \doteq 1{,}293\,\mathrm{kg{\cdot}m^{-3}}\,.$$

Density of a seed of a perfect olive is $\varrho_\mathrm{p} \doteq 1{,}293\,\mathrm{kg{\cdot}m^{-3}}$.

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Problem FoL.35 ... Dan's eggs

7 points

Dan wants to throw a raw egg to Danka in such a way that she can catch it at minimal velocity (in order to minimise the probability that it breaks when caught). He's throwing from a balcony at height h, Danka is at height 0. Due to bad terrain, Danka can only stand at a distance greater than $3h$ from the base of the balcony. Find the elevation angle under which Dan should throw the egg. Do not consider air resistance. *Girls like to grab eggs.*

We choose a coordinate system in which y axis is vertical and points up to Dan, with zero being at the ground level. A throw at Danka's position conforms to equations

$$3h = v_0 t \cos\alpha\,,$$
$$0 = h + v_0 t \sin\alpha - \frac{1}{2} g t^2\,.$$

Factoring out the time variable we get

$$0 = h + 3h\tan\alpha - \frac{9gh^2}{2v_0^2\cos^2\alpha}\,.$$

It follows

$$v_0^2 = \frac{9gh}{2\cos^2\alpha\,(1+3\tan\alpha)} = \frac{9gh}{2\cos^2\alpha + 3\sin 2\alpha}\,.$$

Now we apply energy conservation and seek for an angle that minimizes v_0. Then the impact velocity is minimized as well. In other words, the denominator must be maximized by this angle. Derivative of the denominator with respect to α is

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(2\cos^2\alpha + 3\sin 2\alpha\right) = -2\sin 2\alpha + 6\cos 2\alpha\,.$$

Equating this to zero results in $\tan 2\alpha = 3$. In the interval $(-\pi/2, \pi/2)$ there are two roots, $\alpha_1 \doteq 35.8^\circ$ and $\alpha_2 \doteq -54.2^\circ$. However, for the second value the initial velocity is imaginary, so the correct physical result is 35.8°.

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Problem FoL.36 ... a motivational capacitor

6 points

Vítek has a favourite parallel-plate capacitor. Its plates are separated by distance $d = 2.0\,\mathrm{mm}$ and the area of each plate is $S = 20.0\,\mathrm{cm}^2$. He placed a charge of Q on one plate and a charge of $-Q$ on the other plate, where $Q = 0.1\,\mathrm{nC}$. Then he decided to fill the space between the plates with a dielectric material. As a result, the electric intensity between the plates is no longer constant, but given by the formula

$$E(x) = -\frac{Q\mathrm{e}^{x/d}}{\varepsilon_0 \kappa_0 S}\,,$$

where $\kappa_0 = 1.5$ and x is the distance from the negatively charged plate. Find the capacitance of this capacitor. *Vítek broke a capacitor at his practical class.*

We are dealing with a 1D problem, the electric field can be thus written in the form $E(x) = -\frac{\partial\varphi}{\partial x}$. Capacitance of the capacitor C is defined as $C = Q/\Delta\varphi$, where $\Delta\varphi$ gives the potential difference between the plates and Q is the magnitude of the charge that each plate was given. $E(x)$ is given in the task, so we need to integrate a differential equation

$$\mathrm{d}\varphi = \frac{Q}{\varepsilon_0 \kappa_0 S} \mathrm{e}^{x/d} \, \mathrm{d}x \,.$$

$$\int_{\varphi_1}^{\varphi_2} \mathrm{d}\varphi = \frac{Q}{\varepsilon_0 \kappa_0 S} \int_0^d \mathrm{e}^{\frac{x}{d}} \, \mathrm{d}x \,,$$

$$\Delta\varphi = \frac{Qd}{\varepsilon_0 \kappa_0 S} \left(\mathrm{e} - 1\right) \,.$$

Substituting the result into the definition of capacitance, we get

$$C = \frac{\varepsilon_0 \kappa_0 S}{d \left(\mathrm{e} - 1\right)} \doteq 7.7 \,\mathrm{pF} \,.$$

As we can see, the result is independent of the stored charge. This was to be expected because capacitance is a characteristic of electronic components.

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Problem FoL.37 ... a calm place 5 points

In a vast, empty country, there are two very long, straight, parallel roads $s = 490.0\,\mathrm{m}$ apart. Matěj found himself between these two roads. However, he cannot cross the first road, because there are $n_1 = 0.90$ cars per second flowing through it. The second road has even more traffic, cars flow through it at a rate of $n_2 = 1.60$ cars per second. The velocity of cars is the same on both roads. He had no choice but to wait for death, since there was no crossing in sight. Therefore, he found a place where the total noise from both roads was at a minimum, sat down, and waited… How far from the first road is he sitting? There are no sound barriers, sound travels through the air isotropically and without losses and each car is equally noisy.

Matěj got this idea when he wanted to kill himself after not acing an exam.

How does the noise coming from one road depend on perpendicular distance? A point sound source emits noise isotropically, its intensity I in distance r is $I(r) = K/r^2$, where K is a constant dependent on the intensity of the source.[1] However, the road is not a point source but a line source. If we recall Gauss's law, we can deduce the formula for the intensity of this source $I(r) = k'/r$, where k' is another, unimportant constant. Otherwise, we can integrate the road as a continuous set of point sources

$$I = \int\limits_{-\infty}^{\infty} \frac{k}{y^2 + r^2} \, \mathrm{d}y = \int\limits_{-\infty}^{\infty} \frac{k}{r} \frac{1}{\left(\frac{y}{r}\right)^2 + 1} \frac{\mathrm{d}y}{r} = \frac{k}{r} \int\limits_{-\infty}^{\infty} \frac{1}{t^2 + 1} \, \mathrm{d}t = \frac{k}{r} \left[\arctan t\right]_{-\infty}^{\infty} = \frac{\pi k}{r} \,,$$

where we substituted $y/r = t$ and k is yet another constant, which depends on the linear intensity density of the source.

[1] In the solution we deal with numerous constants whose value is not important to us. We denote those constant as various types of the letter K.

An important property of the sound intensity is its additivity. That is, the total intensity is given by the sum of intensities from all sources (this was already used in the integration). One can easily observe that k and k' are directly proportional to the car flow of respective roads, $k' = \pi k = \mathcal{K}n$.

In the following, the unknown distance of Matěj from the first road is denoted by x. The intensity in distance x is obtained as a sum of intensities from each road,

$$I_{\mathrm{tot}}(x) = \frac{\mathcal{K}n_1}{x} + \frac{\mathcal{K}n_2}{s-x}\,.$$

Now, we need to minimize this function, which is done with the use of its first derivative with respect to x.

$$\frac{\mathrm{d}I_{\mathrm{clk}}(x)}{\mathrm{d}x} = 0$$

$$-\frac{\mathcal{K}n_1}{x^2} + \frac{\mathcal{K}n_2}{(s-x)^2} = 0$$

$$n_1(s-x)^2 = n_2x^2$$

$$\left(\frac{s}{x} - 1\right)^2 = \frac{n_2}{n_1}$$

$$\frac{s}{x} = 1 + \sqrt{\frac{n_2}{n_1}}$$

$$x = s\frac{n_1\sqrt{n_1n_2}}{n_1 - n_2}$$

where the obvious inequality $s/x > 1$ is utilized. Substitution of known values gives us

$$x = \frac{3}{7}s = 210\,\mathrm{m}\,.$$

The second solution, with minus sign, has no physical meaning because $I_{\mathrm{tot}}(x)$ is defined only on the interval between the two roads. This way we avoided the absolute values $|x|$ and $|s-x|$.

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Problem FoL.38 ... finally a complex infinite circuit 5 points

Consider the infinite RLC circuit in figure. The inductance of each coil is $L = 3\,\mathrm{mH}$, the capacity of each capacitor is $C = 4{,}700\,\mathrm{nF}$ and the resistances are $R_1 = 1.5\,\mathrm{k\Omega}$ and $R_2 = 0.3\,\mathrm{k\Omega}$. At which frequency of alternating current is the impedance of the circuit purely real? Neglect phase shift caused by finite propagation speed of electromagnetic field.

Jáchym likes to be complex.

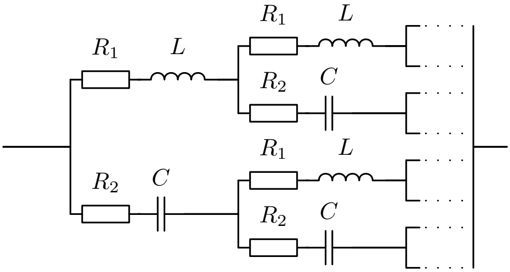
If we denote the impedances of respective components as

$$Z_L = \mathrm{i}\omega L\,,$$

$$Z_C = \frac{1}{\mathrm{i}\omega C}\,,$$

$$Z_{R_1} = R_1\,,$$

$$Z_{R_2} = R_2\,,$$



then for the impedance of the whole circuit we have

$$Z = \frac{(Z_L + Z_{R_1} + Z)(Z_C + Z_{R_2} + Z)}{Z_L + Z_{R_1} + Z + Z_C + Z_{R_2} + Z} \, .$$

Squaring and substitution results in

$$Z^2 = Z_L Z_C + Z_L Z_{R_2} + Z_C Z_{R_1} + Z_{R_1} Z_{R_2} = \frac{L}{C} + R_1 R_2 + \mathrm{i}\left(\omega L R_2 - \frac{R_1}{\omega C}\right) \, .$$

To have a real Z, we need real Z^2 as well, so

$$\omega L R_2 - \frac{R_1}{\omega C} = 0 \qquad \Rightarrow \qquad \omega = \sqrt{\frac{R_1}{R_2 L C}} \, .$$

The frequency is now easily computed from the definition as

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{R_1}{R_2 L C}} \doteq 3.00\,\mathrm{kHz} \, .$$

The impedance of the circuit is real for the frequency $f \doteq 3.00\,\mathrm{kHz}$.

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Problem FoL.39 ... vertical shelf

4 points

There is a copper cuboid with length of $l_0 = 1{,}970.0\,\mathrm{mm}$ and mass of $m = 79.00\,\mathrm{kg}$. If the temperature of copper decreases by $\Delta T = 12\,\mathrm{K}$, the cuboid perfectly fits between two rigid blocks (which do not expand or contract with changes in temperature). We wait for the copper to heat up to its original temperature. What mass may we place on the cuboid before it falls down? The static friction coefficient between copper and the blocks is $f = 0.15$. Assume that the cuboid does not deform when the mass is placed on it.

Jáchym thought up a new way to store things.

We denote the coefficient of linear thermal expansion of copper by α and express the distance between the two blocks as

$$l = l_0 (1 - \alpha \Delta T) \, .$$

With Young's modulus of copper denoted by E, the force exerted by the cuboid on each side of the block is

$$F = SE\frac{l_0 - l}{l_0} = SE\alpha\Delta T\,,$$

where S denotes the cross-section of the cuboid, this value is determined from the known density ϱ of copper

$$S = \frac{m}{\varrho l_0}\,.$$

The maximum magnitude of the static friction force is $F_\mathrm{t} = 2fF$. The maximum mass M that can be placed on the cuboid is then

$$M = \frac{F_\mathrm{t}}{g} - m = m\left(\frac{2fE\alpha\Delta T}{\varrho l_0 g} - 1\right) \doteq 3{,}300\,\mathrm{kg}\,,$$

where we used the values $\alpha = 1.7\cdot 10^{-5}\,\mathrm{K^{-1}}$, $E = 120\,\mathrm{GPa}$ and $\varrho = 8{,}900\,\mathrm{kg\cdot m^{-3}}$. The cuboid can support an additional mass of 3,300 kg.

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Problem FoL.40 ... olive three

5 points

Jáchym is eating olives with eating speed proportional to the square of the number of uneaten olives. A day after he opened a can, there were 32 of them. A day later, there were just 17 left. How many olives were in the can initially? Assume that the number of olives is a continuous variable for Jáchym. *Jáchym dreams only about olives.*

Let's denote the number of olives by N. The eating rate is then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda N^2\,.$$

A simple integration leads to

$$-\frac{1}{N} = \lambda t + C\,.$$

The task tells us that at time $t_1 = 1\,\mathrm{d}$, $N(t_1) = 32$ olives remained, while at time $t_2 = 2\,\mathrm{d}$ only $N(t_2) = 17$ olives remained. Thus, we have two equations for two unknowns. We solve the first for λ,

$$\lambda = -\frac{\frac{1}{N(t_1)} + C}{t_1}\,,$$

and substitute into the second equation:

$$-\frac{1}{N(t_2)} = -\frac{\frac{1}{N(t_1)} + C}{t_1}t_2 + C\,,$$

$$C = \frac{\frac{t_1}{N(t_2)} - \frac{t_2}{N(t_1)}}{t_2 - t_1}\,.$$

Now we can compute the result

$$N(0) = -\frac{1}{C} = \frac{t_1 + t_2}{\frac{t_2}{N(t_1)} - \frac{t_1}{N(t_2)}} = 272\,.$$

There are 272 olives left in the can.

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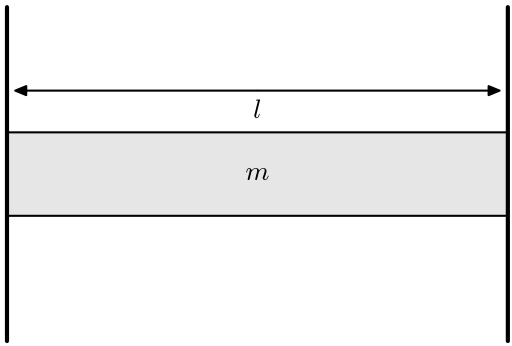
Problem FoL.41 ... elastic loop 6 points

A circular loop with radius $R_0 = 0.45\,\mathrm{m}$ *is placed in a homogeneous static magnetic field with magnitude* $B = 0.83\,\mathrm{T}$ *in such a way that its axis is parallel with the direction of the magnetic field and the loop is compressed by the manetic field. There is an electric current* $I = 16\,\mathrm{A}$ *flowing through the loop. The loop is made from a brass wire with circular cross-section (with radius* $r = 0.254\,\mathrm{mm}$*) and Young's modulus* $E = 98\,\mathrm{GPa}$*. How does the length of the loop reduce (in* mm*)?* *Vašek wanted to know what happens to a magnet in a magnetic field.*



The length of the loop changes from l_0 to l, that is $\Delta l = l - l_0$. Since the cross-section area of a wire is $S = \pi r^2$, the tension in the wire is given by the Hooke's law as

$$\sigma = \frac{\Delta l}{l_0} E = -\frac{F}{S}\,,$$

where F is the force acting inward on each element of the wire. Now, let's choose an infinitesimal section of the wire of length $\mathrm{d}x$. The central angle corresponding to this infinitesimal arc is

$$2\mathrm{d}\varphi = \frac{\mathrm{d}x}{R}\,,$$

where R is the radius of the loop, expressed using $l = 2\pi R$. The initial length of the loop was $l_0 = 2\pi R_0$.

Magnetic force $\mathrm{d}F_\mathrm{m}$ acts upon this element, pointing to the centre of the loop. Then, there is the force F acting on the element from each side. The element is at rest, therefore the forces must cancel out. Also, F can be decomposed into tangential and radial components. The tangential components cancel out each other and the radial component has opposite direction to the magnetic force. This radial force is related to the central angle by

$$\mathrm{d}F_x = F \sin \mathrm{d}\varphi \approx F \mathrm{d}\varphi\,.$$

It is also obvious that $2\mathrm{d}F_x = \mathrm{d}F_\mathrm{m}$ must hold.

The next to last step is to express the magnetic force in known variables. A particle with charge $\mathrm{d}q$ moving with velocity v in the magnetic field B experiences perpendicular force $\mathrm{d}F_m = \mathrm{d}qvB$. Velocity is defined as distance covered in time and electric current is defined as a flow of charge. This leads to

$$\mathrm{d}F_\mathrm{m} = \mathrm{d}qvB = \frac{\mathrm{d}q\,\mathrm{d}x}{\mathrm{d}t} = IB\,\mathrm{d}x\,.$$

Now we have all the ingredients we need to solve this problem. After substituting force F, expressed as

$$F = BIR\,,$$

into the formula for tension, we get

$$\sigma = \frac{\Delta l}{l_0}E = -\frac{BIR}{S}\,.$$

Finally we arrive at

$$\Delta l = \frac{2\pi R_0^2 IB}{R_0 IB - \pi r^2 E}\,.$$

However, we are interested in the shrinkage of the loop, so for the given values we obtained the result $|\Delta l| \doteq 0.85\,\mathrm{mm}$.

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Problem FoL.42 ... astronaut training — 5 points

There is a swing carousel standing on the Earth's surface (in gravity g). With what frequency should it turn so that the visitors riding it would experience net acceleration $2g$? The carousel may be considered a circle with radius $R = 3.5\,\mathrm{m}$ placed $H = 4\,\mathrm{m}$ above the ground with seats attached to the perimeter of this circle by chains. Let's assume that a person sitting in a seat is a point mass at distance $l = 3\,\mathrm{m}$ from the point where the seat is attached to the circle, chains and seats are massless, chains have constant lengths and able to rotate freely around the points where they're attached to the circle. Neglect resistive forces.

Karel was watching the Kosmo series.

In the frame of a person riding on the carousel, the person experiences a gravitational force and a centrifugal force pushing him/her into the seat. Those forces are balanced by the reaction of the seat. The total acceleration can be obtained by dividing the forces by the person's mass. This acceleration is supposed to be equal to $2g$. The centrifugal acceleration then must be

$$a = \sqrt{(2g)^2 - g^2} = \sqrt{3}g\,.$$

At the same time, it holds

$$a = \omega^2 r = 4\pi^2 f^2 r\,,$$

where r is the radius of the seat moving around the central axis and f is the desired frequency. Radius r is given by the sum of the circle radius R and the length of the horizontal projection of the chain. Now, the horizontal length is to the total length as the centrifugal acceleration is to the total acceleration, i.e. in the ratio $\sqrt{3}/2$. So

$$r = R + \frac{\sqrt{3}}{2}l\,.$$

Substituting into the frequency formula, we get the result

$$f = \frac{1}{2\pi}\sqrt{\frac{a}{r}} = \frac{1}{2\pi}\sqrt{\frac{\sqrt{3}g}{R + \frac{\sqrt{3}}{2}l}} \doteq 0.266\,\mathrm{Hz}\,.$$

The carousel would have to rotate with frequency of 0.266 Hz to produce the desired acceleration.

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Problem FoL.43 ... exotic field 6 points

Vítek found a very exotic field in an old textbook. Consider a point-like mass $m = 1.00\,\mathrm{kg}$ at rest in equilibrium in the potential $V(x) = -Ax^{10}\exp(Bx)$, where $A = 2.00\,\mathrm{kg{\cdot}m^{-8}{\cdot}s^{-2}}$ and $B = 0.01\,\mathrm{m^{-1}}$. The mass is then slightly displaced from its equilibrium position. Find the period of small oscillations of the point-like mass around the equilibrium position in this potential field. *Vítek is imagining how Matěj is conducting experiments.*

To solve this problem we have to realize that every smooth enough function can be approximated at its minimum by a parabola. The stiffness of the system can be replaced with $V''(x_0)$, where x_0 is the equilibrium position. As the first step we find this position,

$$\begin{aligned} V'(x) &= -\left(10 + Bx\right) Ax^9 \exp(Bx)\,,\\ V'(x_0) &= 0\,,\\ x_0 &= -\frac{10}{B}\,. \end{aligned}$$

Now we only have to work out the second derivative of the potential and substitute the equilibrium position we just computed. So

$$\begin{aligned} V''(x) &= -\left(Bx + \left(9 + Bx\right)\left(10 + Bx\right)\right) Ax^8 \exp(Bx)\,,\\ V''(x_0) &= \frac{10^9 A}{B^8}\exp(-10)\,. \end{aligned}$$

The period of small oscillations is then determined as

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{V''(x_0)}} = 2\pi\sqrt{\frac{mB^8}{10^9 A}\exp(10)}\,.$$

Plugging in the numerical values we get $T \doteq 2.09 \cdot 10^{-10}\,\mathrm{s}$.

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Problem FoL.44 ... identifying a rod

8 points

Vítek had a cylindrical wire with radius $R = 1.20\,\mathrm{cm}$ made from an unknown material connected to a DC source. Luckily, he knew the energy density of magnetic field $w_{\mathrm{mag}} = 0.50\,\mathrm{J{\cdot}m^{-3}}$ and the energy density of electric field $w_{\mathrm{el}} = 2.50\cdot 10^{-17}\,\mathrm{J{\cdot}m^{-3}}$ immediately above the surface of the conductor. Both fields are parallel with the surface of the conductor. Moreover, using a magnet Vítek found out that the rod is non-magnetic. The conductor is placed in air, the Hall effect is ignored. Find the conductivity of the cylinder's material.

Vítek wanted to make a problem that just needs standard formulas.

With the use of Ohm's law we can express the unknown conductivity in terms of electric field E, which accelerates free charge carries, and current density j as

$$\sigma = \frac{j}{E}\,,$$

where the current density is given by

$$j = \frac{I_R}{S}\,.$$

Here $S = \pi R^2$ denotes the cross-section area up to distance R from the centre of the conductor and I_R is the current passing along the axis of the conductor. The energy densities of the electric and magnetic field are defined as

$$w_{\mathrm{el}} = \frac{1}{2}\varepsilon_0 E^2\,,$$

$$w_{\mathrm{mag}} = \frac{1}{2\mu_0} B^2(R)\,.$$

Electric field E is constant, so we need to find only the magnitude of stationary magnetic field $B(R)$. This can be done with the use of Ampère's law,

$$B(R) = \frac{\mu_0 I_R}{2\pi R}\,.$$

Now by substituting into the definition of w_{mag} and working out the electric current, we get

$$I_R^2 = \frac{8\pi^2 R^2 w_{\mathrm{mag}}}{\mu_0}\,.$$

Rearrangement of the formula defining w_{el} leads to

$$E^2 = \frac{2w_{\mathrm{el}}}{\varepsilon_0}\,.$$

Now, we have all ingredients to express the conductivity in known variables as

$$\sigma = \frac{2}{R}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{w_{\mathrm{mag}}}{w_{\mathrm{el}}}} = \frac{2}{R}\frac{1}{Z_0}\sqrt{\frac{w_{\mathrm{mag}}}{w_{\mathrm{el}}}}\,,$$

where Z_0 is the impedance of vacuum. For given values this yields $\sigma \doteq 6.26\cdot 10^7\,\Omega^{-1}{\cdot}\mathrm{m}^{-1}$. Consulting the tables we conclude that the conductor is made of silver.

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Problem FoL.45 ... doublets

8 points

We used a sodium lamp to symmetrically illuminate two $b = 1.0\,\mu\mathrm{m}$ *wide slits* $a = 0.10\,\mathrm{mm}$ *apart. The light passing through the slits is observed on a screen at distance* $d = 1.0\,\mathrm{m}$*. Determine the ratio of intensity of incident light in the 1*st *minimum to the intensity in the 0*th *maximum.*

Assume that a sodium lamp emits light only on two wavelengths $\lambda_1 = 589.0\,\mathrm{nm}$ *and* $\lambda_2 = 589.6\,\mathrm{nm}$ *with intensity ratio 2:1.* *Xellos likes to enlighten people.*

It is well-known that the zeroth maximum of an interference pattern lies at the axis between the slits ($y = 0$) and the first minimum, with zero intensity, lies at $y = d\lambda/2a$, where λ is the wavelength of the incident light on the screen. This holds for a single wavelength. In this case, there are two but with similar values, so we can safely assume that the first minimum is at $y = d\lambda_1/2a + \Delta y$ where $\Delta y \ll d\lambda_1/2a$.

The intensity on the screen from one wavelength λ up to this first minimum is given approximately as (using $\mathrm{sinc}(x) \equiv \sin x/x$ where I_0 is the intensity at the zeroth maximum)

$$I = I_0 \cos^2\left(\frac{\pi a y}{\lambda d}\right) \mathrm{sinc}^2\left(\frac{\pi b y}{\lambda d}\right) .$$

The second term can be neglected, since $b \ll a$ and therefore the argument of sinc is almost zero at the first maximum, i.e. sinc is approximately equal to one. In the vicinity of the first minimum and for $\lambda \approx \lambda_1$ we can make an estimate

$$\cos\frac{\pi a y}{\lambda d} = \cos\left(\frac{\pi a}{\lambda d}\left(\frac{d\lambda_1}{2a} + \Delta y\right)\right) = \sin\left(\frac{\pi}{2}\frac{\lambda - \lambda_1}{\lambda} - \frac{\pi a}{\lambda d}\Delta y\right) \approx \frac{\pi}{2}\frac{\lambda - \lambda_1}{\lambda} - \frac{\pi a}{\lambda d}\Delta y\,,$$

$$I \approx I_0\left(\frac{\pi}{2}\frac{\lambda - \lambda_1}{\lambda} - \frac{\pi a}{\lambda d}\Delta y\right)^2 .$$

For two wavelengths, the total intensity is $I = I_1 + I_2$, where intensity corresponding to the first wavelength (at the zeroth maximum) is $2I_0$ and I_0 is for the second. Using the approximation above, we have for the first minimum

$$\begin{aligned}\frac{I}{I_0} &= 2\cos^2\left(\frac{\pi a y}{\lambda_1 d}\right)\mathrm{sinc}^2\left(\frac{\pi b y}{\lambda_1 d}\right) + \cos^2\left(\frac{\pi a y}{\lambda_2 d}\right)\mathrm{sinc}^2\left(\frac{\pi b y}{\lambda_2 d}\right) \\ &\approx 2\left(\frac{\pi a}{\lambda_2 d}\Delta y\right)^2 + \left(\frac{\pi}{2}\frac{\lambda_2 - \lambda_1}{\lambda_2} - \frac{\pi a}{\lambda_2 d}\Delta y\right)^2 \\ &\approx 2\left(\frac{\pi a}{\lambda d}\Delta y\right)^2 + \left(\frac{\pi}{2}\delta\lambda - \frac{\pi a}{\lambda d}\Delta y\right)^2 ,\end{aligned}$$

where we denoted one of the wavelengths as λ and introduced the difference $\delta\lambda = |\lambda_1 - \lambda_2|/\lambda$; because the values of wavelengths are similar, the result should depend only on their difference and approximate values (meaning that we can substitute either of the two values for λ). We obtained a quadratic expression for I/I_0 in the form $2A^2x^2 + (C - Ax)^2 = 3A^2x^2 + C^2 - 2ACx$, where unknown x corresponds to Δy and the coefficients are defined as $C = \pi\delta\lambda/2$ and $A = \pi a/\lambda d$. It is easy to show that the minimum of such expression is at $x = \frac{C}{3A}$ and attains the value of $2C^2/3$. The intensity of light in the first minimum is

$$I_{\mathrm{min},\,1} = I_0\frac{2}{3}\left(\frac{\pi}{2}\delta\lambda\right)^2 ,$$

which must be divided by the reference intensity (at the zeroth maximum) $2I_0 + I_0 = 3I_0$. The resulting ratio is

$$\frac{I_{\text{min, 1}}}{I_{\text{max, 0}}} = \frac{2}{9}\left(\frac{\pi}{2}\delta\lambda\right)^2 \doteq 5.7 \cdot 10^{-7}\,.$$

We may notice that the result is heavily dependent on the intensity ration of the two sodium lines, but it is completely independent of the geometry of the experiment!

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Problem FoL.46 ... stellar sail 8 points

At a distance $d = 100\,\mathrm{AU}$ from the Sun, there is a mirror with area $S = 100\,\mathrm{km}^2$ moving away from the Sun with velocity $v = 0.4c$. The mirror is quite strange – it completely reflects light in the range $\langle 400\,\mathrm{nm}, 500\,\mathrm{nm}\rangle$ and is completely transparent for all other wavelengths. Find the force due to solar radiation acting on the mirror, according to an observer at the mirror. Assume that the Sun is an ideal black body with surface temperature $T = 6000\,\mathrm{K}$ and radius $R = 7 \cdot 10^5\,\mathrm{km}$. *Xellos would like to have a Solar Sailer.*

The radiation intensity curve of a black body is described by Planck's law

$$\Delta\Phi(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_BT) - 1}\Delta\nu\,,$$
$$\Delta\Phi(\lambda) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda k_BT) - 1}\Delta\lambda\,,$$

where $\Delta\lambda$ is a small wavelength interval centered around wavelength λ (similarly for frequency $\nu = c/\lambda$). $\Delta\Phi$ is the emitted flux per unit solid angle on this wavelength interval; The solid angle covered by the mirror with area S is $\vartheta \approx S/d^2$ and the incident power (in the given wavelength interval) is given as $\Delta P = \Delta\Phi \cdot \vartheta$. Since the momentum of a photon with relativistic mass m is mc and its energy is mc^2, we can transform the power (energy per time) to force (momentum per time) by diving it by c. The mirror doesn't absorb but reflects the photons, so the relation between incident power P and the force in question (denoted by F) is

$$F = \frac{2P}{c}\,.$$

Now we need to determine P. Since we know the dependence of P on λ, we can determine the total power in the interval $\langle 400\,\mathrm{nm}, 500\,\mathrm{nm}\rangle$ by numerical integration (that is, by division into subintervals and summing the partial powers). But there is a catch: the Planck's law depends on the frame of reference. We will continue to work in an inertial frame with velocity v (slow acceleration during a short time interval can be neglected). The formula for $\Delta\Phi$ then changes to

$$\Delta\Phi'(\nu') = \frac{2h\nu'^3}{c^2}\beta^3\frac{1}{\exp(h\nu'/k_B\beta T) - 1}\Delta\nu'\,.$$

The frequency in mirror's frame of reference is denoted as ν'. We also introduced

$$\beta = \sqrt{\frac{1 - v/c}{1 + v/c}}\,.$$

The relativistic Doppler effect provides us with $\nu' = \nu\beta$ and we can see that the expression for $\Delta\Phi'(\nu)$ looks almost as if we just substituted ν, only that the factor is β^3 instead of β^{-4}. The derivation of $\Delta\Phi'$ can get complicated, so let us only remark that the result must describe a black body radiation curve, but for a different temperature, $T' = \beta T$, and that the factor β^3 comes from spacetime deformation. Using the wavelength λ' in the mirror's frame of reference (keeping in mind that the reflectance is given in this frame) we can write

$$\Delta\Phi'(\lambda') = \frac{2hc^2}{\lambda'^5}\beta^3 \frac{1}{\exp(hc/\lambda' k_B \beta T) - 1}\Delta\lambda' \,.$$

The following conversion to power is also tricky, because from the reference frame of the mirror the Sun is at distance $d' = d\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. The angle under which we observe the solar radiation is then equal to $\vartheta' = \vartheta/\gamma^2 = \vartheta(1 - v^2/c^2)$. In conclusion, the radiation force is

$$F = \frac{4hSc}{d^2}\,(1 - v/c)^{5/2}\,(1 + v/c)^{-1/2} \int_{\lambda_b}^{\lambda_r} \lambda'^{-5} \frac{1}{\exp\left(\sqrt{\frac{c+v}{c-v}}\frac{hc}{\lambda' k_B T}\right) - 1} \mathrm{d}\lambda' \doteq 1.32 \cdot 10^{-22}\,\mathrm{N}\,.$$

This result can be obtained in another way while keeping the required precision. Instead of the integration, we evaluate the integrand at $\lambda' = 450\,\mathrm{nm}$ and multiply it by the length of the interval $100\,\mathrm{nm}$ (a midpoint-rectangular rule). The result will differ only by circa $0.02\,\%$.

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Problem FoL.47 ... throw over a cylinder 6 points

There is a huge cylinder with base radius $R = 10\,\mathrm{m}$ lying on its side. You want to throw a stone from ground level so that the stone flies over the cylinder. What is the minimum required initial velocity of the stone? *Jáchym guessed the original task from a figure in one spanish book.*

We choose the origin of our Cartesian coordinate system so that the cylinder is described by

$$x^2 + (y - R)^2 = R^2\,.$$

The stone we are throwing travels along a parabola

$$y = -ax^2 + c\,, \tag{2}$$

where a, c are unknown, positive constants. The linear term is missing since the parabola must be symmetrical about the vertical axis, which follows from the nature of the problem. We need to fit the circle under the parabola. To minimize the initial velocity, the circle and the parabola must be as close as possible – the touch in two points symmetric about the y axis (there could be possibly just one point of contact at the top of the cylinder, but this option will be ruled out by the following solution). The crossing of the curves can be obtained by substituting the second equation into the first one,

$$a^2x^4 + (1 + 2a\,(R - c))\,x^2 + c\,(c - 2R) = 0\,,$$

and solving this biquadratic equation for x:

$$x^2 = \frac{2a(c-R) - 1 \pm \sqrt{(1+2a(R-c))^2 + 4a^2c(2R-c)}}{2a^2}\,.$$

We want the curves to be tangent and there can be no more than two touching points, one positioned at $-x_\mathrm{t}$ and the second one at x_t. Therefore, only one x_t^2 exists, and so the biquadratic equation must have a zero discriminant. This means that

$$4R^2a^2 + 4(R-c)a + 1 = 0$$

and subsequently

$$c = \frac{4R^2a^2 + 4Ra + 1}{4a}\,. \tag{3}$$

Let's denote the intersection of the parabola and the x axis (the initial coordinate of the stone) by $-x_0$, then the impact point lies at x_0. For the time of travel T, it holds

$$T = \frac{2v_y}{g} = \frac{2x_0}{v_x}\,,$$

which can be rewritten into

$$x_0 = \frac{v_x v_y}{g}\,. \tag{4}$$

The position of the stone in time is described by

$$x = -x_0 + v_x t\,,$$
$$y = v_y t - \frac{1}{2}gt^2\,.$$

We solve the first equation for t, insert it into the second one and get the formula for the path of projectile

$$y = \frac{v_y}{v_x}(x_0 + x) - \frac{g}{2v_x^2}(x_0 + x)^2\,.$$

Substituting for x_0 into the equation (4) leads to

$$y = -\frac{g}{2v_x^2}x^2 + \frac{v_y^2}{2g}\,,$$

which is similar in form to the expression (2). By comparison of these two we get

$$a = \frac{g}{2v_x^2}\,,$$
$$c = \frac{v_y^2}{2g}\,.$$

The initial velocity of the stone is then given by

$$v^2 = v_x^2 + v_y^2 = g\left(\frac{1}{2a} + 2c\right)\,.$$

Substituting for c from expression (3), we obtain

$$v^2 = g\left(2R^2a + 2R + \frac{1}{a}\right)\,.$$

The velocity magnitude v is minimal when v^2 is. The minimum of v^2 is obtained by taking its derivative with respect to a and equating it to zero:

$$\frac{\mathrm{d}v^2}{\mathrm{d}a} = g\left(2R^2 - a^{-2}\right) = 0\,,$$
$$a = 2^{-1/2}R^{-1}\,.$$

Finally, we arrive at the numerical value of the initial velocity $v = \sqrt{2gR(1+\sqrt{2})} \doteq 21.8\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.48 ... evil spring 9 points

Consider a homogeneous spring with rest length $l = 11.3\,\mathrm{cm}$, stiffness $k = 125\,\mathrm{kg{\cdot}s^{-2}}$ and linear density $\lambda = 9.7\,\mathrm{kg{\cdot}m^{-1}}$ placed on a horizontal pad. We fix one of its endpoints in place in such a way that the spring can still freely rotate around this endpoint. Then, we remove the pad. Calculate the difference between the total potential energy of the spring in its new equilibrium position and in the initial position. *Jáchym couldn't figure out the principle of a pen.*

A spring hanging freely in a gravitational field extends more close to the point of suspension than at the bottom. So, the linear density is not a constant anymore. First let us find appropriate coordinates to deal with this problem.

Assume an element of length Δx positioned at point x measured from the point of suspension. Below this elements hangs mass $m(x) = (l-x)\lambda$, so there is downward force $m(x)g$ acting on this element. In consequence the element will extend to

$$\Delta y = \Delta x + \frac{m(x)g}{k'} = \Delta x\left(1 + \frac{(l-x)\lambda g}{kl}\right)\,, \tag{5}$$

where k' is stiffness of the chosen element. This stiffness is proportional to the length of the elements (the reasoning being based on the properties of tension in the spring), that is

$$k' = k\frac{l}{\Delta x}\,.$$

The appropriate coordinate for the elongated state is

$$y = \int_0^x \mathrm{d}y = x + \int_0^x \frac{m(x)g}{k'}\,\mathrm{d}x = x + \frac{g\lambda}{kl}\int_0^x (l-x)\,\mathrm{d}x = \left(1 + \frac{g\lambda}{k}\right)x - \frac{g\lambda}{2kl}x^2\,.$$

we used the fact that $\Delta y/\Delta x$ from equation (5) converges to the derivative $\mathrm{d}y/\mathrm{d}x$ in the zero length limit.

Now let's compute the change in potential energy. In the initial state all points are at height 0, so $E_{g_0} = 0$. After reaching the equilibrium each element $\mathrm{d}x$ with mass $\mathrm{d}m = \lambda\mathrm{d}x$ is in height $-y$. It holds

$$E_g = \int_0^m g\,(-y)\,\mathrm{d}m = -g\lambda\int_0^l \left(\left(1 + \frac{g\lambda}{k}\right)x - \frac{g\lambda}{2kl}x^2\right)\mathrm{d}x = -\left(1 + \frac{2g\lambda}{3k}\right)\frac{g\lambda l^2}{2}\,.$$

Next we have to determine the change in elastic potential energy. Initially, $E_{\mathrm{p}0} = 0$. In the final state each element Δx extends to Δy, which corresponds to energy change

$$\Delta E_{\mathrm{p}} = \frac{1}{2}k'\left(\Delta y - \Delta x\right)^2 = \frac{kl\left(\Delta y - \Delta x\right)^2}{2\Delta x} = \frac{\left(l - x\right)^2 \lambda^2 g^2}{2kl}\Delta x\,.$$

The expression $\Delta E_{\mathrm{p}}/\Delta x$ can be considered as the linear energy density (per unit length of the initial state), which can be integrate to

$$E_{\mathrm{p}} = \int_0^l \mathrm{d}E_{\mathrm{p}} = \frac{g^2\lambda^2}{2kl}\int_0^l \left(l - x\right)^2 \,\mathrm{d}x = \frac{g^2\lambda^2}{2kl}\left[l^2 x - lx^2 + \frac{x^3}{3}\right]_0^l = \frac{g^2\lambda^2 l^2}{6k}\,.$$

Finally we determine the total change in potential energy

$$\Delta E = \Delta E_g + \Delta E_{\mathrm{p}} = E_g - E_{g0} + E_{\mathrm{p}} - E_{\mathrm{p}0} = -\left(1 + \frac{g\lambda}{3k}\right)\frac{g\lambda l^2}{2} \doteq -0.762\,\mathrm{J}\,.$$

At the end let us make a short comment. It is no coincidence that the term

$$-\frac{g\lambda l^2}{2}$$

represents the change of potential energy in the case of zero elongation. Taking the limit of infinite stiffness, $k \to \infty$, we obtain exactly this term. The other term then represents the energy coming from the elongation. We can notice that the released gravitational potential energy was two times as large as the elastic energy stored in the elongated spring.

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Problem M.1 ... Dyson — 3 points

A Dyson sphere is a (so far) hypothetical construction around a star which completely surrounds the star in order to capture all the energy it radiates. Imagine that someone takes our Earth and creates a Dyson sphere from its matter, with our Sun at the centre of the sphere. The outer radius of this sphere is the same as the average Earth-Sun distance $d_{\mathrm{S}} = 149\,600\,000\,\mathrm{km}$. Assume that Earth is a sphere with radius $R_{\mathrm{Z}} = 6\,378\,\mathrm{km}$ and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth. What is the thickness of this Dyson sphere?

Karel likes problems with a Dyson sphere.

Let's use the fact that volume is conserved. The volume of Earth is $V_{\mathrm{Z}} = \frac{4}{3}\pi R_{\mathrm{Z}}^3$. To calculate the volume of the Dyson sphere, we can use the approximation that the thickness h of the sphere is much smaller than its radius. Therefore, we may calculate its volume as the product of its surface area (either inner or outer, it does not matter in this approximation) and thickness

$$V_{\mathrm{D}} \approx S_{\mathrm{D}} h = 4\pi d_{\mathrm{S}}^2 h\,.$$

You can prove by yourself that if we calculated this volume as a difference of volumes of two spheres $V'_\mathrm{D} = \frac{4}{3}\pi d_\mathrm{S}^3 - \frac{4}{3}\pi (d_\mathrm{S} - h)^3$, we would get an almost identical result. Now, we simply use the equation $V_\mathrm{Z} = V_\mathrm{D}$ to get

$$\frac{4}{3}\pi R_\mathrm{Z}^3 = 4\pi d_\mathrm{S}^2 h\,,$$

$$h = \frac{R_\mathrm{Z}^3}{3d_\mathrm{S}^2} \doteq 3.86 \cdot 10^{-3}\,\mathrm{m}\,.$$

We can see that the thickness is in the order of milimetres, fourteen orders of magnitude smaller than d_S. Therefore, we may conclude that our approximation is valid.

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Problem M.2 ... Dyson Reloaded

3 points

Let's consider the Dyson sphere from the previous problem. (Imagine that someone takes our Earth and creates a Dyson sphere from its matter, with our Sun at the centre of the sphere. The outer radius of this sphere is the same as the average Earth-Sun distance $d_\mathrm{S} = 149\,600\,000\,\mathrm{km}$*. Assume that Earth is a sphere with radius* $R_\mathrm{Z} = 6\,378\,\mathrm{km}$ *and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth.) Calculate the gravitational acceleration that could be felt by astronauts standing on its outer surface. The mass of Earth is* $M_\mathrm{E} = 5.97 \cdot 10^{24}\,\mathrm{kg}$ *and the mass of the Sun is* $M_\mathrm{S} = 1.99 \cdot 10^{30}\,\mathrm{kg}$*.* *Karel really likes problems with a Dyson sphere.*

To tackle this problem, we will use the Gauss's law for gravity, which states that the gravitational flux through any closed surface is proportional to the enclosed mass. This implies that the gravitational force is going to be the same as if there was a point-like mass $M_\mathrm{Z} + M_\mathrm{S}$ in the middle of the sphere. Since the gravitational force of the Dyson sphere is way smaller than the gravitational force of the Sun, we can neglect it. Using the Newton's law of universal gravitation, one obtains

$$a_g = \frac{GM_\mathrm{S}}{d_\mathrm{S}^2} \doteq 5.93 \cdot 10^{-3}\,\mathrm{m{\cdot}s^{-2}}\,,$$

where G is the Gravitational constant.

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Problem M.3 ... Dyson Reloaded Reloaded

3 points

For now, let's keep discussing the same Dyson sphere. (Imagine that someone takes our Earth and creates a Dyson sphere from its matter, with our Sun at the centre of the sphere. The outer radius of this sphere is the same as the average Earth-Sun distance $d_\mathrm{S} = 149\,600\,000\,\mathrm{km}$*. Assume that Earth is a sphere with radius* $R_\mathrm{Z} = 6\,378\,\mathrm{km}$ *and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth.) We already know that it would be hard to walk on its outer surface, but could we walk on its inner surface? How many days would a month (i.e. 1/12-th of one orbit around the Sun) take, if the sphere rotated with such an angular velocity that the acceleration* *due to gravity on its equator would be the same as on the equator of the (regular) Earth?*

Karel really likes problems with a Dyson sphere.

Firstly, we must realize that the gravitational force of the Sun (calculated in the previous problem) is very small, compared to the gravitational acceleration $g = 9.81\,\mathrm{m\cdot s^{-1}}$ certainly negligible. Also, the gravitational force inside of the sphere is zero (result of the Gauss's law). Therefore, we only need to use the formula for centrifugal acceleration

$$g \approx \omega^2 d_\mathrm{S}\,,$$

$$\omega \approx \sqrt{\frac{g}{d_\mathrm{S}}}\,,$$

$$t \approx \frac{\pi}{6}\sqrt{\frac{d_\mathrm{S}}{g}} \doteq 64{,}660\,\mathrm{s} \doteq 0.748\,\mathrm{dne}\,,$$

where we used $t = \frac{T}{12} = \frac{2\pi}{12\omega}$ to calculate the period from the angular velocity.

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Problem M.4 ... Dyson Reloaded[3] 3 points

Let's trouble our minds with the Dyson sphere one last time. I promise it's the last time! (Imagine that someone takes our Earth and creates a Dyson sphere from its matter. The outer radius of this sphere is the same as the average Earth-Sun distance $d_\mathrm{S} = 149\,600\,000\,\mathrm{km}$. Assume that Earth is a sphere with radius $R_\mathrm{Z} = 6\,378\,\mathrm{km}$ and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth.) In this case, let's assume that the engineers made a mistake somewhere and did not build the sphere around the Sun. The sphere stays the same (it has the same radius and it is made of the Earth's matter), it does not rotate and there are no stars or planets on the inside or anywhere nearby. What is the minimum amount of pressure (critical stress) the material of Earth must be able to withstand so that the sphere does not collapse under its own weight?

Matěj became infected and now he also likes Dyson spheres.

Let's consider a layer of width h, from which we cut out a little part. Now we calculate the force, by which the little part of area $\mathrm{d}S$ (in the little-part approximaion it has the same outer and inner area) is attracted to the centre. Using Gauss's law we get the relation of gravitational acceleration and location in the layer. Let's consider constant density and denote in this case the inner radius as d_S, then

$$a(x) = G\frac{M_\mathrm{Z} x}{h(d_\mathrm{S}+x)^2} \approx \frac{GM_\mathrm{Z} x}{h d_\mathrm{S}^2}\,,$$

where x is the distance of the point of acceleration a from the inner surface of the sphere and $\frac{M_\mathrm{Z} x}{h}$ is (for linear approximation) the mass closer to the centre.[2]

[2] That is basically the same approximation as we used in the M1 problem, calculating the volume, since the inner mass is $\frac{V_\mathrm{in}}{V} M_\mathrm{Z} = \frac{\frac{4}{3}\pi(d_\mathrm{S}+x)^3 - \frac{4}{3}\pi d_\mathrm{S}^3}{\frac{4}{3}\pi(d_\mathrm{S}+h)^3 - \frac{4}{3}\pi d_\mathrm{S}^3} M_\mathrm{Z} \approx \frac{4\pi d_\mathrm{S} x}{4\pi d_\mathrm{S} h} M_\mathrm{Z} = \frac{x}{h} M_\mathrm{Z}$.

Now, integrating the gravitational force we get the force acting upon this part of the layer

$$\mathrm{d}F = \int\limits_0^h a(x)\varrho\,\mathrm{d}S\mathrm{d}x = \frac{GM_\mathrm{Z}\varrho\mathrm{d}S}{hd_\mathrm{S}^2}\int\limits_0^h x\,\mathrm{d}x = \frac{GM_\mathrm{Z}h\varrho\mathrm{d}S}{2d_\mathrm{S}^2} = \frac{GM_\mathrm{Z}\sigma\mathrm{d}S}{2d_\mathrm{S}^2}\,,$$

where ϱ is density and $\sigma = \varrho h = \frac{M_\mathrm{Z}}{4\pi d_\mathrm{S}^2}$ the area density of the layer.

Using analogy of an air bubble on the water surface, this force per unit volume corresponds to pressure

$$\frac{\mathrm{d}F}{\mathrm{d}S} = p_p = \frac{GM_\mathrm{Z}\sigma}{2d_\mathrm{S}^2}\,.$$

Hence, we can imagine the spheric layer as if it was in a medium of constant hydrostatic pressure p_p, which replaces gravitational forces.

Upon its cross-section acts a force

$$F_p = \pi d_\mathrm{S}^2 p_p = \frac{\pi GM_\mathrm{Z}\sigma}{2} = \frac{GM_\mathrm{Z}^2}{8d_\mathrm{S}^2}\,,$$

This is the force, which pushes two hemispheres together. That force spreads over the area of the cross-section (the area is $2\pi d_\mathrm{S}h$). The material, which the layer is made of must therefore withstand pressure

$$p = \frac{F_p}{2\pi d_\mathrm{S}h} = \frac{GM_\mathrm{Z}^2}{16\pi d_\mathrm{S}^3 h} = \frac{3GM_\mathrm{Z}^2}{16\pi d_\mathrm{S} R_\mathrm{Z}^3} \doteq 3.66\,\mathrm{MPa}\,.$$

Because the layer is very thin, we can consider the pressure inside the material constant.

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Problem E.1 ... oh, these batteries — 3 points

Imagine that you have a fully discharged car battery ($U = 12\,\mathrm{V}$, capacity $60\,\mathrm{Ah}$) and a charging device that can supply the battery with a steady current $I = 3.0\,\mathrm{A}$. If you can charge the battery for $t = 75\,\mathrm{min}$, what will its final charge level be (in percent of total capacity)?

Karel. Don't even ask...

Since the value for the battery capacity is in the units of Ah, let's first convert the time to hours, i. e. $t = 1.25\,\mathrm{h}$. In that time, the device transfers to the battery the charge $Q = It = 3.75\,\mathrm{Ah}$. The maximum capacity of the battery is $Q_\mathrm{max} = 60\,\mathrm{Ah}$, so the percentage reached is $k = Q/Q_\mathrm{max} = 6.25\,\%$. The battery will be charged only to $6.3\,\%$.

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Problem E.2 ... inner resistance — 3 points

If we connect two identical resistors in series to a non-ideal DC voltage source, the efficiency of the source is 0.87. What is the efficiency of the source if we connect the two resistors to it in parallel?

We struggled with a revolution.

Let's denote the inner resistance of the source by R_i and the total resistance of the two resistors in series by R. Then, the efficiency of the source can be expressed as

$$\eta_1 = \frac{R}{R+R_\mathrm{i}}\,.$$

It follows from this equation that

$$R_\mathrm{i} = R\frac{1-\eta_1}{\eta_1}\,.$$

And the efficiency with the parallel connection schema (where the total resistance of the two resistors is $R/4$) is

$$\eta_2 = \frac{\frac{R}{4}}{\frac{R}{4}+R_\mathrm{i}} = \frac{R}{R+4R_\mathrm{i}} = \frac{R}{R+4R\frac{1-\eta_1}{\eta_1}} = \frac{\eta_1}{4-3\eta_1} \doteq 0.626\,.$$

The efficiency of the source is 0.626.

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Problem E.3 ... solar panel incident — 3 points

A horizontal photovoltaic panel generates power $300\,\mathrm{W}$ *when sunlight is incident on it perpendicularly. What power would it generate at 10:00, on a day when the Sun rises at 6:00 and sets at 18:00? At noon, the angle of incidence of sunlight on the panel is* 20° *from the vertical.*

Eating green wasabi, remembering green energy, improperly writing gerunds.

The solar panel efficiency depends on the amount of sunlight received and that can be calculated through the perpendicular projection of sunlight on the given panel. If the original incidental power delivered by the sun is P_0, at noon it's given by $P_\mathrm{p} = P_0 \cos 20^\circ$. The true noon is at 12 pm so the angle that we see the sun under at 10 am differs by $\frac{2}{24}360^\circ = 30^\circ$. This difference is in the direction perpendicular to the zenith distance of the Sun at noon (meaning it will not make a difference in that 20 degree angle towards south (or north if we live on the southern hemisphere)) and we simply need to multiply P_p with another projection factor, this time of 30°. The resulting power at 10 am received by the panel is

$$P = P_0 \cos 20^\circ \cos 30^\circ \doteq 244\,\mathrm{W}\,.$$

This computation is so easy because the light rays are perpendicular to Earth's axis on the equinox. In general case, we would use identities of spherical trigonometry.

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Problem E.4 ... transforming — 3 points

Consider a transformer that has a primary coil with $N_1 = 100$ *turns and a secondary coil with* $N_2 = 300$ *turns. A light bulb with resistance* $R = 25.0\,\Omega$ *is connected to the secondary coil. We want the light bulb to be supplied with power* $P = 200\,\mathrm{W}$*. What should be the effective voltage connected to the primary coil of the transformer?*

Danka was reminiscing about high school problems.

The effective voltages on the primary and the secondary coil are U_1, U_2, respectively. The power delivered to the bulb as a function of the voltage on it is given by

$$P = \frac{U_2^2}{R}\,.$$

The voltage on the transformer is transferred across with the same ratio as is between the number of loops

$$\frac{U_1}{U_2} = \frac{N_1}{N_2}\,.$$

Now what's left, is to express

$$U_1 = \frac{N_1}{N_2} U_2 = \frac{N_1}{N_2}\sqrt{PR} \doteq 23.6\,\mathrm{V}\,.$$

We need to induce a voltage of 23.6 V on the ends of the primary coil.

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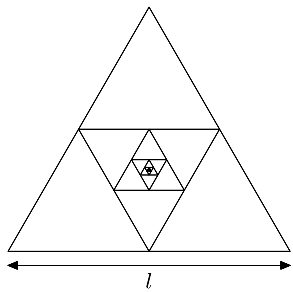
Problem X.1 ... purely fractal

3 points

Matěj likes fractals, so he made one out of infinitely thin wire with linear density $\lambda = 100\,\mathrm{g{\cdot}m^{-1}}$. He started by making an equilateral triangle with side length $l = 10\,\mathrm{cm}$, added the midsegments of this triangle, then added the midsegments of the medial triangle created in the previous step and so on. The result is in figure. What is the total mass of Matěj's fractal?

Matěj likes infinities.



As the midsegment is a half of the length of the side of its original triangle, each added triangle has half the length of a circumference than the previous one. The circumference of the first triangle is $3l$, so the total length of the wire used is

$$3l + \frac{3l}{2} + \frac{3l}{4} + \frac{3l}{8} + \dots = 3l \sum_{n=0}^{\infty} \frac{1}{2^n} = 6l$$

The total mass is consequently $6l\lambda = 60\,\mathrm{g}$.

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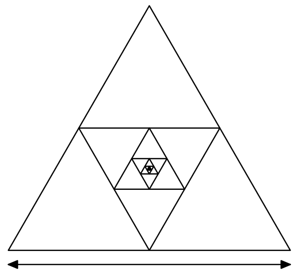
Problem X.2 ... just fractal

3 points



Matěj is still playing with the same fractal (with dimension $l = 10\,\mathrm{cm}$ and linear density $\lambda = 100\,\mathrm{g\cdot m^{-1}}$). He would like to know its moment of inertia with respect to an axis that contains one of the medians of the largest triangle. Find this moment of inertia. *Matěj really likes infinities.*

Moment of inertia of a single equilateral triangle relative to it's median (also its altitude) can be calculated using the formula for the moment of inertia of a homogeneous rod of mass $3l\lambda$ and length $3l$ around the perpendicular axis going through the rod's centre.[3]

The moment of inertia of the largest triangle with respect to the given axis is

$$J_0 = \frac{1}{12} 3l\lambda l^2 = \frac{1}{4}\lambda l^3 \,.$$

Since every additional triangle has sides of half the length, we can express the moment of inertia for the nth triangle as

$$J_n = \frac{1}{4}\lambda \frac{l^3}{(2^n)^3} = \frac{1}{4}\lambda \frac{l^3}{8^n} \,.$$

The altitudes of all triangles lie on the same axis, hence we can use the additive property of moments of inertia and the total moment can be written as the sum

$$J = \sum_{n=0}^{\infty} J_n = \frac{\lambda l^3}{4} \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n = \frac{2\lambda l^3}{7} \doteq 2.9 \cdot 10^{-5}\,\mathrm{kg\cdot m^2}$$

which was evaluated using the formula for an infinite sum of geometric series $\sum_{n=0}^{\infty} a^n = 1/(1-a)$ for $|a| < 1$. We can avoid using the explicit formula through expressing the total moment of inertia as a sum of the moment of the largest triangle and the rest of the fractal (which is half the size and its moment of inertia is $J/8$, as the orientation doesn't matter), which leads to the same result.

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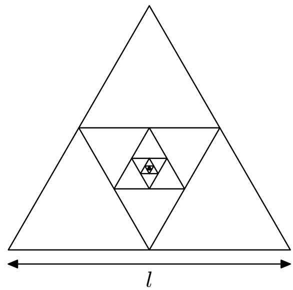
[3] That's because the moment of inertia depends only on the distance of mass from the axis and the triangle has a constant amount of mass at each distance from the axis between 0 and $l/2$

Problem X.3 ... quite fractal

3 points

Now, Matěj hanged his fractal (with dimension $l = 10\,\mathrm{cm}$ and linear density $\lambda = 100\,\mathrm{g{\cdot}m^{-1}}$) from one of the vertices of the largest triangle and let it perform small oscillations around this vertex in the plane that contains the fractal. What period of oscillations did Matěj measure?

Matěj really truly very much does like infinities.



We approach this similarly to the previous problem. First, we calculate the moment of inertia relative to the axis perpendicular to the triangle going through its centre (centroid) – we can talk about a single centroid, because all the triangles and the whole fractal share this centroid (at two thirds of their altitude). The moment of inertia of the largest triangle J_0 consists of three equal moments of the individual sides. Using the parallel axis theorem we have to add the displacement term

$$m\left(\frac{1}{3}\frac{\sqrt{3}}{2}l\right)^2 = \frac{ml^2}{12}\,,$$

to their centre of mass moment of inertia $ml^2/12$ (where $m = \lambda l$). The moment of inertia of the largest triangle is then

$$J_0 = 3\left(\frac{ml^2}{12} + \frac{ml^2}{12}\right) = \frac{1}{2}ml^2 = \frac{1}{2}\lambda l^3\,.$$

Moment of inertia if the nth triangle can be calculated analogously.

$$J_n = \frac{1}{2}\lambda\left(\frac{l}{2^n}\right)^3\,.$$

The total moment of inertia is

$$J_{\mathrm{clk}} = \sum_{n=0}^{\infty} J_n = \frac{1}{2}\lambda l^3 \sum_{n=0}^{\infty}\left(\frac{1}{2^n}\right)^3 = \frac{4}{7}\lambda l^3\,.$$

We can use the parallel axis theorem again to obtain

$$J = J_{\mathrm{clk}} + M\left(\frac{2}{3}\frac{\sqrt{3}}{2}l\right)^2 = \frac{18}{7}\lambda l^3\,,$$

where $M = 6\lambda l$ is the total mass from the first problem. Now what's left is to plug in this value to the formula for period of a physical pendulum

$$T = 2\pi\sqrt{\frac{J}{Mgr}}\,,$$

where $r = l/\sqrt{3}$ is the distance of the centre of mass from the axis of revolution. We arrive to

$$T = 2\pi\sqrt{\frac{3\sqrt{3}l}{7g}} \doteq 0.547\,\mathrm{s}\,.$$

The period of the swings of the fractal is $0.547\,\mathrm{s}$.

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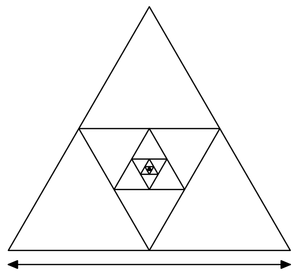
Problem X.4 ... very fractal

3 points



Matěj still likes playing with his fractal (with dimension $l = 10\,\mathrm{cm}$). He found a multimeter and connected it to two vertices of the largest triangle. The resistance of one metre of wire is $1\,000.000\,\Omega$. What resistance did Matěj measure?
Matěj really truly very much does like infinitely infinite infinities.

Let's designate the measured resistance R. In solving this problem, we'll try to make use of the fact that resistance depends on the length of a wire linearly, so if we halve all of the segments of an arbitrary contraption made out of wire, its resistance between arbitrary two points also halves.

If we take away the three outer (largest) sides from Matěj's fractal, we obtain the same fractal just halved in size. Using the statement above, we can claim that the resistance between its vertices is $R/2$. With no effect on the total resistance, we can replace the whole "subfractal" with three resistors of resistance $R/4$ in the star configuration as can be seen in the figure 1.

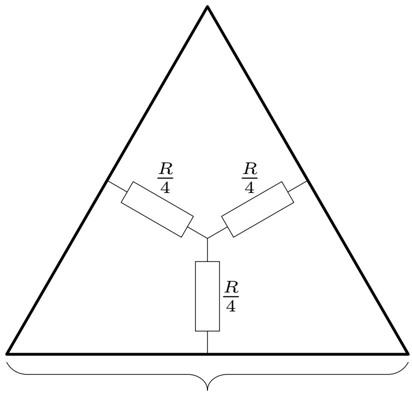


Fig. 1: Replaced subfractal scheme

This is now an easily solvable arrangement of resistors (even symmetrical, so we can ignore the vertical wire of the star). The resistance of half of the outer side is $\lambda l/2$. The total resistance can be expressed as

$$R = \frac{2\frac{\lambda l}{2}\left(2\frac{\lambda l}{2} + \frac{4\frac{R}{4}\frac{\lambda l}{2}}{2\frac{R}{4} + 2\frac{\lambda l}{2}}\right)}{2\frac{\lambda l}{2} + 2\frac{\lambda l}{2} + \frac{4\frac{R}{4}\frac{\lambda l}{2}}{2\frac{R}{4} + 2\frac{\lambda l}{2}}}$$

After a few algebraic steps we obtain a quadratic equation

$$3R^2 + 2\lambda l R - 2\lambda^2 l^2 = 0$$

which solution is (we are only interested in the positive solution, negative resistance has no meaning in physics)

$$R = \frac{-1+\sqrt{7}}{3}\lambda l \doteq 54.858{,}4\,\Omega\,.$$

An alternative approach can be developed from the idea that thanks to the symmetry of the fractal the voltage is the same everywhere on the axis of the triangle and consequently we can replace this axis with a single node S. Then we obtain the resistance $R/2$ between the vertex of the fractal and the node S expressed as

$$\frac{2}{R} = \frac{2}{\lambda l} + \cfrac{1}{\cfrac{\lambda l}{2} + \cfrac{1}{\cfrac{2}{\lambda l} + \cfrac{4}{R}}}\,,$$

because the resistance between the vertex of the inner half-fractal and the node S will be $R/4$. This leads to the identical quadratic equation and to the same result.

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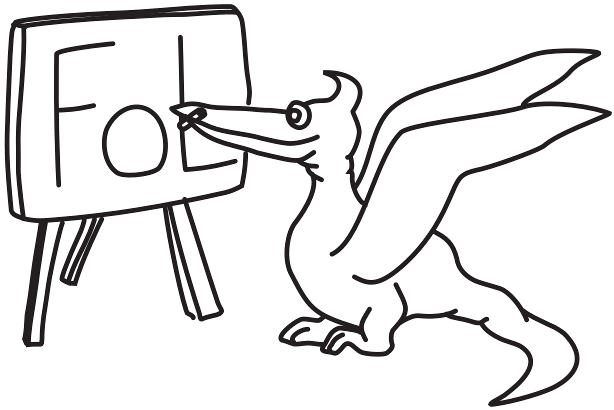
Solutions of 9th Online Physics Brawl



Problem FoL.1 ... inconspicuous escalator

3 points

Suppose that the stairs of an escalator are moving with speed $v = 4.0\,\mathrm{km{\cdot}h^{-1}}$. How much can the speed of the moving belt the passengers hold onto differ from this speed (in percent), such that the passengers cannot tell the difference between the two speeds? The escalator is $d = 30\,\mathrm{m}$ long and a passengers cannot tell the difference if their hand moves by less than $\Delta x = 5\,\mathrm{cm}$ during the whole journey from one end of the escalator to the other end.

Karel was inspired by Dodo's model, which he then disproved with this problem.

The speed of the stairs is $v = 1.11\,\mathrm{m{\cdot}s^{-1}}$. A passenger therefore travels from one end of the escalator to the other end in time $t = d/v = 27\,\mathrm{s}$. Hence, the difference in the speed of the belt can be at most $\Delta v = \Delta x/t = \Delta x \cdot v/d \doteq 0.0018\,\mathrm{m{\cdot}s^{-1}} \doteq 0.0066\,\mathrm{km{\cdot}h^{-1}}$. The speed of the belt can differ from the speed of the stairs by up to $0.17\,\%$ of the speed of the stairs. Notice that the result is independent of the speed v. This is a consequence of choosing a relative difference as our result.

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Problem FoL.2 ... long day

3 points

Consider a boat floating on the equator. How fast (in knots) does it have to move if one day on the ship should last 25 hours?

A 24-hour day is too short for Matěj.

The Earth performs one rotation in 24 hours. In 25 Earth days (i.e. time $t = 25{\cdot}24\,\mathrm{h}$), passengers onboard should experience only 24 boat days (i.e. $t = 24 \cdot 25\,\mathrm{h}$). This corresponds to exactly one journey around the Earth. Let R be the radius of Earth. Then, the velocity of the boat is $v = 2\pi R/t = 18.5\,\mathrm{m \cdot s^{-1}} = 36\,\mathrm{kt}$.

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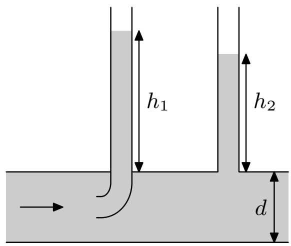
Problem FoL.3 ... pair of pipes

3 points

Water flows without friction in a horizontal pipe with a diameter $d = 10\,\mathrm{cm}$. The pipe is connected to two smaller vertical pipes. The intake of the first pipe faces against the water flow, so it is perpendicular to the horizontal pipe, and the water flowing into it comes to a stop in this pipe. The intake of the second pipe is parallel to the horizontal pipe. The water reaches a height $h_1 = 30.0\,\mathrm{cm}$ in the first smaller pipe, but it only reaches a height $h_2 = 25.0\,\mathrm{cm}$ in the second pipe. What is the velocity of the water flowing in the main pipe?



Jindra was dreaming of water in the summer dry season.

We will start from the Bernoulli equation. All the variables referring to the first smaller pipe are indexed by 1, while variables referring to the second pipe are indexed by 2. The Bernoulli equation states

$$p_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \frac{1}{2}\varrho v_2^2\,.$$

From the problem statement, we deduce that $v_1 = 0$ and v_2 corresponds to the velocity of the water flow in the main tube. The pressures p_1 and p_2 determine the water levels in pipes 1 and 2 respectively.

$$\varrho g h_1 = \varrho g h_2 + \frac{1}{2}\varrho v_2^2$$
$$v_2 = \sqrt{2g(h_2 - h_1)}$$
$$v_2 = 0.99\,\mathrm{m\cdot s^{-1}}$$

The velocity of the water flow is $0.99\,\mathrm{m\cdot s^{-1}}$.

The same principle is used for the design of the Pitot tube, which is used for measuring aircraft speed.

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Problem FoL.4 ... lost heat — 3 points

A calorimeter contains $0.50\,\mathrm{kg}$ *of water at a temperature* $t_1 = 20\,\mathrm{°C}$. *We add another* $0.50\,\mathrm{kg}$ *of water at a temperature* $t_2 = 80\,\mathrm{°C}$ *to the calorimeter. After it has reached thermal equilibrium, the water has temperature* $t_3 = 45\,\mathrm{°C}$. *Determine the heat capacity of the calorimeter. Assume that the calorimeter does not exchange heat with its surroundings.*

Jindra was curious where the heat from water in a calorimeter has gone to.

The hot water that is added to the calorimeter transfers heat to both the cold water and the calorimeter. The specific heat of water is $c = 4180\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$, and we will assume that it is independent of temperature (in the range from $20\,\mathrm{°C}$ to $80\,\mathrm{°C}$, only the third significant digit of c varies). Let C be the heat capacity of the calorimeter. Then

$$cm_2(t_2 - t_3) = cm_1(t_3 - t_1) + C(t_3 - t_1)\,,$$
$$C = cm_2\frac{t_2 - t_3}{t_3 - t_1} - cm_1\,,$$
$$C \doteq 836\,\mathrm{J\cdot K^{-1}}\,.$$

The heat capacity of the calorimeter is $836\,\mathrm{J\cdot K^{-1}}$.

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Problem FoL.5 ... one hundred — 3 points

Jindra wants to know the amount of heat that needs to be supplied to a cube of ice with mass $1.0\,\mathrm{kg}$ *at the temperature* $0.0\,\mathrm{°C}$ *in order to change it into steam at* $100.0\,\mathrm{°C}$. *The specific heat of water is* $1\,\mathrm{kcal\cdot kg^{-1}\cdot K^{-1}}$; *assume that it does not depend on temperature. Look up any other necessary constants.*

Jindra ordered boiled ice in the restaurant.

The heat is necessary for the phase change of ice into water, for heating up the water from 0.0 °C to 100.0 °C and for the phase change of water into steam. Therefore, the heat we need to supply is

$$Q = l_\mathrm{t} m + cm\Delta t + l_\mathrm{v} m\,.$$

The latent heat of fusion of water is $3.337 \cdot 10^5\,\mathrm{J{\cdot}kg^{-1}}$ and the latent heat of vaporization is $2.256 \cdot 10^6\,\mathrm{J{\cdot}kg^{-1}}$. A kilocalorie (kcal) is an old unit of energy and it can be converted into joules as $1\,\mathrm{kcal} \doteq 4\,184\,\mathrm{J}$. The heat supplied to the ice turns out to be $Q \doteq 3.01 \cdot 10^3\,\mathrm{kJ}$.

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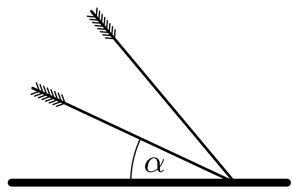
Problem FoL.6 ... archer

4 points

An archer is shooting arrows from a point on a horizontal plane. He manages to land two arrows at essentially the same points, but pointing in different directions. One of the arrows is pointing at an angle $\alpha = 25.00^\circ$ with respect to the horizontal plane. At what angle with respect to the horizontal plane is the second arrow pointing if we assume that both were launched with the same force? *Matěj was shooting arrows.*



It seems that we are not given enough parameters. We have no knowledge of the launch speed nor of the distance of the archer from the point of impact. It will turn out, however, that these are not needed.

The arrow's trajectory can be determined in reverse, starting from the point of impact and launching the arrows with a velocity v. The first arrow is launched at an angle α with respect to the horizontal plane. From the equations of projectile motion, the time of flight is

$$t = \frac{2v\sin\alpha}{g}\,,$$

and the distance at which it lands is

$$s_1 = \cos\alpha vt = \frac{2v^2}{g}\sin\alpha\cos\alpha\,.$$

If we denote the second angle by β, the distance at which the second arrow lands is

$$s_2 = \frac{2v^2}{g}\sin\beta\cos\beta\,.$$

Since the archer launched the arrows from the same place, these distances must be equal

$$\begin{aligned} s_1 &= s_2\,,\\ \frac{2v^2}{g}\sin\alpha\cos\alpha &= \frac{2v^2}{g}\sin\beta\cos\beta\,,\\ \sin\alpha\cos\alpha &= \sin\beta\cos\beta\,,\\ \sin 2\alpha &= \sin 2\beta\,, \end{aligned}$$

The last equation has solutions other than $\alpha = \beta$, since the sine is not an injective function (not one-to-one). Considering the nature of archery, we are only looking for solutions within the interval $(0^\circ, 90^\circ)$. We can simplify it to

$$\cos\left(2\alpha - 90^\circ\right) = \cos\left(90^\circ - 2\beta\right) ,$$
$$2\alpha - 90^\circ = 90^\circ - 2\beta ,$$
$$\beta = \frac{1}{2}\left(180^\circ - 2\alpha\right) = 90^\circ - \alpha = 65.00^\circ .$$

We can see that the orientations of both arrows are symmetric around the angle 45°. It is worth noting that the result is independent of the acceleration due to gravity g, so the same result would be obtained for an archer on a different planet.

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Problem FoL.7 ... strange eclipse

4 points

The maximum duration of a total solar eclipse on the Earth's surface is $t = 449\,\mathrm{s}$. During this time, the stripe of land covered by the total eclipse (the shadow of the Sun) has length $d = 267\,\mathrm{km}$. Find out the maximum duration of a total solar eclipse for a passenger on a plane which flies with a velocity $v = 903\,\mathrm{km/h}$. Neglect the height of the plane above the surface.

Dodo is dreaming about a trip to Argentina.

From the length of the shadow and the duration of the total solar eclipse, we determine the velocity of movement of the shadow on the Earth's surface as $u = d/t$. Then we get the duration of a total solar eclipse T for a passenger on a plane as

$$T = \frac{d}{u - v} = t\frac{d}{d - vt} = t\frac{1}{1 - \frac{vt}{d}} = 777\,\mathrm{s} .$$

The passenger will see the total solar eclipse for about 13 minutes.

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Problem FoL.8 ... neglect causality

4 points

Elisa had a daughter Lottie with Johann. Unfortunately, something strange happened to Lottie. She travelled back in time to the past, where she grew up and married Peter. She and Peter had a daughter Elisa, which is also her mother. Therefore, a time paradox is created. The existence of Lottie is caused by the existence of Elisa and vice versa, we can't claim that their existence has a beginning. Assume that each child inherits exactly half of her genetic information from the father and half from the mother. What percentage of Elizabeth's genetic information originally came from Peter? Johann is not related to Peter.

Matěj was watching a series, but I won't reveal its name to avoid spoilers.

Let's denote the genetic information of each family member by the first letter of their name. In accordance with the assumption in the problem statement, we can write

$$A = \frac{1}{2}P + \frac{1}{2}S\,,$$
$$S = \frac{1}{2}J + \frac{1}{2}A\,,$$

which is a system of equations with four unknowns. Then, we can express

$$A = \frac{1}{3}J + \frac{2}{3}P\,.$$

Elisa actually carries 66.7 % of Peter's genetic information.

We have ignored the fact that this whole problem doesn't make sense from a physics point of view, because everything we know so far shows that travelling back in time is impossible in reality.

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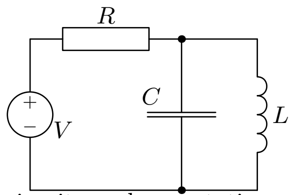
Problem FoL.9 ... fun with RLC 4 points

Consider the circuit in the figure. It consists of a coil with inductance $L = 10.0\,\mathrm{mH}$, a capacitor with capacitance $C = 4.70\,\mathrm{\mu F}$, a resistor with resistance $R = 1.00\,\mathrm{k\Omega}$ and an ideal source of DC voltage $V = 230\,\mathrm{V}$. Calculate the real electric power consumed by the resistor. The utility frequency in Czech sockets is $f = 50\,\mathrm{Hz}$. *Dodo wants to be cruel.*



The source provides **DC** power, hence the coil would (after the circuit reaches a stationary state) behave as a wire and the capacitor as a "hole" in the circuit. Only the resistor remains connected to the power source. The power consumed by the resistor can be calculated as

$$P = VI = \frac{U^2}{R} = 52.9\,\mathrm{W}\,,$$

which is close to the power of a standard light bulb.

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Problem FoL.10 ... an unfair race 4 points

Martin and Ivan had a race to the opposite side of a hill. Both of them ran the same horizontal distance $x = 60\,\mathrm{m}$. However, Ivan ran over the hill, which can be described as an arc of a circle with radius $r = 50\,\mathrm{m}$, while Martin ran along a corridor through the hill. The floor of this corridor is an inclined plane and its angle of inclination with respect to the horizontal plane is $\alpha = 14\,^{\circ}$. How much larger (in percent of Martin's average speed) must Ivan's average speed be if both runners reached the end of the corridor (on the opposite side of the hill) simultaneously? *Martin did not feel like running up a hill.*

The tunnel itself is a chord of the circular hill. If l is the length of the tunnel, then the central angle φ between the beginning and the end of the tunnel satisfies the equation

$$\sin\left(\frac{\varphi}{2}\right) = \frac{l}{2r}\,.$$

The length of such an arc s is

$$s = r\varphi = 2r\arcsin\left(\frac{l}{2r}\right),$$

while the length of the tunnel is

$$l = \frac{x}{\cos\alpha}\,.$$

The ratio of average speeds of both runners is therefore

$$\frac{v_I}{v_M} = \frac{s}{l} = \frac{2r\cos\alpha}{x}\arcsin\left(\frac{x}{2r\cos\alpha}\right).$$

For the given values, $v_I/v_M \approx 1.078$. Ivan's average speed must be higher by approximately 7.8 %.

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Problem FoL.11 ... Humber Bridge 3 points

The Humber Bridge, which runs across the river Humber in England, is one of the largest suspension bridges in the world. It consists of two 155.5 m *high towers (with height measured from the river surface), whose bases are* 1410 m *away from each other (Euclidean distance, not distance on the ground), connected by ropes that are holding a road. Although both towers are vertical, their highest points are further away from each other than their bases. What is the difference of these (Euclidean) distances?* *Matěj was watching Tom Scott.*

The apparent conflict in the problem statement is caused by curvature of the Earth, which causes vertical objects in different places to be non-parallel. Let us imagine an isosceles triangle whose vertices are the bases of the towers and the middle of the Earth. The lengths of its sides are $s = 1410\,\mathrm{m}$, $R = 6378\,\mathrm{km}$ and R. A second triangle would consist of the towers' highest points and the middle of the Earth. The lengths of its sides are s', $R+h$ a $R+h$, where s' is the distance of towers' highest points and h is the height of a tower. Both triangles are obviously similar, so we can easily calculate

$$s' = s\frac{h+R}{R}\,,$$
$$s' - s = \frac{sh}{R} = 3.44\,\mathrm{cm}\,.$$

The highest points of the towers are therefore more than 3 cm further away from each other than their bases. If we thought of distances measured on a sphere instead, we would (from the similarity of circular sectors) come to the same result.

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Problem FoL.12 ... jumping

4 points

Find the angular velocity necessary to land a qaudruple Axel jump. The figure skater takes off from the left forward outside edge (the outer side of the left-leg ice skate's edge while moving forward) and lands on the right back outside edge, while achieving a maximum height $h = 1.0\,\mathrm{m}$ above the ice. *Dodo and the Grand Prix of Figure Skating.*

While jumping to a height h, a figure skater spends a total time t in the air. We can find this time from the equation of motion with constant acceleration (or free fall)

$$h = 1/2g\left(\frac{t}{2}\right)^2 .$$

We can express the time t as

$$t = \sqrt{\frac{8h}{g}} .$$

The skater needs to make four and a half turns, i.e. to rotate by $\varphi = 9\pi$. The angular velocity needed for that can be found as

$$\omega = \frac{\varphi}{t} = 9\pi\sqrt{\frac{g}{8h}} \approx 31.3\,\mathrm{s}^{-1} .$$

As of 27.11.2019, the jump hadn't been performed successfully yet, the only attempt was made by the Russian skater Arthur Dmitriev at Rostelecom Cup.

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Problem FoL.13 ... spider-spider

4 points

Pete the spider is $2\,\mathrm{cm}$ *long and he can weave threads of spider web with different thickness, but he needs to weave a thread whose radius is at least* $50.0\,\mathrm{nm}$ *in order for it to be able to support his own weight. What is the minimum radius of a thread that could support the spider if his size increased to that of a human? Assume that he would be* $2\,\mathrm{m}$ *long, with all his other dimensions increasing accordingly, but the density remaining constant.*

Tomáš was learning mathematics.

Clearly, the mass of the spider increases with the cube of his size (under the assumption that the density is the same). Therefore, after increasing n times in size, the tension in the string increases n^3 times. In our case, $n = 100$. The load capacity of the thread is, however, proportional to the cross-sectional area of the thread, so it increases as the square of the radius. Let m be the ratio of the necessary thread radius to the original thread radius (before the increase in size). Then, we get

$$m^2 = n^3 ,$$

$$m = n^{\frac{3}{2}} = 1000 .$$

The enlarged spider needs to weave a thread with radius at least $50\,\mathrm{\mu m}$. Upon comparison with conventional ropes used to support people, which have radii in the order of centimeters, we can truly appreciate the unbelievable strength of the spider web.

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Problem FoL.14 ... is it colonizable? 4 points

A planet similar to the Earth has a mass $M = 7.166 \cdot 10^{24}\,\mathrm{kg}$, *which is a bit more than the Earth's mass. By coincidence, the first and second cosmic velocities on this planet are the same as on the Earth. What is the gravitational acceleration on its surface?*

Matej can't wait for colonization.

Notice that the first and second cosmic velocities for a given planet depend only on the ratio of its mass and radius. The mass of our planet is $M = 1.20M_\mathrm{Z}$, where M_Z is the mass of the Earth. Analogously, the radius is $R = 1.20R_\mathrm{Z}$ (so that $M/R = M_\mathrm{Z}/R_\mathrm{Z}$). From the formula for gravitational acceleration, it can be seen that we can easily express the result as a multiple of gravitational acceleration on Earth g_Z

$$g = \frac{GM}{R^2} = \frac{1.2GM_\mathrm{Z}}{(1.2)^2R_\mathrm{Z}^2} = \frac{1}{1.2}g_\mathrm{z} \doteq 0.833\,g_\mathrm{z}\,,$$

where G is the gravitational constant. Although the planet has a bigger mass, it has only 83 % of the Earth's gravitational acceleration.

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Problem FoL.15 ... Dan's fléche burnt. 5 points

Jáchym spilled some lava in Minecraft. Let's assume (in order to simplify) that the world is a 2D cube grid with edge length $1\,\mathrm{m}$. *We can simulate the lava by the following set of rules:*

1. *the lava level in the central cube is* $1\,\mathrm{m}$,
2. *the lava level in any block which cannot be reached from the central block by moving through less than 3 other cubes (assume that we can only move between cubes if they have a common face), is* 0,
3. *the lava level in other blocks is given by the arithmetic mean of levels in neighbouring blocks (two blocks are neighbours if they have a common face).*

What is the total volume of the spilled lava?

Jáchym poured a bucket of lava on Dan's fléche in Minecraft.

In the picture 1 we can see an outline of part of the situation. As we can see, our problem has several symmetry axes. That makes our work a lot easier. As a result, we only need to find the lava levels in 5 squares – let's denote them a through e. We get a system of linear equations

$$\begin{aligned}
4a &= 1 + 2d + b\,,\\
4b &= a + 2e + c\,,\\
4c &= b\,,\\
4d &= 2a + 2e\,,\\
4e &= d + b\,.
\end{aligned}$$

Its solution is

$$
\begin{aligned}
a &= \frac{89}{208}\,,\\
b &= \frac{36}{208}\,,\\
c &= \frac{9}{208}\,,\\
d &= \frac{56}{208}\,,\\
e &= \frac{23}{208}\,.
\end{aligned}
$$

The final volume equals $V = (1 + 4a + 4b + 4c + 4d + 8 \cdot 10)\,\mathrm{m^3} = \frac{72}{13}\,\mathrm{m^3} \doteq 5.54\,\mathrm{m^3}$.

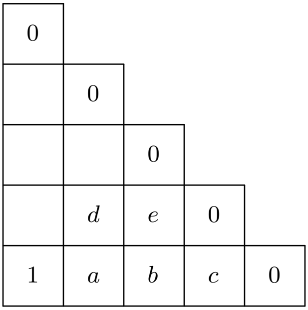

Fig. 1: Outline of part of lava.

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Problem FoL.16 ... one hundred reloaded 4 points

Jindra was interested in the magnitude of error in the problem "one hundred". The specific heat of water is in fact temperature-dependent. Jindra found an empirical relationship for the specific heat $c_{\mathrm{water}} = 3.1832 \cdot 10^{-6} t^4 - 7.7922 \cdot 10^{-4} t^3 + 7.5387 \cdot 10^{-2} t^2 - 2.9190t + 4.2158 \cdot 10^3$ *where the temperature t must be in* °C *and the specific heat is obtained in* $\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. *What is the difference (in percent) between the heat supplied to water with a constant specific heat* $c = 4184.0\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ *and the heat calculated using the empirical relationship, if the mass of this water is* $m = 1.00\,\mathrm{kg}$ *and its temperature increases from* $0.00\,\mathrm{^\circ C}$ *to* $100.00\,\mathrm{^\circ C}$ *in both cases? The sign of the result is important (i.e. if the amount of heat for the case with constant specific heat is lower, we expect a negative sign).* *Jindra received steam instead of boiled ice.*

The problem is relatively straightforward, the only tedious part is evaluating the integral. For the constant specific heat

$$
\begin{aligned}
Q_1 &= cm\Delta t\\
Q_1 &= 4.1840 \cdot 10^5\,\mathrm{J}\,.
\end{aligned}
$$

For the variable specific heat, we find

$$Q_2 = m \int_0^{100} c(t)\mathrm{d}t$$

$$Q_2 = m \left[\frac{3.1832\cdot 10^{-6}}{5}\cdot t^5 - \frac{7.7922\cdot 10^{-4}}{4}\cdot t^4 + \frac{7.5387}{3}\cdot t^3 - \frac{2.9190}{2}\cdot t^2 + 4.2158\cdot 10^3 t\right]_0^{100}$$

$$Q_2 = 4.1900\cdot 10^5\,\mathrm{J}\,.$$

The difference (relative to the case with variable specific heat) is

$$\delta Q = \frac{Q_1 - Q_2}{Q_2}$$

$$\delta Q \doteq -0.14\,\%$$

The value $\delta Q = -0.14\,\%$ is small, so the approximation in the problem "one hundred" was justified.

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Problem FoL.17 ... lazy electron 4 points

Consider the Bohr model of a hydrogen atom – a positively charged nucleus with charge $e = 1.6\cdot 10^{-19}\,\mathrm{C}$ and an orbiting electron with the opposite charge and a much smaller mass $m_e = 9.1\cdot 10^{-31}\,\mathrm{kg}$. What would be the radius of the hydrogen atom (in kilometers) if the period of one orbit of the electron took one day? Do not consider the fact that the hydrogen atom could not exist in such a state (or could it...?).

Matěj was comparing a hydrogen atom with the globe.

In the Bohr model, we can use the classical formulae for the centrifugal force and Coulomb's law and we get

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m_e\omega^2 r\,,$$

$$r = \sqrt[3]{\frac{e^2}{4\pi\varepsilon_0\omega^2 m_e}} = \frac{1}{2\pi}\sqrt[3]{\frac{e^2T^2}{2\varepsilon_0 m_e}} = 3\,600\,\mathrm{m} = 3.6\,\mathrm{km}\,,$$

where $T = \frac{2\pi}{\omega} = 1\,\mathrm{d} = 86\,400\,\mathrm{s}$ is the period of one orbit, ε_0 is the permittivity of vacuum and r is the atomic radius.

N.B. 1: We assumed that the atomic nucleus is static. Correctly, we should include the motion of the nucleus around the centre of mass of the atom. Then, the distance in Coulomb's law would be the distance from the nucleus instead of the centre of mass

$$r' = r\frac{m_e + M}{M}\,,$$

where M is the nuclear mass. As the nuclear mass is about three orders of magnitude larger than the mass of an electron, the result will change by less than 1% (the correction factor would be present in the result with $\frac{2}{3}$ in the exponent).

N.B. 2: We silently neglected Bohr's quantisation condition $L = n\hbar$ (L is the overall angular momentum, $\hbar$ is the reduced Planck constant and n is a natural number). This condition, however, quantises primarily orbits with small radii (and small angular momenta). When the radius increases, the angular momentum increases as $L = m_e \omega r^2 \approx r^{\frac{1}{2}}$ and for extreme cases such as the one in this problem, the value of L is in the order of millions of $\hbar$, so we can consider L to be continuous.

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Problem FoL.18 ... charge in a box — 4 points

A nasty charge has escaped from an imaginary cylinder, so Jindra locked it into a box. The box has dimensions $8.00 \times 8.00 \times 4.00\,\mathrm{cm}$. *Jindra taped the charge into the middle of its square base. The magnitude of the charge is* $Q = 1.00 \cdot 10^{-6}\,\mathrm{C}$. *Find the electric flux through the opposite face of the box.* *However, electric charges don't like Jindra.*

Let us imagine that we have another box (with the same dimensions) and glue it to the original box such that the face with the charge is their common face. These 2 boxes now form a cube with edge length 8 cm. Therefore, 1/6 of the overall electric flux flows through each face. We calculate the overall flux from Gauss's law.

$$\varphi_{\mathrm{tot}} = \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0}$$

The flux through the face opposite to the glued charge is

$$\varphi_{\mathrm{base}} = \frac{1}{6}\varphi_{\mathrm{tot}} = \frac{1}{6}\frac{Q}{\varepsilon_0}\,,$$
$$\varphi_{\mathrm{base}} = 1.88 \cdot 10^4\,\mathrm{V{\cdot}m}\,.$$

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Problem FoL.19 ... helix — 4 points

A positron is moving at a distance $R = 10\,\mathrm{cm}$ *from a wire carrying a current* $I = 2.5\,\mathrm{A}$. *The current generates a magnetic field, which causes the positron to orbit around the wire. The component of the positron's velocity in the direction of the current is* $v = 1.5\,\mathrm{m{\cdot}s^{-1}}$ *and the radial component is zero. Under the condition that the movement is stable, determine the period of the positron's orbit around the wire (in miliseconds).*
Let $e = 1.6 \cdot 10^{-19}\,\mathrm{C}$, $\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{N{\cdot}A^{-2}}$, $m_e = 9.1 \cdot 10^{-31}\,\mathrm{kg}$. *You spin me right round.*

The centripetal force acting on the positron is the Lorentz force

$$\boldsymbol{F} = -e\vec{v} \times \vec{B}\,,$$
$$F = evB(R)$$

where $B(R)$ is the magnitude of the magnetic field at the distance R from the wire, given by Ampére's law

$$B(R) = \frac{\mu_0}{2\pi}\frac{I}{R}$$

The centripetal force determines the angular velocity ω

$$F = m_e\omega^2 R$$

where the period of one orbit T satisfies

$$\omega = \frac{2\pi}{T}$$

Therefore

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_e R}{F}} = 2\pi\sqrt{\frac{m_e R}{evB(r)}} = 2\pi\sqrt{\frac{2\pi m_e R^2}{ev\mu_0 I}} = 2\pi R\sqrt{\frac{m_e}{\frac{\mu_0}{2\pi}evI}} \approx 1.7\,\mathrm{ms}$$

Štěpán Marek
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Problem FoL.20 ... charge inside a cylinder 6 points

In a thought experiment, Jindra managed to catch an electric charge $Q = 1.00\cdot 10^{-6}\,\mathrm{C}$ *into a cylinder which does not interact with the charge. The charge remains stationary in the centre of the cylinder. The cylinder has radius* $r = 4.00\,\mathrm{cm}$ *and height* $v = 6.00\,\mathrm{cm}$*. Determine the electric flux through the top base of the cylinder.* *Jindra likes electric charges.*

The electric flux is

$$\varphi = \int_S \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S}\,.$$

Gauss's law states that the electric flux through a closed surface is proportional to the charge enclosed by this surface

$$\varphi = \oint_S \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0}$$

Imagine the circumscribed sphere around the cylinder (the sphere that contains the perimeters of both bases). The flux through the top base of the cylinder is equal to the flux through the spherical cap over this base. Furthermore, the vector of the electric field is normal to the surface of the sphere and it has a constant magnitude on this surface. The area of the spherical cap is

$$S_{\mathrm{cap}} = 2\pi h R\,,$$

where h is the height of the spherical cap and R is the radius of the sphere, in our case $R = \sqrt{r^2 + (v/2)^2} = 5.00\,\mathrm{cm}$ and $h = R - v/2 = 2.00\,\mathrm{cm}$. The ratio of the electric flux through the spherical cap to the flux through the entire sphere is the same as the ratio of S_{cap} to the surface area of the entire sphere

$$\frac{\varphi_{\mathrm{cap}}}{\varphi_{\mathrm{sphere}}}=\frac{\varphi_{\mathrm{cap}}}{\frac{Q}{\varepsilon_0}}=\frac{S_{\mathrm{cap}}}{S_{\mathrm{sphere}}}=\frac{h}{2R}\,,$$

$$\varphi_{\mathrm{cap}}=\frac{h}{2R}\frac{Q}{\varepsilon_0}\,,$$

$$\varphi_{\mathrm{cap}}=2.26\cdot 10^4\,\mathrm{V{\cdot}m}\,.$$

The electric flux through the top base of the cylinder is therefore $\varphi_{\mathrm{cap}}=2.26\cdot 10^4\,\mathrm{V{\cdot}m}$.

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Problem FoL.21 ... African Sun 4 points

Mišo went on another train adventure – he wanted to cross the entire Africa by train. In the middle of September, before arriving to Nairobi, he saw a reflection of the just-rising Sun in the window on the right while sitting in the direction of the route (facing in this direction). At that time, the train was moving exactly south. 10 minutes later, the train was moving south-southwest. Determine the angle (in degrees) by which the reflection of the Sun with respect to the wagon deviated from the original position from Mišo's point of view.

Dodo was going home by a night train.

The train turned by the angle between the south and south-southwestern direction, which is $\alpha=22.5\,^\circ$. During that time, the Sun was also moving on the sky. Taking into account that this event happened near the equator and the date was close to the solar equinox, we could make an approximation that the Sun rose exactly in the East and ascended perpendicularly to the horizon with an angular velocity $\omega=15\,^\circ/\mathrm{h}$. Therefore, after 10 minutes, it rose to the angle $\beta=2,5^\circ$ above the horizon. These two angles are measured in perpendicular directions, so we can use the Pythagorean theorem as an approximation and we get that in total, the Sun turned by the angle

$$\gamma=\sqrt{\beta^2+\alpha^2}=22.64\,^\circ\,,$$

with respect to the train. The correct solution needs to use the law of cosines for a spherical triangle and then we get $\gamma=\arccos\left(\cos\alpha\cos\beta\right)=22.63\,^\circ$. The last step is finding out how this affects the change in the direction at which Mišo observes the reflection. It turns out that it is exactly the angle γ.

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Problem FoL.22 ... Red hot nickel... cable? 4 points

A heater has a nickel heating filament with a temperature coefficient of resistance $\alpha=0.0068\,\mathrm{K}^{-1}$ and with a resistance $R_0=80\,\Omega$ at the room temperature $T_0=293\,\mathrm{K}$. After turning on the heater, the filament becomes red-hot at the temperature $T=1100\,\mathrm{K}$ thanks to a current $I=$ $=2\,\mathrm{A}$ passing through it. What is the total surface of the filament (in m^2), assuming that the filament transfers heat only by radiation (acts as a black body)?

Jirka is already thinking about how to warm up in winter.

The power of the hot filament as a resistor is $P = RI^2$. This power will be the same as the radiant flux transmitted by the filament to its surroundings $\Phi = \sigma T^4 \cdot S$. However, we still do not know the resistance of the hot filament. This can be calculated using the formula $R = R_0(1 + \alpha\Delta t)$, where we substitute $T - T_0$ for Δt. We get the equation

$$\sigma T^4 \cdot S = R_0[1 + \alpha(T - T_0)]I^2 \,,$$

from which we simply express the required area

$$S = \frac{R_0[1 + \alpha(T - T_0)]I^2}{\sigma T^4} \,.$$

Now, all we have to do is substitute the numeric values

$$S = \frac{80\,\Omega[1 + 0.0068\,\mathrm{K}^{-1} \cdot (1100\,\mathrm{K} - 293\mathrm{K})] \cdot (2\,\mathrm{A})^2}{5.67 \cdot 10^{-8}\,\mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4} \cdot (1100\,\mathrm{K})^4} = 0.025\,\mathrm{m}^2 \,.$$

The filament's surface area is $0.025\,\mathrm{m}^2$.

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Problem FoL.23 . . . hanging here

5 points

Two springs with identical length $l = 5.0\,\mathrm{m}$ and stiffness $k = 1.0.10^5\,\mathrm{N{\cdot}m^{-1}}$ are stretched horizontally between two vertical walls (at the distance $2l$) in such a way that the springs are connected in the middle. Next, we hang a mass $m = 3.0\,\mathrm{kg}$ at the point where the springs are connected. What is the height h by which the point of connection drops? You may assume that $h \ll l$ and $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

The tension is rising.

The length of each spring under the load is $l' = \sqrt{l^2 + h^2}$, so each spring is extended by

$$\Delta l = l' - l = \sqrt{l^2 + h^2} - l = l\left(\sqrt{1 + \frac{h^2}{l^2}} - 1\right) \approx l\left(1 + \frac{h^2}{2l^2} - 1\right) = \frac{h^2}{2l} \,.$$

The vertical component F_v of the force with which the springs are acting on the mass is

$$F_v = \frac{h}{\sqrt{l^2 + h^2}} k\Delta l = \frac{kh^3}{2l^2}\frac{1}{\sqrt{1 + \frac{h^2}{l^2}}} \approx \frac{kh^3}{2l^2}\left(1 - \frac{h^2}{2l^2}\right) \approx \frac{kh^3}{2l^2} \,.$$

The two springs together balance out with the force of gravity F_g acting on the mass, and hence

$$F_g = 2F_v \,,$$

$$mg = \frac{kh^3}{l^2} \,,$$

$$h = \sqrt[3]{\frac{mgl^2}{k}} \approx 0.19\,\mathrm{m} \,.$$

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Problem FoL.24 ... doggy 6 points

Suppose that we are on a hypothetical planet Caniculium orbiting the white dwarf Sirius B, which is the weaker component of the Sirius binary system. From the orbit of this planet, a hybrid eclipse of Sirius A can be observed (in a hybrid eclipse, the stars have the same angular diameter). Find out the proportion of landscape illumination (power incident on the surface of the planet) during the eclipse compared with the situation just before the eclipse starts. The effective surface temperatures of these stars are $T_A = 9940\,\mathrm{K}$ $T_B = 25\,000\mathrm{K}$.

Dodo thinks that the Moon is not shining by itself.

A hybrid eclipse occurs when both stars involved in the eclipse have the same angular diameters, so their radii R_A, R_B and distances from the planet r_A, r_B have to satisfy

$$\frac{R_A}{r_A} = \frac{R_B}{r_B} \,.$$

We can determine the total radiant flux of a star from the Stephan-Boltzmann law as

$$L = 4\pi R^2 \sigma T^4 \,.$$

This power is radiated to a whole sphere with radius r, so the radiant flux per $1\,\mathrm{m}^2$ received by the surface of the planet is

$$F = \frac{L}{4\pi r^2} = \frac{R^2}{r^2}\sigma T^4 \,.$$

We are interested in the flux ratio between the situations where only Sirius B illuminates the planet and where both stars illuminate the planet

$$w = \frac{F_B}{F_A + F_B} = \left(1 + \frac{F_A}{F_B}\right)^{-1} = \left(1 + \frac{\frac{R_A^2}{r_A^2}\sigma T_A^4}{\frac{R_B^2}{r_B^2}\sigma T_B^4}\right)^{-1} = \frac{1}{1 + \frac{T_A^4}{T_B^4}} \doteq 0.976 \,.$$

During the eclipse, the landscape is illuminated by $w \doteq 0.976$ of the light before the eclipse.

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Problem FoL.25 ... ethanol emissions 3 points

We want to completely burn $V_\mathrm{e} = 1.00\,\mathrm{dm}^3$ *of liquid ethanol in an oxygen rich atmosphere, so that the main product of the combustion is carbon dioxide* (CO_2)*. All the released carbon dioxide should be contained in a vessel at the standard atmospheric pressure and temperature* $0\,^\circ\mathrm{C}$*. What is the required volume of such a vessel? (Assume that we can eliminate all other reaction products.)*

Karel was thinking about experiments.

Let's denote the density and molecular weight of ethanol by $\varrho = 789\,\mathrm{kg{\cdot}m^{-3}}$ and $M_\mathrm{m} = 46.07\,\mathrm{g{\cdot}mol^{-1}}$ respectively. Then, the molar amount of ethanol is

$$n_\mathrm{e} = \frac{V_\mathrm{e}\varrho}{M_\mathrm{m}} \,.$$

One molecule of ethanol contains two carbons, so the amount of carbon dioxide will be twice the amount of ethanol, $n_\mathrm{o} = 2n_\mathrm{e}$. The last thing we need to do is multiply the molar amount of CO_2 and the molar volume $V_\mathrm{m} = 22.41\,\mathrm{dm^3\cdot mol^{-1}}$ of a gas at the given standard conditions and we get

$$V = n_\mathrm{o} V_\mathrm{m} = \frac{2V_\mathrm{e}\varrho V_\mathrm{m}}{M_\mathrm{m}} \doteq 0.768\,\mathrm{m^3}\,.$$

To contain all the carbon dioxide produced by burning $1\,\mathrm{l}$ of ethanol, we would need a vessel with volume $V \doteq 0.768\,\mathrm{m^3}$.

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Problem FoL.26 ... jump into the unknown 4 points

A very thin cylinder with a length $l = 32.7\,\mathrm{cm}$ is in a vertical position with its bottom base at the height $10\,l$ above a water surface. What depth below the water surface does the lowest point of the cylinder reach after we let it fall? The density of the liquid is three times higher than the density of the cylinder. Neglect any drag, friction and surface tension.

Jáchym fell into the water in Minecraft.

Let us introduce a vertical coordinate x pointing downwards, with the origin at the surface of the water. The cylinder is subject to a force of gravity $F_g = mg$ and a buoyant force

$$F_\mathrm{v} = \begin{cases} 0\,, & x \in (-nl,\,0) \\ -xS\varrho g\,, & x \in (0,\,l) \\ -lS\varrho g, & x \in (l,\,\infty)\,, \end{cases}$$

where $n = 10$, S is the cross-sectional area of the cylinder and ϱ is the density of water. The buoyant force is negative because it is directed in the direction of decreasing x. Also, let's denote $k = 3$; then, the density of the cylinder is ϱ/k and $m = lS\varrho/k$ holds.

The bottom of the cylinder dives to an unknown depth h, where it stops. At that point, it has zero kinetic energy, so its potential energy is the same as at the beginning. Hence, the total work performed by gravity and buoyancy during the movement is zero. This can be expressed by the equation

$$\int_{-nl}^{h} (F_g + F_\mathrm{v})\,\mathrm{d}x = 0\,.$$

We divide this integral into three parts, because the buoyant force behaves differently at three different intervals, and we get

$$\int_{-nl}^{0} mg\,\mathrm{d}x + \int_{0}^{l} (mg - xS\varrho g)\,\mathrm{d}x + \int_{l}^{h} (mg - lS\varrho g)\,\mathrm{d}x = 0\,,$$

$$[mgx]_{-nl}^{0} + \left[mgx - \frac{1}{2}x^2 S\varrho g\right]_0^l + [mgx - lS\varrho gx]_l^h = 0\,.$$

From this, we can easily express

$$h = \frac{l}{2}\frac{2nm + lS\varrho}{lS\varrho - m}\,.$$

Now we just substitute for m and we can write the result

$$h = \frac{l}{2}\frac{2n + k}{k - 1} = \frac{23}{4}l \doteq 1.88\,\mathrm{m}\,.$$

The cylinder sinks to the depth 1.88 m. Thanks to all simplifications in the problem statement, the result does not depend on its area or the shape of the base.

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Problem FoL.27 ... rolling cone

5 points

On a horizontal surface, a cone with a base radius $r = 10.0\,\mathrm{cm}$ *and height* $H = 100.0\,\mathrm{cm}$ *made from a material with density* $\varrho = 1\,253\,\mathrm{kg{\cdot}m^{-3}}$ *is slipping with negligible rolling friction. The angular speed of rotation of the cone around the axis perpendicular to the surface and passing through its apex is* $\omega = 2.50\,\mathrm{rad{\cdot}s^{-1}}$*. Determine the magnitude of the force which the cone exerts on the surface.* *Dodo's head was spinning.*

The center of gravity of the cone moves uniformly along a circle, so the net force acting on the cone must be directed from the center of gravity perpendicularly to the axis of rotation. If the distance of the center of gravity from the axis is s, we have $F_\mathrm{c} = m\omega^2 s$ for the centripetal force. The cone is subject to the force of gravity F_g and the reactive force from the pad F. The horizontal component of the force from the pad is thus $F_1 = -F_\mathrm{c}$ and its vertical component is $F_2 = -F_g = -mg$. Overall, its magnitude is

$$F = \sqrt{F_1^2 + F_2^2} = m\sqrt{\omega^4 s^2 + g^2} = \frac{\pi}{3}\varrho r^2 h \sqrt{\frac{9\omega^4 h^4}{16(h^2 + r^2)} + g^2}\,,$$

where we used the Pytagorean theorem and the fact that the center of gravity of the cone divides the line segment connecting the apex and the centre of the base in a 3:1 ratio.

When we evaluate the equation, we get $F \doteq 143\,\mathrm{N}$.

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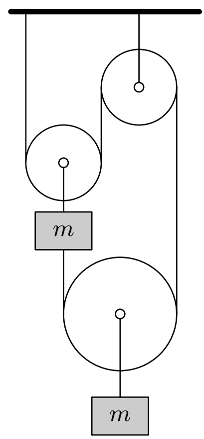
Problem FoL.28 ... pulleys

5 points



Find the acceleration of the lower pulley and its direction (the result should be negative if the pulley accelerates downwards or positive if it accelerates upwards). Neglect moments of inertia of the pulleys.
Matěj wanted to invent an infinite system of pulleys, but it didn't work.

Let us begin with the fact that the tensile force along the length of the rope is uniform and equal to F at each point. For the acceleration of the first pulley, we obtain

$$a_1 = \frac{F}{m} - g\,,$$

and for the lower one, we get

$$a_2 = \frac{2F}{m} - g\,.$$

Now we only have to realise that the rope is inelastic, so $a_1 = -2a_2$.

Solving these 3 equations, we get

$$a_1 = -2/5g\,,$$
$$a_2 = 1/5g = 1.96\,\mathrm{m{\cdot}s^{-2}}\,.$$

The acceleration of the lower pulley is $a_2 = 1.96\,\mathrm{m{\cdot}s^{-2}}$.

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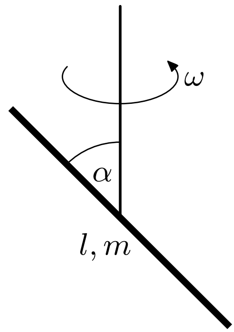
Problem FoL.29 ... small torque

5 points

A thin homogeneous rod is attached to an axis of rotation passing through the rod's centre. The angle between the rod and the axis of rotation is $\alpha = 45^\circ$. The rod has mass $m = 40\,\mathrm{g}$, length $l = 30\,\mathrm{cm}$ and it rotates with an angular velocity $\omega = 25\,\mathrm{rad{\cdot}s^{-1}}$. What is the magnitude of the torque by which the rod acts on its point of contact with the axis?

Jindra's problem with the inertia tensor didn't pass, so he invented this.



In the rotating reference frame connected with the rod, there is a centrifugal force acting on the rod, which pushes its mass from the axis of rotation. If the rod was perpendicular to the axis of rotation, the centrifugal forces acting on both ends of the rod would be collinear and their torque would be zero.

Let's denote the linear mass density of the rod by τ. The centrifugal force acting on an element of the rod is

$$\mathrm{d}F_\mathrm{o} = \mathrm{d}m\omega^2 r_x = \tau\omega^2 s\sin\alpha\mathrm{d}s\,.$$

The total torque can be calculated by integration of these infinitesimal forces' torques over the whole length of the rod.

$$\mathrm{d}M_\mathrm{o} = r_y\mathrm{d}F_\mathrm{o} = s\cos\alpha\mathrm{d}F_\mathrm{o} = \tau\omega^2\sin\alpha\cos\alpha s^2\mathrm{d}s$$
$$M_\mathrm{o} = \tau\omega^2\sin\alpha\cos\alpha\int_{-l/2}^{l/2} s^2\mathrm{d}s = \tau\omega^2\sin\alpha\cos\alpha\left[\frac{1}{3}s^3\right]_{-l/2}^{l/2}$$
$$M_\mathrm{o} = \frac{1}{24}m\omega^2 l^2\sin 2\alpha$$
$$M_\mathrm{o} = 0.09375\,\mathrm{N{\cdot}m}\,.$$

The torque acting on the point of connection between the rod and the axis is $0.09375\,\mathrm{N{\cdot}m}$.

The vector of the torque points perpendicularly to the plane containing the rod and the axis. In general, the vectors of angular momentum and angular velocity don't have to be parallel. In that case, the vector of angular momentum rotates around the axis. In order to prevent the rod from turning until it's perpendicular to the axis, the point of connection has to exert on it the same torque with the opposite direction. This torque causes the rotation of angular momentum in accordance with Newton's second law (for rotational motion) $M = \frac{\mathrm{d}L}{\mathrm{d}t}$.

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Problem FoL.30 ... asteroid with impact parameter

5 points

An asteroid very, very far from the Sun (we may assume it is infinitely far) has a speed $v = 3.9\,\mathrm{km\cdot s^{-1}}$. If there was no gravitational influence on the asteroid, the smallest distance between the asteroid and the Sun would be $1\,\mathrm{au}$. What will be the smallest distance between the asteroid and the Sun when the Sun's gravitational force is acting on the asteroid?

Karel was thinking about alien spacecrafts.

The angular momentum of the asteroid at infinity can be expressed as $L_1 = mr_1v_1$, where m is the mass of the asteroid, $v_1 = v$ and r_1 is the smallest distance between the Sun and the asteroid in the case with a straight-line trajectory.

For a real hyperbolic trajectory, the angular momentum is conserved, and at the asteroid's perihelium, the angular momentum is $L_2 = mr_2v_2$, where r_2 is the smallest distance of the asteroid from the Sun, so we get

$$r_1v_1 = r_2v_2\,.$$

The other conserved quantity is the total mechanical energy. The gravitational potential energy is defined as

$$E_\mathrm{p} = -\frac{GMm}{r}\,,$$

where $M = 1.99\cdot 10^{30}\,\mathrm{kg}$ is the mass of the Sun and we set E_p to zero at infinity. From the energy conservation,

$$v_1^2r_2^2 + 2GMr_2 - r_1^2v_1^2 = 0\,,$$

and by substituting for v_1, we get

$$v_1^2r_2^2 + 2GMr_2 - r_1^2v_1^2 = 0\,,$$

which has two solutions

$$r_2 = \frac{-GM \pm \sqrt{G^2M^2 + r_1^2v_1^4}}{v_1^2}\,.$$

Because distance from the Sun is always positive, we choose the + sign and get the final result

$$r_2 = \frac{-GM + \sqrt{G^2M^2 + r_1^2v_1^4}}{v_1^2} \doteq 1.3\cdot 10^6\,\mathrm{km}\,.$$

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Problem FoL.31 ... defective spherical mirror

5 points

An astronomer wants to photograph a star using a Newtonian telescope. The primary mirror of the telescope is spherical (not parabolic) with a radius of curvature $R = 1.00\,\mathrm{m}$ and transverse diameter $d = 10.0\,\mathrm{cm}$. The astronomer placed a CCD sensor in the focal plane at a distance $R/2$ from the mirror's vertex. What will be the radius of the star's image on the chip? Assume that the star is a point source of light.

Jindra was thinking about imperfections of the world...

The spherical mirror is influenced by the so-called spherical aberration - the rays reflected from the edge of the mirror intersect at a different point than the rays reflected closer to the optical axis. The resulting image will be blurred. However, the law of reflection still holds, so we can calculate this effect. A ray coming from the star at infinity, parallel to the optical axis and at a distance x from the axis, forms an angle $\alpha = \arcsin(x/R)$ with the normal to the mirror's surface at the point of its reflection. The angle between the reflected beam and the optical axis is 2α and the ray crosses the optical axis at a distance $s = R/(2\cos\alpha)$ from the centre of curvature. The distance between the point where the beams passing very close to the optical axis intersect and the point where the beams reflected from the edge of the mirror intersect is

$$\Delta s = \frac{R}{2}\left(\frac{1}{\cos\alpha_m} - 1\right) ,$$

where $\alpha_\mathrm{m} = \arcsin(d/(2R))$, numerically $\alpha_\mathrm{m} \doteq 2.87^\circ$, and $\Delta s \doteq 6.26 \cdot 10^{-4}\,\mathrm{m}$. The reflected rays create a circle on the CCD chip and its radius is

$$\varrho = \Delta s \tan(2\alpha_m) .$$

The numerical result is $\varrho = 62.9\,\mathrm{\mu m}$.

A parabolic mirror would solve the spherical aberration. On the other hand, manufacturing a spherical surface is easier and cheaper. Therefore, only mirrors with a larger diameter are made parabolic.

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Problem FoL.32 ... quantum Kepler 5 points

As you surely know, Kepler's third law is the relationship between the semi-major axis and the orbital period of celestial bodies $a^3/T^2 = \mathrm{const.}$ This relationship can be also derived from the nature of the gravitational field. The electric field is also inversely proportional to the square of distance, so the situation is analogous to the gravitational field. Derive how Kepler's third law would look for the system of an electron and a proton if they behaved according to the classical laws of physics. What would be the ratio of the third power of the semi-major axis and the square of the orbital period?

Jindra wondered if hydrogen orbitals could be explained using Platonic bodies.

Let the distance between the proton and the electron be a. They're orbiting their common center of gravity and its distance from the electron is

$$r_\mathrm{e} = \frac{m_p}{m_p + m_e} a .$$

The electron and proton attract each other with a force

$$F_\mathrm{E} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{a^2} .$$

At the same time, the attractive electric force acts as the centripetal force curving the paths of the electron and proton.

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{a^2} = m_\mathrm{e}\omega^2 r_\mathrm{e} = m_\mathrm{e}\frac{4\pi^2}{T^2}\frac{m_\mathrm{p}}{m_\mathrm{p} + m_\mathrm{e}} a$$

$$\frac{a^3}{T^2} = \frac{e^2(m_\mathrm{p} + m_\mathrm{e})}{16\pi^3\varepsilon_0 m_\mathrm{e} m_\mathrm{p}} \tag{1}$$

Since we want to know the result with high precision, at least 6 significant digits, the constants we plug into the formula need to have at least 6 significant digits too.[1]

$$
\begin{aligned}
e &= 1.602\,176\,634 \cdot 10^{-19}\,\mathrm{C}\\
m_\mathrm{e} &= 9.109\,383\,7015(28) \cdot 10^{-31}\,\mathrm{kg}\\
m_\mathrm{p} &= 1.672\,621\,923\,69(51) \cdot 10^{-27}\,\mathrm{kg}\\
\pi &\doteq 3.141\,592\,654\\
\varepsilon_0 &= 8.854\,187\,8128(13) \cdot 10^{-12}\,\mathrm{F{\cdot}m}
\end{aligned}
$$

After plugging into the equation (1), we get

$$\frac{a^3}{T^2} = 6.418\,74\,\mathrm{m^3{\cdot}s^{-2}}\,.$$

For the system of an electron and a proton (a hydrogen atom), the ratio is $a^3/T^2 = 6.418\,74\,\mathrm{m^3{\cdot}s^{-2}}$. Just for comparison, for the system of the Earth and the Sun, it is $a^3/T^2 = 3.36 \cdot 10^{18}\,\mathrm{m^3{\cdot}s^{-2}}$.

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Problem FoL.33 ... tense tree — 6 points

Jáchym cut down a homogeneous cylindrical tree with height $l = 30.0\,\mathrm{m}$. The trunk of the tree is still partially connected to the stump, so the top of the tree follows a circular trajectory. Find the distance from the stump to the point where the normal stress in the tree is zero at the moment when the angle between the tree and the ground is $\alpha = 47^\circ$. The connection between the trunk and the stump isn't causing any energy losses. *Jáchym fell(ed a tree).*

Let's denote the distance from the stump to the point in question by x. Then, the center of gravity of the top segment of the tree with length $(l - x)$ and mass m_x is located at the distance $r = (l + x)/2$ from the stump. There is a vertical force of gravity $F_g = m_x g$ acting on this segment and the projection of its vector on the tree is $F_{g_t} = F_g \sin\alpha$. There needs to be a centripetal force $F_\mathrm{d} = m_x \omega^2 r$ acting on the segment to make it move along a circular arc, where ω is the angular velocity of the tree's rotation. At the distance x, there will be zero normal stress only when both forces acting on the top segment of the tree in the radial direction are equal. In other words, $F_{g_n} = F_\mathrm{d}$ must hold.

We can easily work out the value of angular velocity because the rotational energy of the tree is equal to the decrease of its potential energy

$$
\begin{aligned}
\frac{1}{2} J\omega^2 &= mg\left(l - l\sin\alpha\right)/2\,,\\
\omega^2 &= \frac{3g\left(1 - \sin\alpha\right)}{l}\,,
\end{aligned}
$$

[1] `https://en.wikipedia.org/wiki/List_of_physical_constants`

where we used the moment of inertia of a bar with respect to its endpoint

$$J = \frac{1}{3} m l^2 \,.$$

Now we just substitute the given values into the force balance equation and we get

$$x = \frac{5 \sin \alpha - 3}{3\,(1 - \sin \alpha)} l \doteq 24.4\,\mathrm{m} \,.$$

It is worth noting that the final formula does not make good sense for too small or too large angles α, because the distance x is either negative or higher than l in those cases. In our case, the value of α was chosen reasonably and we get $x = 24.4\,\mathrm{m}$.

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Problem FoL.34 ... slim line

5 points

Jindra is returning from a space mission and is moving towards the Earth with the speed $v = 0.300\,c$. Danka, whose rest mass is $50.0\,\mathrm{kg}$, is looking forward to meeting Jindra after a long time, so she set off in a rocket towards Jindra's spaceship. However, Danka has been on a diet. She doesn't want Jindra to be able to measure her relativistic mass being greater than $60.0\,\mathrm{kg}$. What maximal speed u with respect to the Earth can Danka fly? The answer is the ratio between u and c. *For Jindra, travelling around Earth is not enough.*

Danka's relativistic mass, measured by Jindra in his reference frame, must not exceed $60.0\,\mathrm{kg}$. Let us denote the relative velocity of Danka with respect to Jindra as w. Jindra, from his frame of reference, observes that Earth is approaching him with velocity of $v = 0.300\,\mathrm{c}$ and Danka approaching him at a higher speed w. The velocities u and v are added together relativisticly, so

$$w = \frac{v + u}{1 + \frac{uv}{c^2}} \,. \tag{2}$$

The relationship between rest mass and relativistic mass, as measured by Jindra, is

$$m = \frac{m_0}{\sqrt{1 - (\frac{w}{c})^2}} \,.$$

If Jindra is supposed to observe that Danka's weight is less than $60.0\,\mathrm{kg}$, then the maximal velocity at which she can move towards him is $w = c\sqrt{1 - (m_0/m)^2} = 0.553\,\mathrm{c}$. We express velocity u from equation (2) as

$$u = \frac{w - v}{1 - \frac{wv}{c^2}} \doteq 0.303\,\mathrm{c} \,.$$

Therefore, Danka can travel towards Jindra at maximum speed $u = 0.303\,\mathrm{c}$.

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Problem FoL.35 ... overheated flywheel

4 points

By how many degrees Kelvin will a flywheel heat up by stopping? The initial frequency of rotation of the flywheel is $f = 120\,\mathrm{Hz}$*. Assume that half of its kinetic energy will be used for heating it up. The flywheel is a homogeneous cylinder with a radius* $r = 15\,\mathrm{cm}$ *and it is made of steel with a specific heat capacity* $c = 450\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$*. Hint: we don't need to know any other quantities, only the value of* π*.*

Karel was discussing IYPT tasks.

The kinetic energy of the flywheel is

$$E = \frac{1}{2} J\omega^2 \,.$$

We know that $\omega = 2\pi f$ and the moment of inertia of a cylinder is $J = mr^2/2$, therefore

$$E = \pi^2 m r^2 f^2 \,.$$

We can determine the change in temperature from the expression for heat $Q = mc\Delta t$, where Δt is the change in temperature. From the problem statement, we know that we should only consider half of the dissipated energy, so we can calculate

$$\frac{E}{2} = Q \quad \Rightarrow \quad mc\Delta t = \frac{1}{2}\pi^2 m r^2 f^2 \quad \Rightarrow \quad \Delta t = \frac{\pi^2 r^2 f^2}{2c} \doteq 3.6\,\mathrm{K} \,.$$

The flywheel will heat up by about 3.6 K when it completely stops.

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Problem FoL.36 ... looping

6 points

Imagine a biker who wants to ride around a thin vertical circular looping (a cylindrical surface such that its axis is horizontal) with a radius $R = 10.0\,\mathrm{m}$*. Therefore, the looping has no entry or exit and the biker starts from the lowest point with a constant angular acceleration. What is its minimum value such that the motorcycle can drive through the looping while always keeping in contact with the looping (without falling)? Use* $g = 9.81\,\mathrm{kg\cdot s^{-2}}$*.*

Matěj has driven through a looping by car many times.

Let's solve the problem from the perspective of the biker. There are two forces acting on him: the centrifugal force and gravitational force. These forces can be separated into components parallel to the direction of velocity (tangent) and components perpendicular to the velocity (normal). Our first condition is that the total normal force must always push the biker to the track, otherwise he would fall off the track. Next, instead of forces, we will work only with accelerations so that we don't have to consider the biker's weight.

Let the radius of the looping be R. In the rotating system associated with the biker, there is a centrifugal acceleration and and Euler's acceleration, which act only in the tangential direction. The Coriolis acceleration is equal to zero because the radial velocity is equal to zero. The centrifugal acceleration has always only the normal component $a_\mathrm{c} = \omega^2 R = \varepsilon^2 t^2 R$, where $\omega = \varepsilon t$ is the angular velocity of the biker and ε is the angular acceleration of his motorcycle (with respect to the center of the cylinder). The normal component of the force of gravity is $a_\mathrm{n} = g\cos\varphi$, where φ is the angle describing the biker's position (zero at the start and $\varphi = \pi$ at the highest point). We can write down the condition of the biker sticking to the track as

$$
\begin{aligned}
a_\mathrm{n} + a_\mathrm{c} &\geq 0\,,\\
g\cos\varphi + \varepsilon^2 t^2 R &\geq 0\\
g\cos\varphi + 2\varepsilon R\varphi &\geq 0
\end{aligned}
$$

where using substitution $\varphi = \frac{\varepsilon t^2}{2}$ we get

$$\varepsilon \geq -\frac{g}{2R}\frac{\cos\varphi}{\varphi}\,. \tag{3}$$

This condition must hold for all φ, so we are looking for φ_m such that the expression $\frac{\cos\varphi_\mathrm{m}}{\varphi_\mathrm{m}}$ has the minimal value. Using the first derivative of the expression above, we get

$$
\begin{aligned}
\frac{-\varphi\sin\varphi - \cos\varphi}{\varphi^2} &= 0\,,\\
\varphi\tan\varphi &= -1\,,\\
\varphi &= -\cot\varphi\,.
\end{aligned}
$$

This type of equations cannot be solved analytically, but we can use a calculator, since the result is not needed with a high number of significant digits. After a few iterations, we find out that the lowest positive value that satisfies the equation is approximately $\varphi_\mathrm{m} = 2.798\,3\ldots$ Surprisingly, the critical point is not required to lie at the top of the looping.

After substituting this value into the equation (3), we get the minimum possible acceleration

$$a = \left(0.01683\,\mathrm{m}^{-1}\right) g = 0.16506\,\mathrm{rad{\cdot}s^{-2}}\,.$$

With a smaller acceleration, the biker would fall off the track.

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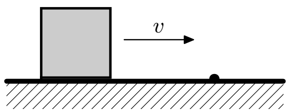
Problem FoL.37 ... bumps on the road

6 points



A homogeneous cube with edge length $a = 1.0\cdot 10^{-2}$ m is happily sliding with a velocity v on a horizontal plane without friction such that one face (hence two) is perpendicular to the direction of movement. Directly in front of it, there is a low stopper, and the cube collides with it perfectly inelastically; the point of impact is in the middle of its bottom front edge. Find the minimum velocity v such that the cube flips over the bump (stopper). *Matej calculated this during a college party.*

Consider the conservation of angular momentum with respect to the axis which coincides with the edge that collided with the bump, since the cube started rotating around this axis as a result. Due to the fact that all points are moving with the velocity v before the collision and the direction of the velocity of the center of mass is at the distance $a/2$ from this axis, the angular momentum before the collision is

$$L = \frac{1}{2}mav\,,$$

where m is the mass of the cube. The moment of inertia of the cube with respect to the axis of rotation is, using Steiner's theorem, the sum of the moment of inertia of a cube around an axis passing through its center of mass and $m(a/\sqrt{2})^2$, because the center of mass is at the distance $a/\sqrt{2}$ from the axis,

$$J = \frac{1}{6}ma^2 + \frac{1}{2}ma^2 = \frac{2}{3}ma^2 .$$

At the moment of collision, the block is turning around the bump with the angular velocity

$$\omega = \frac{L}{J} .$$

Its kinetic energy is

$$E = \frac{1}{2}J\omega^2 = \frac{L^2}{2J} .$$

This energy has to be high enough for it to go around the bump, what means to lift the center of gravity by

$$\frac{\sqrt{2}-1}{2}a ,$$

so

$$E = \frac{\sqrt{2}-1}{2}amg .$$

After substituting this value into the previous formula for energy, we get

$$\frac{L^2}{2J} = \frac{\sqrt{2}-1}{2}amg ,$$
$$\frac{3}{8}v^2 = \left(\sqrt{2}-1\right)ag ,$$
$$v = \sqrt{\frac{8}{3}\left(\sqrt{2}-1\right)ag} = 0.3292\,\mathrm{m{\cdot}s^{-1}} .$$

This is the critical velocity at which the cube flips over.

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Problem FoL.38 ... speed merchant — 4 points

Dan is driving on a straight road with a velocity $v_\mathrm{d} = 90\,\mathrm{km{\cdot}h^{-1}}$. In the distance $d = 110\,\mathrm{m}$ in front of him, he notices a cyclist riding with a velocity $v_\mathrm{c} = 18\,\mathrm{km{\cdot}h^{-1}}$. Because there are cars coming from the opposite direction, Dan can't drive around the cyclist, so he has to hit the brakes. His response time is $t_\mathrm{r} = 0.50\,\mathrm{s}$. What is the lowest allowed value of acceleration (deceleration due to the brakes) of the car such that Dan doesn't crush the cyclist? The car decelerates uniformly. *Jindra rode with Dan in a car once and had enough.*

The problem is easiest to solve if we use the reference frame connected with the cyclist. The velocity with which the car is approaching the cyclist is $v_\mathrm{r} = 72\,\mathrm{km{\cdot}h^{-1}} = 20\,\mathrm{m{\cdot}s^{-1}}$. Dano won't run over the cyclist if he brakes to zero (in the cyclist's reference frame) just behind him. Before Dano hits the brakes, the car travels a distance

$$d_1 = v_\mathrm{r} t_\mathrm{r}\,.$$

Numerically, $d_1 = 10\,\mathrm{m}$. In that moment, the distance between the car and the cyclist is $d_2 = 100\,\mathrm{m}$. We get the minimum acceleration from equations of motion with uniform acceleration

$$v_\mathrm{r}^2 = 2ad_2\,,$$

$$a = \frac{v_\mathrm{r}^2}{2d_2}\,.$$

The minimum acceleration is $a = 2.0\,\mathrm{m\cdot s^{-2}}$.

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Problem FoL.39 ... slowly thrown rock 6 points

Ida is one of the few asteroids in the main asteroid belt that has its own natural satelite. The moon's name is Dactyl. We haven't yet measured the shape of its orbit very precisely. Suppose that its distance from Ida in the pericenter is $r_\mathrm{p} = 100\,\mathrm{km}$ and in the apocenter is $r_\mathrm{a} = 200\,\mathrm{km}$. Dactyl's mass is $m_\mathrm{D} = 10^{10}\,\mathrm{kg}$, the mass of Ida is $m_\mathrm{I} = 5\cdot 10^{16}\,\mathrm{kg}$. Imagine an astronaut with a mass $m = 100\,\mathrm{kg}$ on the surface of Dactyl. In the pericenter, the astronaut jumps up in the direction of motion of Dactyl with a velocity $v = 5\,\mathrm{m\cdot s^{-1}}$. Determine the displacement of the apocenter distance of Dactyl caused by the jump (a positive number means that this distance increases). Neglect gravitational effects of the astronaut, i.e. treat it as a two-body problem.

Mirek gains inspiration at conference lectures.

We can estimate the new speed of Dactyl (in the pericenter, immediately after the jump) from the law of momentum conservation as

$$v'_\mathrm{p} = v_\mathrm{p} - \frac{m}{m_\mathrm{D}} v\,.$$

The next part of our solution is based on the laws of conservation of angular momentum and conservation of energy using the distances in the apocenter and pericenter. It is necessary to realize that if we subtract the weight of the astronaut from the weight of Dactyl, its orbit remains unchanged. Therefore, we may use conservation laws written with the mass of Dactyl

$$m_\mathrm{D} r_\mathrm{p} v_\mathrm{p} = m_\mathrm{D} r_\mathrm{a} v_\mathrm{a}\,,$$

$$-\frac{G m_\mathrm{D} m_\mathrm{I}}{r_\mathrm{p}} + \frac{1}{2} m_\mathrm{D} v_\mathrm{p}^2 = -\frac{G m_\mathrm{D} m_\mathrm{I}}{r_\mathrm{a}} + \frac{1}{2} m_\mathrm{D} v_\mathrm{a}^2$$

and after the astronaut jumps, we get the same pair of equations, but with new velocities v'_p instead of v_p, v'_a instead of v_a, and distances r'_a instead of r_a and $r'_\mathrm{p} = r_\mathrm{p}$. It is obvious that the new pericenter is the same as the original one, because the direction of velocity remains unchanged and the satellite moves perpendicularly to the line Dactyl-Ida only in case of being in the pericenter or apocenter (and it can not be in the new apocenter, since the change in the velocity was very small).

From the first set of laws (non-primed variables), we can express the velocity in the pericenter as

$$v_\mathrm{p} = \left(\frac{2Gm_\mathrm{I}}{r_\mathrm{p}\left(1 + \frac{r_\mathrm{p}}{r_\mathrm{a}}\right)} \right)^{1/2} = 6.670\,\mathrm{m{\cdot}s^{-1}}\,,$$

therefore we also know the velocity v'_p. Furthermore, we can approximate

$$(v'_\mathrm{p})^2 \approx v_\mathrm{p}^2 - \frac{2m}{m_\mathrm{D}} v v_\mathrm{p}\,.$$

Similarly, we can express r'_a from the second set of laws (primed variables) as

$$r'_\mathrm{a} = -\frac{v'^2_\mathrm{p} r_\mathrm{p}}{v'^2_\mathrm{p} - \frac{2Gm_\mathrm{I}}{r_\mathrm{p}}} \approx r_\mathrm{a} \left(1 - \frac{4Gm_\mathrm{I} m v}{v_\mathrm{p} \left(v_\mathrm{p}^2 - \frac{2Gm_\mathrm{I}}{r_\mathrm{p}} \right) r_\mathrm{p} m_\mathrm{D}} \right).$$

Now we can substitute v_p into the equation, but the expression for r'_a does not change much. Furthermore, we can see that v_p does not appear anywhere where subtraction of close numbers would occur, so we can use the numeric value calculated above and together with values given in the problem statement, we get

$$r'_\mathrm{a} - r_\mathrm{a} \doteq -4.45 \cdot 10^{-8} r_\mathrm{a} \doteq -9.0 \cdot 10^{-3}\,\mathrm{m}\,.$$

The apocenter of Dactyl would move by 0.9 cm closer to Ida. A relative change by 10^{-8} was expected, since the velocity of the astronaut's jump and the velocity of Dactyl in the pericenter are similar and the ratio of their masses is $m/m_\mathrm{D} = 10^{-8}$.

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Problem FoL.40 ... aesthetic balls 6 points

We have two small charged balls with the same mass hanging on strings (with negligible mass) from one point at the ceiling. Both balls have the same charge and they are in a homogeneous gravitational field. The angle between the strings is 10°. Then, we double the charge of one ball and increase the charge of the other ball k times. Find the value of k which is necessary if we want the angle between the strings to be 25°. *Karel heard about Danka's plans.*

Let's denote the initial charges, the masses of the balls and the lengths of the strings by q, m and l respectively. The initial slope of the string is $\alpha_0 = 5°$. The system is stationary and therefore, the net force acting on each ball must be zero. To satisfy this, we only need the vector sum of the force of gravity and electrostatic force to point in the direction of the string. The forces are

$$F_g = mg\,,$$

$$F_\mathrm{e} = \frac{1}{4\pi\varepsilon} \frac{q^2}{4l^2 \sin^2 \alpha_0}\,,$$

and then we get

$$\frac{F_\mathrm{e}}{F_g} = \tan \alpha_0 \,.$$

After simplification, we can write

$$\frac{1}{4\pi\varepsilon} \frac{q^2}{4l^2 mg} = \tan \alpha_0 \sin^2 \alpha_0 \,.$$

Now we change the angle to $\alpha = 12.5^\circ$. All constants remain the same, the only difference is that q^2 changes to $2kq^2$. From the simplified equation, we can easily express

$$k = 4\pi\varepsilon \frac{4l^2 mg}{2q^2} \tan \alpha \sin^2 \alpha = \frac{\tan \alpha \sin^2 \alpha}{2 \tan \alpha_0 \sin^2 \alpha_0} \,,$$

which gives the result $k \doteq 7.81$.

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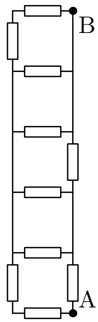
Problem FoL.41 ... disgusting resistive skyscraper — 7 points

Determine the resistance between the points A and B if each resistor has resistance $R = 3141.59\,\Omega$.

Matej was reminiscing about Physics Olympiad preparation.



The problem can be solved in a classical way according to the rules of resistor assembling[2] or according to Kirchhoff's laws. We would get the same result, but we won't explain these approaches because it's tedious and there are plenty of problems solved this way.

We can also do it with a non-traditional, but for this particular problem faster, approach. First, we will redraw the electrical circuit into an equivalent form in the figure. We deliberately drew resistors with different sizes because we can notice that each resistor corresponds to one square in the figure. Imagine that the whole square in this figure is made up of material with constant conductivity per area. If we apply electric voltage between its upper and lower side[3] then due to homogeneity, each horizontal line corresponds to a conductor in the original wiring of resistors, i.e. an equipotential. This implies that the height of each sub-square corresponds to the voltage on the corresponding resistor. By analogy, the width of each square[4] corresponds to the electric current passing through the corresponding resistor. However, these are all squares, each resistor has the same voltage to current ratio and therefore, all resistors have the same resistance. This fulfills the conditions given in the problem statement. The figure 2 then shows that the total resistance of the circuit is equal to the resistance of one resistor because the whole square also has the same aspect ratio.

Note: In this procedure, it is necessary to follow the rule that the aspect ratio of each individual rectangle (in our case, all of them are squares) is proportional to the resistance on the corresponding resistor and the rectangles must form one large rectangle without any holes.

[2] Using star-delta transformation is necessary.
[3] I.e. we connect wires with the given potential difference to both sides.
[4] In general, rectangles could also occur here.

Otherwise, we cannot use this procedure so simply. Then, the resulting resistance is derived from the aspect ratio of the total rectangle. This problem was created by exactly the opposite procedure to the solution mentioned above - we made a problem which is solvable this way - and therefore, it is easy to reverse this procedure, but generally, it doesn't have to work.

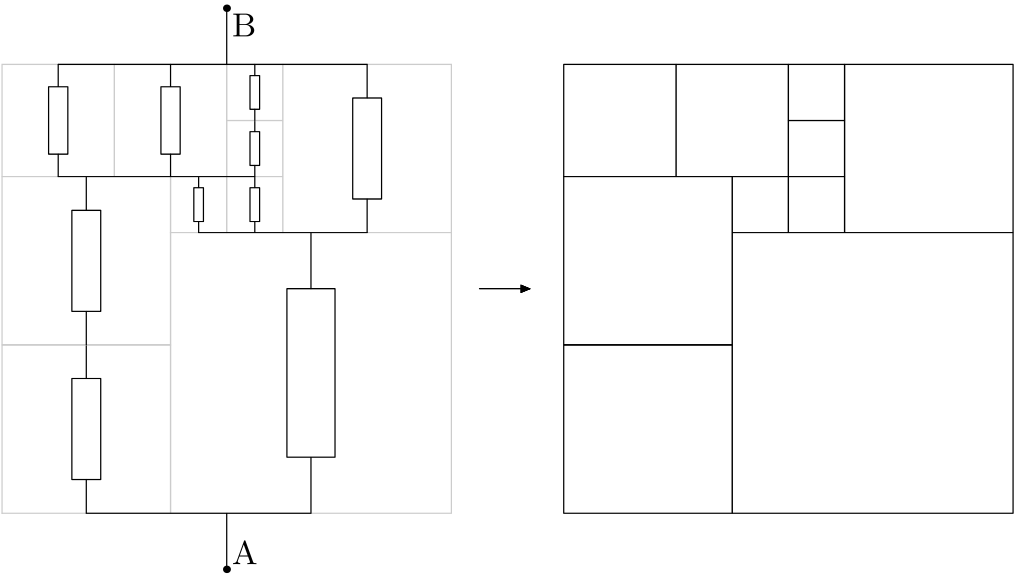


Fig. 2: Equivalent form of the circuit

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Problem FoL.42 ... manoeuvre

6 points

Imagine that we are operating a spacecraft on a circular orbit around a star and we wish to double the radius of our circular orbit. We choose to achieve this by a specific manoeuvre: we maintain a constant velocity – with a constant magnitude and a constant angle $\theta = 5^\circ$ between the velocity vector and the tangent line to the local circular orbit. How many orbits (not necessarily an integer) do we make before we reach the desired distance from the star?

Mirek came up with a problem which turned out to be too difficult for elementary school.

Let us solve this problem in polar coordinates. The velocity vector is given by $\boldsymbol{v} = v_0(\sin\theta, \cos\theta)$, where v_0 is the constant magnitude of the velocity – the value of this constant has obviously no effect on the result of this problem, so we set it to 1. The radial motion, as a time-dependent quantity, is described by a simple differential equation

$$\mathrm{d}r = v_0 \sin\theta \mathrm{d}t$$

with the unique solution $r(t) = r_0 + v_0 t \sin\theta$, where r_0 is the initial radial distance. The manoeuvre will take the time $\tau = r_0/\left(v_0 \sin\theta\right)$. The tangential motion is described by a slightly more complicated equation

$$\mathrm{d}\varphi = \frac{v_0 \cos\theta}{r(t)} \mathrm{d}t\,.$$

During the manoeuvre, the angular coordinate changes by

$$\Delta\varphi = \int_0^\tau \mathrm{d}\varphi = \int_0^\tau \frac{v_0 \cos\theta}{r_0 + v_0 t \sin\theta} \mathrm{d}t = \frac{1}{\tan\theta} \left[\ln\left(r_0 + v_0 t \sin\theta\right)\right]_0^\tau = \frac{\ln 2}{\tan\theta}\,.$$

For the given numerical value of θ, this amounts to $\Delta\varphi/(2\pi) = 1.26$ orbits.

Another approach to this problem is to recall that the requirement on constant θ defines a specific class of spirals – the trajectory is a spiral described by a formula

$$\ln r = \ln a \; + b\varphi$$

and called the logarithmic spiral. Here, a is an irrelevant constant and $b = \tan\theta$, as can be shown by differentiating the defining equation.

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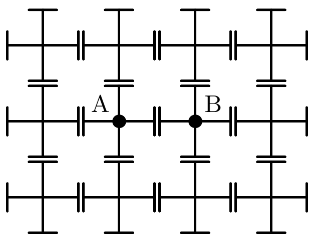
Problem FoL.43 ... capacitor grid

6 points

Consider a square lattice made from capacitors. Part of the lattice is shown in the figure. The capacitance of each capacitor is $C = 100.0\,\mu\mathrm{F}$. Find the total capacitance between two adjacent vertices.



Matěj would like to have an infinite grid at home.

It is convenient to solve the problem using a trick, as is the case with almost every problem where we are dealing with an infinite number of components and some kind of symmetry. We will use the principle of superposition for the electric field and separate the problem into two cases, which we will then combine.

In the first case, we add an external charge Q at the vertex A (while doing nothing with the vertex B) and this charge distributes uniformly between the four adjacent capacitors, because the lattice is symmetrical under rotation by $90\,^\circ$. On each of these capacitors, we now have a charge $Q/4$, that is, on the side connected with the vertex A, there is a charge $Q/4$, and on the opposite side, there is an induced charge $-Q/4$.

In the second case, we add a charge $-Q$ at the vertex B and do nothing with the vertex A. We get an analogous situation, with charges $-Q/4$ on the four corresponding capacitors.

Now let us combine these two situations. According to the superposition principle, electric charge, potential and some other electromagnetic quantities are additive, so if we sum up several solutions,[5] we'll get another valid solution of our problem. This implies that if we put a charge Q in the vertex A and a charge $-Q$ in the vertex B, then the charge on the capacitor between these two vertices will be $Q/2$ (i.e. sum of $Q/4$ and $-Q/4$, where we changed the second sign due to the fact that the opposite side of a capacitor induces the opposite charge). Using the definition of capacitance, we can calculate the voltage between A and B

$$U_{\mathrm{AB}} = \frac{Q}{2C}\,.$$

If we connect the whole grid to an electric circuit through the vertices A and B, it will behave as one large capacitor with capacity C_{tot}. Because we know the voltage between these two vertices when there is a charge Q on each of them, this capacity can be easily calculated,

$$C_{\mathrm{tot}} = \frac{Q}{U_{\mathrm{AB}}} = 2C = 200.0\,\mathrm{\mu F}\,.$$

After some further thinking, we would realize that if an infinite grid of capacitors shows a rotational symmetry, the capacity between two neighboring vertices depends only on the number of edges N adjacent to every vertex as $C_{\mathrm{tot}} = \frac{N}{2}C$. In our case, $N = 4$. If you're interested, there is a similar problem in our archive – problem EG from Fyziklani2019 – with a similar question, but for a triangular grid and with resistors instead of capacitors.

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Problem FoL.44 ... the fourth velocity 5 points

The particle velocity distribution in a gas is described by the Maxwell-Boltzmann distribution

$$f(v) = \sqrt{\frac{2}{\pi}}\left(\frac{M}{RT}\right)^{3/2} v^2 \mathrm{e}^{-\frac{Mv^2}{2RT}}\,.$$

There are several ways to determine an average velocity. We define the most probable velocity $v_{\mathrm{p}} = \sqrt{\frac{2RT}{M}}$, the mean absolute velocity $v_{\mathrm{s}} = \sqrt{\frac{8RT}{\pi M}}$ and the mean quadratic velocity $v_{\mathrm{k}} = \sqrt{\frac{3RT}{M}}$. For the oxygen O_2 at temperature $20\,^\circ\mathrm{C}$, these velocities have values $390\,\mathrm{m\cdot s^{-1}}$, $440\,\mathrm{m\cdot s^{-1}}$ and $478\,\mathrm{m\cdot s^{-1}}$ respectively. However, we can also calculate the n-th moment of the Maxwell-Boltzmann distribution for any n and thus find some other average speed. What is the value of the mean cubic absolute velocity of oxygen under the same conditions?

An oxygen molecule hit Jindra's eye.

[5] In this case, we define a solution as a stable distribution of charge.

The mean cubic velocity is determined by calculating the third moment of the Maxwell-Boltzmann distribution and taking the cube root of it.

$$v_{\mathrm{cu}} = \sqrt[3]{\int_0^\infty v^3 f(v)\mathrm{d}v}\,,$$

$$v_{\mathrm{cu}} = \sqrt[6]{\frac{2}{\pi}}\left(\frac{M}{RT}\right)^{1/2}\sqrt[3]{\int_0^\infty v^5 \mathrm{e}^{-\frac{Mv^2}{2RT}}\mathrm{d}v}\,,$$

$$\text{substituce:}\quad \tfrac{Mv^2}{2RT} = t \qquad \Rightarrow \qquad v^2 = \frac{2RTt}{M}\,,$$

$$\tfrac{Mv}{RT}\mathrm{d}v = \mathrm{d}t\,,$$

$$v\mathrm{d}v = \frac{RT}{M}\mathrm{d}t\,,$$

$$v_{\mathrm{cu}} = \sqrt[3]{4}\sqrt[6]{\frac{2}{\pi}}\left(\frac{RT}{M}\right)^{1/2}\sqrt[3]{\int_0^\infty t^2\mathrm{e}^{-t}\mathrm{d}t}\,.$$

You probably recognized the gamma function in the integral with the root at first. The gamma function is a generalization of the factorial to all real (and also complex) numbers except zero and negative integers, which it is not defined for. The gamma function is defined as $\Gamma(n) = \int_0^\infty x^{n-1}\mathrm{e}^{-x}\mathrm{d}x$. And what do the gamma function and factorials have in common? If n is some natural number, then $\Gamma(n) = (n-1)\,!$. The integral under the root is $\Gamma(3) = (3-1)\,! = 2$.

$$v_{\mathrm{cu}} = 2\sqrt[6]{\frac{2}{\pi}}\left(\frac{RT}{M}\right)^{1/2}$$

After plugging in the numerical values,

$$v_{\mathrm{cu}} = 512\,\mathrm{m{\cdot}s^{-1}}\,.$$

The mean cubic velocity of the oxygen molecules is $512\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.45 ... inside a spherical mirror 6 points

Imagine two concave spherical mirrors with a common axis, such that the mirrors are two parts of a sphere with a radius $r = 1\,\mathrm{m}$. When we place a point light source in the centre of the sphere, each mirror forms a reflected image of the source which coincides with the source itself. Find the distance of the source from the centre such that it coincides with its image formed after three reflections (but does not coincide with the image formed after one reflection).

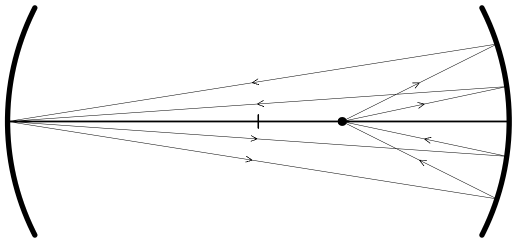
Matěj was inspired by a Vsauce video.

Let's imagine an optical axis passing through the centre of the sphere and through the source. We can replace the hollow sphere by two concave mirrors at distances r from the centre. Obviously, their focal distances are $\frac{r}{2}$.

We will use symmetry to solve this problem. After the first reflection, the first image should form at the distance r from the centre of the sphere and on the opposite side from the source (i.e. exactly in the centre of the second spherical mirror). Because of that, when beams are "sent" from this image and reflected for the second time, they return in the exactly opposite direction, which means that the first image is projected directly on itself (since it is located directly on the surface of the mirror). The third reflection sends the beams back to the source. Let's denote the distance of the light source from the centre of the sphere by x. We can write the mirror equation as

$$\frac{1}{r-x}+\frac{1}{2r}=\frac{2}{r}\,,$$

and its solution is $x=\frac{r}{3}$.



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Problem FoL.46 ... rotary tree — 5 points

Jachym cut down a tree that can be approximated by a thin homogeneous cylinder with length $l = 30.0\,\mathrm{m}$. The tree is still partly connected to the stump, so during its fall, the tip of the tree travels along a circular arc. There is a stone with negligible height lying on the ground at a distance $l/9$ away from the stump. Just before the tree hits the ground, it hits the rock in a perfectly inelastic manner. This breaks the connection between the tree and the stump and the tree continues to move as a free body. With what speed does the tip of the tree hit the ground? The connection to the stump or its breaking don't involve any losses in energy. *This year, Jáchym is going to get his Christmas tree from a forest.*

The moment of inertia of the tree with respect to a horizontal axis crossing the stump is

$$J_0=\frac{1}{3}ml^2\,,$$

where m is mass of the tree. During free fall, its center of mass would drop from the height $h = 1/2$ directly above the ground, which corresponds to a potential energy drop by $E = mgh$. This energy would transform into rotational energy, so the angular velocity of the tree right before impacting the stone is

$$\omega_0=\sqrt{\frac{2E}{J_0}}=\sqrt{\frac{3g}{l}}\,.$$

In the moment of impact, part of this rotational energy dissipates. However, the angular momentum with respect to the horizontal axis crossing the point of impact remains unchanged –

let's denote it by L. The point that hit the rock stops, therefore, the tree starts to rotate around it. Let's denote the moment of inertia of the tree with respect to the axis crossing the point of impact by J. Then, the angular velocity of the motion around the point of impact would be

$$\omega = \frac{L}{J}\,.$$

The velocity of the top of the tree can be easily calculated as $v = \omega\,(l - a)$, where $a = l/9$. Just before the impact, the velocity of a section of the tree at a distance x from the stub is equal to $\omega_0 x$. A section of the tree with length $\mathrm{d}x$ therefore has momentum $\mathrm{d}p = \lambda\omega_0 x\mathrm{d}x$, where $\lambda = m/l$ is the linear density of the tree. The angular momentum with respect to a point at a distance a from the stub is

$$L = \int_0^l (x - a)\,\mathrm{d}p = \int_0^l x\mathrm{d}p - a\int_0^l \mathrm{d}p = L_0 - a\,\lambda\int_0^l \omega_0 x\mathrm{d}x = L_0 - a\,\lambda\omega_0\frac{l^2}{2} = L_0 - \frac{1}{2}alm\omega_0\,,$$

where L_0 is the original angular momentum $J_0\omega_0$. The new moment of inertia can be calculated as

$$J = \int_0^l (x - a)^2\,\mathrm{d}m = \lambda\int_0^l (x - a)^2\,\mathrm{d}x = J_0 - alm + a^2 m\,.$$

Now we only need to use all the equations together and we get

$$v = \omega\,(l - a) = \frac{L}{J}\,(l - a) = \frac{L_0 - \frac{1}{2}alm\omega_0}{J_0 - alm + a^2 m}\,(l - a) = \frac{20}{19}\omega_0 l = \frac{20}{19}\sqrt{3gl} \doteq 31.3\,\mathrm{m{\cdot}s^{-1}}\,.$$

The top of the tree hits the ground with the velocity $31.3\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.47 ... neutron rack

7 points

Find the smallest distance from the surface of a neutron star at which a free-falling steel rod can exist, pointing towards the star, without being ripped in half. The star has mass $M = 1.80M_\odot$ and radius $R = 10.0\,\mathrm{km}$. The ultimate tensile strength of the rod is $\sigma = 800\,\mathrm{MPa}$, its density is $\varrho = 7\,900\,\mathrm{kg{\cdot}m^{-3}}$ and its length is $L = 1.00\,\mathrm{m}$. Neglect deformation of the rod which occurs before fracturing and consider only Newtonian mechanics.

Dodo was reading a sci-fi from 1967.

A famous similar problem deals with estimating the maximum length of a hanging cylinder (made of specified material) at which it doesn't fracture as a result of gravity. In our case, the rod is "hanging" in its center of gravity and stressed by tidal forces. Let's denote the height of the rod above the surface of the star by h. From the difference of gravitational forces affecting the rod at two points with distance l from each other, we get

$$a = \frac{GM}{(r + h)^2} - \frac{GM}{(r + h + l)^2} \approx \frac{GM}{(r + h)^3}2l\,,$$

which is our tidal acceleration.

The total tensile force acting on the material in the rod's center of gravity can be calculated by integration over layers with thickness $\mathrm{d}x$ and weight $\mathrm{d}m = \varrho\mathrm{d}x$, where S is the cross section of the rod. Let's describe the situation from the point of view of the center of mass of the rod, where we set $x = 0$. We consider the inhomogeneity of the field to be low enough that we can estimate the center of mass to be in the middle of the rod.

$$F = \int_0^{\frac{L}{2}} a\mathrm{d}m = \int_0^{\frac{L}{2}} \frac{GM\mathrm{d}m}{(r+h)^3} 2x = \int_0^{\frac{L}{2}} \frac{GM\varrho S}{(r+h)^3} 2x\mathrm{d}x\,,$$

$$F = \frac{GM\varrho S}{(r+h)^3} x^2 \Big|_0^{\frac{L}{2}} = \frac{GM\varrho S}{4\,(r+h)^3} L^2\,.$$

The force F acts away from the star on the farther half of the rod and symmetrically, it acts towards the star on the closer half of the rod. The tensile strength and force are connected by the equation $F = \sigma S$. After substituting for the force, we can calculate h and, assuming $r = R$, define the critical distance from the surface as

$$r = \sqrt[3]{\frac{GM\varrho L^2}{4\sigma}} - R \doteq 74\,\mathrm{km}\,.$$

We would also like to validate our approximations. In the first approximation, we introduce an uncertainity with magnitude 10^{-5}. Next, let's estimate the error in the position of the center of mass. We will calculate the "moment" of distribution of the force acting on the rod M_x around the middle of the rod

$$M_x = \int a(x)x\mathrm{d}m \approx \int_{\frac{-L}{2}}^{\frac{L}{2}} \left(F(0) + \frac{GM}{(r+h)^3} 2x \right) \varrho S x\mathrm{d}x$$

$$M_x = 0 + \frac{2GM}{3\,(r+h)^3} \varrho S x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{GM}{6\,(r+h)^3} \varrho S L^3\,.$$

In the center of mass, the value above should be zero. We can compare it with a rod in a homogeneous gravitational field a with respect to a point displaced from the middle of the rod by $s \ll L$

$$M_x(s) = \int_{\frac{-L}{2}-s}^{\frac{L}{2}-s} a\ \varrho S x\mathrm{d}x = a\varrho S \left(\left(\frac{-L}{2} - s \right)^2 - \left(\frac{L}{2} - s \right)^2 \right) \approx a\ \varrho S 2Ls\,.$$

Comparing with the tidal acceleration formula mentioned above $a = \frac{GM}{(r+h)^2}$, we have

$$s = \frac{L^2}{12\,(r+h)} \approx 1 \cdot 10^{-6}\,\mathrm{m}\,.$$

We can see that the approximations used in the solution are correct. The approximations given in the problem statement are worse, because the distance from the center of the neutron star equals approx. 17 Schwarzschild radii of the star (it would be necessary to consider the consequences of relativistic effects), and steel can stretch by tens of percents before tearing.

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Problem FoL.48 ... sunbathing session 6 points

The average solar irradiance of the Earth (power incident on the Earth's surface) oscillates with a period of about 11 *years. A big part of the variance constitutes of changes in the UV part of the EM spectrum. Assume, for the purposes of this problem, that the Sun in the solar maximum is a blackbody with a temperature* $T = 5.800\,\mathrm{K}$. *In the solar minimum, the spectral characteristic is the same, except for a region of wavelengths between* $10\,\mathrm{nm}$ *and* $300\,\mathrm{nm}$, *which is missing. Determine the total drop in power emitted by the Sun between the maximum and minimum, in percent of the value in the maximum.*

Mirek gains inspiration at conference lectures.

Black body radiation is described by Planck's law, which gives the specific spectral intensity of radiation as

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}}-1}\,.$$

Since we only want to find the ratio of incident powers with a different range of frequencies, we only need to integrate over frequency (i.e. we do not calculate the exact power, but power per incident area). We can neglect the constant before the integral and substitute frequency by energy in electronvolts. Hence, the desired power drop is

$$\delta P = \frac{\int_{x_0}^{x_1}\frac{x^3\mathrm{d}x}{\mathrm{e}^x-1}}{\int_0^\infty\frac{x^3\mathrm{d}x}{\mathrm{e}^x-1}}\,,$$

where the upper and lower limits of the variable x in the integral are obtained by converting from wavelengths using the formula

$$x = \frac{hc}{\lambda k_\mathrm{B}T}\,.$$

In the denominator of the expression for δP, we can identify the definition of the ζ-function, so the denominator equals $\Gamma(4)\zeta(4) = \pi^4/15$. A more detailed solution is

$$\int_0^\infty\frac{x^3\mathrm{d}x}{\mathrm{e}^x-1} = \int_0^\infty x^3\mathrm{e}^{-x}\sum_0^\infty\mathrm{e}^{-nx}\mathrm{d}x = \sum_0^\infty(n+1)^{-4}\int_0^\infty((n+1)\,x)^3\,\mathrm{e}^{-(n+1)x}\mathrm{d}\,(n+1)\,x =$$
$$= \sum_1^\infty n^{-4}\Gamma(4) = \Gamma(4)\zeta(4)\,.$$

The numerator can be calculated in approximate form as

$$\int_{x_0}^{x_1}\frac{x^3\mathrm{d}x}{\mathrm{e}^x-1} \approx \int_{x_0}^\infty\frac{x^3\mathrm{d}x}{\mathrm{e}^x} = \mathrm{e}^{-x_0}\left(x_0^3+3x_0^2+6x_0+6\right)\,,$$

since $x_0 \doteq 8.269 \gg 1$ and $x_1 \gg x_0$. Thus, we get the result $\delta P \doteq 3.3\,\%$.

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Problem FoL.49 ... diabolical spring 8 points

Let's consider a straight elastic rubber rod (i.e. non-circular rubber band) with rest length $l = 9.57\,\mathrm{cm}$, spring constant $k = 11.08\,\mathrm{N{\cdot}m^{-1}}$, mass $m = 24.53\,\mathrm{g}$ and cross-sectional area $S = 5.70\,\mathrm{cm^2}$. One end of the rod is connected with a pad underneath it (not vertically glued to the pad, i.e. the only constraint is that this end cannot move away from the pad). Now, we fill the space around the rubber rod with water; the density of water is $\varrho = 998\,\mathrm{kg{\cdot}m^{-3}}$. What will the new equilibrium length of the rod be? Assume that the cross-sectional area of the rod always remains constant.

You thought the last year's evil spring problem was hard? I will show you hard... – Jáchym

The density of the spring is $m/lS \doteq 450\,\mathrm{kg{\cdot}m^{-3}} < \varrho$, so in the equilibrium state, the spring will vertically float right above the pad.

For the initial state (without influence of any external forces), we introduce an x-coordinate with the origin at the lower end of the rod; for the final state, we use a similar y-coordinate. Now we are looking for a function $y(x)$ which maps a part of the original rod at a coordinate x to its distance from the pad y after immersion into water. The solution of this problem will be the value $y(l)$.

A small part of the rod at a coordinate x (at a height $y(x)$) is pulled upward by a force equal to the buoyant force acting on the segment of the rod above this part. Symbolically written, this force is

$$F_\mathrm{v}(x) = S\varrho g\left(y(l) - y(x)\right) .$$

However, this segment of the rod is also influenced by gravity. Unlike buoyancy, the force of gravity depends on the original length of the segment, instead of the volume, which is directly proportional to its new equilibrium length. Obviously,

$$F_g(x) = mg\frac{l-x}{l} .$$

The last force acting on this segment of the rod (downwards) is the elastic force. If a very small part of the spring right underneath this segment has an original length Δx, we can model it by a spring with a spring constant[6]

$$k_{\Delta x} = k\frac{l}{\Delta x} .$$

Its current length is $\Delta y \approx y(x+\Delta x) - y(x)$, which corresponds to the force

$$F_\mathrm{p}(x) = k_{\Delta x}\left(\Delta y - \Delta x\right) = kl\left(y'(x) - 1\right)$$

in the limit $\Delta x \to 0$. The net force acting on any segment of the spring must be zero, which means that we need to solve a simple differential equation

$$\begin{aligned} F_\mathrm{v} - F_g &= F_\mathrm{p} , \\ S\varrho g\left(y(l) - y\right) - mg\frac{l-x}{l} &= kl\left(y' - 1\right) , \\ y' + Ay &= \frac{B}{l}x + Ay(l) - B + 1 , \end{aligned}$$

[6] This can be proved by splitting a whole spring stretched by a force F into two smaller springs, which also have to exert forces equal to F on each other.

which we simplified by introducing substitutions

$$A = \frac{S\varrho g}{kl}\,,$$
$$B = \frac{mg}{kl}\,.$$

The homogeneous solution is

$$y_\mathrm{H} = C\mathrm{e}^{-Ax}\,,$$

where C is some constant. Now we are looking for a unique particular solution in the form of a polynomial with degree one, or $y_\mathrm{P} = ax + b$. By substituting y for y_P, we get

$$\begin{aligned} a + Aax + Ab &= \frac{B}{l}x + Ay(l) - B + 1\,,\\ a &= \frac{B}{Al} = \frac{m}{S\varrho l}\,,\\ b &= y(l) + \frac{1}{A}\left(-B + 1 - \frac{B}{Al}\right)\,. \end{aligned}$$

The result is

$$y = y_\mathrm{H} + y_\mathrm{P} = C\mathrm{e}^{-Ax} + ax + b\,,$$

where we can determine the constant C from the condition $y(0) = 0$. It gives $C = -b$, so

$$y = -b\mathrm{e}^{-Ax} + ax + b\,.$$

In addition, substituting $x = l$ must give $y(l)$. This leads to the formula

$$y(l) = \frac{1}{A}\left(B + \left(1 - \frac{B}{Al}\right)\left(\mathrm{e}^{Al} - 1\right)\right) = \frac{m}{S\varrho} + \frac{k}{S\varrho g}\left(l - \frac{m}{S\varrho}\right)\left(\mathrm{e}^{\frac{S\varrho g}{k}} - 1\right) \doteq 11.15\,\mathrm{cm}\,,$$

and that's the final result of this problem.

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Problem FoL.50 ... the charge and the pentagram 9 points

Jindra has had enough with the free charges so he decided to try some magic. He took a regular pentagon with side length 1 *and cut out a pentagram. Then he caught a free point charge* $Q = 1.00\cdot 10^{-6}\,\mathrm{C}$ *and placed it directly above the centre of the pentagram at a distance* $\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{10}}$. *What is the total electric flux through the pentagram?* *Jindra still doesn't give up...*

First, let us describe the geometry of a pentagon (and a pentagram). The inner angle at each vertex of a regular pentagon is $\frac{3\pi}{5}$. The central angle between adjacent vertices (each angle vertex-centre-adjacent vertex) is $\frac{2\pi}{5}$. The angle at each outer vertex of a pentagram is $\frac{\pi}{5}$. The inner vertices of a pentagram are also the vertices of a smaller pentagon. This pentagon will be important later, let's call it simply "smaller pentagon". The centre of the smaller pentagon is identical with the centre of the larger pentagon.

The side length of the larger pentagon is 1, so the radius of its circumscribed circle is

$$R_\mathrm{O} = \frac{1}{2\sin\frac{\pi}{5}}\,.$$

The circle inscribed in the smaller pentagon has a radius

$$\varrho = R_\mathrm{O}\cos\frac{2\pi}{5} = \frac{\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}\,. \tag{4}$$

Since we know that[7] $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and $\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}$, we can plug them into the equation (4).

$$\varrho = \frac{\frac{\sqrt{5}-1}{4}}{2\sqrt{\frac{5-\sqrt{5}}{8}}} = \frac{\sqrt{5}-1}{8}\cdot\sqrt{\frac{8}{\sqrt{5}(\sqrt{5}-1)}} = \frac{1}{2}\sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}$$

$$\varrho = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{10}} \tag{5}$$

Bingo! The distance z between the charge and the centre of the pentagram is the same as the radius of the circle inscribed in the smaller pentagon ϱ.

Now let's look at the directions of the vectors of electric field on the surface of the pentagram. The distance between the charge and the centre of the pentagram is $z = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{10}}$. Consider a point on the pentagram's surface with a distance r from the centre. From the Pythagorean theorem, we obtain $R = \sqrt{z^2 + r^2}$ for the distance of this point from the charge. The magnitude of the electric field at this point is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2 + r^2}\,.$$

The magnitude of the component perpendicular to the pentagram is

$$E_\perp = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2 + r^2}\cdot\frac{z}{\sqrt{z^2 + r^2}}$$

$$E_\perp = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{\left(z^2 + r^2\right)^{\frac{3}{2}}}\,. \tag{6}$$

The electric flux through the pentagram can be obtained by solving the integral $\int_S E_\perp \mathrm{d}S$ over the surface area of the pentagram. Let's divide the surface into two parts to make the calculation

[7] Really, try to plug the values into a calculator. One possible way to derive these expressions for goniometric functions is to use the formulae for $\sin 2x$, $\sin 3x$ and $\sin 4x$ obtained from DeMoivre's theorem to arrive at

$$\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} + \sin\frac{8\pi}{5} = 0\,.$$

This equality simply describes the fact that if we walk around a pentagon, we will return to the original point. The individual terms can be interpreted as y-coordinates of the vectors describing the sides of the pentagon. After a few simplifications, we obtain

$$2\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}(4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1) = 0\,.$$

The value of $\cos\frac{2\pi}{5}$ can be obtained by solving a quadratic equation.

easier. Section A is the area inside the circumscribed circle of the inner pentagon. There, we can calculate the electric flux easily. Section B is the part of the pentagon outside this circle. The radius of the circumscribed circle of the smaller pentagon is $r_\mathrm{O} = \varrho/\cos\frac{\pi}{5} = 4\varrho/(\sqrt{5}+1)$.

The electric flux through the section A is the same as the flux through the spherical cap above it.[8] The full sphere would have a radius $R_\mathrm{K} = \sqrt{r_\mathrm{O}^2 + z^2}$. The height of the spherical cap is $h = R_\mathrm{K} - z$. The electric flux is then

$$\varphi_1 = \frac{Q}{\varepsilon_0}\frac{S_\mathrm{vrchlík}}{S_\mathrm{koule}} = \frac{Q}{\varepsilon_0}\frac{\sqrt{r_\mathrm{O}^2+z^2}-z}{2\sqrt{r_\mathrm{O}^2+z^2}}\,,$$

and we plug $z = \varrho$ and $r_\mathrm{O} = 4\varrho/(\sqrt{5}+1)$ into this to get

$$\varphi_1 = \frac{Q}{2\varepsilon_0} - \frac{Q}{2\varepsilon_0}\frac{1}{\sqrt{\left(\frac{4}{\sqrt{5}+1}\right)^2+1}}$$

$$\varphi_1 = \frac{Q}{2\varepsilon_0} - \frac{Q}{2\varepsilon_0}\frac{\sqrt{5}+1}{\sqrt{22+2\sqrt{5}}}\,. \tag{7}$$

The flux through the section B can be obtained by integrating over the circular arcs corresponding to each outer vertex of the pentagram. If the radius of a circular arc is r, its length is

$$l_1 = 2r\left(\frac{2\pi}{5} - \arccos\frac{\varrho}{r}\right).$$

There are five vertices, so the total length is

$$l = r\left(4\pi - 10\arccos\frac{\varrho}{r}\right).$$

The length l lies between $\frac{2\pi\varrho}{\cos\frac{\pi}{5}}$ for $r = \frac{\varrho}{\cos\frac{\pi}{5}}$ and $l = 0$ for $r = \frac{\varrho}{\cos\frac{2\pi}{5}}$. The flux through the section B can be calculated as the integral

$$\varphi_2 = \int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} E_\perp l \,\mathrm{d}r$$

$$\varphi_2 = \frac{Qz}{4\pi\varepsilon_0}\int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} \frac{r\left(4\pi - 10\arccos\frac{\varrho}{r}\right)}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathrm{d}r$$

$$\varphi_2 = \frac{Qz}{\varepsilon_0}\int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathrm{d}r - \frac{5Qz}{2\pi\varepsilon_0}\int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} \frac{r\arccos\frac{\varrho}{r}}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathrm{d}r\,.$$

[8] The integrals $\varphi_1 = \int_0^\varrho E_\perp 2\pi r \mathrm{d}r = \frac{Qz}{2\varepsilon_0}\int_0^\varrho \frac{r}{(z^2+r^2)^{3/2}}\mathrm{d}r$ are trivial and left as an exercise to the reader. The results will be the same (which is to be expected).

We calculate each integral separately.

$$\int \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathrm{d}r = -\int \mathrm{d}t = -t + C = -\frac{1}{\sqrt{z^2+r^2}} + C$$

$$\text{substituce:}\quad \frac{1}{\sqrt{z^2+r^2}} = t$$

$$\frac{r\mathrm{d}r}{\left(z^2+r^2\right)^{\frac{3}{2}}} = -\mathrm{d}t$$

$$\frac{Qz}{\varepsilon_0}\int \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathrm{d}r = -\frac{Qz}{\varepsilon_0}\frac{1}{\sqrt{z^2+r^2}} + C \tag{8}$$

The other integral is

$$\int \frac{r\arccos\frac{\varrho}{r}}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathrm{d}r = -\frac{\arccos\frac{\varrho}{r}}{\sqrt{z^2+r^2}} + \int \frac{\varrho}{r\sqrt{z^2+r^2}\sqrt{r^2-\varrho^2}}\mathrm{d}r$$

$$\text{per partes:}\quad u = -\frac{1}{\sqrt{z^2+r^2}} \qquad u' = \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}$$

$$v = \arccos\frac{\varrho}{r} \qquad v' = \frac{-1}{\sqrt{1-\left(\frac{\varrho}{r}\right)^2}}\frac{-\varrho}{r^2} = \frac{\varrho}{r\sqrt{r^2-\varrho^2}}$$

The resulting integral cannot be solved in terms of elementary functions. However, we are lucky. We know that $z=\varrho$. After we plug it in, the integral becomes solvable. Magic!

$$\int \frac{\varrho}{r\sqrt{r^4-\varrho^4}}\mathrm{d}r = \frac{1}{\varrho^2}\int \frac{\mathrm{d}r}{\frac{r}{\varrho}\sqrt{\left(\frac{r}{\varrho}\right)^4-1}} = \frac{1}{2\varrho}\int \frac{2\frac{r}{\varrho^2}\mathrm{d}r}{\left(\frac{r}{\varrho}\right)^2\sqrt{\left(\frac{r}{\varrho}\right)^4-1}} = \frac{1}{2\varrho}\int \frac{\sinh t\mathrm{d}t}{\cosh t\sqrt{\cosh^2 t-1}} =$$

$$\text{substituce:}\quad \left(\frac{r}{\varrho}\right)^2 = \cosh t$$

$$\frac{2r}{\varrho^2}\mathrm{d}r = \sinh t\mathrm{d}t$$

$$= \frac{1}{2\varrho}\int \frac{\mathrm{d}t}{\cosh t} = \frac{1}{2\varrho}\int \sqrt{1-\tanh^2 t}\mathrm{d}t = \frac{1}{2\varrho}\int \frac{1-\tanh^2 t}{\sqrt{1-\tanh^2 t}}\mathrm{d}t = \frac{1}{2\varrho}\int \frac{\mathrm{d}y}{\sqrt{1-y^2}} =$$

$$\text{substituce:}\quad \tanh t = y$$

$$\frac{1}{\cosh^2 t}\mathrm{d}t = \mathrm{d}y$$

$$(1-\tanh^2 t)\mathrm{d}t = \mathrm{d}y$$

$$= \frac{1}{2\varrho}\arcsin y + C = \frac{1}{2\varrho}\arcsin\tanh t + C = \frac{1}{2\varrho}\arcsin\frac{\sqrt{\cosh^2 t-1}}{\cosh t} + C = \frac{1}{2\varrho}\arcsin\frac{\sqrt{\left(\frac{r}{\varrho}\right)^4-1}}{\left(\frac{r}{\varrho}\right)^2} + C$$

Thus we can write

$$\int \frac{\varrho}{r\sqrt{r^4-\varrho^4}}\mathrm{d}r = \frac{1}{2\varrho}\arcsin\frac{\sqrt{r^4-\varrho^4}}{r^2} + C$$

and plug this into the original integral

$$\int \frac{r \arccos \frac{\varrho}{r}}{(\varrho^2 + r^2)^{\frac{3}{2}}} \mathrm{d}r = -\frac{\arccos \frac{\varrho}{r}}{\sqrt{\varrho^2 + r^2}} + \frac{1}{2\varrho} \arcsin \frac{\sqrt{r^4 - \varrho^4}}{r^2} + C\,. \tag{9}$$

Now we can combine the equations (8) and (9) and calculate the electric flux through the tips of the pentagram.

$$\varphi_2 = \frac{Q\varrho}{\varepsilon_0} \left[-\frac{1}{\sqrt{\varrho^2 + r^2}} + \frac{5}{2\pi} \frac{\arccos \frac{\varrho}{r}}{\sqrt{\varrho^2 + r^2}} - \frac{5}{2\pi} \frac{1}{2\varrho} \arcsin \frac{\sqrt{r^4 - \varrho^4}}{r^2} \right]_{\frac{\varrho}{\cos \frac{\pi}{5}}}^{\frac{\varrho}{\cos \frac{2\pi}{5}}}$$

We can factor ϱ out of the square roots in the denominators and cancel them out. After further manipulation, we obtain

$$\varphi_2 = \frac{Q}{\varepsilon_0} \left(\frac{1}{2\sqrt{1 + \frac{1}{\cos^2 \frac{\pi}{5}}}} - \frac{5}{4\pi} \left(\arcsin \sqrt{1 - \cos^4 \frac{2\pi}{5}} - \arcsin \sqrt{1 - \cos^4 \frac{\pi}{5}} \right) \right).$$

Into this, we can plug the values of goniometric functions $\cos \frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and $\cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$.

$$\varphi_2 = \frac{Q}{2\varepsilon_0} \left(\frac{\sqrt{5}+1}{\sqrt{22 + 2\sqrt{5}}} - \frac{5}{2\pi} \left(\arcsin \sqrt{\frac{25 + 3\sqrt{5}}{32}} - \arcsin \sqrt{\frac{25 - 3\sqrt{5}}{32}} \right) \right)$$

The total electric flux is the sum of φ_1 and φ_2

$$\begin{aligned}
\varphi &= \frac{Q}{2\varepsilon_0} - \frac{Q}{2\varepsilon_0} \frac{\sqrt{5}+1}{\sqrt{22 + 2\sqrt{5}}} + \\
&+ \frac{Q}{2\varepsilon_0} \left(\frac{\sqrt{5}+1}{\sqrt{22 + 2\sqrt{5}}} - \frac{5}{2\pi} \left(\arcsin \sqrt{\frac{25 + 3\sqrt{5}}{32}} - \arcsin \sqrt{\frac{25 - 3\sqrt{5}}{32}} \right) \right) \\
\varphi &= \frac{Q}{2\varepsilon_0} \left(1 - \frac{5}{2\pi} \left(\arcsin \sqrt{\frac{25 + 3\sqrt{5}}{32}} - \arcsin \sqrt{\frac{25 - 3\sqrt{5}}{32}} \right) \right) \\
\varphi &= \frac{Q}{2\varepsilon_0} \left(1 - \frac{5}{2\pi} \arcsin \frac{1}{16} \sqrt{110 - 2\sqrt{145}} \right) \\
\varphi &= 2.87 \cdot 10^4 \,\mathrm{V{\cdot}m}\,,
\end{aligned}$$

where we utilized some trigonometric identities for arcsine.

Numerical solution

It is clear that the solution presented above is algebraically demanding and lengthy. Since we only need the numerical result, the problem can also be solved numerically. For example, in the program Geogebra, we can project a triangle constituting one tenth of the pentagram onto a sphere with its centre at the point charge, which passes through its tips (see the picture).

Let's label its radius r. Geogebra will calculate the lengths of circular arcs a, b, and c, from which we can, using the equation

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{r}{2}\right)\tan\left(\frac{r-a}{2}\right)\tan\left(\frac{r-b}{2}\right)\tan\left(\frac{r-c}{2}\right)}\,, \tag{10}$$

calculate the solid angle E that is taken up by the triangle from the perspective of the point charge. On a sphere, the flux is constant, so it is sufficient to multiply the area by the electric field strength in the distance r.

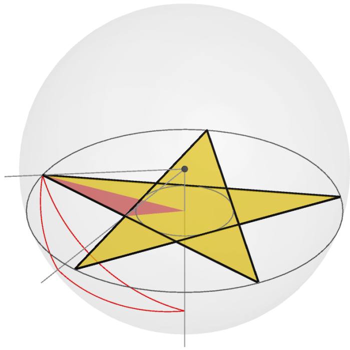

Fig. 3: Geometry of charge and pentagram

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Problem M.1 ... Rick's microplanet 3 points

Imagine that you are hiding from the Galactic Federation on a planet which has the same gravitational acceleration on the equator as the Earth ($a_g = 9.83\,\mathrm{m\cdot s^{-2}}$), as well as the same centrifugal acceleration on the equator ($a_c = 0.034\,\mathrm{m{\cdot}s^{-2}}$), but its radius is only $R = 5\,\mathrm{km}$. How long would one day on this planet be?

Consider the planet to be an ideal homogeneous sphere. The period of the planet's orbit around its star is negligible compared to the length of the day.

Karel and Matěj were wondering where Rick and Morty's family escaped.

Thanks to the fact that we can neglect the period of the planet's orbit, we can simply calculate the length of the day using the centrifugal force $F_c = ma_c = mv_r^2/R = m/omega_r^2 R$, where v_r is the velocity of a point on the planet's equator and ω_r is the angular velocity of its rotation. We can express the angular velocity from the period of rotation T as $\omega_r = 2\pi/T$. We get

$$a_c = \frac{4\pi^2}{T^2}R \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{R}{a_c}} \doteq 40\,\mathrm{min}\,.$$

The period of the planet's rotation, i.e. the length of one day there, would be 40 minutes.

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Problem M.2 ... interdimensional density 3 points

Imagine that you are still hiding on the planet from the previous problem. This planet has the same gravitational acceleration on the equator ($a_\mathrm{g} = 9.83\,\mathrm{ms^{-2}}$) and the same centrifugal acceleration ($a_\mathrm{c} = 0.034\,\mathrm{ms^{-2}}$) as the Earth, but its radius is only $R = 5\,\mathrm{km}$. What is the density of this planet?

Assume that the planet is an ideal homogeneous sphere.

Karel was watching where Rick and Morty's family escaped.

We obtain the density of the planet from the gravitational acceleration on its surface

$$a_g = G\frac{M}{R^2}$$

where $G = 6.67\cdot 10^{-11}\,\mathrm{kg^{-1}\cdot m^3\cdot s^{-2}}$ is the gravitational constant. The mass can be expressed as $M = \varrho V = 4\pi R^3\varrho/3$, where V is the volume of the planet.

$$a_g = G\frac{4\pi}{3}\varrho R \qquad \Rightarrow \qquad \varrho = \frac{3}{4\pi}\frac{a_g}{GR} \doteq 7.04\cdot 10^6\,\mathrm{kg\cdot m^{-3}}$$

The density of the planet would have to be about 700 times higher than the density of water. By the way, this corresponds to the total mass of the planet

$$M = \frac{4}{3}\pi\varrho R^3 = \frac{4}{3}\pi R^3\frac{3}{4\pi}\frac{a_g}{GR} = \frac{a_g R^2}{G} \doteq 3.7\cdot 10^{18}\,\mathrm{kg}\,,$$

which would be more than 6 orders of magnitude smaller than the mass of our Earth.

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Problem M.3 ... squanchy frisbee 4 points

Consider the planet from the previous problems. This planet has the same gravitational acceleration on the equator ($a_\mathrm{g} = 9.83\,\mathrm{ms^{-2}}$) and the same centrifugal acceleration ($a_\mathrm{c} = 0.034\,\mathrm{ms^{-2}}$) as the Earth, but its radius is only $R = 5\,\mathrm{km}$. Life on the planet is boring, so you throw an object eastward along the equator. You want it to travel around the planet and return back to you from the west. How long does it take the object to come back?

Assume that the planet is an ideal homogeneous sphere and air drag is negligible. On this planet, the stars disappear on the western horizon.

Matej had nobody to play with.

For the object to be able to orbit around the planet, it has to move with the first cosmic velocity, which is

$$v_\mathrm{k} = \frac{GM}{R} = \sqrt{a_\mathrm{g}R}\,,$$

where we expressed the mass of the planet as $M = \frac{a_\mathrm{g}R^2}{G}$. Don't forget that when we stand at the equator, we are rotating together with the planet, with the velocity $v_\mathrm{o} = \sqrt{a_o R}$. Try to think about why are rotating eastward. This means that we need to subtract our velocity from the absolute velocity of the object v_k. The time it takes the object to travel around the planet is

$$t = \frac{2\pi\sqrt{R}}{\sqrt{a_g} - \sqrt{a_o}} = 150.6\,\mathrm{s}\,.$$

By the way, the object would need to be thrown with the velocity $v = \sqrt{a_\mathrm{g} R} - \sqrt{a_\mathrm{o} R} = 208\,\mathrm{m\cdot s^{-1}}$, which is more than two times faster than the fastest recorded tennis serve.

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Problem M.4 ... wubba lubba dub dub — 4 points

We'll stay on the planet from the previous problems. This planet has the same gravitational acceleration on the equator ($a_\mathrm{g} = 9.83\,\mathrm{ms^{-2}}$) and the same centrifugal acceleration ($a_\mathrm{c} = 0.034\,\mathrm{ms^{-2}}$) as the Earth, but its radius is only $R = 5\,\mathrm{km}$. How high above the surface would the orbit of a geostationary satellite be, if it existed at all? If it does not exist, fill in 0 as the result.

Assume that the planet is an ideal homogeneous sphere.

Karel was watching where Rick and Morty's family escaped.

Let's focus on the geostationary orbit above the equator. A satellite on such an orbit is staying above one fixed place. We'll start from the equality of the gravitational and centrifugal force. Using this, we'll determine the distance from the center of the planet r for which a circular orbit would be geostationary. If it is smaller than R, then the planet does not have a geostationary orbit. If it is greater, we can determine the distance from the surface as $r - R$.

$$m\omega^2 r = G\frac{mM}{r^2} \qquad \Rightarrow \qquad \frac{4\pi^2}{T^2} = \frac{GM}{r^3} \qquad \Rightarrow \qquad r^3 = \frac{GMT^2}{4\pi^2}$$

We need to express r using the given quantities:

$$r^3 = \frac{G}{4\pi^2}\frac{a_g R^2}{G}\frac{4\pi^2 R}{a_o} = \frac{a_g}{a_c}R^3 \quad \Rightarrow \quad r = \sqrt[3]{\frac{a_g}{a_c}}R \doteq 33\,\mathrm{km}\,.$$

A geostationary orbit does exist, at the height approximately 28 km above the surface of our planet.

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Problem E.1 ... breakers — 3 points

The FYKOS-bird has two cylindrical copper wires, each with radius $r = 1\,\mathrm{mm}$ and resistivity $\varrho = 1.69 \cdot 10^{-8}\,\Omega\cdot\mathrm{m}$. He wants to make an extension cord, but he connects two ends of one wire to a socket by mistake, so the wire short-circuits the socket. What is the maximum length of the wire which activates the circuit breaker? The voltage in the socket is 230 V and the maximum allowed current through the breaker is 16 A. *Mišo short-circuited.*

Let's use the formula for the resistance of a wire with cross-sectional area S and length l

$$R = \frac{\varrho l}{S} = \frac{\varrho l}{\pi r^2}\,,$$

where we used $S = \pi r^2$. The current can be expressed from Ohm's law

$$I = \frac{U}{R} = \frac{\pi r^2 U}{\varrho l} \qquad \Rightarrow \qquad l = \frac{\pi r^2 U}{\varrho I} \doteq 2\,670\,\mathrm{m}\,,$$

where we substituted for the voltage in the socket and the critical value of the current. If we stick the wire in the socket and its length is greater than 2 670 m, nothing should happen. You can try that at home if you don't believe it.

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Problem E.2 ... Fykonacci 3 points

The FYKOS-bird was playing with some resistors. He connected two points A and B by a resistor with resistance $R_1 = 1.600\,\Omega$. He thought it was too much, so he added another resistor with the same resistance $R_2 = 1.600\,\Omega$ between these points, parallel to the first one. Yet, he was still not satisfied, so he added a third resistor $R_3 = R_1 + R_2 = 3.200\,\Omega$, parallel to the others. Then he went bonkers and started adding more and more resistors parallel to the previous ones, with resistances $R_4, R_5, \dots$ described by the formula $R_n = R_{n-2} + R_{n-1}$, where $n \in N$. What was the final resistance between points A and B?

Jáchym does not enjoy physics, so he tries to create some math problems.

Let us denote the n-th element of the Fibonacci sequence by F_n, where $F_1 = 1$ and $F_2 = 1$. Therefore $R_n = F_n R_1$. All the resistors are connected in parallel, which means that the total resistance R satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots = \sum_{n=1}^{\infty} \frac{1}{R_n} = \frac{1}{R_1} \sum_{n=1}^{\infty} \frac{1}{F_n}\,.$$

The Fibonacci sequence increases roughly as an exponential curve, so it is no surprise that the sum of its reciprocal values converges. We don't need to calculate it analytically, its value can be found on the internet,[9] or we can estimate it using numerical methods. In any case, the result is

$$\sum_{n=1}^{\infty} \frac{1}{F_n} \doteq 3.359886\dots\,.$$

Now we can easily see that the total resistance is $R = 0.4762\,\Omega$.

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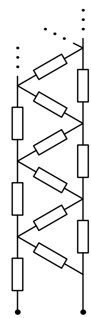
Problem E.3 ... triangular ladder 4 points

The FYKOS-bird wanted to climb onto a certain resistive skyscraper which you might also encounter. Therefore, he built a triangular ladder (infinite, of course), made wholly of resistors with resistance $R = 1.0\,\Omega$. Calculate the resistance between the legs of the ladder.

Lego was infinitely sad because none of his other tasks made it through.



The principle of solving tasks with infinite resistive networks is to denote the result by R_v. Then we only find a part of the ladder which is the same as the original ladder, while having in mind that the ladder is infinite.

[9] https://en.wikipedia.org/wiki/Fibonacci_number

In our case, if we disconnect the first vertical resistor and the first diagonal resistor, the final diagram will be the same (just mirrored, but the resistance does not depend on this). Even if we cut away another 2 resistors, the result stays the same, we will just spend more time on cutting. This infinite network will have a resistance R_v. When we connect a resistor with resistance R_v back to the other two resistors (i.e. parallel to the diagonal one), we will get a circuit with the same resistance as the original one. We get the equation

$$R_\mathrm{v} = R + \frac{RR_\mathrm{v}}{R + R_\mathrm{v}}\,,$$
$$0 = R_\mathrm{v}^2 - RR_\mathrm{v} - R^2\,.$$

This is a quadratic equation, so we can solve it easily by using the discriminant formula. Among the two roots we get, we naturally care only about the positive one, whose value is

$$R_\mathrm{v} = \frac{1+\sqrt{5}}{2}R \doteq 1.62\,R\,,$$

where the value of the fraction is called the golden ratio.

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Problem E.4 ... nonlinear FYKOS-bird 4 points

The FYKOS-bird was tired of connecting resistors and so he decided to connect himself in series with two batteries, each with voltage $U_\mathrm{b} = 4.50\,\mathrm{V}$, and a resistor with resistance $R = 50.0\,\Omega$. The bird behaves as a non-linear component with the current–voltage characteristic shown in the figure 4. The cut-in voltage of the bird is $U_0 = 1.50\,\mathrm{V}$, the saturation voltage of the bird is $U_\mathrm{n} = 6.00\,\mathrm{V}$ and the saturation current through the bird is $I_\mathrm{n} = 250\,\mathrm{mA}$. In each of the three parts of the current–voltage characteristic, the bird behaves as a linear component. Determine the current flowing through the bird. *Dodo remembered labs at midnight.*

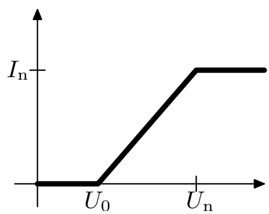
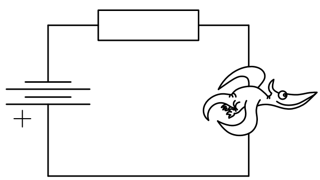


Fig. 4: I-V curve



The voltage across the circuit $U_\mathrm{z} = 2U_\mathrm{b} = 9.0\,\mathrm{V}$ is divided between the voltage across the resistor U_R and the voltage across the FYKOS-bird U_F.

$$U_\mathrm{z} = U_R + U_\mathrm{F}\,.$$

Ohm's law applies to the voltage across the resistor, and hence $U_R = RI$, where I is the current through the circuit. Expressing I in terms of other variables, we have

$$I(U_\mathrm{F}) = \frac{U_\mathrm{z} - U_\mathrm{F}}{R}\,,$$

where $I(U_F) = I$ is the current through the bird given by its current–voltage characteristic. The simplest solution to the problem is a graphical one. Then we only need to find the intersection of the current–voltage curve with the line $I(U) = \frac{U_\mathrm{z} - U}{R}$. From the graphical solution, it is clear that this intersection lies within the second part of the characteristic, which is described by the equation

$$I(U_\mathrm{F}) = I_\mathrm{n} \frac{U_\mathrm{F} - U_0}{U_\mathrm{n} - U_0}\,.$$

After substitution, we arrive at the result

$$\frac{U_\mathrm{z} - U_\mathrm{F}}{R} = I_\mathrm{n} \frac{U_\mathrm{F} - U_0}{U_\mathrm{n} - U_0}\,,$$

$$U_\mathrm{F} = \frac{I_\mathrm{n} R U_0 + U_\mathrm{z}\left(U_\mathrm{n} - U_0\right)}{I_\mathrm{n} R + U_\mathrm{n} - U_0}\,,$$

from which we get the current

$$I = I_\mathrm{n} \frac{U_\mathrm{z} - U_0}{I_\mathrm{n} R + U_\mathrm{n} - U_0} \doteq 110\,\mathrm{mA}\,.$$

Problem X.1 ... Chernobyl

3 points

Soon after the reactor no. 4 exploded, a radiation of 5 roentgens per hour was measured at a distance of $800\,\mathrm{m}$ *from the reactor. How much would have been measured at the distance of* $200\,\mathrm{m}$ *from the reactor? Neglect the effects of air and environment on the propagation of radiation.*

Matěj was watching the series.

Let us begin with the fact that any dose of radiation is inversely proportional to the square of distance. Therefore, when the distance from the source is multiplied by 4, the dose decreases 16 times, so

$$16 \cdot 5\,\mathrm{R{\cdot}h^{-1}} = 80\,\mathrm{R{\cdot}h^{-1}}\,.$$

The radiation at the distance of $200\,\mathrm{m}$ equals $80\,\mathrm{R{\cdot}h^{-1}}$.

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Problem X.2 ... hold my graphite

3 points

An unsuspecting firefighter grabs a piece of graphite with an activity of $1.00 \cdot 10^{10}\,\mathrm{Bq}$*, lying near the now defunct reactor. How many particles will go into (or through) his hand if the firefighter holds the graphite for* $10\,\mathrm{seconds}$*? Consider the chunk of graphite to be a sphere with a diameter of* $5\,\mathrm{cm}$ *and assume that the firefighter's hand covers one fifth of its surface when holding it. Assume further that each particle which leaves the piece of graphite corresponds to the decay of one nucleus.*

Matěj was chilling while watching memes.

The unit becquerel tells us how many radiation particles a given object emits per one second. We can simply multiply the activity by time, and of course, we cannot forget the factor of one fifth, which takes into consideration the part of the surface covered by the hand

$$\frac{1}{5} \cdot 10^{10}\,\mathrm{Bq} \cdot 10\,\mathrm{s} = 2 \cdot 10^{10}\,.$$

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Problem X.3 ... not great, not terrible 3 points

Calculate the activity of a destroyed nuclear reactor. We measure 10000 radiation particles per minute per square decimeter at a distance of 5.00 km *from the reactor. Neglect the effects of air and environment on the propagation of radiation.* *Matěj wanted to set the answer to* 3.6 R.

The activity of an emitter indicates the number of emitted radiation particles per a given time unit, so we get $A = (10\,000\,\mathrm{min^{-1}{\cdot}dm^{-2}}) \cdot 4\pi(5\,\mathrm{km})^2 \doteq 5.24 \cdot 10^{12}\,\mathrm{s^{-1}}$.

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Problem X.4 ... radio-activity 3 points

By how many percent is the radiation around a destroyed nuclear power plant reduced after a 2.00m *thick sarcophagus is built around it? Assume that* 3.00 dm *of the material which the sarcophagus is made of can capture, on average,* 50.0 % *of the radiation.*
Matěj is worried about his safety.

Only

$$0.5^{2/0.3} \doteq 0.0098$$

of the original radiation passes through the sarcophagus, so radiation drops by 99.0 %. Let us add that the sarcophagus is primarily built to prevent radioactive material from leaking into the air.

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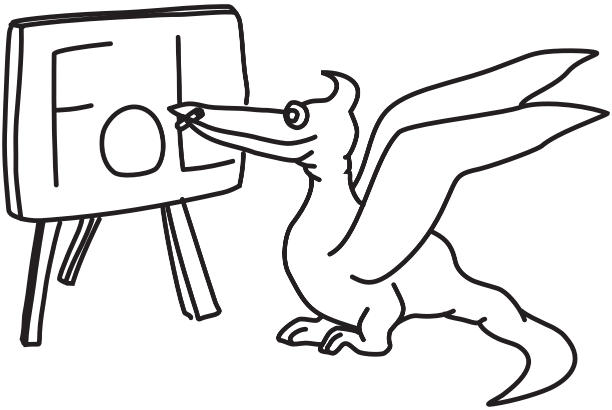
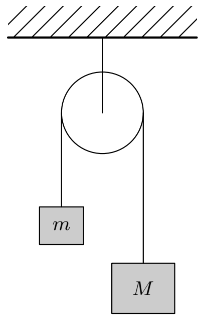
Solutions of 10th Online Physics Brawl



Problem FoL.1 ... (hopefully just) easy pulleys

3 points



Suppose that we have a pulley and some weights hanging from a cord as shown in the figure. Assume that both the cord and the pulley are ideal and massless, $M = 2.0\,\mathrm{kg}$ and $m = 1.0\,\mathrm{kg}$. Find the acceleration of the weight M (in the downwards direction). *Lego wants to discover the simplest pulley problem that half of all teams still fail.*

The cord as well as the pulley are massless (assumption from the problem statement), therefore the forces that the cord exerts on both weights are equal. Let us denote each by T. To describe this, we can write a system of two equations

$$a_1 M = Mg - T\,,$$
$$a_2 m = mg - T\,,$$

where both accelerations are oriented downwards.

We subtract the equations and set $a_1 = -a_2$, because the cord is ideal, which means that it does not change its length. We get

$$a_1 = \frac{M-m}{M+m} g \doteq 3.3\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem FoL.2 ... two circuits

3 points

Legolas found several old identical resistors and connected them in series. Such a circuit had a total resistance $R_\mathrm{S} = 10.0\,\mathrm{k\Omega}$. After that, he decided to connect the resistors in parallel and measured a resistance $R_\mathrm{P} = 1.0\,\Omega$. What was the resistance of a single resistor?

Lego made himself a lego out of resistors.

Let R denote the desired resistance of one resistor and n the total number of the resistors. Then we get

$$R_\mathrm{S} = nR\,,$$
$$R_\mathrm{P} = \frac{R}{n}\,.$$

By multiplying these two equations, we get rid of n and the remaining equation is $R^2 = R_\mathrm{S} R_\mathrm{P}$, from which we obtain the result $R = \sqrt{R_\mathrm{S} R_\mathrm{P}} = 100\,\Omega$. We need $n = 100$ such resistors.

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Problem FoL.3 ... felling trees 3 points

A coniferous tree could be (from a mechanical point of view) approximated by a homogeneous right circular cone with height $h = 40\,\mathrm{m}$ *and radius at the base* $r = 1.0\,\mathrm{m}$. *Find the maximal angle by which its axis may be displaced from the vertical axis before it starts to fall due to its weight.* *Dodo was procrastinating on Youtube.*

A rigid body begins to fall when it is displaced in such a way that its centre of mass is no longer straight above its base. The centre of mass of a cone is located at height $h/4$ above the base. The maximal angle at which the tree does not fall yet is the same as the angle between the vertical and the line connecting the centre of mass and the edge of the base. Its magnitude is calculated using the formula $\tan\Phi = \frac{r}{h/4}$, therefore $\Phi = \arctan\frac{4r}{h} \doteq 5.7^\circ$. If the tree is tilted more, it will fall.

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Problem FoL.4 ... the nearest asteroid 3 points

On 16th August 2020, an asteroid (later called 2020 QG) has been recorded as the closest asteroid (spotted so far) that flew by the Earth without colliding with it. At the nearest point of approach, it was only $2\,950\,\mathrm{km}$ *above Earth's surface and it had a velocity* $v = 12.3\,\mathrm{km{\cdot}s^{-1}}$. *How much higher was its velocity compared to the escape velocity at that height above Earth's surface? Find the ratio* v/v_{esc}. *Karel made a problem out of news from the website astro.cz.*

The escape velocity is given by the condition that total energy of the object in the gravitational field shall equal zero. Therefore

$$\frac{1}{2}mv^2 - G\frac{Mm}{r} = 0\,,$$

where M denotes the mass of the Earth, G is gravitational constant and v and r is the velocity of the object and its distance from the centre of Earth, respectively. By expressing the velocity we get

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2GM}{R+h}}\,,$$

where r was substituted with the sum of radius of Earth and height of the flyby (measured from the ground). If we divide these velocities, we get

$$\frac{v}{v_{\mathrm{esc}}} = v\sqrt{\frac{R+h}{2GM}} \doteq 1.33\,.$$

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Problem FoL.5 ... unstable 3 points

Suppose that we have a cuboid with dimensions $a = 20\,\mathrm{cm}$, $b = 30\,\mathrm{cm}$, $c = 50\,\mathrm{cm}$ and density $\rho = 620\,\mathrm{kg{\cdot}m^{-3}}$. One of its faces with dimensions a and c is lying on a horizontal surface in a homogeneous gravitational field. How stable is it with respect to rotation around one of its bottom edges with length c? Find the smallest amount of energy needed to turn it over. Assume $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Danka's stuff was falling down.*

To overturn the cuboid, we only have to get its center of gravity above the c edge and just barely behind it. This change of the cuboid's position requires an increase in the cuboid's potential energy. This can be counted as the difference in the potential energy of the cuboid's center of gravity. Let's place the zero energy level on the horizontal plane. In the beginning the cuboid has a potential energy of

$$E_{\mathrm{p}1} = \rho abcg \frac{b}{2}\,.$$

When its center of gravity is above the c edge, it will be in h above the plane, where

$$h = \frac{1}{2}\sqrt{a^2 + b^2}\,.$$

Then it will have a potential energy of

$$E_{\mathrm{p}2} = \rho abcgh\,.$$

The stability is the difference of these two energies, therefore

$$\Delta E = E_{\mathrm{p}2} - E_{\mathrm{p}1} = \Delta E = \frac{1}{2}\rho abcg\left(\sqrt{a^2+b^2} - b\right) \doteq 5.5\,\mathrm{J}\,.$$

The stability of the cuboid is therefore $5.5\,\mathrm{J}$.

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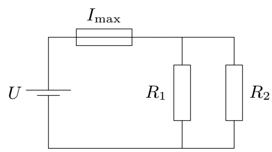
Problem FoL.6 ... safe power 3 points

There are two appliances with resistances $R_1 = 500\,\Omega$ and $R_2 = 2\,000\,\Omega$ respectively, connected in parallel to the terminals of a battery through a fuse with a maximum allowed current $I_{\max} = 500\,\mathrm{mA}$. What is the maximum power we could get from the circuit? *Dodo must pay attention at the dormitory.*



The relation between the voltage and the current is described by the Ohm's law $U = RI$, where R is the total resistance of the connected appliances. The total resistance of the appliances connected in parallel is

$$R = \frac{R_1 R_2}{R_1 + R_2} = 400\,\Omega\,.$$

We can calculate the power of an appliance from the voltage on them and the current that flows through them as $P = UI = RI^2$. Maximum power implies maximum current, which is the current that blows the fuse. Plugging in the numerical values, we get $P = 100\,\mathrm{W}$.

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Problem FoL.7 ... the ink-blooded prince 3 points

When Harry Potter did not behave, professor Umbridge would punish him by forcing him to write with his own blood. He needed 1.21 µl of blood on average to write one lowercase letter. A book of school rules has 259 standard pages, each of which contains 1 488 letters on average. However, each thirty-seventh letter is a capital letter, which means it consumes three times more ink than a lowercase letter. Assuming that Harry has 5 l of blood, how many copies of the rules could be made from him? Jáchym found the whole book series considerably illogical.

For each 37 letters of the school rules the blood consumption equals writing 39 lower case letters. Let $k = 39/37$ denote that ratio. Let V_a, n_p, n_l and V_H denote the remaining quantities respectively. The volume of the blood required to write on set of school rules is

$$V_\mathrm{b} = n_\mathrm{p} n_\mathrm{l} k V_\mathrm{a} \,.$$

Therefore the number of copies equals

$$n_\mathrm{c} = \frac{V_\mathrm{H}}{n_\mathrm{p} n_\mathrm{l} k V_\mathrm{a}} \doteq 10.2 \,.$$

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Problem FoL.8 ... slowed down train 4 points

While arriving at a station, a train decelerates evenly. Its braking distance is $s = 75\,\mathrm{m}$ and during the penultimate (second to last) second before stopping, it drives a distance $l = 2.25\,\mathrm{m}$. What is its initial velocity v_0 before it begins to brake?
Verča took advantage of a train delay to think of new problems.

The train moves with evenly decelerated motion and during the next to last second it drives the same distance as during the second second of a train accelerating from rest with the same magnitude of acceleration. To make this solution more illustrative, let us mark the beginning and the end of the next to last second as t_1 and t_2 respectively. The distance l could be written as

$$l = \frac{1}{2} a t_2^2 - \frac{1}{2} a t_1^2 = \frac{1}{2} a \left(t_2^2 - t_1^2\right) \,.$$

We can express acceleration a from this equation as

$$a = \frac{2l}{t_2^2 - t_1^2} \,.$$

We will plug it into the formula for total distance

$$s = \frac{1}{2} a t^2 = \frac{1}{2} \frac{v_0^2}{a} \,,$$

from which we can already express the initial velocity as

$$v_0 = \sqrt{2as} = \sqrt{\frac{4ls}{t_2^2 - t_1^2}} \,.$$

Numerical evaluation for values given in the task results in $v_0 = 15\,\mathrm{m\cdot s^{-1}}$.

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Problem FoL.9 ... we pump oil 4 points

Find the efficiency of a pump, if the electric current flowing through it has an effective value $I = 125\,\mathrm{mA}$ *when attached to a standard power supply of voltage* $U = 230\,\mathrm{V}$*. Our pump pumps oil with density* $\rho = 870\,\mathrm{kg{\cdot}m^{-3}}$ *through pressure difference* $\Delta p = 120\,\mathrm{hPa}$ *with volumetric flow rate* $Q = 0.83\,\mathrm{ml{\cdot}s^{-1}}$.

Dodo wanted to get a rotary vane pump.

The efficiency η is defined as the ratio between the useful work done (in our case mechanical work against pressure force) and energy supplied (by electric power). The electric network supplies the pump with energy

$$W_1 = UIt\,.$$

in time t. The pump uses it to push the liquid through the given pressure difference, where the useful work is given by the product of the pressure force F_p and the distance s, along which the force is exerted

$$W_2 = F_p s = S\Delta p s = \Delta p V\,,$$

where S is the cross-section of the pump. The distance s is, in fact, the length of the part of the liquid which flowed through the pump in time t, so $V = Ss$ is the volume that flowed through. Plugging it in the equation we obtain

$$\eta = \frac{\Delta p V}{UIt} = \frac{\Delta p Q}{UI} \doteq 0.000\,35\,,$$

where we used the formula for the flow rate $V = Qt$. The second part of the problem is solvable by dimensional analysis as well.

In this task, we made a numerical error. We decided that the fairest resolution is to not count any points gained or lost because of this problem. We apologize for the troubles.

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Problem FoL.10 ... homogeneous air 4 points

From the point of view of classical optics, every substance is homogeneous. However, today we know that everything consists of particles. Determine the number of particles (molecules) of air that are, under standard conditions, contained in a cube, where the length of one edge of this cube corresponds to the wavelength of the yellow D-line of sodium.

Dodo couldn't sleep again.

The wavelength of the D-line is approximately $\lambda = 590\,\mathrm{nm}$. Let us first determine the mass of the air contained in our cube

$$m = \rho V = \rho\lambda^3\,,$$

where $\rho = 1.29\,\mathrm{kg{\cdot}m^{-3}}$ is the density of the air. We will determine the number of particles using the average molar mass of air $M_\mathrm{m} = 29.0\,\mathrm{g{\cdot}mol^{-1}}$ and the Avogadro constant $N_\mathrm{A} = 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}$ as

$$N = \frac{mN_\mathrm{A}}{M_\mathrm{m}} = \frac{\rho\lambda^3 N_\mathrm{A}}{M_\mathrm{m}} \doteq 5\ 500\ 000\,.$$

Therefore, in the cube where the edge corresponds to the wavelength of the sodium D-line there are millions of molecules.

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Problem FoL.11 ... our old clock 4 points

Our old clock is battery-powered and the capacity of the battery is $E = 1.2\,\mathrm{Wh}$. The hour hand is $r_\mathrm{h} = 5\,\mathrm{cm}$ long and the minute hand is $r_\mathrm{m} = 8\,\mathrm{cm}$ long. The linear density of each hand is $\tau = 10\,\mathrm{g{\cdot}m^{-1}}$. The efficiency of the clockwork is $\eta = 7\,\%$. How long does it take for the clock to stop? The hands rotate continuously and any potential energy released during descending is lost. *Michal was late due to a delayed clock.*

When the watch hands rotate continuously the clockwork does not have to accelerate them. However, it needs to compensate the gravitational force during their movement up. Energy which the clockwork has to spend for one cycle of its hand can be calculated as the difference of the potential energy between the highest and the lowest point. For the minute hand we have

$$\Delta E_\mathrm{m} = mg\Delta h = r_\mathrm{m}\tau g\left(\frac{r_\mathrm{m}}{2} - \frac{-r_\mathrm{m}}{2}\right) = r_\mathrm{m}^2\tau g\,.$$

and for the hour hand

$$\Delta E_\mathrm{h} = r_\mathrm{h}^2\tau g\,.$$

The task does not provide information about the starting position of watch hands so we assume consistent power as follows

$$P = P_\mathrm{m} + P_\mathrm{h} = \frac{\Delta E_\mathrm{m}}{3\,600\,\mathrm{s}} + \frac{\Delta E_\mathrm{h}}{12\cdot 3\,600\,\mathrm{s}} \doteq 1.7\cdot 10^{-7}\mathrm{W} + 5.7\cdot 10^{-9}\mathrm{W} = 1.8\cdot 10^{-7}\mathrm{W}\,.$$

The battery could power the watch for

$$t = \frac{E\eta}{P} \doteq 47\cdot 10^4\mathrm{h}\,.$$

Now we can see that it really does not matter that we assumed stable power output of watch, because the period of $12\,\mathrm{h}$ is much smaller than the expected battery lifetime. Even with such a low efficiency the watch could work for over 50 years.

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Problem FoL.12 ... cough cough 4 points

A deadly virus with a long incubation period has spread around the world and already infected one ten-thousandth of the total population. Hard-working researchers haven't found any cure yet, but managed to develop a test which can decide whether a given person is infectious or not. It returns the positive result in $99.99\,\%$ of infected cases. However, it returns the false positive result in $0.03\,\%$ healthy cases. It may seem that the test is quite reliable. We have chosen and tested a random person. The test returned a positive value. Estimate the probability that the chosen person is infected. *Matěj felt sick during statistical physics practice.*

Let $p_\mathrm{P} = 0.000\,1$ denote the probability that a random person is infected and $p_\mathrm{N} = 0.999\,9$ the probability that he is healthy. Furthermore let $p_\mathrm{PP} = 0.999\,9$ denote the probability that an infectious person will have positive test and $p_\mathrm{NP} = 0.000\,3$ that a healthy person will have positive test.

The desired probability can be calculated as the ratio between the number of all cases in which is infectious person correctly tested (this equals the product $p_\mathrm{P}p_\mathrm{PP}$)to the number of all people with positive test, which leads to the formula

$$\frac{p_\mathrm{P}p_\mathrm{PP}}{p_\mathrm{P}p_\mathrm{PP} + p_\mathrm{N}p_\mathrm{NP}} = \frac{1}{4}\,.$$

On the other hand, even though the test seemed to be quite reliable, we obtained the result, that a person with a positive test is infectious only with 25 % probability. Do not give up hope. . .

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Problem FoL.13 . . . uneven illumination 4 points

Danka was sitting at a round table with a radius $R = 1.0\,\mathrm{m}$ and noticed that the edge of the table was illuminated much less than the center. What is the difference between illuminance in the center and at the edge of the table if the only light source in the room is a light bulb with luminous intensity $I = 120\,\mathrm{cd}$ hanging $h = 1.5\,\mathrm{m}$ above the center of the table? The light bulb is an isotropic source of light and the ceiling is black. *Danka couldn't see her books well.*

Illuminance E is a photometric quantity defined as the luminous flux incident on a unit area. If we have a point source of light with luminous intensity I at the distance r away from a surface, we can calculate the illuminance as

$$E = \frac{I}{r^2}\cos\alpha\,,$$

where α is the angle between the normal of the surface and the direction of the incident light rays. In the centre of the table, the rays arrive perpendicularly, that is $\cos\alpha = 1$. At the edge of the table however, the rays arrive at an angle α and

$$\cos\alpha = \frac{h}{\sqrt{h^2 + R^2}}\ ,.$$

The distance between the edge of the table and the light source is

$$l = \sqrt{R^2 + h^2}\,.$$

The difference of illuminance between the centre and the edge of the table is therefore

$$\delta E = \frac{I}{h^2} - \frac{I}{R^2 + h^2}\frac{h}{\sqrt{h^2 + R^2}}\,,$$

$$\delta E = I\left(\frac{1}{h^2} - \frac{h}{\left(R^2 + h^2\right)^{\frac{3}{2}}}\right) \doteq 22.6\,\mathrm{lx}\,.$$

The difference of illuminance between the centre and the edge of the table is 22.6 lx.

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Problem FoL.14 ... autumn in a train 4 points

A train is climbing an icy track uphill. The steepest section of track which the train is still able to climb has a slope angle $\alpha = 1.75^\circ$. When the train gets over the slope, it reaches a station where tracks are horizontal and still icy. What is the shortest distance at which the train can stop from a speed $v = 52\,\mathrm{km{\cdot}h^{-1}}$? *Dodo was waiting for a train.*

During the ascent, it is necessary to have a condition on static friction $F_\mathrm{f} \leq fF_\mathrm{n}$. If only the gravity force F_G affects the train, we use its components in the equations $F_\mathrm{f} = F_\mathrm{G}\sin\alpha$ and $F_\mathrm{n} = F_\mathrm{G}\cos\alpha$. This implies a relation for the slope and the coefficient of friction $\tan\alpha \leq f$.

During the deceleration a, the maximum friction force allowed before the wheels start slipping is given by condition $F_\mathrm{n} = F_\mathrm{G}$ and $F_\mathrm{f} = ma$. Using the condition from above we get $a \leq fg$ where g is gravitational acceleration. The shortest distance at which the train stops is when $a = fg = g\tan\alpha$ and the distance is

$$s = \frac{1}{2}at^2 = \frac{v^2}{2a} = \frac{v^2}{2g\tan\alpha} \doteq 348\,\mathrm{m}\,.$$

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Problem FoL.15 ... the power of a waterfall 4 points

A $h = 30\,\mathrm{m}$ high waterfall has flow rate $Q = 1.2\,\mathrm{m^3{\cdot}s^{-1}}$. Find the total force with which water impacts the ground under the waterfall. Assume that the water quickly flows away from the point of impact and the depth of water under the waterfall is negligible.

Dodo is reminiscing about National park Plitvička jezera.

To solve this problem, we will use Newton's second law in the formulation with momentum

$$F = \frac{\mathrm{d}p}{\mathrm{d}t}\,,$$

which states, that the force is defined as as the time derivative of momentum. In time $\mathrm{d}t$, the river bed decelerates falling water with mass $\mathrm{d}m = \rho\,\mathrm{d}V = \rho Q\,\mathrm{d}t$ from the impact velocity v to zero. We will denote the impact velocity by comparing kinetic and potential energy in a homogeneous gravitational field

$$mgh = \frac{1}{2}mv^2\,, \quad v = \sqrt{2gh}\,,$$

where g is gravitational acceleration. Plugging in, we obtain the force

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{v\,\mathrm{d}m}{\mathrm{d}t} = \frac{v\rho Q\,\mathrm{d}t}{\mathrm{d}t} = \rho Q\sqrt{2hg} \doteq 29\,\mathrm{kN}\,.$$

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Problem FoL.16 ... burning coal

4 points

We burn $m = 213\,\mathrm{mg}$ of pure carbon in a closed vessel of volume $V = 20\,\mathrm{l}$ filled with air. Once the temperature equilibrium between the vessel and its surroundings is restored, we measure the pressure in the vessel. Find the ratio of the pressure after burning to the pressure before burning. *Dodo wanted to be malicious.*

A chemical reaction occurring while burning carbon in an environment with enough oxygen is described by equation

$$C + O_2 \longrightarrow CO_2 \,.$$

Since carbon is solid, one mole of gas transforms into one mole of other gas. If we assume that the gas is ideal, the ideal gas law

$$pV = nRT \,,$$

must hold before the reaction as well as after it once the equilibrium is reached again. Consider that the volume V has not changed, nor temperature T (in balance with the surrounding) nor the amount of substance n. Therefore the pressure hasn't changed either. The solution to the task above is $p/p_0 = 1$. We should make sure there is enough oxygen in the vessel. In the case of room conditions, a mole of gas has a volume of approximate $24\,\mathrm{l}$. Therefore there are approximately $0.2\,\mathrm{mol}$ of oxygen molecules. The amount of our carbon is $n = m/M_\mathrm{m} \approx 0.02\,\mathrm{mol}$, which means there is enough oxygen.

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Problem FoL.17 ... heat and phases

4 points

How much higher is the amount of heat required to let ice boil away (Q_b) compared to the amount of heat necessary to only melt it (Q_m)? Find $k = Q_\mathrm{b}/Q_\mathrm{m}$. Our ice is taken from a fridge with inner temperature $t = -18\,^\circ\mathrm{C}$. The specific heat capacity of liquid water is $c = 4\,180\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ and that of ice is $c_0 = 2\,090\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. The enthalpy of fusion of ice is $l_1 = 334\,\mathrm{kJ\cdot kg^{-1}}$ and the enthalpy of vaporization of water is $l_2 = 2.26\,\mathrm{MJ\cdot kg^{-1}}$. We are interested in the points when the ice fully melts to water (at $0\,^\circ\mathrm{C}$) and when the water fully vaporizes (at $100\,^\circ\mathrm{C}$). *Karel was wondering about the possibility of scalding above a kettle.*

Let us divide the process into four parts. Q_1 is the heat necessary to warm the ice up to $0\,^\circ\mathrm{C}$ (the temperature difference is $\Delta t_1 = 18\,^\circ\mathrm{C}$). To calculate it we will use the equation $Q_1 = mc_0\Delta t_1$, where m is the mass of the ice and c_0 is the specific heat capacity of the ice.

Q_2 is the heat necessary to change ice to water at $0\,^\circ\mathrm{C}$. We will calculate it using the equation $Q_2 = ml_1$, where we use the enthalpy of fusion (after all, it's about melting).

Q_3 is the heat necessary to warm the water from $0\,^\circ\mathrm{C}$ up to $100\,^\circ\mathrm{C}$ (the temperature difference is $\Delta t_2 = 100\,^\circ\mathrm{C}$). To calculate it, we will use analogous equation as for Q_1, i.e. $Q_3 = mc\Delta t_2$. However, now we used the specific heat capacity of water and a different temperature difference.

Q_4 is the heat necessary to change water to steam at $100\,^\circ\mathrm{C}$. The calculation is again analogous to the calculation of Q_2, but now we use the enthalpy of vaporization (after all, this is about vaporization). The equation is therefore $Q_4 = ml_2$.

Now, if we look closely at the situation, we will find that $Q_\mathrm{b} = Q_1 + Q_2 + Q_3 + Q_4$ and that $Q_\mathrm{m} = Q_1 + Q_2$. In total, after canceling out m we have

$$\frac{Q_\mathrm{b}}{Q_\mathrm{m}} = \frac{c_0\Delta t_1 + l_1 + c\Delta t_2 + l_2}{c_0\Delta t_1 + l_1} \,.$$

After numerical substitution we will find out that we have to supply 8.21 times more heat to the ice to boil it away than to melt it.

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Problem FoL.18 ... boiling

4 points

Danka put $V = 1.5\,\mathrm{l}$ of water with a temperature $T_0 = 25\,\mathrm{^\circ C}$ in a pot and set it to boil on a stove with wattage (electric power consumption) $P = 1\,200\,\mathrm{W}$. The efficiency of heating the pot with water (i.e. how much heat is transferred to the pot with water) is $\eta = 0.69$ and the heat capacity of the pot is $C = 500\,\mathrm{J{\cdot}K^{-1}}$. How long does Danka need to wait till the water starts to boil? Find the necessary constants in tables. *Danka had to wait a long time while cooking.*

We assume the water to have the following properties: density $\rho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$, specific heat capacity $c_\mathrm{w} = 4\,180\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ and boiling point of $T_\mathrm{w} = 100\,\mathrm{^\circ C}$.The heat needed to be absorbed by the pot and the water is

$$Q = V\rho c_\mathrm{w}\left(T_\mathrm{w} - T_0\right) + C\left(T_\mathrm{w} - T_0\right) .$$

The stove transmits this heat in time t and therefore

$$Q = \eta P t\,.$$

Thus

$$\eta P t = \left(T_\mathrm{w} - T_0\right)\left(V\rho c_\mathrm{w} + C\right) \quad .$$

Final formula for the time is

$$t = \frac{\left(T_\mathrm{w} - T_0\right)\left(V\rho c_\mathrm{w} + C\right)}{\eta P} \doteq 613\,\mathrm{s} \approx 10\,\mathrm{min}\,.$$

Danka will wait approximately 10 minutes for the water to start boiling.

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Problem FoL.19 ... internal resistance

4 points

Consider two identical resistors connected in series to a non-ideal voltage source. In this case, the efficiency of the voltage source is 0.87. Find the efficiency of the voltage source, i.e. the ratio between the power consumed by our resistors and the total power supplied by the source, when the two resistors are connected in parallel instead. We assume model the non-ideal voltage source like an ideal voltage source (not a current source) with an internal resistor connected in series. *Matěj wanted to save some money – so he tried saving electricity.*

The efficiency of a power supply source can be calculated as the ratio of the total power to the power on attached resistors. To calculate the power on a resistor, we will use the formula $P = RI^2$, where I is the current flowing through a resistor with resistance R. Let R_i denote the internal resistance of the power supply source and R the total resistance of two resistors in series. Then, for the efficiency of the connection in series, the equation

$$\eta_1 = \frac{RI^2}{(R + R_\mathrm{i})I^2} = \frac{R}{R + R_\mathrm{i}}\,,$$

holds. We used the fact that the current is the same on all devices connected in series. We denote R_i from the equation as

$$R_\mathrm{i} = R\frac{1-\eta_1}{\eta_1}\,.$$

Parallel connection of two resistors can be substitued by one resistor with half resistance. Therefore, two identical resistors connected in parallel have four times smaller resistance compared to serial connection. For the efficiency of the parallel connection we obtain

$$\eta_2 = \frac{\frac{R}{4}I^2}{\left(\frac{R}{4}+R_\mathrm{i}\right)I^2} = \frac{\frac{R}{4}}{\frac{R}{4}+R_\mathrm{i}} = \frac{R}{R+4R_\mathrm{i}} = \frac{R}{R+4R\frac{1-\eta_1}{\eta_1}} = \frac{\eta_1}{4-3\eta_1} \doteq 0.626\,,$$

where we used the formula for R_i.

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Problem FoL.20 ... raft voyage 4 points

A raft consists of twelve cylinder-shaped wooden logs. Each log is $l = 8.00\,\mathrm{m}$ *long and has a radius* $r = 12.0\,\mathrm{cm}$*. When loaded with* $m = 70\,\mathrm{kg}$ *freight, it floats in such a way that the logs stick* $s = 3.0\,\mathrm{cm}$ *above the water level. What is the density of the wood?*

Jarda was wondering about possible improvements for a trip on water.

The force of gravity, caused by the raft and its freight, must be balanced by the force of buoyancy, which, according to Archimedes' principle, equals the force of gravity of the water displaced by the immersed part of the body

$$F_\mathrm{bu} = V_\mathrm{i}\rho_0 g\,,$$

where ρ_0 is the density of water, g is the gravitational acceleration and V_i is the volume of the immersed part of the raft. For the force of gravity, using the formula for the volume of a cylinder, we have

$$F_G = mg + V\rho g = mg + 12\pi r^2 l\rho g\,,$$

where ρ is the density of the wood. The remaining task is to estimate the immersed volume of the raft. From the geometrical point of view, these are twelve bodies, each of which was made by cutting off a cylinder perpendicularly to its base. The base of each body is a circular segment with the surface S. The formula for the surface can be found in the literature as

$$S = r^2 \arccos\frac{r-h}{r} - (r-h)\sqrt{2hr-h^2}\,,$$

where $h = 2r - s$. After plugging it into the formula for the volume and enumeration we obtain the volume of the immersed part $V_\mathrm{i} = 12Sl \doteq 4.03\,\mathrm{m^3}$.

If we compare the forces mentioned above we can express the density of the wood as

$$\rho = \frac{\rho_0 V_\mathrm{i} - m}{12\pi r^2 l} \doteq 910\,\mathrm{kg{\cdot}m^{-3}}\,.$$

Therefore it is wood with quite high density, but still, we can float on it with a sufficient margin. For the conditions given, a single log with density $\rho \doteq 730\,\mathrm{kg{\cdot}m^{-3}}$ would be enough to carry a person.

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Problem FoL.21 ... mysterious ball 4 points

We have a ball with radius $r = 2.0\,\mathrm{cm}$, which has an unknown mass and is made from an unknown material, hanging from a massless spring. When submerged in water with density $\rho = 1\,000\,\mathrm{kg\cdot m^{-3}}$, the elongation of the spring drops to 80% of its value before the ball is submerged. What is the mass of the ball?

Verča was reminiscing about experiments in high school.

The difference in elongation of the string is caused by the force of buoyancy acting on the submerged ball. The relevant forces can be expressed as

$$0.8F_G = F_G - F_\mathrm{b}\,,$$

where F_G is the force of gravity and F_b the force of buoyancy. Now we need to plug the formulas for forces in and express mass m as

$$0.8mg = mg - V\rho g\,,$$
$$m = \frac{20}{3}\pi r^3 \rho\,.$$

Using the numerical values given we obtain the mass of the ball as $m \doteq 168\,\mathrm{g}$.

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Problem FoL.22 ... illuminated 5 points

Danka was sitting in a futuristic train, which was traveling at a constant speed $v = 1.0\cdot 10^5\,\mathrm{km\cdot s^{-1}}$. The train track was perpendicular to a long straight road and along this road, lamps were placed with constant spacing $l = 5.0\,\mathrm{m}$. Danka crossed the intersection of the track and the road. In the landscape's reference frame, all the lamps were turned on at once when the train was at a distance $d_0 = 2.0\,\mathrm{km}$ from the intersection. One lamp is at the intersection, let's give it index 0. What time passes in Danka's reference frame between the moments when light from lamp 0 reaches her and when light from a lamp with index 100 reaches her? Use the exact value of speed of light.

Danka was traveling by train.

The light from the lamp with index 0 reaches Danka in time t_0. In that moment she is in the distance $d_0 + vt_0$ from the crossing. For time t_0

$$t_0 = \frac{d_0}{c - v}\,,$$

holds. Let t_{100} denote the time when light from lamp with index 100 reaches Danka and let r be the distance between the lamp and the position of Danka at time t_{100}. It satisfies

$$r = \sqrt{n^2 l^2 + (d_0 + vt_{100})^2}\,,$$

where $n = 100$. Then

$$t_{100} = \frac{r}{c} = \frac{\sqrt{n^2 l^2 + (d_0 + vt_{100})^2}}{c}\,,$$

holds. This equation rewrites as a quadratic equation for t_{100}, with solutions

$$t_{100} = \frac{vd_0 \pm \sqrt{c^2{d_0}^2 + (c^2 - v^2)\, n^2 l^2}}{c^2 - v^2}\,.$$

The physically correct one is the positive one. Furthermore, let us calculate the difference $\Delta t = t_{100} - t_0$. We obtain $\Delta t = 2.06 \cdot 10^{-7}\,\mathrm{s}$. However, this time difference is in the reference frame connected with the countryside. Since Danka moves in that reference frame, we need to transform it into the reference frame connected with the train (let us denote it with apostrophe), in which is Danka stationary and her proper time therefore equals the coordinate time t'. We will begin with the transformation formula $t' \to t$

$$t = \gamma \left(\frac{v}{c^2} x' + t'\right)\,,$$

where v is the velocity of one reference frame towards the other, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and if Danka has constant x' (coordinate in reference frame connected with the train), for the time difference we obtain

$$\Delta t' = \frac{1}{\gamma}\Delta t \doteq 1.94 \cdot 10^{-7}\,\mathrm{s}\,.$$

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Problem FoL.23 ... a ball

5 points

Anička is playing with a ball of yarn with a radius $R = 5\,\mathrm{cm}$, which is formed by a $l = 100\,\mathrm{m}$ long string. She stands on an inclined plane with a slope angle $\alpha = 3^\circ$. She kicks the ball along the slope upwards, but one end of the string stays stuck at the point where she kicked it away, so the string is unraveling as the ball travels up the slope until the whole ball unravels and only the straight string remains on the slope. Find the smallest initial velocity of the ball needed to reach the state described above. *Matěj read a children's book.*

Let's start with the law of conservation of energy. Once the ball is fully unraveled (i.e. there is only a straight string), its centre of mass is exactly in one half of its length, which is at height $\frac{l}{2}\sin\alpha$. In the beginning, the centre of mass of the ball is at height $R\cos\alpha$. Therefore, the difference in potential energy is $\Delta E = Mg\left(\frac{l}{2}\sin\alpha - R\cos\alpha\right)$. Let v denote the velocity of the kick. The initial kinetic energy of translation is $\frac{1}{2}Mv^2$ and the kinetic energy of rotation is $\frac{1}{2}J\frac{v^2}{R^2}$, where $J = \frac{2}{5}MR^2$. Therefore, we obtain

$$\Delta E = \frac{1}{2}Mv^2 + \frac{1}{5}Mv^2 = \frac{7}{10}Mv^2\,,$$

$$Mg\left(\frac{l}{2}\sin\alpha - R\cos\alpha\right) = \frac{7}{10}Mv^2\,,$$

$$v = \sqrt{\frac{5g}{7}\left(l\sin\alpha - 2R\cos\alpha\right)} \doteq 6.00\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem FoL.24 ... a traveler's problem

5 points

A traveler needs to catch his train and does not have much time. The train station is $s = 1\,000\,\mathrm{m}$ far from him. Unfortunately, he finds himself in the fields, where he has to force his way out with velocity $v_\mathrm{f} = 3.0\,\mathrm{km{\cdot}h^{-1}}$. There is a road leading to the station as well, and on this road, he could jog with an average velocity $v_\mathrm{r} = 7.0\,\mathrm{km{\cdot}h^{-1}}$. However, the road is $l = 600\,\mathrm{m}$ far from him. Find the optimal angle α (measured with respect to a perpendicular to the road) such that the traveler reaches the station as soon as possible if he starts walking towards the road at this angle. *Verča went hiking.*

We will begin with calculation of the time of travel depending on the angle α and other parameters as

$$t = \frac{l}{v_\mathrm{f}\cos\alpha} + \frac{\sqrt{s^2 - l^2} - l\tan\alpha}{v_\mathrm{r}}\,.$$

In order to reach minimal time possible we will differentiate the formula with respect to α and put it equal zero

$$\frac{\mathrm{d}t}{\mathrm{d}\alpha} = \frac{l}{v_\mathrm{f}}\cdot\frac{\sin\alpha}{\cos^2\alpha} - \frac{l}{v_\mathrm{r}}\cdot\frac{1}{\cos^2\alpha} = 0\,.$$

The desired angle expresses as $\alpha = \arcsin\left(v_\mathrm{f}/v_\mathrm{r}\right) \doteq 25.4^\circ$.

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Problem FoL.25 ... a different parallel-plate capacitor

5 points

Consider a capacitor consisting of a square conductive plate with surface area $S = 6.0\,\mathrm{cm^2}$ and a parallel, infinite and grounded conductive plane at a distance $d = 1.1\,\mathrm{mm}$. Find the capacitance of such a capacitor, defined as the ratio of charge on the plate to potential on the plate. *Originally, Vašek prepared a problem with a capacitor with infinite capacitance.*

Both conductive planes create equipotential surfaces. As usual, we choose electrostatic potential φ such that equals zero on the grounded conductor. Let Q denote the electric charge of the square plate and φ_r the electrostatic potential on it. When looking for electrostatic potential in half-space containing the square plate bounded by the infinite plane (let us denote it right half-space), the problem reduces to solving Poisson's equation

$$\Delta\varphi = -\frac{\rho}{\varepsilon_0} \tag{1}$$

in given half-space with boundary condition $\varphi = 0$ on the infinite plane, where ρ is the charge density and ε_0 is vacuum permittivity. This kind of problems is often solved using the *method of images*. We will consider the infinite plane to be a plane of symmetry. If all charges from the right half-space were mirrored to the left half-space with the opposite charge, we would get a new electrostatic problem which solution equals the solution of the original problem in the right half-space. It should be noted that mirroring and change of signs cause the charge on the plane to be zero. Overall it causes the charge density ρ to be antisymmetric when mirrored through the plane of symmetry. Consider that the Poisson equation (1) is linear – therefore potential φ exists and it is antisymmetric when mirrored and just like the charge density ρ is zero on the plane of symmetry. These observations form the basis of the *method of images*.

In our particular problem, the method of images gives electrostatic problem with two identical square plates in the distance $2d$, from which one has potential φ_r and charge Q and the second one has potential $-\varphi_r$ and charge $-Q$. These plates together form a parallel-plate capacitor. Considering that their distance $2d$ is significantly less than their characteristic size $\sqrt{S}$, we may approximate their capacitance as

$$C' = \varepsilon_0 S/2d\,.$$

Now we need to realise, that such capacitor has potential difference twice the difference between plate and the infinite plane and so capacitance, as defined in task, is $C = 2C' = 4.8\,\mathrm{pF}$.

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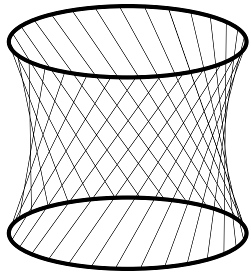
Problem FoL.26 ... oscillating hoop — 5 points



Suppose that we have two identical hoops, each with a radius $R = 1.0\,\mathrm{m}$. The upper hoop is in a fixed position, while the lower one is attached to the upper one by several massless cords; each of these cords has the same length $l = 2.0\,\mathrm{m}$ and hangs vertically. Let the mass of the lower hoop be $m = 1.0\,\mathrm{kg}$. If we rotate it a bit around the vertical axis and release, what is its period of small oscillations?

Lego modified a mathematical problem into a physics problem.

As usual, there are plenty of ways how to calculate the period of small oscillations. Here we will use a less well known but a very effective method.

The main idea is to express the energy of our system as a sum of kinetic and potential energy. We will use the angle of displacement from the equilibrium φ as a variable to the relations.

The kinetic energy is simply

$$E_\mathrm{k} = \frac{1}{2} I\omega^2 = \frac{1}{2} mR^2\dot{\varphi}^2\,.$$

The potential energy is slightly more difficult. For sufficiently small φ we can assume that the cord attachment points have moved from their initial positions by $R\varphi$. Applying the Pythagorean theorem we get the new distance between the hoops $\sqrt{l^2 - R^2\varphi^2}$. If we assume $R\varphi \ll l$, we can approximate the relation as

$$l\sqrt{1 - \left(\frac{R\varphi}{l}\right)^2} \approx l\left(1 - \frac{1}{2}\left(\frac{R\varphi}{l}\right)^2\right) = l - \frac{R^2\varphi^2}{2l}\,.$$

Therefore, compared to the initial position (zero displacement was at the distance l below the upper hoop), the lower hoop is lifted by

$$\Delta h = \frac{R^2\varphi^2}{2l}\,.$$

The difference in potential energy is

$$E_\mathrm{p} = mg\Delta h = \frac{1}{2}\frac{mgR^2}{l}\varphi^2\,.$$

Consider that kinetic and potential energy of simple harmonic oscillator is described by equations

$$E_\mathrm{k} = \frac{1}{2} m \dot{x}^2 \,,$$
$$E_\mathrm{p} = \frac{1}{2} k x^2 \,.$$

The period is $T = 2\pi\sqrt{m/k}$. Now we just need to notice that the formulas, which we have derived for our system, have the same "pattern" as these for SHO (the only difference is that our variable is angle, not position, but that is not an issue). Let us denote the expressions equivalent to m and k as m_ef and k_ef, respectively, and plug them into the equation for the period of small oscillations. We obtain

$$T = 2\pi\sqrt{\frac{m_\mathrm{ef}}{k_\mathrm{ef}}} = 2\pi\sqrt{\frac{mR^2}{\frac{mgR^2}{l}}} = 2\pi\sqrt{\frac{l}{g}} \doteq 2.84\,\mathrm{s}\,.$$

We probably could have guessed the result straight away...

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Problem FoL.27 ... oscillating 5 points

Suppose that we have a homogeneous square plate with edge length 1 m. *We drill a tiny hole in it, hammer a nail into a wall, hang the plate from the nail through this hole and let the plate oscillate in the vertical plane in which it lies. How far from the centre of the square do we have to drill the hole to maximise the frequency of small oscillations?*

Matěj had some spare metal sheet and didn't know what to do with it.

Let us use the equation for physical (or *compound*) pendulum

$$\omega = \sqrt{\frac{mgl}{J}}\,,$$

where m is the mass of the plate, l is the distance from the axis to the centre of gravity of the plate and J is the moment of inertia of a square with respect to the rotation axis. The moment of inertia can be calculated using the parallel axis theorem, since we know the moment of inertia of a square with respect to the centre of mass $J_0 = \frac{1}{6}ma^2$, where a is the length of its side. For the desired moment of inertia we can write

$$J = J_0 + ml^2 = \frac{1}{6}ma^2 + ml^2\,.$$

We obtain

$$\omega^2 = \frac{gl}{\frac{1}{6}a^2 + l^2}$$

and put the derivative with respect to l equal zero giving

$$g\left(\frac{1}{6}a^2 + l^2\right) - 2gl^2 = 0\,,$$

from which we can calculate the required distance

$$l = \frac{1}{\sqrt{6}} a \doteq 0.408\,\mathrm{m}\,.$$

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Problem FoL.28 ... lego and dice 1.0 — 4 points

We have two small cuboids, one lying on the other – the lower cuboid has mass $M = 2.0\,\mathrm{kg}$, the upper one $m = 1.0\,\mathrm{kg}$. The coefficient of friction between the lower cuboid and the surface under it is 0, the coefficient of static friction between the cuboids is $f_\mathrm{s} = 0.50$ and the coefficient of kinetic friction between them is $f_\mathrm{k} = 0.20$.

Find the magnitude of the force we must exert on the lower cuboid that causes it to move with a constant acceleration $a = 10\,\mathrm{m\cdot s^{-2}}$.

Lego was helping his friend with physics, so he at least took some ideas for new problems.

The maximum lateral force that the lower cuboid can impart on the upper is $F_\mathrm{max} = mgf_\mathrm{s} \doteq 5\,\mathrm{N}$. In this case, the upper cuboid would accelerate with the acceleration $a_\mathrm{max} = F_\mathrm{max}/m \doteq 5\,\mathrm{m\cdot s^{-2}}$, which is less than the acceleration required by the problem task. Therefore, we are interested in dynamic friction.

The magnitude of the force of friction between those two cuboids is simply $F_\mathrm{k} = mgf_\mathrm{k}$. Therefore, if we denote F to be the force that we impart on the lower cuboid, the resulting acceleration will be equal

$$a = \frac{F - mgf_\mathrm{k}}{M}\,.$$

Now we only need to express $F = aM + mgf_\mathrm{k} \doteq 22\,\mathrm{N}$.

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Problem FoL.29 ... lego and dice 2.0 — 5 points

We have two small cuboids, one lying on the other – the lower cuboid has mass $M = 2.0\,\mathrm{kg}$, the upper one $m = 1.0\,\mathrm{kg}$. The coefficient of friction between the lower cuboid and the surface under it is 0, the coefficient of static friction between the cuboids is $f_\mathrm{s} = 0.50$ and the coefficient of kinetic friction between them is $f_\mathrm{k} = 0.20$.

Find the magnitude of the force we must exert on the lower cuboid that causes it to move with a constant acceleration $a = 1.0\,\mathrm{m\cdot s^{-2}}$.

Lego was helping his friend with physics, so he at least took some ideas for new problems.

Consider that the upper cuboid will not accelerate faster than the lower, which implies that the maximal acceleration of the upper cuboid is a. To achieve that, force $am = 1\,\mathrm{N}$ is necessary.

The maximal force that the lower cuboid can impact the upper with is $F_\mathrm{max} = mgf_\mathrm{s} \doteq 5\,\mathrm{N}$, which is more than is necessary, therefore the upper cuboid will accelerate with the acceleration a as well.

Now we need to subtract the force that the lower cuboid imparts on the upper from the force we impart on it. Alternatively, it is enough to realise that these cuboids behave as a single object, which means

$$a = \frac{F}{M+m}\,.$$

Now we will only need to express the force as $F = (M+m)\,a = 3.0\,\mathrm{N}$.

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Problem FoL.30 ... impact accelerator 5 points

We have a large number of point masses in a series in one line. The first point mass has a mass $M_0 = 1\,\mathrm{kg}$ and every subsequent point mass has a mass equal to $70\,\%$ of the previous one. The first point mass starts to move towards the second one with a kinetic energy $E_0 = 50\,\mathrm{J}$. All collisions are perfectly elastic. Which point mass will be the first to have a speed larger than one percent of the speed of light? Neglect any relativistic effects.

Jarda wanted to improve upon CERN technology.

During perfectly elastic collisions, we can use the law of conservation of momentum

$$Mu = Mv_M + mv_m$$

and the law of conservation of energy

$$\frac{1}{2}Mu^2 = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2\,,$$

where M is the mass of the more massive of two points, $m = 0.7M$ is the mass of the less massive point, u is the speed of the more massive point before the collision, v_M is it's speed after the collision, and v_m is the speed of the lighter point after the collision. From the equations we get

$$v_m = \frac{2M}{M+m}u = \frac{2}{1.7}u\,.$$

Apparently, the speed of a point after a collision is simply a multiple of the speed of the previous point. After n collisions, the speed of the last (fastest) point is

$$v_{\max} = \left(\frac{2}{1.7}\right)^n u_0\,,$$

where

$$u_0 = \sqrt{\frac{2E_0}{M_0}} = 10\,\mathrm{m{\cdot}s^{-1}}$$

is the speed of the first mass point before the first collision. Now we plug in $v_{\max}$ and get n as

$$n = \frac{\ln\left(\frac{0.01c}{u_0}\right)}{\ln\left(\frac{2}{1.7}\right)} \doteq 77.6\,.$$

For the index of the first point that surpasses one percent of lightspeed, we need to round up and add one (we get to the nth point after $(n-1)$ collisions), that is 79.

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Problem FoL.31 ... jump on!

5 points

Imagine a sufficiently large cuboid with mass $M = 32.5\,\mathrm{kg}$ lying on the ground. We throw a smaller cuboid with mass $m = 11.7\,\mathrm{kg}$ on it in such a way that right before the impact, the vertical component of its velocity is almost zero, while the horizontal component is $v = 19.2\,\mathrm{m{\cdot}s^{-1}}$. The coefficient of friction between the larger cuboid and the ground is $f_1 = 0.13$, the coefficient of friction between the two cuboids is $f_2 = 0.69$. What is the total distance covered by the larger cuboid? *Jachym wondered wherever is possible to land with a plane.*

The friction force $f_2 mg$ will affect the smaller cuboid until its velocity reaches the velocity of the lower cuboid (let's denote it v_1 and the time when this happens t_1). From the magnitude of this force it follows that the acceleration is $a_2 = -f_2 g$.

The smaller cuboid affects the bigger one with the same friction force. The bigger cuboid is also affected by the friction force between it and the ground, which is $f_1 (m + M) g$. This force decelerates it, while the friction force between it and the smaller cuboid accelerates it. In total, it's acceleration is

$$a_1 = \frac{f_2 mg - f_1 (m + M) g}{M}\,,$$

therefore its velocity at t_1 will be $v_1 = a_1 t_1$. For the small block, on the other hand, we have $v_1 = v + a_2 t_1$; from this equation we can express time as

$$t_1 = \frac{v}{a_1 - a_2} = \frac{Mv}{(f_2 - f_1)(m + M) g}\,.$$

During this time, the bigger cuboid will travel the distance of

$$x_1 = \frac{1}{2} a_1 t_1^2\,.$$

During the rest of the trajectory, the cuboids move as one body, so their acceleration is $a_3 = -f_1 g$. Deceleration from v_1 to zero will take them

$$t_3 = -\frac{v_1}{a_3} = -\frac{a_1 t_1}{a_3}\,.$$

During this time they travel the distance of

$$x_3 = v_1 t_3 + \frac{1}{2} a_3 t_3^2 = -\frac{1}{2} a_3 t_3^2 = -\frac{a_1^2 t_1^2}{2a_3}\,.$$

Therefore, overall covered distance will be

$$x = x_1 + x_3 = \frac{1}{2} a_1 t_1^2 - \frac{a_1^2 t_1^2}{2a_3} = \frac{1}{2} a_1 t_1^2 \left(1 - \frac{a_1}{a_3}\right) = \\ = \frac{mv^2}{2g(m + M)} \frac{f_2 m - f_1 (m + M)}{f_1 (f_2 - f_1)(m + M)} \doteq 3.60\,\mathrm{m}\,.$$

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Problem FoL.32 ... disgusting piston 5 points

In a thermally insulated cylinder with an internal cross section $S = 500\,\mathrm{cm}^2$ and a height $l = 50\,\mathrm{cm}$, there is a resistor with resistance $R = 120\,\Omega$. The cylinder is otherwise filled with air at a temperature $T_0 = 20\,\mathrm{^\circ C}$ and pressure $p_0 = 101\,\mathrm{kPa}$, and the same kind of air surrounds the cylinder. A current $I = 200\,\mathrm{mA}$ flows through the resistor. A base of the cylinder breaks away when pushed with a force exceeding $F = 500\,\mathrm{N}$. After what time does that happen?

Jarda wanted to break something using air.

For the base to break, the pressure difference must reach

$$\Delta p = \frac{F}{S}\,.$$

At the beginning, both outside and inside the piston, there is the pressure p_0. The pressure inside the cylinder must then rise to $p_0 + \Delta p$, that is $1 + \Delta p/p_0$ times. Since the process will be isochronic, this is also the ratio of the increase in temperature.

The resistor heats the air with the power $P = I^2R$. All of the energy will be converted to the internal energy of the gas, giving

$$Q = \Delta U\,,$$
$$I^2Rt = mc_V\Delta T\,,$$

where m ist the mass of the gas which we are heating up, c_V is it's specific heat capacity at constant volume which, for air, is approximately $0.72\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$, and ΔT is the difference in temperatures at the beginning and the end of the process. We already know that the temperature must increase by the factor $1 + \Delta p/p_0$, that is by $\Delta T = T_0\Delta p/p_0$.

Now we only need to find the mass of gas in the cylinder, which is simply $m = \rho V = \rho Sl$, where $\rho = 1.20\,\mathrm{kg{\cdot}m^{-3}}$ (the density of air in normal conditions). Finally we have

$$t = \frac{\rho Slc_VT_0F}{Sp_0I^2R} = \frac{\rho lc_VT_0F}{p_0I^2R} \doteq 130\,\mathrm{s}\,.$$

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Problem FoL.33 ... 4D scouring pad 5 points

Find the resistance between two neighbouring vertices of a four-dimensional cube made of wire. Each edge of the cube has a resistance $R = 1\,000\,\Omega$.

Karel was wondering about multi-dimensional budgeting problems.

A 4D cube can be visualised as two 3D cubes where their relevant vertices are connected. Two neighbouring vertices in the 4D cube are e.g. two vertices neighbouring in the 4D cube but from different 3D cubes. Let us discuss, how could the current flow from one vertex (we will denote it A) into the second one (Z).

The first option, of course, is through the edge connecting them, which has the resistance R. One could see that this trajectory is parallel to the others.

If the current does not flow from A through the edge connecting two 3D cubes, it must flow through the edge of the 3D cube to one vertex of this cube which is neighbour to A. There are three such vertices, and they are interchangeable. Therefore, we will use the often-used trick – imagine, that these vertices are connected perfectly conductively (but still no current will flow through these connections) and let B denote one of them. From A to B there are three resistors with the resistance R in parallel, therefore for the resistance the following will hold (for any n resistors)

$$\frac{1}{R_n} = n \cdot \frac{1}{R} \quad \Rightarrow \quad R_n = \frac{R}{n}\,.$$

For $n = 3$ we obtain that the resistance between A and B equals $R/3$.

In vertex B the current has two possible paths again – through the edge connecting the two 3D cubes or "further" in the original cube. In the first case, it always flows to a vertex neighbouring with Z. We can connect these vertices into one (similarly as we did with B) and denote it Y. Consider that the resistance between Y and Z, as well as the resistance between B and Y, equals $R/3$. In the second case the current has two possible paths (since it can not flow back to A) in each of the three vertices equivalent to B, which means six edges in total. But it will always flow into one of three vertices of the first cube, which is in the distance of two edges from A. As before, we will connect these three vertices into a single one (let us denote it C) and the resistance between B and C is $R/6$.

There are two possibilities in the vertex C. The current may either flow into the second 3D cube – through resistance $R/3$ and end in the vertex X, where the resistance between X and Y equals $R/6$; or it can continue in the first 3D cube – through resistance $R/3$ and it will end in the vertex which is opposite to A – let us denote it D.

From the vertex D the current can flow only into the vertex of the second 3D cube, which is opposite to Z. Let us denote it W. Then the resistance between D and W is R and between W and X again $R/3$. Therefore we can calculate the resistance between C and X as the "direct" (R'_{CX}) path and the path through D and W (R_{CDWX}) in parallel

$$R_{CX} = \frac{R'_{CX} R_{CDWX}}{R'_{CX} + R_{CDWX}} = \frac{\frac{R}{3}\left(\frac{R}{3} + R + \frac{R}{3}\right)}{3\frac{R}{3} + R} = \frac{5}{18}R\,.$$

After that we can similarly calculate the resistance between B and Y as

$$R_{BY} = \frac{R'_{BY} R_{BCXY}}{R'_{BY} + R_{BCXY}} = \frac{\frac{R}{3}\left(\frac{R}{6} + \frac{5}{18}R + \frac{R}{6}\right)}{\frac{R}{3} + \frac{R}{6} + \frac{5}{18}R + \frac{R}{6}} = \frac{11}{51}R\,.$$

Finally we will obtain the resistance between A and Z as

$$R_{AZ} = \frac{R'_{AZ} R_{ABYZ}}{R'_{AZ} + R_{ABYZ}} = \frac{R\left(\frac{R}{3} + \frac{11}{51}R + \frac{R}{3}\right)}{R + \frac{R}{3} + \frac{11}{51}R + \frac{R}{3}} = \frac{15}{32}R \doteq 469\,\Omega\,.$$

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Problem FoL.34 ... levitation by firearm 5 points

Consider a body with mass $M = 12.1\,\mathrm{kg}$*, which is kept in the air by sustained fire from AK-47. Calculate the rate of fire necessary for the body to hover and constantly oscillate between two points low above the ground. All the impacts are elastic, the mass of each bullet is* $m = 7.93\,\mathrm{g}$ *and the bullets are flying straight up with speed* $v = 715\,\mathrm{m{\cdot}s^{-1}}$*.*

Don't forget to take cover and move only when the enemy is reloading.

Any drop of velocity of the bullets associated with gravity can be neglected due to the low heights involved. We can therefore place the lowest point of the body's oscillatory motion to a height 0. At this height the body is moving downwards with a velocity v_0. In a perfectly elastic collision, both momentum and energy are conserved. If the velocity of the body after a collision is v_0' (this time moving upwards), the laws of conservation can be written as

$$mv - Mv_0 = -mv' + Mv_0'\,,$$
$$\frac{1}{2}mv^2 + \frac{1}{2}Mv_0^2 = \frac{1}{2}mv'^2 + \frac{1}{2}Mv_0'^2\,,$$

where v' is the downwards velocity of the bullet after the collision. However, we can notice that the mechanical energy is conserved so the velocity v_0 has the same magnitude as v_0'. Substituting $v_0 = v_0'$, the second equation becomes $v^2 = v'^2$. The solution $v = -v'$ does not make any physical sense (it would be as if the collision never happened), so we must have $v = v'$. From the first equation, we obtain

$$v_0 = \frac{m}{M}v\,.$$

Let the fall of the body down from the highest point take a time t, then $v_0 = gt$. The time it takes the body to move up is the same as the time of the fall so we are dealing with a period

$$T = 2t = \frac{2v_0}{g} = \frac{2mv}{Mg}\,.$$

Now we can just calculate the frequency (the rate of fire)

$$f = \frac{1}{T} = \frac{Mg}{2mv} \doteq 10.5\,\mathrm{s^{-1}}\,.$$

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Problem FoL.35 ... burning phosphorus 5 points

Assume that we burn $m = 213\,\mathrm{mg}$ *of phosphorus dust in a closed vessel with volume* $V = 20\,\mathrm{l}$ *filled with air. Once the temperature equilibrium between the vessel and its surroundings is restored, we measure the pressure in the vessel. Find the ratio of the pressure after burning to the pressure before burning the phosphorus.* *Dodo wanted to be malicious, v. 2.*

We start from the equation

$$\mathrm{4P + 5O_2 \longrightarrow 2P_2O_5}\,,$$

i.e. five moles of gas and four moles of phosphorus change into a solid product. The amount of substance of oxygen used in the reaction is therefore

$$n_{\mathrm{O}_2} = \frac{5}{4} n_\mathrm{P} = \frac{5m}{4M_\mathrm{m}} \doteq 0.008\,59\,\mathrm{mol}\,,$$

where $M_\mathrm{m} = 31.0\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of the phosphorus atoms. We will use the ideal gas law in the form of

$$\frac{pV}{Tn} = \mathrm{const}\,,$$

where the amount of substance of the gas and the pressure changes during the examined process. At the beginning there was $n_1 = V/V_\mathrm{mol} \doteq 0.892\,9\,\mathrm{mol}$ in the vessel. We obtain the ratio of the pressures as

$$\frac{p_2}{p_1} = \frac{n_2}{n_1} = 1 - \frac{n_{\mathrm{O}_2}}{n_1} = 1 - \frac{5mV_\mathrm{mol}}{4M_\mathrm{m}V} \doteq 0.990\,4\,.$$

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Problem FoL.36 ... mixed current 4 points

Consider an electronic component which behaves like a resistor with resistance $R = 42\,\Omega$. We let electric current flow through it; this current has an alternating part and a direct part. The alternating part is harmonic with a frequency $f = 50.0\,\mathrm{Hz}$. The minimum value of the current is $12\,\mathrm{mA}$ and the maximum value is $42\,\mathrm{mA}$. The current does not change direction. What is the average power consumed by the component? *Karel wanted to combine.*

Let us begin with the discussion about what is the current like. The problem statement reads, that it has both direct and harmonic components, which can be written as $I = I_\mathrm{DC} + I_\mathrm{AC}\sin(2\pi ft)$. In general, the formula should contain the phase as well, but since we are interested in the average power, it does not matter at all. Furthermore the problem statement mentiones that the current does not change direction, which means $I_\mathrm{DC} > I_\mathrm{AC}$. We can write the following system of equations

$$I_\mathrm{DC} + I_\mathrm{AC} = I_\mathrm{max} = 42\,\mathrm{mA}\,,$$
$$I_\mathrm{DC} - I_\mathrm{AC} = I_\mathrm{min} = 12\,\mathrm{mA}\,,$$

from which we can express

$$I_\mathrm{DC} = \frac{I_\mathrm{max} + I_\mathrm{min}}{2} = 27\,\mathrm{mA}\,,$$
$$I_\mathrm{AC} = \frac{I_\mathrm{max} - I_\mathrm{min}}{2} = 15\,\mathrm{mA}\,.$$

We now know what current flows through the appliance. The remaining task is to denote the power being dissipated at time t. Since the power is the product of the current and the voltage, we get

$$P(t) = I(t)U(t) = RI(t)^2 = R\left(I_\mathrm{DC}^2 + 2I_\mathrm{DC}I_\mathrm{AC}\sin(2\pi ft) + I_\mathrm{AC}^2\sin^2(2\pi ft)\right)\,.$$

The power changes periodically with the period $T = 1/f$, therefore if we want to calculate the average power, it is enough to integrate it through one period and divide it by the length of one period

$$\overline{P} = \frac{1}{T}\int_0^T R\left(I_{\mathrm{DC}}^2 + 2I_{\mathrm{DC}}I_{\mathrm{AC}}\sin\left(2\pi ft\right) + I_{AC}^2\sin^2\left(2\pi ft\right)\right)\mathrm{d}t =$$
$$= fR\left[tI_{\mathrm{DC}}^2 - \frac{1}{\pi f}I_{\mathrm{DC}}I_{\mathrm{AC}}\cos\left(2\pi ft\right) + I_{\mathrm{AC}}^2\left(\frac{t}{2} - \frac{\sin\left(4\pi ft\right)}{8\pi f}\right)\right]_0^{1/f} =$$
$$= R\left(I_{\mathrm{DC}}^2 + \frac{I_{\mathrm{AC}}^2}{2}\right) \doteq 35\,\mathrm{mW}\,.$$

The evaluation of the integral could be simplified using the fact that the integral of a sine through the whole period is zero and the integral of the square of a sine through an interval equal to a multiple of the half-period is one half of the interval length.

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Problem FoL.37 ... stopping a train 5 points

There is a train with mass $m = 500\,\mathrm{t}$ and speed $v = 100\,\mathrm{km{\cdot}h^{-1}}$ heading towards a superhero standing on the tracks. The train is at distance $s = 1\,300\,\mathrm{m}$ from the superhero, who wants the train to stop just in front of him. However, the only thing available to him is a laser pointer. What power must his laser have to achieve this feat? The front of the locomotive is a mirror.

Jarda thought up the plot for a new film.

Let's first calculate what the deceleration of the train must be if it stops just shy of the superhero. The velocity of the train is much smaller than the speed of light, so we can consider this to be movement under a constant (negative) acceleration and write

$$s = \frac{1}{2}at^2 = \frac{1}{2}vt\,,$$

i.e. the superhero needs to stop the train in $t = 2s/v$. This means that the train needs to decelerate at a rate of

$$a = \frac{v}{t} = \frac{v^2}{2s}\,.$$

The force that needs to act on the train is

$$F = ma = m\frac{v^2}{2s}\,.$$

Now for the laser. The momentum of a single photon can be expressed as $p = h/\lambda$, where h is Planck's constant and λ is the wavelength of the photon. The energy of the photon is $E = ch/\lambda = cp$, where c is the speed of light (in fact, the momentum is defined as E/c, which is a direct consequence of the formula for relativistic energy and the fact that a photon has no rest mass).

We therefore have a simple relationship between the momentum of photons traveling in the laser and the energy needed for the train to stop. It remains to find the momentum of photons that need to leave the laser per unit time. We can use the well known formula

$$F = \frac{\Delta p}{\Delta t}\,,$$

but here we need to take care. The problem statement says that there is a mirror at the front of the locomotive, i.e. the photons are not absorbed, but rather reflected. The change of momentum the photons undergo upon impact with the train is double the magnitude of their momentum when traveling. Symbolically,

$$m\frac{v^2}{2s} = 2\frac{\Delta p_{\mathrm{laser}}}{\Delta t}\,,$$

where we only substituted for the necessary force.

At the end, we only need to realize that power is energy per unit time

$$P = \frac{\Delta E_{\mathrm{laser}}}{\Delta t} = c\frac{\Delta p_{\mathrm{laser}}}{\Delta t} = cm\frac{v^2}{4s} \doteq 22.2\,\mathrm{TW}$$

We can note that even though lasers in use today can reach powers up to several petawatts, but they are only emitting pulses of light. For continuous lasers, the highest power used tends to be around tens of kilowatts.

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Problem FoL.38 ... solar cube — 5 points

Imagine that we have a satellite in the shape of a perfect cube with its surface covered by solar panels. We position it in such a way that it orbits a star with a circular trajectory much larger than the star. The satellite is able to rotate in any way we want. Find the ratio of the maximum to the minimum radiant power absorbed by the satellite. Assume that all incident radiation is absorbed. *Karel saw a solar cube on the Internet.*

At a long distance from the star, light rays are be parallel, i.e. the wavefronts are parallel planes perpendicular to the rays. The power absorbed by the satellite is directly proportional to the cross-sectional area of the satellite (from the point of view of the rays). Therefore, the solution of the problem is the ratio between the areas of maximal and minimal projections of a cube onto a plane.

Let $\mathbf{z}$ be a unit vector in the direction of the radiation. Then the area of the projection satisfies

$$S = \int_\omega |\mathbf{z}\cdot \mathrm{d}\mathbf{S}|\,,$$

where ω is the part of the surface of the cube which is exposed to the radiation and $\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S$, where $\mathbf{n}$ denotes the normal vector of an infinitesimal surface $\mathrm{d}S$. We want to calculate that integral for all possible positions of the cube with respect to $\mathbf{z}$, so we can choose the highest and lowest value.

Consider that no more than three faces of the cube can be exposed simultaneously. Also, all exposed faces must have one vertex in common. Let us denote these faces by A, B and C and their normal vectors by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

Furthermore, the dot product of $\mathbf{z}$ and a given normal (mentioned above) is zero for all faces which are not exposed to the radiation. We may claim that there are always exactly three faces exposed, but some of them have normals perpendicular to the rays and therefore do not add anything to the total surface. The set ω from the integral therefore represents the union of faces A, B and C. We obtain

$$S = \int_A |\mathbf{z}\cdot\mathbf{a}|\,\mathrm{d}S + \int_B |\mathbf{z}\cdot\mathbf{b}|\,\mathrm{d}S + \int_C |\mathbf{z}\cdot\mathbf{c}|\,\mathrm{d}S\,.$$

The expression $\mathbf{z}\cdot\mathbf{n}$ is constant on each face. For a cube with unit edges, we get

$$S = |\mathbf{z}\cdot\mathbf{a}| + |\mathbf{z}\cdot\mathbf{b}| + |\mathbf{z}\cdot\mathbf{c}|\ .$$

The vector of the radiation may be divided into three components – each in the direction of one coordinate vector – let us denote them by $\mathbf{z}_a$, $\mathbf{z}_b$ and $\mathbf{z}_c$ respectively. The expression for the surface area simplifies to

$$S = |\mathbf{z}_a|\,|\mathbf{a}| + |\mathbf{z}_b|\,|\mathbf{b}| + |\mathbf{z}_c|\,|\mathbf{c}| = |\mathbf{z}_a| + |\mathbf{z}_b| + |\mathbf{z}_c|\ ,$$

where we used the fact that normal vectors have unit length. The vector of radiation has unit length as well and assuming orthogonality of $\mathbf{z}_a$, $\mathbf{z}_b$ and $\mathbf{z}_c$ (which holds because of orthogonality of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$), we get

$$1 = |\mathbf{z}_a|^2 + |\mathbf{z}_b|^2 + |\mathbf{z}_c|^2\ .$$

Notice that the equation resembles the geometric mean of magnitudes of vectors $\mathbf{z}$. Similarly, the formula for calculation of the surface area can be understood as their arithmetic mean. It is possible to obtain the maximal estimate of the surface area from the inequality between the geometric and arithmetic mean, the minimum can be (with luck) estimated as well.

However, if we do not know the inequality yet, we must continue our work. Let $x = |\mathbf{z}_a|$ and $y = |\mathbf{z}_b|$. Plugging them both into the formula for area, we obtain

$$S = x + y + \sqrt{1 - x^2 - y^2}\,.$$

Let us examine extrema inside intervals of permissible values $x \in \langle 0,\, 1\rangle$, $y \in \left\langle 0,\, \sqrt{1-x^2}\right\rangle$. The partial derivatives of the function $S(x, y)$, which are

$$\frac{\partial S}{\partial x} = 1 - \frac{x}{\sqrt{1-x^2-y^2}}\,, \quad \frac{\partial S}{\partial y} = 1 - \frac{y}{\sqrt{1-x^2-y^2}}\,,$$

must both equal zero. Solving the system of equations results in $x = y$ and then a quadratic equation

$$3x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$$

giving the only solution. The resulting surface area is $S_1 = \sqrt{3}$.

The second option is to find extrema on the boundaries of the intervals. For $y = 0$ we have

$$S = x + \sqrt{1-x^2}\,,$$

which is a simple function whose maximum (on the permissible interval for x) is $S_2 = \sqrt{2}$ for $x = \frac{1}{\sqrt{2}}$ and whose minimum is $S_3 = 1$ for $x \in \{0,\, 1\}$.

For the second boundary condition $y = \sqrt{1 - x^2}$, we receive

$$S = x + \sqrt{1 - x^2}\,,$$

which was already examined before.

We have examined all possibilities and obtained all possible candidates for extrema. Since $S_1 > S_2$ we can claim that the maximum of the function $S(x, y)$ on the given interval is $\sqrt{3}$, while the minimum is 1. The desired solution is the ratio of the maximum to the minimum, which equals $\sqrt{3}$.

Finally, it should be pointed out that it is possible to find the solution much easier using intuition – it was enough to imagine a cross-section of the cube when cut by some plane. The minimal value corresponds to the situation when the wavefronts are parallel with one face – the maximal value is reached if the cross-section is a hexagon. However, these are only estimates, not mathematical proofs.

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Problem FoL.39 ... optimal direction — 6 points

We have a rigid body with mass $M = 1.0\,\mathrm{kg}$ lying on a horizontal surface. The coefficient of friction between the body and the surface below it is $f = 0.4$. What is the maximum acceleration which the body can reach if we exert a force $F = 5\,\mathrm{N}$ on it? *Lego was moving boxes.*

Assume that the force vector, together with vector in horizontal direction, form the angle φ. Therefore, the resulting net force acting on the body will be horizontal (since the vertical component is compensated by the surface below) and its magnitude will equal

$$F_\mathrm{r} = F\cos\varphi - f\left(Mg - F\sin\varphi\right)\,.$$

Our only freedom is in the choice of angle φ, so we find the derivative of F_v with respect to φ and put the result equal 0.

$$-F\sin\varphi + fF\cos\varphi = 0$$
$$f = \tan\varphi$$

We obtain $\varphi = \arctan f \doteq 0.38$. If we plug the angle into F_r and consider that $a_\mathrm{r} = F_\mathrm{r}/M$, we obtain $a_{\max} \doteq 1.5\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.40 ... watch out, Danka is throwing 6 points

Danka wants to throw a cricket ball on flat ground. Find the launch angle (measured with respect to the horizontal plane) which she should choose in order to throw the ball as far as possible. Danka is $h = 1.6\,\mathrm{m}$ tall and throws with a velocity $v = 4.5\,\mathrm{m{\cdot}s^{-1}}$.

Throwing a cricket ball has never been Danka's favourite sport.

Let us choose a coordinate system, where x denotes the horizontal axis and y is the vertical one. Let φ denote the angle under which Danka throws the ball, measured from the horizontal axis. The force, which acts on the ball, acts in $-y$ direction only, therefore the horizontal velocity $v_x = v\cos\varphi$ is constant. The horizontal distance satisfies

$$x = v\cos\varphi\; t\,.$$

The vertical position will change according to the formula for motion with constant acceleration giving

$$y = h + v\sin\varphi\; t - \frac{1}{2}gt^2\,.$$

By elimination of time from these two equations we obtain height y as a function of x

$$y = h + x\tan\varphi - \frac{gx^2}{2v^2\cos^2\varphi}\,.$$

We are interested in the moment when the ball hits the ground, which means $y = 0$. Let d denote the horizontal coordinate of the point of impact. We will use the formula $\cos^{-2}\varphi = \tan^2\varphi + 1$, plug it to the equation and obtain

$$0 = h + d\tan\varphi - \frac{gd^2}{2v^2} - \frac{gd^2\tan^2\varphi}{2v^2}\,.$$

This is quadratic equation not only in d, but also in $\tan\varphi$. Let us write down the variant where $\tan\varphi$ is the desired unknown variable

$$-\tan^2\varphi\frac{gd^2}{2v^2} + d\tan\varphi - \frac{gd^2}{2v^2} + h = 0\,.$$

Now a number of tiny and simple physics ideas. For most of possible values of d we will find two angles that lead to the distance given. If we increase d, the difference between these two angles approaches zero – for $d = d_{\max}$ there is only one angle which leads to such distance. In terms of mathematics, the discriminant equals zero. Let us calculate it from the quadratic equation for $\tan\varphi$ and put it equal 0. This leads to

$$d_{\max}^2 + 4\frac{gd_{\max}^2}{2v^2}\left(h - \frac{gd_{\max}^2}{2v^2}\right) = 0\,,$$

from which we express $d_{\max}$

$$d_{\max} = \frac{v}{g}\sqrt{2gh + v^2}\,.$$

Now we continue solving the quadratic equation for $\tan\varphi$, which has reduced to

$$\tan\varphi = \frac{v^2}{gd}\,.$$

Plugging in $d = d_{\max}$ we obtain

$$\tan\varphi = \frac{v}{\sqrt{2gh + v^2}} = 0.63\,.$$

The desired optimal angle is 32°. Consider that for $h > 0$ the optimal angle is always smaller than 45° and with increasing h it approaches zero.

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Problem FoL.41 ... laser at a dormitory — 4 points

Jáchym shines a laser from a dormitory window towards the ground. Matěj measures the radiation of the laser on the ground, $\Delta h = 42.0\,\mathrm{m}$ below Jáchym. The laser shines with a frequency f_0 but Matěj measures a different frequency f. What is the ratio $|f - f_0|\,/f_0$?

Karel repeatedly heard about a black hole being observed.

A photon with frequency f has energy $E = hf$, where h is the Planck's constant. As the photon moves down in the gravitational field, it loses its potential energy and $\Delta E = mg\Delta h$, where m is the "mass" of the photon. While it may seem strange to talk about mass of a photon (which has zero rest-mass), we know from relativity that $E = mc^2$ and therefore

$$m = \frac{hf}{c^2} \approx \frac{hf_0}{c^2}\,.$$

A change in energy therefore leads to a change in frequency. The change of potentional energy is

$$E - E_0 = \Delta E \quad \Rightarrow \quad h\left(f - f_0\right) = mg\Delta h \approx \frac{hg\Delta h}{c^2} f_0\,.$$

Finally we obtain the required ratio as

$$k = \frac{|f - f_0|}{f_0} \approx \frac{g\Delta h}{c^2} \doteq 4.58 \cdot 10^{-15}\,.$$

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Problem FoL.42 ... pyramid — 6 points

Imagine a wooden (density $\rho_\mathrm{w} = 600\,\mathrm{kg{\cdot}m^{-3}}$) symmetrical square-base pyramid floating on the surface of a calm lake. A part of the pyramid with height $h = 6.0\,\mathrm{cm}$ is under the water surface. We impact the pyramid in the vertical direction. Find the frequency of small oscillations of the pyramid. The density of the water is $\rho_\mathrm{l} = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$. Assume that these oscillations do not affect the height of water level in the lake. The pyramid has an ideal shape, it is homogeneous and pointing downwards (i.e. the apex is lower than the base).

Vítek wondered about global warming.

Firstly we will choose the coordinate system such that the origin is in the apex of the pyramid when the pyramid remains at rest. Next, before we make the impact, the system is in equilibrium, which means that buoyant force equals the weight. If s denotes the surface of a cross-section at the height h and S denotes the surface of the base at the height H (where H is the height of the whole pyramid) and if we consider that the weight of the pyramid equals $M = 1/3\rho_\mathrm{w}SH$, we get

$$sh\rho_\mathrm{l} = SH\rho_\mathrm{w}\,. \tag{2}$$

If we look at a vertical cross-section along the pyramid altitude, looking the angles between the vertical altitude and the horizontal base we get

$$\frac{h}{a} = \frac{H}{A}\,, \frac{s}{S} = \frac{h^2}{H^2}\,,$$

where A and a denote the length of the base at given heights. Using this relation and (2) we can find the equation

$$h^3\rho_\mathrm{l} = \rho_\mathrm{w}H^3\,.$$

The motion of the pyramid after that small vertical impact could be described using equation

$$M\ddot{z} = -\rho_\mathrm{l}g\,\mathrm{d}V = -\rho_\mathrm{l}gsz\,,$$

where $\mathrm{d}V \approx sz$ for sufficiently small z is infinitesimal volume difference of the part which is under water in addition (compared to the system in equilibrium). Plugging in for the mass we get

$$\frac{1}{3}\rho_\mathrm{w}SH\ddot{z} = -\rho_\mathrm{l}gsz\,,$$

$$\ddot{z} + \frac{3g\rho_\mathrm{l}h^2}{\rho_\mathrm{w}H^3}z = 0\,.$$

The equation above is the equation of a simple harmonic oscillator, therefore for the frequency of small oscillations, we can write

$$\Omega^2 = \frac{3g\rho_\mathrm{l}h^2}{\rho_\mathrm{w}H^3} = \frac{3g}{h}\,,$$

$$f = \frac{1}{2\pi}\sqrt{\frac{3g}{h}}\,.$$

Using numerical values from the problem statement, we obtain the desired frequency $f \doteq 3.52\,\mathrm{Hz}$.

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Problem FoL.43 ... pole dance

6 points

We have two coaxial cylinders – the smaller one has an outer radius $r = 1.0\,\mathrm{cm}$ and the bigger one has an inner radius $R = 10\,\mathrm{cm}$. The cylinders are connected by a rod positioned in the radial direction. On the rod, next to the surface of the smaller cylinder, there is a small bead. There is one more small ball on the surface of the smaller cylinder. We spin the whole system as a rigid body around the common axis with an angular velocity $\omega = 2.0\,\mathrm{rad\cdot s^{-1}}$ and then release the bead and the ball simultaneously. The bead can only move on the rod; the ball can move freely. What is the absolute difference between the times when the ball and the bead hit the inner surface of the larger cylinder? *Jarda got a bit flummoxed from this.*

We will calculate the motion of the ball (the one which is not on the rod) first. If we assume both cylinders rotate with the angular velocity ω, the ball will move with the velocity $r\omega$ in the direction tangential to the smaller cylinder. When we release the ball, it will not change its velocity and move in a straight line until it hits the larger cylinder. The question is how long is the straight line. We obtain its length using the Pythagorean theorem as $s_1 = \sqrt{R^2 - r^2}$. Therefore, the time between the release of the ball and the moment when it hits the larger cylinder is

$$t_1 = \frac{\sqrt{R^2 - r^2}}{r\omega} \doteq 5.0\,\mathrm{s}\,.$$

Let us investigate the bead on the rod now. Since it is on the rod, it moves with the constant angular velocity ω even after its release from the cylinder. We will be using the reference frame connected to the rotating cylinders (and most importantly, with the rod). In this reference frame, the bead will be acted on by the centrifugal force, which is given as $F_{\mathrm{cen}} = m\omega^2 r(t)$, where $r(t)$ denotes the position of the bead on the rod in time t. By dividing the formula by m we get the acceleration of the bead as a function of time t. Consider that the acceleration is the second derivative of $r(t)$, so we get the differential equation

$$\ddot{r}(t) = \omega^2 r(t)\,.$$

Similar equations are most easily solved using the so-called characteristic polynomial. This means that we assume that the solution is exponential function $r(t) = r_0 \mathrm{e}^{\lambda t}$. We plug it into the equation and get

$$\lambda^2 r_0 \mathrm{e}^{\lambda t} = \omega^2 r_0 \mathrm{e}^{\lambda t}\,.$$

First of all, consider that $r_0 = 0$ would satisfy the equation. This makes sense – if the bead's initial position was on the axis, there would not be any force to push it. However, its initial position is on the smaller cylinder and therefore such a solution is not interesting for us. We also know, that an exponential is never zero, therefore we can cancel it out. We cancel out r_0 as well. From the remaining equation, we obtain $\lambda = \pm\omega$. If we solve the differential equation of order n, we get n solutions. The next step is to write the solution as sum of individual solutions. In our case

$$r(t) = r_1 \mathrm{e}^{\omega t} + r_2 \mathrm{e}^{-\omega t}\,.$$

This is general solution of the motion of the bead on the rod rotating with the angular velocity ω. The remaining task is to investigate how will it move in our case. We know that the position at time $t = 0$ is r and the velocity at time $t = 0$ is zero. We get a system of equations

$$\begin{aligned} r &= r_1 + r_2 \\ 0 &= \omega r_1 - \omega r_2\,, \end{aligned}$$

which holds for $r_1 = r_2 = r/2$. Plugging this to the general solution, we get

$$r(t) = \frac{r}{2}\left(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t}\right) = r\cosh\omega t\,.$$

If we had not noticed the cosh, it would not have been a mistake, but then we would have to find the resulting time somehow else – using the substitution $x = \mathrm{e}^{\omega t}$ – we would get a quadratic equation or using a numerical calculation.

The remaining task is to find the time when the bead hits the larger cylinder. Mathematically said, for which t_2 the formula $R = r(t_2)$ holds? From the formula above we obtain

$$t_2 = \frac{1}{\omega}\operatorname{argcosh}\frac{R}{r} \doteq 1.5\,\mathrm{s}\,.$$

Subtracting the times we get the desired time difference between the impacts of the ball and the bead $\Delta t = t_1 - t_2 \doteq 3.5\,\mathrm{s}$.

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Problem FoL.44 ... a fall into the unknown 7 points

We have an infinite grounded conductive plane in free space without any gravitational fields. At a distance $d = 1.00\,\mathrm{m}$ from the plane, there is a small ball with mass $m = 2.00\,\mathrm{g}$ and electric charge with magnitude $q = 4.00\,\mathrm{\mu C}$. We release the ball. How long does it take for the charge to drop on the plane?

Jarda knew that it would fall eventually, but he wanted to know when exactly.

Firstly, we want to find the force that the plane exerts on the charge. A charge with the opposite sign and the same magnitude is electrostatically induced on the plane. In the equilibrium state, the charges do not move inside the plane, therefore no force is exerted on them, the potential on the plane is constant and since we are free to add a constant to potential, we choose the potential to be zero on the plane. The net potential is the sum of the potential φ_p due to charges of the plane and the potential φ_q due to the charge q. Therefore

$$\varphi_q + \varphi_\mathrm{p} = 0\,,$$
$$\varphi_\mathrm{p} = -\varphi_q\,.$$

The potential on the plane is the same as if there was an opposite charge $-q$ on the opposite side of the plane at the same distance d. Now we can substitute all induced charges on the plane by a single charge $-q$. The force that the charge q exerts on $-q$ is

$$F = \frac{1}{4\pi\varepsilon}\frac{q^2}{(d+d)^2} = \frac{1}{16\pi\varepsilon}\frac{q^2}{d^2}\,.$$

Because the charges have opposite signs, the charge q is attracted to the plane. This calculation is based on the method of image charges. We can see that the force is inversely proportional to the square of the distance from the plane to the charge. When we have utilized the plane as a mirror, why not utilise motion of planets too? Gravitational force is, similarly to our force F, proportional to r^{-2}. The motion of the charge, therefore, satisfies Kepler's laws of planetary motion with changed constants. Kepler's third law is

$$\frac{a^3}{T^2} = \frac{MG}{4\pi^2}\,.$$

We need to use our constant instead of a constant derived from Newton's gravitational law, the physical behaviour of the system remains unchanged otherwise. The equation becomes

$$\frac{a^3}{T^2} = \frac{q^2}{64\pi^3\varepsilon m}.$$

What is this equation good for? The charge's trajectory will not be an ellipse, but a straight line perpendicular to the plane. However, if we assume that it is an infinitely thin ellipse, with its foci almost in the original and final point of the trajectory, then such an ellipse approximates the straight line segment on which the charge moves. This ellipse has a semi-major axis a with length $\frac{d}{2}$. The period equals

$$T = \sqrt{\frac{64\pi^3 \varepsilon m \left(\frac{d}{2}\right)^3}{q^2}} = \sqrt{\frac{8\pi^3 \varepsilon m d^3}{q^2}}\,.$$

The move from the initial point to the plane lasts only half a period, therefore the time, which is the final solution to this problem, equals

$$t = \frac{T}{2} = \sqrt{\frac{2\pi^3 \varepsilon m d^3}{q^2}} \doteq 0.262\,\mathrm{s}\,.$$

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Problem FoL.45 ... spring of knowledge — 7 points

Jáchym owns a well of knowledge, which has depth $h = 32\,\mathrm{m}$ and a constant circular cross-section with radius $r = 1.5\,\mathrm{m}$. At the bottom of the well, in the middle, there is a point which radiates knowledge uniformly into the whole space around it with power P_0. Knowledge propagates in space similarly to light rays. The reflectance of the sides of the well is $k = 0.42$. The bottom of the well of knowledge does not reflect anything, it absorbs knowledge perfectly. Let P denote the total knowledge power radiated out of the well. Find the ratio P/P_0.

Wells are inexhaustible sources of ideas.

The problem has radial symmetry. Imagine one of the possible two-dimensional cross-sections – the well is projected into a rectangle with dimensions $2r \times h$, while the source of knowledge is in the middle of the bottom side. Let φ denote the angle between a knowledge ray and the vertical. Then all rays with angles from $\varphi_0 = 0$ to

$$\varphi_1 = \arctan \frac{r}{h}$$

exit the well without any reflection and therefore with their original intensity (power). In general, rays with the angle between φ_i and φ_{i+1}, where

$$\varphi_i = \arctan \frac{(2i-1)\,r}{h}\,,$$

reflect i-times in total (let us call them rays in the i-th zone). Their intensity is k^i times the original intensity.

In order to solve the problem, we need to find out what power corresponds to each zone. Imagine a ball with a radius R and with its center in the source of knowledge. The flux of knowledge is uniform on its surface and has the magnitude

$$I_0 = \frac{P_0}{4\pi R^2}\,,$$

which is the total power divided by the surface area. The power in each zone equals $I_0 S_i$, where S_i is the part of the ball's surface coresponding to the zone between angles φ_i and φ_{i+1}. We can calculate it as

$$S_i = \int_{\varphi_i}^{\varphi_{i+1}} 2\pi R \sin\varphi R \,\mathrm{d}\varphi = 2\pi R^2 \left[-\cos\varphi\right]_{\varphi_i}^{\varphi_{i+1}} = 2\pi R^2 \left(\cos\varphi_i - \cos\varphi_{i+1}\right) .$$

The total power leaving the well in the i-th zone is

$$p_i = k^i I_0 S_i = P_0 \frac{k^i}{2} \left(\cos\varphi_i - \cos\varphi_{i+1}\right) ,$$

while the desired ratio of powers is the sum

$$\eta = \frac{P}{P_0} = \sum_{i=0}^{\infty} \frac{p_i}{P_0} = \frac{1}{2}\left(1 - \frac{(1-k)}{k} \sum_{i=1}^{\infty} k^i \cos\varphi_i\right) .$$

Using the trigonometric identity

$$\cos\arctan x = \left(1 + x^2\right)^{-\frac{1}{2}}$$

we can modify the formula to

$$\eta = \frac{1}{2}\left(1 - \frac{(1-k)}{k} \sum_{i=1}^{\infty} k^i \left(1 + \left(\frac{(2i-1)\,r}{h}\right)^2\right)^{-\frac{1}{2}}\right) .$$

The sum must be calculated numerically; it is approximately 0.716. The ratio of power radiated out of the well to total radiated power is $\eta \doteq 5.53 \cdot 10^{-3}$.

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Problem FoL.46 ... an incredible space battle 8 points

Imagine us in the middle of a fight in space between civilization 1 and civilization 2. A battle cruiser of civilization 1 has just launched a rocket on a battle cruiser of civilization 2. The cruisers are $s = 5.00\,\mathrm{km}$ from each other and do not move with respect to each other. The properties of the rocket are: initial mass $m_0 = 5.00\,\mathrm{t}$, engine thrust $T = 1.50 \cdot 10^5\,\mathrm{N}$, specific impulse (exhaust velocity with respect to the rocket) $u = 3.00\,\mathrm{km{\cdot}s^{-1}}$. The engines are set to maximum thrust from launch till the moment of impact. Find the velocity of the rocket right before it crashes into the cruiser of civilization 2. *Jindra was watching Star Wars.*

The velocity of the rocket is described by the Tsiolkovsky equation

$$v(t) - v_0 = u \ln \frac{m_0}{m(t)} . \tag{3}$$

The initial velocity v_0 is zero and the rocket's mass depends on time according to the formula $m(t) = m_0 - Rt$, where R is the fuel mass flow rate

$$R = \frac{T}{u} .$$

Numerically, it is $R = 50.0\,\mathrm{kg{\cdot}s^{-1}}$. Substituting it into the equation (3), we get

$$v(t) = u \ln \frac{m_0}{m_0 - Rt} \,. \tag{4}$$

By integrating the equation (4) we obtain the formula for the distance covered at a given time. The initial condition is $s(t = 0) = 0$, therefore

$$s(t) = ut + ut \ln \frac{m_0}{m_0 - Rt} - \frac{m_0 u}{R} \ln \frac{m_0}{m_0 - Rt} \tag{5}$$

After numerically solving the equation (5), we get the time when the rocket reaches the distance $s = 5.00\,\mathrm{km}$. Let's plug it into the equation (4) and calculate the rocket's velocity at the point of collision. We get $t \doteq 17.7\,\mathrm{s}$ and $v \doteq 584\,\mathrm{m{\cdot}s^{-1}}$. We can tell that the warring civilizations are not very advanced, since their rockets move very slowly compared to the speed of light. Therefore, we could calculate the solution safely without using the special theory of relativity.

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Problem FoL.47 ... rotational pump 8 points

Vítek would like to get some water from his well, but he does not want to keep pulling the bucket up. Therefore, he gradually stirred the water around faster and faster until, at an angular velocity $\omega = 11\,\mathrm{rad{\cdot}s^{-1}}$, the water started flowing out of the well all by itself. Vítek knows the depth of the well (from the top edge to the ground at the bottom) $h = 47\,\mathrm{m}$. The well has a circular cross-section with a radius $r_0 = 1.6\,\mathrm{m}$. What was the height of the water column (from the bottom of the well to the water surface) before Vítek started spinning the water around? Jáchym heard that problems about wells were popular in Fyziklání 2020.

Let's introduce cylindrical coordinates in which we label the horizontal radial distance from the axis of the well as r, an angle of rotation with respect to the axis of the well as φ and the height above the bottom as z.

The forces acting on a small volume of rotating water with mass m and at distance r from the axis are centrifugal force $F_\mathrm{o} = m\omega^2 r$ and force of gravity $F_g = mg$, plus some buoyant hydrostatic forces. The surface is nothing else than a region with a constant potential energy. The potential energy due to the centrifugal force can be calculated as

$$E_\mathrm{c}(r) = \int_0^r -F_\mathrm{c}(x)\,\mathrm{d}x = -\int_0^r m\omega^2 x\,\mathrm{d}x = -\frac{1}{2}m\omega^2 r^2 \,.$$

The minus sign comes from the fact that we are integrating against the force which we would need to exert to counteract the centrifugal force. The potential energy due to the force of gravity is

$$E_g(z) = \int_0^z F_g\,\mathrm{d}x = mgz \,.$$

As we said, the water surface is a surface with constant potential. (There are also no buoyant forces on the surface.) For every point on the surface with coordinates r_s, z_s, the following holds

$$E_\mathrm{c}(r_\mathrm{s}) + E_g(z_\mathrm{s}) = \mathrm{const} \,,$$

From this condition, we obtain the height of the surface as a function of radial distance

$$z_\mathrm{s}(r) = \frac{\omega^2}{2g} r^2 + z_0 \,,$$

where z_0 is the height of the water surface in the center of the well. Of course, this only holds if the surface is always above the bottom of the well. If there wasn't enough water in the well, the parabola we obtained could intersect the bottom of the well, but for now, let's assume that this is not the case. We can later check whether this assumption holds.

The volume of the water does not change when it is spinning. THe initial volume was $V = \pi r_0^2 z_\mathrm{v}$, where z_v is the original height of the water column. After spinning, we have

$$V = \int_0^{r_0} \int_0^{2\pi} z_\mathrm{s}(r) r \; \mathrm{d}\varphi \, \mathrm{d}r = \int_0^{r_0} 2\pi z_\mathrm{s}(r) r \; \mathrm{d}r = 2\pi \int_0^{r_0} \left(\frac{\omega^2}{2g} r^3 + z_0 r \right) \, \mathrm{d}r = \pi \left(\frac{\omega^2}{4g} r_0^4 + z_0 r_0^2 \right) .$$

From this, we can express the height of water in the center

$$z_0 = z_\mathrm{w} - \frac{\omega^2}{4g} r_0^2 \,.$$

We know from the problem statement that at the given angular velocity, the water just started to flow out of the well. We can express that as $z_\mathrm{s}(r_0) = h$. From this condition, we get

$$h = \frac{\omega^2}{2g} r_0^2 + z_0 = \frac{\omega^2}{2g} r_0^2 + z_\mathrm{w} - \frac{\omega^2}{4g} r_0^2 \,.$$

The original height of the water in the well that we are looking for is then

$$z_\mathrm{w} = h - \frac{\omega^2}{4g} r_0^2 \doteq 39\,\mathrm{m} \,.$$

At the end, we just check that in this case $z_0 \doteq 31\,\mathrm{m}$, which means our assumption was correct.

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Problem FoL.48 ... ideal centrifuge 9 points

Imagine a tube filled with ideal gas with molar mass $M_m = 36\,\mathrm{g{\cdot}mol^{-1}}$. The length of the tube is $r_0 = 1.00\,\mathrm{m}$, its thickness is negligible compared to its length. We spin the tube around the axis perpendicular to the tube and passing through one of its bases, with an angular velocity $\omega = 451.00\,\mathrm{s^{-1}}$. The gas inside will settle in equilibrium, at a temperature $T = 300\,\mathrm{K}$ along with the tube. Find the distance between the axis of rotation and the centre of mass of the air in the tube. The tube is rotating in a horizontal plane. *Jáchym wanted to separate air.*

Let us choose a coordinate describing distance from the axis of rotation up to the end of the tube, i.e. from 0 to r_0, denoted by r. In some section $\mathrm{d}r$, we have gas with mass $\mathrm{d}m = \lambda(r)\,\mathrm{d}r$. When the equilibrium is reached, the temperature in the tube must be the same as at the beginning (since the tube is thermally isolated, the process must be adiabatic). From the equation of state, we express the pressure in a given section

$$p = \frac{\mathrm{d}nRT}{\mathrm{d}V} = \frac{\mathrm{d}nRT}{S\,\mathrm{d}r} = \frac{\mathrm{d}mRT}{M_\mathrm{m} S\,\mathrm{d}r} = \frac{RT}{M_\mathrm{m} S} \lambda \,,$$

where S is the cross-sectional area of the tube. From this formula, we get

$$\mathrm{d}p = \frac{RT}{M_\mathrm{m} S}\lambda' \,\mathrm{d}r$$

There must be a centripetal force

$$\mathrm{d}F_\mathrm{d} = \mathrm{d}m\omega^2 r\,,$$

exerted on a given section of the tube. This force must be caused by difference in pressure. Therefore $S\,\mathrm{d}p = \mathrm{d}F_\mathrm{d}$ holds. Putting it together, we get

$$\frac{RT}{M_\mathrm{m}}\lambda' \,\mathrm{d}r = \mathrm{d}m\omega^2 r\,,$$

which is a differential equation

$$\lambda' = \frac{\omega^2 M_\mathrm{m}}{RT}\lambda r = 2k^2\lambda r\,,$$

where k is a wisely-defined constant

$$k = \sqrt{\frac{\omega^2 M_\mathrm{m}}{2RT}} \doteq 1.212\,\mathrm{m}^{-1}\,.$$

The solution of the equation is

$$\lambda = A\mathrm{e}^{k^2 r^2}\,,$$

where A is a constant. We calculate the center of mass

$$T_r = \frac{1}{m}\int_0^{r_0} r\,\mathrm{d}m = \frac{\int_0^{r_0} r\lambda\,\mathrm{d}r}{\int_0^{r_0}\lambda\,\mathrm{d}r} = \frac{A\int_0^{r_0} r\mathrm{e}^{k^2r^2}\,\mathrm{d}r}{A\int_0^{r_0}\mathrm{e}^{k^2r^2}\,\mathrm{d}r}$$

and substituting $x = kr$, we get the result

$$T_r = \frac{1}{2k}\frac{\int_0^{kr_0} 2x\mathrm{e}^{x^2}\,\mathrm{d}x}{\int_0^{kr_0}\mathrm{e}^{x^2}\,\mathrm{d}x} = \frac{1}{2k}\frac{\left[\mathrm{e}^{x^2}\right]_0^{kr_0}}{\int_0^{kr_0}\mathrm{e}^{x^2}\,\mathrm{d}x} = \frac{1}{2k}\frac{\mathrm{e}^{k^2r_0^2}-1}{\int_0^{kr_0}\mathrm{e}^{x^2}\,\mathrm{d}x} = \frac{1}{k}\frac{\mathrm{e}^{k^2r_0^2}-1}{\sqrt{\pi}\,\mathrm{erfi}(kr_0)} \doteq 0.629\,\mathrm{m}\,,$$

where erfi is the imaginary error function.

Statistical solution

It is possible to solve the problem using statistical physics as well, but the ability to work with statistical ensembles is required.

The one-particle Hamiltonian is

$$H_1(X_1) = \frac{\mathbf{p}^2}{2m} - \boldsymbol{\omega}\cdot\mathbf{L} = \frac{\mathbf{p}^2}{2m} - \omega\mathbf{n}_3\cdot\mathbf{x}\times\mathbf{p} = \frac{\mathbf{p}^2 - 2m\omega\mathbf{p}\cdot\mathbf{n}_3\times\mathbf{x}}{2m} =$$
$$= \frac{(\mathbf{p}-m\omega\mathbf{n}_3\times\mathbf{x})^2}{2m} - \frac{m\omega^2}{2}(\mathbf{n}_3\times\mathbf{x})^2 = \frac{\mathbf{p}'^2}{2m} - \frac{m\omega^2}{2}r^2\,,$$

where X_1 is microstate of one particle, m is its mass and vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{p} = (p_1, p_2, p_3)$ represent its coordinates and momentum. The vector $\mathbf{n}_3$ is the unit vector in the direction of the third coordinate axis, which we choose to be identical with the axis of rotation. We also define a new coordinate $\mathbf{p}' = \mathbf{p} - m\omega \mathbf{n}_3 \times \mathbf{x}$, which is shifted momentum. As we will see later, the shift has no effect during integration. Compared to the Hamiltonian of a free particle, there is also a potential term which decreases with increasing distance from the axis. This potential represents the centrifugal force exerted on the particles in a corotating system. The variable r is the distance from the axis of rotation and it satisfies $r^2 = x_1^2 + x_2^2$.

Since this is a system with constant temperature, we describe it using a canonical statistical ensemble. We divide the tube into small layers with thickness Δr, perpendicular to the axis of the tube. We will focus on one layer, which is at a distance r from the axis of rotation. The one-particle partition function is

$$Z_1(r) = \frac{1}{h^3}\int \mathrm{e}^{-\beta H_1(X_1)}\,\mathrm{d}X_1 = \frac{1}{h^3}\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\beta p'^2}{2m}}\,\mathrm{d}p'\right)^3 S\Delta r \mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}} = \frac{V}{\kappa^3}\mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}}\,,$$

where the integral is over all possible states X_1 of one particle, $\beta = \frac{1}{k_\mathrm{B}T}$ and h is Planck's constant. We used an auxiliary constant

$$\kappa = h\sqrt{\frac{2\pi m}{\beta}}$$

and denoted the volume of a layer by $V = S\Delta r$. Since we are working with an ideal gas, the particles do not interact with each other and therefore we can write the total partition function as

$$Z(r) = \frac{Z_1(r)^{N(r)}}{N(r)!}\,,$$

where $N(r)$ is the number of particles in the given layer. From statistical physics, we know that the result is connected to Helmholtz free energy $F = -\frac{1}{\beta}\ln Z$ and we also know that the chemical potential $\mu(r)$ can be calculated as the derivative of F with respect to the number of particles N. We can write

$$\mu(r) = \frac{\partial F}{\partial N} = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial N} = -\frac{1}{\beta}\left(\ln\frac{\mathrm{e}Z_1}{N} - \frac{\mathrm{e}Z_1}{N}\right) \approx \frac{1}{\beta}\ln\frac{N}{\mathrm{e}Z_1} = \frac{1}{\beta}\ln\left(\frac{N\kappa^3}{\mathrm{e}V}\mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}}\right)\,,$$

where we used the approximation

$$N! \approx \left(\frac{N}{\mathrm{e}}\right)^N$$

and neglected the term $\frac{\mathrm{e}Z_1}{N}$, because it is zero in the thermodynamic limit. Now, we use the fact that the equilibrium between individual layers requires $\mu(r) = \mu_0$ to be constant. This means that the whole formula inside the logarithm must be constant as well. If we denote the volume density of particles $n(r) = \frac{N}{V}$, we obtain

$$\frac{N\kappa^3}{\mathrm{e}V}\mathrm{e}^{-\frac{\beta m\omega^2 r^2}{2}} = \mathrm{const}\,,$$

$$n(r) = n_0\mathrm{e}^{\frac{\beta m\omega^2 r^2}{2}}\,,$$

where n_0 is the volume density of particles at the axis of rotation.

The remaining task is to calculate the position of the center of mass

$$T_r = \frac{\int_0^{r_0} rn(r)\,\mathrm{d}r}{\int_0^{r_0} n(r)\,\mathrm{d}r} = \frac{\int_0^{r_0} r\mathrm{e}^{k^2r^2}\,\mathrm{d}r}{\int_0^{r_0} \mathrm{e}^{k^2r^2}\,\mathrm{d}r}\,,$$

where $k = \sqrt{\frac{\beta m\omega^2}{2}}$. We obtain the same result as in the case above.

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Problem FoL.49 ... falling impulse 9 points

There is a thread with a length $l = 1.00\,\mathrm{m}$ and a total mass $m_\mathrm{v} = 1.00\,\mathrm{g}$ hanging from the ceiling. On its bottom end, a small weight with unknown mass m is hung. We send an impulse down the thread from the top and simultaneously drop a point mass from the ceiling. We notice two things:

1. *the impulse is traveling along the thread with a constant speed,*
2. *the impulse reaches the bottom end of the thread at the same time as the point mass.*

What is the mass of the little weight? Everything is taking place in homogeneous gravity with acceleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Jirka was unable to figure out how this idea struck his mind.*

Let's use x as a coordinate of distance measured from the top of the thread. The bottom end of the thread is at $x = l$. The impulse is traveling along the thread with speed

$$v(x) = \sqrt{\frac{F(x)}{\lambda(x)}} = \mathrm{const},$$

where $F(x)$ is the force of tension within the thread at a distance x and $\lambda(x)$ is the linear density of the thread at the same point. We know that the speed of the impulse is constant along the whole length of the thread. We can find the force $F(x)$ by integrating the infinitesimal contributions from pieces of thread hanging between x and l

$$F(x) = \left(m + \int_x^l \lambda(t)\,\mathrm{d}t\right) g\,.$$

We express the force from the first equation and substitute into the second one

$$\frac{v^2}{g}\lambda(x) = m + \int_x^l \lambda(t)\,\mathrm{d}t = m + L(l) - L(x)\,, \tag{6}$$

where L is the indefinite integral of λ. We differentiate this equation by x and obtain the differential equation

$$\frac{v^2}{g}\lambda' = -\lambda\,,$$

solvable by the method of separation of variables. We get

$$\lambda = \lambda_0 \mathrm{e}^{-\frac{g}{v^2}x}\,,$$

where λ_0 is an unknown constant which we need to infer from boundary conditions. We can see that the linear density of the thread decreases exponentially with length. The total mass of the thread is

$$m_\mathrm{t} = \int_0^l \lambda_0 \mathrm{e}^{-\frac{g}{v^2}t}\,\mathrm{d}t = \frac{\lambda_0 v^2}{g}\left(1 - \mathrm{e}^{-\frac{gl}{v^2}}\right),$$
$$\lambda_0 = \frac{m_\mathrm{t} g}{v^2\left(1 - \mathrm{e}^{-\frac{gl}{v^2}}\right)}.$$

In the equation (6), we set $x = 0$ and express the mass of the little weight

$$m = \frac{v^2}{g}\lambda_0 - \int_0^l \lambda\,\mathrm{d}x\,.$$

We've already calculated this integral once, the result is m_t. We substitute for λ_0 and get

$$m = \frac{v^2}{g}\cdot\frac{m_\mathrm{t} g}{v^2\left(1 - \mathrm{e}^{-\frac{gl}{v^2}}\right)} - m_\mathrm{t} = m_\mathrm{t}\left(\frac{1}{1 - \mathrm{e}^{-\frac{gl}{v^2}}} - 1\right) = \frac{m_\mathrm{t}}{\mathrm{e}^{\frac{gl}{v^2}} - 1}\,.$$

Now we can use the information that the impulse reaches the end in the same time t as the point mass falling from height l. We have

$$l = vt\,,$$
$$l = \frac{1}{2}gt^2\,,$$

but the velocity v is not the final velocity of the point mass, but the speed of the impulse, i.e. the mean velocity of the point mass during fall. From that follows

$$v^2 = \frac{gl}{2}\,.$$

Putting everything together, we get

$$m = \frac{m_\mathrm{t}}{\mathrm{e}^2 - 1} \doteq 0.157\,\mathrm{g}.$$

The mass of the little weight is roughly 0.157 g.

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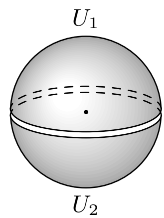
Problem FoL.50 ... a problem with potential 3 points

Lego took two hemispherical shells with the same radii and connected them in a non-conducting way such that together they formed a sphere. Then he set a potential $U_1 = 100\,\mathrm{V}$ on one hemisphere and a potential $U_2 = -100\,\mathrm{V}$ on the other hemisphere. What is the potential in the middle of the sphere?

Lego knew that he has a potential. . .



The problem can be solved using Laplace's equation, of course. However, let's wait with that for now…

Let's denote the desired potential in the center of the sphere by φ. If we exchange U_1 and U_2, the signs of all potentials given in the problem change. Therefore, the signs of all potentials in the solution change as well and φ changes to $-\varphi$ (we can also look at it as if there were aliens who define the signs of electric charge in the exactly opposite way; this would be the problem they would need to solve).

However, if we exchange the potentials on the spherical shell, it is also the same as if we only turned the sphere over (or if we looked at it from the opposite side). And that cannot have any impact on the potential in its center. We get the equation $\varphi = -\varphi$, which has only one solution, 0.

Therefore, the potential in the center of the sphere is $\varphi = 0\,\mathrm{V}$.

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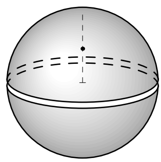
Problem FoL.51 … another problem with potential 9 points

Lego took two hemispherical shells with the same radii and connected them in a non-conducting way such that together they formed a sphere. Then he set a potential $U_1 = 100\,\mathrm{V}$ on one hemisphere and a potential $U_2 = -100\,\mathrm{V}$ on the other hemisphere. Find the potential in the middle of a straight line connecting the centre of the sphere and the apex of the positively charged hemisphere (depicted in the figure)?

… and he also thought that it is greater than 0.



Numerical solution

The electric potential everywhere except for the regions with non-zero charge (therefore everywhere inside of the sphere) satisfies Laplace's equation

$$\Delta\Phi = 0\,.$$

For some basic idea, it is enough to realise that the Laplace operator is a sum of second derivatives in all (three) directions. In theory, we could discretize the sphere as a 3D grid. Then, using differentiation of the Laplace equation, we would obtain that the potential at each point is equal to the arithmetic mean of potentials at the neighbouring 6 points, and could let the computer iterate the given formula (with fixed potential on the border) until it converges. However, to obtain the result with the required precision, the calculation would be very time-consuming (without an external server, it could even last longer than our competition). Therefore, as usual in electrostatics problems, we use symmetry. The Laplace operator in spherical coordinates equals

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\Phi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\,.$$

If we choose the z-axis as identical with the symmetry axis of the problem, the potential does not depend on the φ coordinate (written as $\Phi = \Phi(r,\theta)$), i.e. the third term is zero. Most importantly, we can then discretize the problem using two variables only. Therefore, we have a 2D grid where each point is described with two indices i, j and its potential $\Phi(r, \theta) = \Phi(ih_r, jh_\theta) = f(i, j)$, where we defined a function f for clarity; h_r, h_θ are constants denoting the length of one step in the direction of r or θ. If we index our points starting from 0, then $h_r = R/i_{\max}$ and $h_\theta = \pi/2j_{\max}$, where R is the radius of the sphere, $i_{\max}$ is the largest index i and $j_{\max}$ is the largest index j, because we know from the solution of the previous problem that the potential in the plane connecting both hemispheres is everywhere equal to 0. Then we only need to find the potential inside the hemisphere which is interesting for us, i.e. where $\theta \in \langle 0, \pi/2 \rangle$.

In the direction of increasing r, we have discrete points with spacing h_r. The derivative of fr at a point i, j is

$$[f(i,j)r(i)]' = \frac{f(i+1,j)r(i+1) - f(i,j)r(i)}{h_r}\,,$$

where we used only the definition of a derivative and omitted the detail that h shall be infinitely small. To be precise, we have written the so-called forward derivative, which somewhat describes the derivative of f in the region between i and $i+1$. It is useful to keep this in mind. The second derivative is the derivative of the derivative of f. Let us think about it for a while. Instead of using two forward derivatives, which would mean comparing the derivative between i and $i+1$ with the one between $i+1$ and $i+2$ (this does not seem to be the correct way to express the second derivative in the point i), we use the backward derivative (symbolically described as $f'(i) = (f(i) - f(i-1))/h$, which is, of course, only another first derivative) instead. Next, we plug in $r(i) = ih_r$ and calculate

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(fr) = \frac{1}{r(i)}\frac{[f(i,j)r(i)]' - [(f(i-1,j)r(i-1)]'}{h_r} = \\ = \frac{f(i+1,j)(i+1) - 2f(i,j) + f(i-1,j)(i-1)}{ih_r^2}\,.$$

Similarly (with slightly more difficulty), we differentiate the second term

$$\frac{1}{r^2(i)\sin\theta(j)}\frac{\partial}{\partial\theta}\left(\sin\theta(j)\frac{\partial f}{\partial\theta(j)}\right) = \\ = \frac{\sin\left[\left(j+\frac{1}{2}\right)h_\theta\right](f(i,j+1) - f(i,j)) - \sin\left[\left(j-\frac{1}{2}\right)h_\theta\right][f(i,j) - f(i,j-1)]}{\sin(jh_\theta)\, i^2h_r^2h_\theta^2}\,.$$

The expression $\sin\left[\left(j+\frac{1}{2}\right)h_\theta\right]$ may look quite weird, but it results from our desire to get the value of the second derivative "exactly" between j and $j+1$.

The remaining task is to use the fact that the sum of these two terms equals 0 and express

$$f(i,j) = \frac{\sin(jh_\theta)\, ih_\theta^2\left[f(i-1,j)(i-1) + f(i+1,j)(i+1)\right]}{2i^2h_\theta^2\sin(jh_\theta) + \sin\left[\left(j+\frac{1}{2}\right)h_\theta\right] + \sin\left[\left(j-\frac{1}{2}\right)h_\theta\right]} + \\ + \frac{f(i,j-1)\sin\left[\left(j-\frac{1}{2}\right)h_\theta\right] + \sin\left[\left(j+\frac{1}{2}\right)h_\theta\right]f(i,j+1)}{2i^2h_\theta^2\sin(jh_\theta) + \sin\left[\left(j+\frac{1}{2}\right)h_\theta\right] + \sin\left[\left(j-\frac{1}{2}\right)h_\theta\right]}\,.$$

This holds for the potential at all points except these with non-zero charge. Such points are on the hemispheres. However, the potential on the hemispheres is given by the task, so these are our boundary conditions.

We discretize the space into a 2D grid, pass through all its points and calculate the formula given above for each of them. We repeat this over and over again until $f(i,j)$ converges. The code in Python is attached.

```
import numpy as np
Fi=np.zeros((101,101))
Fi2=np.zeros((101,101))
h=np.pi/200
b=True
eps=10**(-3)
F=100
while b:
   b=False
   for i in range(101):
     for j in range(101):
       if (i==100):
         Fi2[i,j]=F
       elif (i==0):
         Fi2[i,j]=0
       elif (j==100):
         Fi2[i,j]=0
       elif (j==0):
         Fi2[i,j]=(np.sin(j*h)*i*h*h*(Fi[i-1,j]*(i-1)+Fi[i+1,j]*(i+1))
+2*np.sin((j+0.5)*h)*Fi[i,j+1])/(2*i*i*h*h*np.sin(j*h)+2*np.sin((j+0.5)*h))
         if abs(Fi2[i,j]-Fi[i,j])>eps:
           b=True
       else:
         Fi2[i,j]=(np.sin(j*h)*i*h*h*(Fi[i-1,j]*(i-1)+Fi[i+1,j]*(i+1))+
Fi[i,j-1]*np.sin((j-0.5)*h)+np.sin((j+0.5)*h)*Fi[i,j+1])/
(2*i*i*h*h*np.sin(j*h)+np.sin((j+0.5)*h)+np.sin((j-0.5)*h))
         if abs(Fi2[i,j]-Fi[i,j])>eps:
           b=True
   for x in range(101):
     for y in range(101):
       Fi[x,y]=Fi2[x,y]
print(Fi[50,100])
```

The result after rounding is 66 V.

Analytical solution

Let us use the general solution of Laplace's equation $\Delta\Phi = 0$ in spherical coordinates

$$\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{ml}r^{l} + B_{lm}r^{-l-1}\right)Y_{lm}(\theta,\varphi)\,,$$

where $Y_{lm}(\theta, \varphi)$ are spherical harmonics and A_{ml} and B_{ml} are constants determined from the initial conditions. We will choose the coordinates in such a way that the ray $\theta = 0$ intersects the apex of one hemisphere. In such a case, the coordinate φ characterizes the azimuthal symmetry of the problem and therefore the potential is independent from it, which means that only components with $m = 0$ remain in the sum and the solution with azimuthal symmetry satisfies

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta) ,$$

where $P_l(x)$ are Legendre polynomials. We look for potential inside the ball with radius R, which does not diverge in the centre (for $r = 0$), therefore $B_l = 0$ for all l. The problem has simplified into

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) . \tag{7}$$

The potential V is specified on the sphere $r = R$, which leads to

$$V(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \begin{cases} U_0 & \theta \in \left\langle 0, \frac{\pi}{2}\right) \\ -U_0 & \theta \in \left(\frac{\pi}{2}, \pi\right\rangle \end{cases} . \tag{8}$$

Our objective is to find constants A_l such that the equation holds for each $\theta \in \langle 0, \pi\rangle$. We use orthogonality of Legendre polynomials given by equation

$$\int_{-1}^{1} P_l(x) P_k(x)\, \mathrm{d}x = \frac{2}{2l+1} \delta_{lk} .$$

We multiply the equation (8) by $P_k(\cos\theta)$ and integrate, which gives

$$\begin{aligned} \frac{2A_l R^l}{2l+1} &= \int_{-1}^{1} V(\theta) P_l(\cos\theta)\, \mathrm{d}\cos\theta , \\ A_l &= \frac{2l+1}{2R^l} \int_{0}^{\pi} V(\theta) P_l(\cos\theta) \sin\theta\, \mathrm{d}\theta . \end{aligned} \tag{9}$$

In the next step, we calculate the integral

$$\begin{aligned} \int_0^{\pi} V(\theta) P_l(\cos\theta)\sin\theta\, \mathrm{d}\theta &= \int_{-1}^{0} (-U_0)\, P_l(x)\, \mathrm{d}x + \int_0^1 U_0 P_l(x)\, \mathrm{d}x = \\ = U_0 \left(-\int_0^1 P_l(-x)\, \mathrm{d}x + \int_0^1 P_l(x)\, \mathrm{d}x\right) &= U_0 \left(1 - (-1)^l\right) \int_0^1 P_l(x)\, \mathrm{d}x . \end{aligned} \tag{10}$$

For even l, $A_l = 0$ holds, and for odd l we need to calculate the last integral. To do this, we use several properties of Legendre polynomials, namely the knowledge of the values on the borders of the interval of integration $P_l(1) = 1$, $P_l(0) = \frac{(-1)^{\frac{l}{2}}}{2^l} \binom{l}{\frac{l}{2}}$ for even l and the integral formula for Legendre polynomials

$$\int P_l(x) = \frac{P_{l+1}(x) - P_{l-1}(x)}{2l+1} .$$

Using these properties we get

$$\begin{aligned}\int_0^1 P_l(x)\,\mathrm{d}x &= \frac{P_{l+1}(1)-P_{l-1}(1)}{2l+1} - \frac{P_{l+1}(0)-P_{l-1}(0)}{2l+1} = \\ &= -\frac{1}{2l+1}\left(\frac{(-1)^{\frac{l+1}{2}}}{2^{l+1}}\binom{l+1}{\frac{l+1}{2}} - \frac{(-1)^{\frac{l-1}{2}}}{2^{l-1}}\binom{l-1}{\frac{l-1}{2}}\right) = \\ &= \frac{(-1)^{\frac{l-1}{2}}}{(2l+1)2^{l-1}}\left(\frac{1}{2^2}\binom{l+1}{\frac{l+1}{2}} + \binom{l-1}{\frac{l-1}{2}}\right) = \\ &= \frac{(-1)^{\frac{l-1}{2}}}{2^{l+1}}\frac{(l-1)(l+1)(l-2)!}{\left(\left(\frac{l+1}{2}\right)!\right)^2} = \frac{(-1)^{\frac{l-1}{2}}}{2^{l+1}l}\binom{l+1}{\frac{l+1}{2}}.\end{aligned}$$

We plug this result into (10) and then into (9) and (7), which lets us explicitly express the potential inside the sphere using Legendre polynomials as

$$\begin{aligned}\Phi(r,\theta) &= \sum_{l=0}^{\infty}\frac{2l+1}{2R^l}U_0\left(1-(-1)^l\right)\frac{(-1)^{\frac{l-1}{2}}}{2^{l+1}l}\binom{l+1}{\frac{l+1}{2}}r^l P_l(\cos\theta) \\ &= U_0\sum_{l\,\mathrm{odd}}\frac{(-1)^{\frac{l-1}{2}}(2l+1)}{2^{l+1}l}\binom{l+1}{\frac{l+1}{2}}\frac{r^l}{R^l}P_l(\cos\theta) \\ &= U_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+2}}\frac{4n+3}{2n+1}\binom{2n+2}{n+1}\frac{r^{2n+1}}{R^{2n+1}}P_{2n+1}(\cos\theta) \\ &= U_0\sum_{n=0}^{\infty}\frac{(-1)^n(4n+3)}{2^{2n+1}(n+1)}\binom{2n}{n}\frac{r^{2n+1}}{R^{2n+1}}P_{2n+1}(\cos\theta).\end{aligned}$$

Now we plug in the coordinates of the point where we want to calculate the potential, which are $\theta = 0$, $r = \frac{R}{2}$, and get

$$\begin{aligned}\Phi\left(\frac{R}{2},0\right) &= U_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}}\frac{4n+3}{n+1}\binom{2n}{n}\frac{1}{2^{2n+1}} \\ &= U_0\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n+1}}\frac{4n+3}{n+1}\binom{2n}{n} \\ &= U_0\left(2-\frac{3}{\sqrt{5}}\right) \doteq 0.658\,U_0.\end{aligned}$$

Halfway from the centre to the positively charged apex of the hemisphere, the potential is $\Phi \doteq 66\,\mathrm{V}$. It is enough to evaluate the first three terms of the series to obtain the result with sufficient precision, i.e. the first two significant figures.

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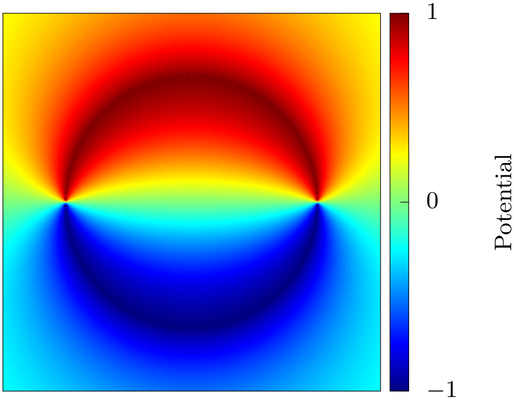

Fig. 1: Visualization of the potential between the hemispheres (normalized).

Problem M.1 ... at the train station

3 points

Legolas was trying to catch a train. He was running with a speed $v = 3.0\,\mathrm{m\cdot s^{-1}}$ when he noticed that one of his shoelaces was untied. Therefore, he stopped and spent $t = 10.0\,\mathrm{s}$ tying it. He then jumped on an escalator, which was moving with a speed $u = 1.0\,\mathrm{m\cdot s^{-1}}$ (on which he ran with speed v as well). Suddenly, he facepalmed, realizing he could have made one more step, tied the shoelace on the escalator and saved some time. But how much exactly?

You do not need to know the exact length of the escalator – it is sufficient to know that it was long enough for Legolas to manage to tie his shoelace on it. *Lego missed the train.*

We're interested in the difference of time between these two scenarios. In the first one, Legolas stops moving for a time t and then travels on the escalator with velocity $u + v$. In the second scenario, he first travels on the escalator for the time t with velocity u and then he traverses the rest of the escalator with velocity $u + v$ (the problem statement clarifies that there is some distance left).

The velocity of traversing the second part is the same in both scenarios. This means we are only interested in the time difference after traversing the first part of the escalator, which is the distance Legolas covered while tying his shoelaces in the second scenario. The distance of this point from the entrance to the escalator is $l = ut$.

In the second scenario, he reaches this point in time t. In the first one, he's first stationary for time t and then covers the distance l in time

$$t_l = \frac{l}{u+v} = t\frac{u}{u+v}\,.$$

Hence, the time difference is

$$\Delta t = (t + t_l) - t = t_l = t\frac{u}{u+v} = 2.5\,\mathrm{s}\,.$$

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Problem M.2 ... at the airport — 3 points

Dano and Danka are in a hurry to catch a plane to Mexico, so they both run on a moving walkway (in the direction of its movement). Both of them have the same step length, which is exactly 1 m. *Dano runs twice as fast as Danka and he has to take 28 steps to run from one end of the walkway to the other end, while Danka needs to take just 21 steps. How long is the walkway?* *Legolas wanted to catch a plane.*

We denote the velocity of the walkway by v and the frequency of Danka's steps by f (the step length is the same, so Dano's frequency is $2f$). From Dano's point of view, the length of the walkway is $s_{\mathrm{Dano}} + s_{\mathrm{ww}}$, where s_{Dano} is the distance Dano walked on the walkway, that is $s_{\mathrm{Dano}} = 28\,\mathrm{m}$, and s_{ww} is the distance the walkway traveled while Dano was standing on it. Dano spent the time $28/(2f)$ on the belt, so the length of the walkway is $s_{\mathrm{Dano}} + 14v/f$. We can do the same calculation for Danka, to obtain the length of the walkway as $s_{\mathrm{Danka}} + 21v/f$, where $s_{\mathrm{Danka}} = 21\,\mathrm{m}$. Comparing these two relationships, we obtain $v/f = 1\,\mathrm{m}$, and substituting into (either of) the formulas for the length of the walkway, we determine the length of the walkway as $42.0\,\mathrm{m}$.

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Problem M.3 ... under the ground — 3 points

Matěj likes to travel by metro. One day, he got off at a station which was $h = 50\,\mathrm{m}$ *deep under the surface. He walked upwards with a speed* $v = 1\,\mathrm{m\cdot s^{-1}}$ *on an escalator, which itself moves upwards with a speed* $u = 1.5\,\mathrm{m\cdot s^{-1}}$. *What work has Matěj done by walking upwards? Assume that his mass is* $m = 60\,\mathrm{kg}$. *Matěj likes escalators and the metro.*

The total work done was $W = mgh$. Part of it was done by Matěj by walking upwards, part was done by the escalator. Because the work is directly proportional to the traveled height h, it will be split in the same ratio as the heights covered by each of them. The time of motion is the same for both Matěj and the escalator and they move in the same direction, so the height is directly proportional to the speeds of movement. The height covered by the effort of the person is

$$h_{\mathrm{M}} = h\frac{v}{v+u}\,,$$

whilst the escalator lifts him by the height

$$h_{\mathrm{e}} = h\frac{u}{v+u}\,.$$

From this we can calculate the work done by Matěj as

$$W_{\mathrm{M}} = mgh_{\mathrm{M}} = mgh\frac{v}{v+u} \doteq 11\,800\,\mathrm{J}\,.$$

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Problem M.4 ... in the shopping mall 3 points

You have traveled by train, by metro and by plane to visit a shopping mall, which is, of course, equipped with many escalators. On a switched-on escalator moving upwards, the time it takes you to walk up and down again is $t_{\mathrm{on}} = 50\,\mathrm{s}$. On a switched-off escalator, it takes $t_{\mathrm{off}} = 30\,\mathrm{s}$. What is the speed of the stairs of a switched-on escalator, if you walk with a speed $v_{\mathrm{u}} = 1.0\,\mathrm{m{\cdot}s^{-1}}$ upwards and $v_{\mathrm{d}} = 2.0\,\mathrm{m{\cdot}s^{-1}}$ downwards?

Legolas runs up and down escalators in metro stations.

We can find the length of the escalator from the case when it is switched off as

$$t_{\mathrm{off}} = \frac{l}{v_{\mathrm{u}}} + \frac{l}{v_{\mathrm{d}}} \quad \Rightarrow \quad l = \frac{t_{\mathrm{off}} v_{\mathrm{u}} v_{\mathrm{d}}}{v_{\mathrm{u}} + v_{\mathrm{d}}}\,.$$

For the switched-on escalator, we label its speed as u and we have

$$t_{\mathrm{on}} = \frac{l}{v_{\mathrm{u}} + u} + \frac{l}{v_{\mathrm{d}} - u} \quad \Rightarrow \quad l = \frac{t_{\mathrm{on}}}{v_{\mathrm{u}} + v_{\mathrm{d}}} \left(v_{\mathrm{u}} + u\right)\left(v_{\mathrm{d}} - u\right)\,.$$

We can therefore construct an equation

$$\begin{aligned}
\frac{t_{\mathrm{on}}}{v_{\mathrm{u}} + v_{\mathrm{d}}} \left(v_{\mathrm{u}} + u\right)\left(v_{\mathrm{d}} - u\right) &= \frac{t_{\mathrm{off}} v_{\mathrm{u}} v_{\mathrm{d}}}{v_{\mathrm{u}} + v_{\mathrm{d}}}\,,\\
v_{\mathrm{u}} v_{\mathrm{d}} + u\left(v_{\mathrm{d}} - v_{\mathrm{u}}\right) - u^2 &= \frac{t_{\mathrm{off}} v_{\mathrm{u}} v_{\mathrm{d}}}{t_{\mathrm{on}}}\,,\\
u^2 - \left(v_{\mathrm{d}} - v_{\mathrm{u}}\right) u + \left(\frac{t_{\mathrm{off}}}{t_{\mathrm{on}}} - 1\right) v_{\mathrm{u}} v_{\mathrm{d}} &= 0\,.
\end{aligned}$$

We've obtained a quadratic formula for u, the solutions of which are

$$u = \frac{v_{\mathrm{d}} - v_{\mathrm{u}} \pm \sqrt{(v_{\mathrm{d}} + v_{\mathrm{u}})^2 - 4 v_{\mathrm{d}} v_{\mathrm{u}} \frac{t_{\mathrm{off}}}{t_{\mathrm{on}}}}}{2}\,,$$

and we are interested in the solution with the plus sign (the one with the minus sign is negative), which is approximately $u \doteq 1.5\,\mathrm{m{\cdot}s^{-1}}$.

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Problem E.1 ... statistical 1 - intro 3 points

In statistical physics, we can describe a system of particles enclosed in an unchanging volume. We call this a cannonical system and the probability that we find the system in a certain state n follows Boltzmann's Law

$$p_n = \frac{1}{Z} \mathrm{e}^{-\frac{E_n}{k_\mathrm{B} T}} \,,$$

where E_n is the energy of the system in the state n, T is the temperature of the system and $k_\mathrm{B} = 1.381 \cdot 10^{-23}\,\mathrm{m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}}$ is the Boltzmann constant. The value of Z is chosen in such a way that the sum of p_n over all the states equals one. Typically, there is a large number of states a system can reside in, but in this simple problem, let's work with a system with only three distinguishable states ($n = 1, 2, 3$) with energies $E_1 = 1.00 \cdot 10^{-20}\,\mathrm{J}$, $E_2 = 2E_1$ and $E_3 = 3E_1$. Determine the value of Z at a temperature $T_0 = 275\,\mathrm{K}$.

Matěj missed statistical physics among FYKOS problems.

Let's start with the fact that the sum of all probabilities is equal to 1

$$1 = p_1 + p_2 + p_3 = \frac{1}{Z} \left(\mathrm{e}^{-\frac{E_1}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_2}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_3}{k_\mathrm{B} T}} \right) \,,$$

$$Z = \mathrm{e}^{-\frac{E_1}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_2}{k_\mathrm{B} T}} + \mathrm{e}^{-\frac{E_3}{k_\mathrm{B} T}} \doteq 0.0774 \,.$$

The specific value of Z usually does not have any physical meaning. It is simply a normalizing constant. Only its derivatives will be physically relevant, as we shall see in the subsequent problems.

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Problem E.2 ... statistical 2 - basics 3 points

Now, let's examine a canonical system which has only two possible states. The difference of energies between these two states is exactly $\Delta E = 10^{-20}\,\mathrm{J}$. For example, you can imagine a molecule that exists either in its ground state or an excited state, but this description works for any general two-state system. What is the probability of finding the system in the state with higher energy if the temperature of the system is $T_0 = 275\,\mathrm{K}$?

Note: use knowledge from the previous problem. *Because the intro was not enough.*

According to the problem statement, we have only two available states with energies E_1 and $E_2 = E_1 + \Delta E$. In this case, we can easily plug in all (i.e. both) cases into the condition that the sum of all probabilities equals 1, so we get

$$p_1 + p_2 = 1 \,,$$

$$\frac{1}{Z} \mathrm{e}^{-\frac{E_1}{k_\mathrm{B} T}} + \frac{1}{Z} \mathrm{e}^{-\frac{E_1 + \Delta E}{k_\mathrm{B} T}} = 1 \,,$$

$$1 + \mathrm{e}^{-\frac{\Delta E}{k_\mathrm{B} T}} = Z \mathrm{e}^{\frac{E_1}{k_B T}} \,.$$

Substituting into Boltzmann's law we obtain

$$p_2 = \frac{1}{Z\mathrm{e}^{\frac{E_1}{k_B T}}}\mathrm{e}^{-\frac{\Delta E}{k_B T}} = \frac{\mathrm{e}^{-\frac{\Delta E}{k_B T}}}{1+\mathrm{e}^{-\frac{\Delta E}{k_\mathrm{B} T}}} = \frac{1}{1+\mathrm{e}^{\frac{\Delta E}{k_\mathrm{B} T}}} \doteq 0.067\,0\,.$$

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Problem E.3 ... statistical 3 - the quantum way 4 points

Earlier, you had to express Z, which you can calculate using

$$Z = \sum_n \mathrm{e}^{-\frac{E_n}{k_\mathrm{B} T}}\,,$$

where the sum goes over all possible states (such as the two in the previous problem). Then you had to use Boltzmann's Law to calculate probabilities. The sum Z is called the partition function.

Now let's consider a more complicated system - the quantum harmonic oscillator, which can be found in many different states. The energy of the n-th state is $\left(n+\frac{1}{2}\right)\hbar\omega$, where $n \in \mathbb{N}_0$ and $\hbar\omega = 10^{-21}$ J is the parameter of the oscillator. Determine the probability that the system is in the ground state ($n = 0$). The temperature of the system is $T_0 = 275\,\mathrm{K}$.

Hint $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ *for $q \in (0,1)$.* *This is where the fun begins.*

We calculate the partition function

$$Z = \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{\left(n+\frac{1}{2}\right)\hbar\omega}{k_\mathrm{B} T}} = \mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B} T}} \sum_{n=0}^{\infty} \left(\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B} T}}\right)^n = \frac{\mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B} T}}}{1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B} T}}}$$

and substitute it into Boltzmann's law

$$p_0 = \frac{1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B} T}}}{\mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B} T}}}\mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B} T}} = 1-\mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B} T}} \doteq 0.231\,5\,.$$

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Problem E.4 ... statistical 4 - the second quantum one 4 points

In the previous problem, you calculated the partition function of a quantum harmonic oscillator with energies $\left(n+\frac{1}{2}\right)\hbar\omega$, where $\hbar\omega = 10^{-21}$ J. It's actually possible to obtain all the different thermodynamic properties of the system just using the partition function. For example, the internal energy of the system $\overline{E}$ (the expected value of energy) can be calculated as

$$\overline{E} = k_\mathrm{B} T^2 \frac{\partial}{\partial T} \ln\left(Z(T)\right).$$

Determine the internal energy of the quantum harmonic oscillator from the previous problem in the units of $\hbar\omega$. The temperature of the system is $T_0 = 275\,\mathrm{K}$. This was supposed to be a problem with a second partial derivative, but it would probably be too hard.

We differentiate the partition function and get

$$\overline{E} = k_\mathrm{B}T^2 \frac{\partial}{\partial T} \ln\left(Z(T)\right) = k_\mathrm{B}T^2 \frac{1}{Z} \frac{\partial Z}{\partial T} =$$

$$= k_\mathrm{B}T^2 \frac{1}{Z} \frac{\frac{\hbar\omega}{2k_\mathrm{B}T^2} \mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B}T}} \left(1 - \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right) + \frac{\hbar\omega}{k_\mathrm{B}T^2} \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}} \mathrm{e}^{-\frac{\hbar\omega}{2k_\mathrm{B}T}}}{\left(1 - \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right)^2} =$$

$$= \hbar\omega \frac{\frac{1}{2}\left(1 - \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right) + \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}}{\left(1 - \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}\right)} = \frac{1}{2}\hbar\omega \frac{1 + \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}}{1 - \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}} = \frac{1}{2}\hbar\omega \coth \frac{\hbar\omega}{2k_\mathrm{B}T} \doteq 3.819\,7\,\hbar\omega\,.$$

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Problem X.1 ... the strongman and the weight 3 points

Imagine a $m = 10\,\mathrm{kg}$ weight attached to the end of a $d = 1.0\,\mathrm{m}$ long (and very light) rod. How many times bigger is the force our biceps needs to exert to hold the rod with the weight steady, compared to simply holding it in hand? Assume that our forearm is $l = 30\,\mathrm{cm}$ long. We hold the rod in such a way that it forms a "straight-line extension" of our forearm.

Dodo was carrying a pan full of water.

The force that the biceps must exert can be calculated from an equlibrium of forces on a lever, with the elbow joint serving as a pivot. By changing the position of the weight, we extend the length of the lever arm. We obtain the ratio of forces

$$\frac{F_2}{F_1} = \frac{l + d}{l} \doteq 4.3\,.$$

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Problem X.2 ... (not) breathing 3 points

In a single day, approximately $V = 11\,000\,\mathrm{l}$ of air pass through human lungs. Find the average speed of air flowing through the larynx, assuming that its diameter is $d = 40\,\mathrm{mm}$.

Dod and his treacherous factors of two.

To solve this, we use the relation for volumetric flow rate of fluids Q

$$Q = Sv$$

where S is the cross-sectional area and v is the flow speed. The flow rate can be determined from the volume of air exchanged. The volume V has to flow both into the lungs and out of them in a day, so the average flow rate is $Q = 2V/1\,\mathrm{day}$. The cross-section of the larynx has an area

$$S = \pi \left(\frac{d}{2}\right)^2 \,.$$

By expressing the flow speed and substituting for S and Q, we get

$$v = \frac{Q}{S} = \frac{8V}{\pi d^2 \cdot 1\,\mathrm{day}} \doteq 0.20\,\mathrm{m{\cdot}s^{-1}} \,.$$

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Problem X.3 ... the pressure of a ballerina 3 points

A ballerina that weighs $m = 50\,\mathrm{kg}$ is landing on a big toe after jumping into the height $h = 0.5\,\mathrm{m}$. At its narrowest point, the last phalanx bone of the big toe has a diameter $d = 1.3\,\mathrm{cm}$. Find the highest stress across this bone during the landing. Assume that the ballerina decelerates upon impact with a constant force for $t = 0.5\,\mathrm{s}$. *Dodo stood on one toe.*

The stress can be calculated from its definition as a force across a cross-sectional area

$$p = \frac{F}{S} = \frac{4F}{\pi d^2} \,.$$

The force transferred through the bone is given by the sum of the weight of the ballerina and the force used for deceleration, which can be obtained from Newton's 2nd law as the rate of change of the momentum

$$F_\mathrm{d} = \frac{mv}{t} = \frac{m\sqrt{2gh}}{t} \,,$$

into which we substituted the free fall velocity right before impact.

Substituting the expression for the force into the equation for stress, we obtain

$$p = \frac{4\left(mg + \frac{m\sqrt{2gh}}{t}\right)}{\pi d^2} \doteq 6.06\,\mathrm{MPa} \,.$$

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Problem X.4 ... saliva 3 points

Fykos organizers are very skillful and managed to create $100\,\mathrm{g}$ a substance such that $99\,\%$ of its mass is water. Unfortunately, after a while, some water evaporated away and now only $98\,\%$ of the sample is water. What is the current mass of the sample? *Lego likes Vsauce2.*

The key is to realize that the mass of the non-water part of the sample is conserved. In the beginning, it is $100\,\% - 99\,\% = 1\,\%$ of the total mass. Since the sample weighed $100\,\mathrm{g}$, the non-water part weighs $100\,\mathrm{g} \cdot 1\,\% = 1\,\mathrm{g}$.

After drying out, the non-water part makes up $100\,\% - 98\,\% = 2\,\%$ of the total mass. The whole sample must then weigh $1\,\mathrm{g}/2\,\% = 50\,\mathrm{g}$.

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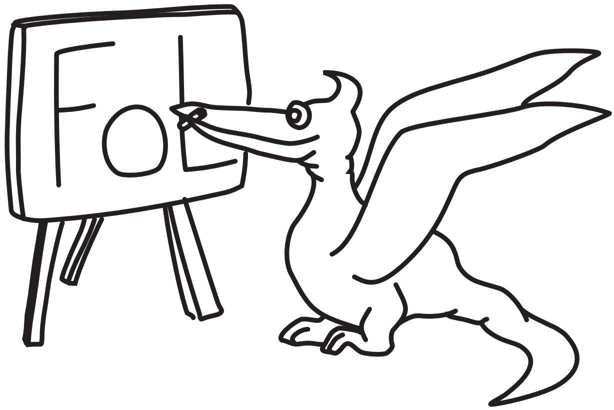
Solutions of 11^{th} Physics Brawl Online



Problem 1 ... an unfortunate trolleybus 3 points

A trolleybus of total mass $M_\mathrm{t} = 15\,\mathrm{tonnes}$ (without any passengers) and of volume $V_\mathrm{t} = 90\,\mathrm{m^3}$ is entering a bridge over a river. Unfortunately, the bridge is being reconstructed, and careless workers forgot to put up no entry signs. The trolleybus falls into the river. What is the maximum percentage of the trolleybus volume that can be occupied by passengers such that after the fall, the trolleybus ends up floating to the surface? Consider the density of the human body to be the same as the density of water. The trolleybus is airtight.

Verča heard that there was a demand for a trolleybus problem.

Let us denote the sought ratio as x. The final mass of trolleybus loaded with passengers can be expressed as $M_\mathrm{t} + x\rho_\mathrm{w}V_\mathrm{t}$. If the trolleybus ought to float, the buoyancy force acting on a fully submerged trolleybus has to be at least as great as the gravity. Thus, we get the equation

$$(M_\mathrm{t} + x\rho_\mathrm{w}V_\mathrm{t})\,g = V_\mathrm{t}\rho_\mathrm{w}g\,.$$

From the expression above, we can easily express the ratio

$$x = \frac{V_\mathrm{t}\rho_\mathrm{w} - M_\mathrm{t}}{V_\mathrm{t}\rho_\mathrm{w}} = 1 - \frac{M_\mathrm{t}}{V_\mathrm{t}\rho_\mathrm{w}}\,.$$

After substituing the numbers, we get $x \doteq 83.3\,\%$.

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Problem 2 ... our antique clock 3 points

An antique pendulum clock has to be wound up every Sunday at the same time to show the correct time during the whole week. While winding up, one has to raise a weight $m = 5.6\,\mathrm{kg}$ by $h = 31\,\mathrm{cm}$ to enable the clockwork to drive the pendulum. The pendulum is of length $l = 64\,\mathrm{cm}$ and almost all its mass is located at the lower end. How much energy (on average) is dissipated during one swing?

Jarda is late sometimes.

The pendulum in time $t = 1\,\mathrm{week} = 604\,800\,\mathrm{s}$ "use up" potential energy of the weight, which is

$$E_\mathrm{p} = mgh \doteq 17.0\,\mathrm{J}\,,$$

therefore, the dissipation power is $P = \frac{E_\mathrm{p}}{t} \doteq 28.1\,\mathrm{\mu W}$. Regarding the problem description, we assume the pendulum to be mathematical

$$T = 2\pi\sqrt{\frac{l}{g}} \doteq 1.60\,\mathrm{s}\,.$$

Thus, the disipated energy during one swing is

$$E = P\frac{T}{2} = \frac{\pi m h}{t}\sqrt{lg} \doteq 22.6\,\mathrm{\mu J}\,.$$

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Problem 3 ... flywheel 3 points

Let us assume that we have a homogeneous disc (flywheel) of a mass $M = 10m$ and radius R. There is a fly called Ľubo of a mass m sitting on the flywheel at a distance $0.75R$ from the center of the disc, while the disc (together with Ľubo) is rotating with an angular frequency $\omega_0 = 1.20\,\mathrm{rad{\cdot}s^{-1}}$. Ľubo's friend Slavo, who has the same mass as Ľubo, suddenly sits at the edge of the disc, while he had zero angular momentum with respect to the axis of rotation of the disc. Assume that both flies are point masses and that the system (the flywheel + flies) is not subjected to any external torques. What is the angular frequency of the disc after Slavo sits on it? *Flies had been flying around Tomáš's head.*

Let denote the numerical factors as $\alpha = 10$ and $\beta = 0.75$.

The moment of inertia of the disc is $I = \frac{1}{2}MR^2$. In our special case, the initial moment of inertia includes Ľubo, and thus

$$I_0 = \frac{1}{2}MR^2 + I_\mathrm{L} = \frac{1}{2}\alpha mR^2 + \beta^2 mR^2 = \left(\frac{1}{2}\alpha + \beta^2\right) mR^2\,.$$

The final moment of inertia after Slavo's arrival is

$$I_1 = I_0 + I_\mathrm{S} = \left(\frac{1}{2}\alpha + \beta^2 + 1\right) mR^2\,.$$

We will find the final angular frequency by the law of conservation of angular momentum

$$\begin{aligned} L_0 &= L_1\,,\\ I_0\omega_0 &= I_1\omega_1\,,\\ \omega_1 &= \omega_0\frac{I_0}{I_1}\,,\\ \omega_1 &= \omega_0\frac{\frac{1}{2}\alpha + \beta^2}{\frac{1}{2}\alpha + \beta^2 + 1}\,,\\ \omega_1 &\doteq 0.848\omega_0 \doteq 1.02\,\mathrm{rad{\cdot}s^{-1}}\,. \end{aligned}$$

After Slavo's arrival, the disc flywheel will have angular frequency $\omega_1 \doteq 1.02\,\mathrm{rad{\cdot}s^{-1}}$.

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Problem 4 ... roofing 3 points

Martin is standing on a roof. The coefficient of static friction between his shoes and roof tiles is 0.7. To what percentage of the initial value does the effective coefficient of friction between Martin and the roof decrease when Martin sits down, if doing so transfers 60 % of his weight from shoes to his trousers? The coefficient of static friction between his trousers and roof tiles is 0.4. *Martin was learning to be a roofer.*

In the first case, the effective coefficient is clearly $f_1 = 0.7$. In the second case, $1 - w = 40\,\%$ of Martin's weight takes the coefficient $f_1 = 0.7$ and $w = 60\,\%$ of his weight takes the coefficient $f_2 = 0.4$. Thus, the effective coefficient in the second case is

$$f = 0.4 \cdot 0.7 + 0.6 \cdot 0.4 = 0.52\,.$$

We assume constant roof inclination at the spot where Martin stood and sat. We could express the resulting coefficient as

$$f = \frac{F_\mathrm{f}}{F_\mathrm{G}} \frac{f_1 F_1 + f_2 F_2}{F_1 + F_2} = f_1(1 - w) + f_2 w\,.$$

The asnwer is the ratio of the two values $\frac{0.52}{0.7} \doteq 0.74$, what is approximately $74\,\%$.

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Problem 5 ... spring in an airplane 3 points

An ideal spring oscillator is placed in a climbing airplane that has a load factor 3g. How does the frequency of its oscillations change? Express your answer as the ratio of the original frequency to the new one. *Vojta was oscillating in an airplane.*

The frequency of an ideal spring oscillator depends only on the spring stiffness and mass of a suspended object; therefore, the frequency does not change when put airborne. Thus, the answer is 1.00 times.

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Problem 6 ... focusing light 4 points

We have an aquarium and a thin converging lens with a small diameter, which has an optical power of $\varphi = 4.20\,\mathrm{D}$ in the air. We place a point light source on the optical axis of the lens at a great distance from the lens. The lens is placed $d = 4.20\,\mathrm{cm}$ from the aquarium in such a way that its optical axis is perpendicular to the wall of the aquarium. The aquarium is large enough, its walls are made of thin glass, and it is full of water. What is the distance from the center of the lens at which the rays coming from the source are focused?

Karel was thinking about optics.

We can assume, that the light falling on the lens comes from infinity, the refracted rays therefore point to the focus. So if there were no aquarium, answer would be, that the rays intersect in the distance

$$f = \frac{1}{\varphi} \doteq 23.8\,\mathrm{cm}\,.$$

However in our situation, rays refract again in the distance d. Water is optically denser medium, refracted rays will thus be closer to normal, which means that the point of their intersection will be located further than in air. Thanks to the small diameter of the lens, we are only interested in rays close to optical axis and so we can guess[1], that the ratio of those distances will match the ratio of indices of refraction. So we get

$$\frac{x}{f - d} = \frac{n}{n_0} \quad \Rightarrow \quad x = \frac{n}{n_0}\left(f - d\right)\,,$$

[1] This is faster and less correct variant of the solution. Better explanation follows.

where x denotes the distance from the aquarium wall in which the rays intersect. The total distance from the lens to the point of intersection is thus

$$s = d + x = d + \frac{n}{n_0}(f - d) \doteq 30.3\,\mathrm{cm}\,.$$

Rays intersect 30.3 cm from the optical center of the lens. With the required precision, we considered air's index of refraction to be equal to the index of refraction of vacuum, that is $n_0 \doteq 1$. Let's now look at more detailed explanation of why does our solution work.
We will analyze the whole situation with a help of a picture 1, in which the angles are highlighted for better idea. Axis o is the optical axis of our lens.

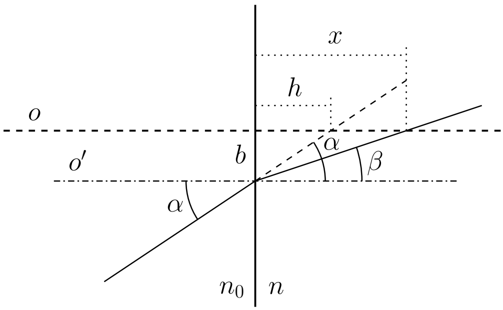

Fig. 1: Schematic representation of the refraction at the interface of air (n_0) and aquarium (n) – dashed line o denotes axis of the lens, o' denotes the normal of incidence, full line illustrates the ray, $h = f - d$ is the distance, at which would the rays at the same distance from axis o (denoted as b) intersect without an interface, and x is the distance of the actual intersection

We're interested in what happens at the interface of air and water. Although the aquarium is made out of glass and thus there should be two refractions, its wall is thin so the first refraction can be neglected. The ray arrives at the optical interface at the angle α in the distance b from the lens.

The normal is denoted as o' and the angle of refraction is β. If there were no aquarium, the rays would have intersected at the distance $f - d$ from the interface. In our situation, they will however intersect in the distance again denoted as x.

So we've described the picture and now let's focus on the solution itself. Consider two specific triangles in the picture, from which we can express the tangents of angles α and β as follows

$$\tan\alpha = \frac{b}{f - d}\,, \quad \tan\beta = \frac{b}{x}\,.$$

We rearrange the second expression to $b = x\tan\beta$, which we substitute into the first one and then solve for x

$$x = (f - d)\,\frac{\tan\alpha}{\tan\beta}\,.$$

Now we prepare the Snell's law (law of refraction) into the form suitable for substitution.

$$n_0 \sin\alpha = n\sin\beta \qquad \Rightarrow \qquad \beta = \arcsin\left(\frac{n_0}{n}\sin\alpha\right)\,.$$

$$x = (f - d)\,\frac{\tan\alpha}{\tan\!\left(\arcsin\!\left(\frac{n_0}{n}\sin\alpha\right)\right)}\,.$$

At this point, it would be useful to use following identity, which holds for all z from the domain of $\arcsin x$ function,

$$\tan\arcsin z = \frac{z}{\sqrt{1-z^2}}\,.$$

In our situation $z = \frac{n_0}{n}\sin\alpha$. We'll also make use of $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and we get

$$\begin{aligned} x &= (f-d)\tan\alpha\left(\frac{\frac{n_0}{n}\sin\alpha}{\sqrt{1-\frac{n_0^2}{n^2}\sin^2\alpha}}\right)^{-1} \\ &= (f-d)\,\frac{n}{n_0}\,\frac{\frac{\sin\alpha}{\cos\alpha}}{\sin\alpha}\sqrt{1-\frac{n_0^2}{n^2}\sin^2\alpha} \\ &= \frac{n}{n_0}\,(f-d)\,\frac{1}{\cos\alpha}\sqrt{1-\frac{n_0^2}{n^2}\sin^2\alpha}\,. \end{aligned}$$

By this, we've even obtained an exact result for rays falling on the interface at an arbitrary angle. However now we can notice, that thanks to the small diameter of the lens, all the rays are close to the axis o. Angle α is thus small, so we could set $\alpha \approx 0$, which also means $\sin\alpha \approx 0$ and $\cos\alpha \approx 1$. So we get

$$x = \frac{n}{n_0}\,(f-d)\;,$$

which is exactly what we wanted to show. The rest of the solution, that is adding the distance from the lens to the aquarium, is then analogous.

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Problem 7 ... induced voltage 3 points

Consider a region with homogeneous magnetic field. The region has a rectangular cross-section with sides $a = 3.0\,\mathrm{m}$, $b = 2.0\,\mathrm{m}$ and the magnetic induction vector $B = 1.0\cdot 10^{-3}\,\mathrm{T}$ is perpendicular to this cross-section. We place a straight wire in parallel to the side a, such that its free ends lie outside the region with the magnetic field. We connect these ends to a voltmeter.

We then move the wire in uniform linear motion at velocity $v = 0.20\,\mathrm{m\cdot s^{-1}}$ along the side b, i.e. in a direction perpendicular to the magnetic induction vector and to the side a. What voltage does the voltmeter show when the wire passes through the magnetic field?

Jindra came up with a problem longer than its solution.

The voltage is calculated as the ratio of a change in the magnetic induction flux $\mathrm{d}\Phi$ through the loop with respect to an infinitesimal time change $\mathrm{d}t$

$$U = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}SB}{\mathrm{d}t} = \frac{\mathrm{d}x\cdot aB}{\mathrm{d}t} = Bva = 6.0\cdot 10^{-4}\,\mathrm{V}\,.$$

The same result can be obtained by using the formula for induced voltage $U = Bvl$.

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Problem 8 ... a constant atmosphere 3 points

What would be the thickness of the Earth's atmosphere if it had constant density $\rho = 1.29\,\mathrm{kg{\cdot}m^{-3}}$ *everywhere? Assume that the Earth is round and that the atmosphere has a mass of* $m = 5.157 \cdot 10^{18}\,\mathrm{kg}$.

Karel is always speculating.

The volume of the atmosphere of a constant density ρ is

$$V = \frac{m}{\rho} \doteq 3.998 \cdot 10^{18}\,\mathrm{kg}\,.$$

We assume the Earth's radius to be $R_\oplus = 6.378 \cdot 10^6\,\mathrm{m}$. While it is the equatorial radius, it is sufficient enough for the required accuracy of this computation. By using the radius, we get the Earth's surface area $S = 4\pi R_\oplus^2 \doteq 5.11 \cdot 10^{14}\,\mathrm{m}^2$.

Regarding the atmosphere being relatively thin, we can assume the top and the bottom surface area of the atmosphere to be the same. The thickness of the atmosphere h is then the ratio of the atmosphere's volume and Earth's surface area

$$h = \frac{V}{S} = \frac{m}{4\pi\rho R_\oplus^2}\,,$$

$$h \doteq 7.8 \cdot 10^3\,\mathrm{m}\,.$$

Thus, the constant atmosphere would be of thickness 7 800 m.

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Problem 9 ... molar mumble 4 points

An enthomologist caught a "mol" and he wants to dry it. Calculate its mass after evaporation of all the water, if you know that a particular ball of one mole of living "mols"

- *contains* $n = 3.346 \cdot 10^{19}\,\mathrm{mol}$ *of water molecules,*
- *has surface area equal to twenty four times the area of Moldova (including Transnistria),*
- *has density* $\rho = 27.4\,\mathrm{kg{\cdot}m^{-3}}$.

Translator's note: "mol" is a Czech word for a clothing moth.

Vojta was admiring Vivaldi's Summer's tremolos.

Let N be the number of "mols" in the ball (numerical value of Avogadro's constant in mol^{-1}). First we will compute the mass of a living *mol*

$$m_\mathrm{m} = \frac{\rho V}{N_\mathrm{A}} = \frac{\rho \frac{4}{3}\pi r^3}{N_\mathrm{A}}\,,$$

whereas

$$r = \sqrt{\frac{24 S_\mathrm{M}}{4\pi}} = \sqrt{\frac{6 S_\mathrm{M}}{\pi}}\,,$$

where $S_\mathrm{M} = 33\,843\,\mathrm{km}^2$ is the area of Moldova. Overall we have

$$m_\mathrm{m} = \frac{4\pi\rho}{3N_\mathrm{A}}\left(\frac{6S_\mathrm{M}}{\pi}\right)^{\frac{3}{2}} = \frac{8\rho}{N_\mathrm{A}}\sqrt{\frac{6S_\mathrm{M}^3}{\pi}} \doteq 3.13\,\mathrm{mg}\,.$$

The mass of the water in the *mol* will be

$$m_\mathrm{w} = \frac{nM_{\mathrm{H_2O}}}{N_\mathrm{A}} \doteq 1.00\,\mathrm{mg}\,,$$

Thus

$$m = m_\mathrm{m} - m_\mathrm{w} = 2.13\,\mathrm{mg}\,.$$

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Problem 10 ... don't lean out of the windows

4 points

A dignitary is taking a fiacre ride. While moving at a speed $v_\mathrm{k} = 2.0\,\mathrm{m{\cdot}s^{-1}}$, he's eating an apple. When finished eating, he throws the apple core out of the window such that it gains horizontal velocity $v_\mathrm{h} = 3.0\,\mathrm{m{\cdot}s^{-1}}$ perpendicular to the direction of travel. The core then proceeds to hit an unsuspecting peasant in the head just when he is $d = 1.0\,\mathrm{m}$ away from the dignitary and is not moving. At what speed did the apple core hit the peasant's head? Consider the core to be a point mass. *Lego often travels by train.*

The core's velocity will consist of three components, which can be considered independent. The first component will be the speed v_k in the direction of travel (let us denote this direction as x). The second component will represent the speed v_h at which it was thrown out of the window (in the horizontal direction perpendicular to x – let's denote this direction as y) and the third, vertical component (the direction z) gained thanks to the gravity.

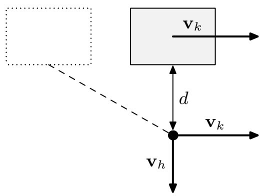

Fig. 2: Illustration of fiacre's and apple core's movements.

We know that the distance between peasant and fiacre at the time of impact was equal to d. Note that the core and the fiacre must have traveled the same distance in the x direction. This implies that in the time of impact, the distance between fiacre and peasant must have been minimal (see the illustration 2), which means the core must have traveled exactly distance d in the y direction. Thanks to this observation, we're able to compute the time of its flight $t = \frac{d}{v_\mathrm{k}}$, from which we can subsequently determine the speed in z direction

$$v_z = \frac{gd}{v_\mathrm{h}}\,.$$

Now we can finally use the Pythagorean theorem to compute the final speed as

$$v = \sqrt{\left(\frac{gd}{v_\mathrm{h}}\right)^2 + v_\mathrm{k}^2 + v_\mathrm{h}^2} \doteq 4.9\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem 11 ... a healthy mind in a healthy body 4 points

Let a flexible rope be fastened to a wall at one end. Danka is pulling the other end. She is standing on a small rug that lies freely on the floor. What is the maximum distance (measured from the wall) at which Danka can be pulling the rope, such that the elasticity of the rope does not drag Danka back towards the wall? The coefficient of static friction between the rug and the floor is $f = 0.45$, while it is much larger between Danka's feet and the rug. Danka weighs $m = 55\,\mathrm{kg}$ and the mass of the rug can be considered negligible. Assume that the rope is being pulled horizontally, its free length is $l_0 = 1.97\,\mathrm{m}$ and its stiffness is $k = 164\,\mathrm{N{\cdot}m^{-1}}$.

Danka was exercising.

An elastic force of a rope is given by the extension with respect to the proper length as $F_\mathrm{e} = k(l - l_0)$, where l is the length of the rope. Static friction force grows with an increasing elastic force in a way that both forces cancel out. However, this applies only until the point when we reach a maximum static friction force, which we can calculate as

$$F_\mathrm{f} = fF_\mathrm{n} = fmg\,.$$

where F_f is a friction force in a direction opposite to the force exerted by the rope. F_n is the force exerted by Danka and the rug on the ground. As we know, there exists a critical point where the elastic force finds its balance with the maximum static friction force

$$k(l - l_0) = fmg\,.$$

We isolate the length of the rope and plug in the corresponding values

$$l = l_0 + \frac{fmg}{k}\,,$$

$$l \doteq 3.45\,\mathrm{m}\,.$$

Danka can extend up the rope to 3.45 m from the wall.

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Problem 12 ... insidiously smoky 4 points

A typical ionization smoke detector contains a sample of $^{241}\mathrm{Am}$ that corresponds to a radioactivity of $1\,\mathrm{\mu Ci}$. How many such detectors at the minimum do we need to dismantle to get a sufficient mass of $^{241}\mathrm{Am}$ to start a nuclear chain reaction? The critical mass of $^{241}\mathrm{Am}$ corresponds to $60\,\mathrm{kg}$.

Pepa always steals problems from textbooks.

We look up the half-life of $^{241}\mathrm{Am}$ in the tables, $T = 432.6$ years. Specific activity of the sample can be calculated as

$$a = \frac{1}{m}\left|\frac{\mathrm{d}N}{\mathrm{d}t}\right| = \frac{\lambda N}{m} = \frac{\lambda N}{NA_\mathrm{R}u}\,,$$

where $A_\mathrm{R} = 241$ is relative atomic mass of $^{241}\mathrm{Am}$, $u = 1.661 \cdot 10^{-27}\,\mathrm{kg}$ is the atomic mass constant, and λ is the exponential decay constant of $^{241}\mathrm{Am}$, i.e. $\frac{\ln 2}{T}$.

We need to convert the activity of the sample to units of SI, which is Becquerel ($\mathrm{Bq} = \mathrm{s}^{-1}$). Becquerel is related to Curie (Ci) as follows: $\mathrm{Ci} = 3.7 \cdot 10^{10}\,\mathrm{Bq}$. The reason we had to convert the units was because we determined the specific activity of americium in units $\mathrm{Bq{\cdot}s^{-1}}$. Then, for a mass of one sample from the detector we get

$$m = \frac{3.7 \cdot 10^4\,\mathrm{Bq}}{a} \doteq 0.29\,\mathrm{\mu g}\,.$$

The critical mass of 60 kg thus corresponds to approximately $60\,\mathrm{kg}/0.29\,\mathrm{\mu g} = 207$ billion detectors.

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Problem 13 ... an electron source 4 points

Consider a point source that emits electrons of speed $v = 20\,000\,\mathrm{km{\cdot}s^{-1}}$ in the plane yz, and is placed in a homogeneous magnetic field which has magnitude B and is pointing in the direction of the x axis. The electron source is surrounded by a tube of radius $R = 10\,\mathrm{cm}$. The axis of symmetry of the tube passes through the electron source and is parallel to the y axis. What is the minimum magnitude of the magnetic field B which prevents the emitted electrons from touching the tube? *Kiko has been reading famous "green lecture notes".*

The magnetic component of the Lorentz force acting on an electron is

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B}\,,$$

where $e \doteq 1.6 \cdot 10^{-19}\,\mathrm{C}$ is the elementary charge. Magnetic induction $\mathbf{B}$ is oriented in the direction of x axis, i.e. perpendicular to the arbitrary velocity vector $\mathbf{v}$ of plane yz. It follows that for the magnitude of Lorentz force we have

$$F = evB\,.$$

The Lorentz force always lies in the plane yz and is perpendicular to the velocity $\mathbf{v}$. It means that electron will move in a circle of the radius r. For the centripetal acceleration, the following equation holds

$$\frac{v^2}{r} = a = \frac{F}{m_e} = \frac{eBv}{m_e}\,.$$

To prevent the electrons from touching the tube, the electron trajectory radius r can be at most half of the radius of the tube R. Now, we can express the minimum magnitude of the magnetic field

$$B = \frac{2m_e v}{eR} \doteq 2.3\,\mathrm{mT}\,.$$

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Problem 14 ... is it cold outside? 4 points

We are designing a thermometer. We want its range to be from $-40\,°\mathrm{C}$ *up to* $+50\,°\mathrm{C}$*, with the scale lines at every* $1\,°\mathrm{C}$*. To distinguish the values on the thermometer properly just by eye from a distance of* $2\,\mathrm{m}$*, we need the angular distances between adjacent lines to be at least* $4'$*. The radius of the capillary tube of the thermometer is* $0.2\,\mathrm{mm}$*. What is the minimum volume of alcohol that the thermometer needs to contain if it has to be able to show the full range of temperatures? We use alcohol with the volumetric thermal expansion coefficient* $\beta = 1.1\cdot 10^{-3}\,\mathrm{K^{-1}}$*.* *Jarda will sell you a thermometer one day.*

The spacing between the scale lines must be at least

$$d = l\tan\theta \approx l\theta \doteq 2.3\,\mathrm{mm}\,,$$

where $l = 2\,\mathrm{m}$ is the distance we are looking from and $\theta = 4'$ is the angle at which we have to see the gap between the lines. Let $\delta = 1\,°\mathrm{C}$ be the smallest distance between two scale lines.
To be able to display the full range of temperatures, the thermometer has to measure at least

$$h = d\frac{\Delta T}{\delta} \doteq 21\,\mathrm{cm}\,.$$

The alcohol has to be able to fill out the volume of

$$\Delta V = \pi r^2 h \doteq 26\,\mathrm{mm^3}\,,$$

as the temperature changes by $\Delta T = 90\,°\mathrm{C}$. The change of volume with respect to temperature is $\Delta V = V\beta\Delta T$, V being the initial volume and $\beta = 1.1\cdot 10^{-3}\,\mathrm{K^{-1}}$ is the given volumetric thermal expansion coefficient of the alcohol. Thus

$$V = \frac{\Delta V}{\beta\Delta T} = \frac{\pi r^2 l\theta}{\beta}\,,$$
$$V \doteq 266\,\mathrm{mm^3}\,.$$

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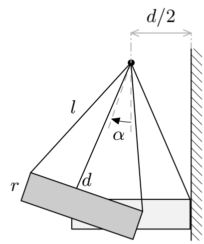
Problem 15 ... battering ram 5 points

We decided to conquer the Prague Castle. We will destroy the front gate with a battering ram in the shape of a thick homogeneous cylinder of length $d = 2.0\,\mathrm{m}$ *and radius* $r = 25\,\mathrm{cm}$*, with its axis placed horizontally. We attach cables with lengths* $l = 2.0\,\mathrm{m}$ *to the upper points of both bases of the cylinder and fasten the other ends to a common point right above the center of the cylinder. With the ram assembled in this way, we come to the gate and place the ram in such a way that one of its bases touches the gate. Then we incline the ram* $\alpha = 20°$ *around the hanging point and finally release it. At what speed will it hit the gate?*



Jarda wants to be the president of FYKOS.

We can proceed from the law of conservation of energy. When we deflect the ram, we increase the position of the cylinder's center of gravity a bit, so we increase the potential energy

$$E_\mathrm{p} = mgh = mgs\left(1 - \cos\alpha\right)\,,$$

where $\alpha = 20^\circ$ and $s = r + \sqrt{l^2 - \frac{d^2}{4}}$ is the distance of the center of gravity from the axis of its rotation. This energy is converted to rotational $E_\mathrm{r} = \frac{1}{2}J\omega^2$, where ω is the angular frequency of a rotation and J is the moment of inertia of the cylinder relative to the selected axis, which we yet need to calculate. The cylinder can still be approximated as a thin rod at given dimensions, so its moment of inertia with respect to the axis passing through the center of gravity perpendicular to the axis of symmetry is

$$J_s = \frac{1}{12}md^2\,.$$

The error of this approximation is about five percent, but because we still have to use Steiner's parallel axis theorem, this error will no longer be important. The resulting moment of inertia by Steiner's theorem is

$$J = J_s + ms^2 = \frac{1}{12}md^2 + m\left(r + \sqrt{l^2 - \frac{d^2}{4}}\right)^2\,.$$

The speed at which the battering ram hits the gate is then

$$v = \omega s = \sqrt{\frac{2mgs\left(1-\cos\alpha\right)}{J}}s = \sqrt{\frac{2gs\left(1-\cos\alpha\right)}{\frac{1}{12}d^2 + \left(r + \sqrt{l^2 - \frac{d^2}{4}}\right)^2}}s = 1.47\,\mathrm{m\cdot s^{-1}}\,.$$

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Problem 16 ... night-time thermodynamics lesson 4 points

Verča got lost on a trip and decided to stay in an old shack overnight. The outside temperature dropped to $t_0 = 10.0\,^\circ\mathrm{C}$ at night, but Verča kept the inside temperature at $t_1 = 16.0\,^\circ\mathrm{C}$. However, the shack had a rectangular hole in the wall of dimensions $a \times b$, where $a = 0.50\,\mathrm{m}$ and $b = 0.30\,\mathrm{m}$. To save some of the escaping heat, Verča plugged the hole with two bricks of cross-sections $S_1 = a \times (b/3)$ and $S_2 = a \times (2b/3)$, and of thermal conductivity coefficients $\lambda_1 = 0.80\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$ and $\lambda_2 = 1.30\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$, respectively. The lengths of both bricks are the same as the thickness of the wall $s_1 = 15\,\mathrm{cm}$, so they filled in the hole perfectly. Nevertheless, it would be nicer to be able to look outside. What thermal conductivity coefficient λ would a homogeneous glass panel of thickness $s_2 = 3.0\,\mathrm{mm}$ need to have if it conducted heat from the shack in the same way as the two bricks? The glass panel would be inset in the hole parallelly to the wall (like a window). Assume that the rest of the shack insulates perfectly.

Verča gets lost in other places than just lectures.

The quantity we want to preserve is the thermal flux, which we can generally calculate as

$$q = \frac{t_1 - t_0}{R}\,,$$

where $t_1 - t_0$ is the temperature difference and R is the thermal resistance of the body

$$R = \frac{d}{\lambda S}\,,$$

while d is the thickness of the body through which we conduct the heat, S is the cross-section of the body, and λ is its coefficient of thermal conductivity. Because the temperature difference is the same in both cases, we only need to compare the thermal resistances of the bricks and the glass block. The resistances compound in the same way as in electrical circuits, so for the total thermal resistance of the parallelly laid bricks, the following expression holds

$$\frac{1}{R_{\mathrm{c1}}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{ab(\lambda_1 + 2\lambda_2)}{3s_1} \quad \Rightarrow \quad R_{\mathrm{c1}} = \frac{3s_1}{ab(\lambda_1 + 2\lambda_2)}$$

and the thermal resistance of the glass block is

$$R_{\mathrm{c2}} = \frac{s_2}{ab\lambda}\,.$$

From the equality

$$R_{\mathrm{c1}} = R_{\mathrm{c2}}$$

we get

$$\frac{3s_1}{2\lambda_2 + \lambda_1} = \frac{s_2}{\lambda}\,,$$

from which we can express the result

$$\lambda = \frac{s_2(2\lambda_2 + \lambda_1)}{3s_1}\,.$$

We can see that nothing but the values of the individual factors and thicknesses affected the result, as all other quantities were the same in both cases. Numerically we get $\lambda = 0.022\,7\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$.

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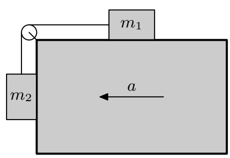
Problem 17 ... moving pulleys on a wagon 5 points

We have a wagon in the shape of a cuboid, which is moving with an acceleration of $a = 1.2\,\mathrm{m{\cdot}s^{-2}}$. On the top of the wagon, a smaller cuboid of mass $m_1 = 15\,\mathrm{kg}$ is placed, and it is connected by a rope over a pulley with another cuboid of mass $m_2 = 10\,\mathrm{kg}$, which is hanging on the rope in front of the wagon, touching its front wall. With what acceleration (with respect to the wagon) are the small cuboids going to move? The static friction coefficient between the small cuboids and the wagon is $f = 0.15$. Assume that the rope and pulley are massless.



Legolas truly enjoys adjusting his own problems.

Since the rope and the pulley are massless, the magnitude of the force by which the rope pulls the block m_2 upwards is the same as the magnitude of the force that pulls the cuboid m_1 to the left. Let's denote the magnitude of this force as T.

In addition to the force T, friction acts on the cuboid m_1 (the normal force from the wagon and the gravitational force cancel each other out), too. The magnitude of the force of friction is $F_\mathrm{t}^1 = fm_1g$. This force acts against acceleration, so the resulting equation of motion for this cuboid will be

$$T - fm_1g = m_1a_1 \,,$$

where we assume that the block will accelerate more than the wagon and thus will slide to the left. If our assumption were incorrect, we would end up with a negative result. But beware! It would not be enough to simply change its sign, as it would mean that we assumed the wrong direction of acceleration all the time, and we would expect the opposite sign of friction in the entire subsequent calculation! Why do we assume this direction? From a simple estimate that $m_1a < m_2g$.

In the system accelerating together with the wagon, the cuboid m_2 presses against the wall of the wagon with force m_2a, which is also the force by which the wall pushes it back (it means that in the system connected to the wagon, this block will not accelerate horizontally, as we would expect). Thus, the friction force $F_\mathrm{t}^2 = fm_2a$, the gravitational force $F_\mathrm{g}^2 = m_2g$, and the tensile force of the rope T act on the cuboid in the vertical direction. Consistently with the first cuboid, let's assume that this one also accelerates downwards, and we obtain the equation

$$m_2g - T - fm_2a = m_2a_2 \,.$$

In these two equations, the accelerations and the force T are unknown, but since both blocks are tied to one tensioned rope, their accelerations will be the same in the system connected to the wagon. However, a_1 is the resulting acceleration, so we still need to express the acceleration of the cuboid m_1 in the system connected to the wagon a_1'. We can do this either by adding a fictitious force or by noticing that $a_1 = a + a_1'$. Either way, we will get

$$T - fm_1g - m_1a = m_1a_1' \,.$$

Subsequently, we use the information that in this system $a_2 = a_1'$, we add up the equations to eliminate the unknown T and express the resulting acceleration as

$$m_2g - m_1a - fm_2a - fm_1g = (m_1 + m_2)\,a_2$$

$$a_2 = \frac{m_2g - m_1a - fm_2a - fm_1g}{m_1 + m_2} \doteq 2.2\,\mathrm{m{\cdot}s^{-2}} \,.$$

Of course, we could have started working in an accelerating system with the wagon since the beginning, in which $m_{1,2}a$ would be interpreted as the inertial force acting on the bodies in this system. After a very similar (and probably a little shorter) calculation, we would reach the same result.

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Problem 18 ... beware of the fountain 4 points

There is a circular fountain at a town square with a diameter of $D = 6\,\mathrm{m}$. At its center, a spring of water gushes vertically upwards to a height of $h = 4\,\mathrm{m}$. What is the lowest speed a short gust of wind has to blow at so that the gushing water falls outside the fountain? Assume that the wind blows horizontally and transfers $20\,\%$ of its speed to water drops.

Danka was observing children swimming in a fountain.

The longer the drops of water will be in the air, the further they can fly. Hence, it is best for the gust of wind to blow right after the water gushes out. This way, the drops will gain a horizontal component of velocity $v = 0,2\,v_w$, where v_w is the speed of wind we are searching. The time T that the waterdrop spends above ground is double the time t it takes for it to fall from the highest point h of its trajectory so that we can express it as

$$T = 2t = 2\sqrt{\frac{2h}{g}}\,,$$

where g is the gravitational acceleration. The borderline case, where the drop hits directly the edge of the fountain occurs, if the drop surpasses in the horizontal direction the radius of the fountain, that is to say $vT = D/2$. Therefore, the wind must blow at minimum at a speed of

$$v_w = \frac{5}{4}\sqrt{\frac{g}{2h}}D \doteq 8.3\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem 19 ... Beryllgläser 5 points

We have two thin lenses of the same shape. The first one is made out of plastic with a refractive index of $n_\mathrm{p} = 1.67$ and has an optical power of $\varphi_\mathrm{p} = 4.20\,\mathrm{D}$. The second lens is made out of beryl and has a refractive index of $n_\mathrm{b} = 1.57$. What is the optical power of the second lens? Both of the lenses are surrounded by air.

Karel heard that eyeglasses were made out of beryl in old times, which inspired its German name, Brille.

We can compute the optical power for a thin lens as

$$\varphi = \left(\frac{n_1}{n_2} - 1\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\,,$$

where n_2 is the refractive index of the surrounding environment, in our case air with a refractive index of $n_\mathrm{a} = 1$, n_1 is the refractive index of a lens (either plastic or beryl) and r_1, r_2 are the radii of curvature of the lens. However, radii aren't really important for us, because we know, that both of the lenses have the same shape; therefore, the value $(1/r_1 + 1/r_2)$ remains the same for both lenses. We can determinate this expression from the value of the optical power of the plastic lens

$$\varphi_\mathrm{p} = (n_\mathrm{p} - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$
$$\frac{\varphi_\mathrm{p}}{n_\mathrm{p} - 1} = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\,.$$

Now we only need to subtitute this into the expression for the beryl lens

$$\varphi_\mathrm{b} = (n_\mathrm{b} - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = (n_\mathrm{b} - 1)\frac{\varphi_\mathrm{p}}{n_\mathrm{p} - 1} = 3.57\,\mathrm{D}\,.$$

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Problem 20 ... filling the bathtub 3 points

Robert of mass $m = 60\,\mathrm{kg}$ and of the same density as water wanted to measure the volumetric flow rate of water in the tap while filling his bathtub. The bathtub has a rectangular base of dimensions $150\,\mathrm{cm} \times 75\,\mathrm{cm}$. Robert filled the bathtub with water to the height $h = 20\,\mathrm{cm}$, when he had $p = 80\,\%$ of his body under the water. It took him $t = 8\,\mathrm{min}$ to fill the bath. Find the volumetric flow rate of water in the tap. *Robert spends too much time in the bathtub.*

The volume of the bathtub to the water level is $V_\mathrm{b} = abh = 0.225\,\mathrm{m}^3$. However, Robert dislodges the water of the same volume as his submerged part of his body, which is $V_\mathrm{s} = pm/\rho = 0.048\,\mathrm{m}^3$. The volume of the water in the bathtub is $V = V_\mathrm{b} - V_\mathrm{s} = 0.177\,\mathrm{m}^3$.

Thus, the volumetric flow rate of the water in the tap is

$$Q = \frac{V}{t} \doteq 22\,\mathrm{l{\cdot}min^{-1}}\,.$$

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Problem 21 ... Lipno and relativity 5 points

Jarda found out that running is physically demanding. He wants to accelerate himself to the speed of $99\,\%$ of the speed of light in vacuum; however, he would need lots of energy. To imagine such an amount of energy, he calculated how many times he would have to heat and evaporate the entire Lipno reservoir, which has a volume of 310 million cubic meters. Assume the initial water temperature is always $20\,^\circ\mathrm{C}$. Jarda weighs $75\,\mathrm{kg}$. If Jarda's calculations are correct, what number did he get? *Jarda ran out of energy.*

At the speed v which is so close to the speed of light c, the total energy of a moving object is given by a relativistic relation

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}\,,$$

where m_0 is the rest mass. Now that Jarda is at rest and he's writing a solution to Physics Brawl Online problem, his energy is $E_0 = m_0 c^2$, so the energy he needs to accelerate himself is "only"

$$\Delta E = E - E_0 = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right) = 6.09 m_0 c^2\,.$$

Now, we are going to calculate how much heat is needed for one cycle of heating and evaporation of Lipno. It's the sum

$$Mc_\mathrm{w}\Delta T + Ml = \rho V \left(c_\mathrm{w}\left(T_\mathrm{w} - T_0\right) + l\right) ,$$

where $M = \rho V$ is the mass of water in Lipno, $V = 310 \cdot 10^6\,\mathrm{m^3}$ is the volume of Lipno, $\rho = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water, c_w is specific heat capacity of water, l is the enthalpy of vaporization of water and $T_\mathrm{w} - T_0 = 100\,^\circ\mathrm{C} - 20\,^\circ\mathrm{C} = 80\,^\circ\mathrm{C}$ is the required temperature difference. In the end, we just express the number of these evaporation cycles as

$$n = \frac{6.09 m_0 c^2}{\rho V \left(c_\mathrm{w}\left(T_\mathrm{w} - T_0\right) + l\right)} = 51.3 \doteq 52\,.$$

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Problem 22 ... the intersections of Lissajous curves 5 points

There is a point that oscillates along the x axis according to the equation $x = x_\mathrm{m}\sin 5\omega t$ and along the y axis according to $y = y_\mathrm{m}\sin 6\omega t$, where x_m and y_m are amplitudes (which are not necessarily the same), t is time and 5ω and 6ω are angular frequencies of oscillations. How many intersection points of the motion graph are there in the xy plane?

Karel loves to create problems on Lissajous figures.

The quickest approach to finding the solution is to plot a graph in arbitrary software, supporting parametric equations. One of the most direct options is to use Wolfram Alpha [2], which is available for free. We plot a graph for $x_\mathrm{m} = y_\mathrm{m} = 1$ as these values only scale the height and width of our graph but don't change the number of intersections. We similarly consider $\omega = 1$, because it only tells us how fast the point travels through the graph. Due to the remark that the movement repeats to infinity, even very slow oscillations plot the whole graph. If we get a graph similar to the one we see in the figure 3, it only remains to calculate the points where the Lissajous curve intersects with itself. If we go by "intersection columns", we add $4+5+4+5+4+5+4+5+4+5+4 = 49$. If we took it line by line, we would get the same result $5+6+5+6+5+6+5+6+5 = 49$. So the graph of motion has altogether 49 points of intersection in different locations.

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Problem 23 ... conductor in a hurry 5 points

At the beginning of every rehearsal session of the symphonic orchestra, all the players tune their instruments. The process begins with a piano or an oboe giving off the tone A (in this problem, we assume it has a frequency $f_\mathrm{A} = 443\,\mathrm{Hz}$), and the other musicians tune their instruments according to them. Imagine that the conductor is still running around during the process with velocity $v_\mathrm{c} = 3\,\mathrm{m{\cdot}s^{-1}}$. Suppose that when running from the oboist to the pianist, the sound waves of their instruments interfere. What beat frequency is he going to hear?

Vojta was tuning his cello unsuccessfully.

[2] https://tinyurl.com/lissajous-intersections

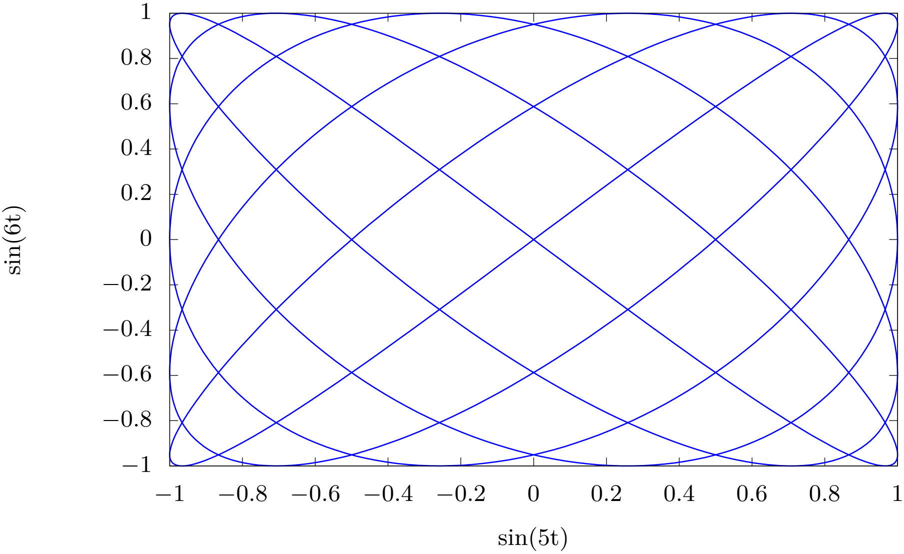

Fig. 3: Plotted Lissajous figure.

We will determine the beat frequency f_b as the difference of the two frequencies that reach the conductor

$$f_\mathrm{b} = f_\mathrm{p} - f_\mathrm{o} = f_\mathrm{A}\left(\frac{c + v_\mathrm{c}}{c}\right) - f_\mathrm{A}\left(\frac{c - v_\mathrm{c}}{c}\right) = 2f_\mathrm{A}\frac{v_\mathrm{c}}{c} \doteq 7.74\,\mathrm{Hz}\,.$$

Alternatively, we might interpret the situation as if the conductor was running along a standing wave formed by the interference of the two tones. The conductor then hears the beat every time he passes through one of the antinodes, which are half a wavelength apart, that is $\frac{c}{2f_\mathrm{A}}$. The beat frequency then is the inverse of time T required for the conductor to travel this distance

$$f_\mathrm{b} = \frac{1}{T} = 2f_\mathrm{A}\frac{v_\mathrm{c}}{c}\,,$$

which gives the same result.

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Problem 24 ... height does not matter 4 points

Consider a classical mathematical pendulum – a point mass on a massless string. Such a pendulum is placed in a hot air baloon and is left to rise through the atmosphere. The temperature t of the surrounding air decreases linearly with the height above the sea level h as $t = t_0 - kh$, where $t_0 = 25\,{}^\circ\mathrm{C}$ and $k = 0.007\,\mathrm{K{\cdot}m^{-1}}$. At the same time, however, the magnitude of the Earth's gravitational field slowly decreases as the height increases. We want the period of oscillations of the pendulum to remain independent of the height h. What should the (non-zero) value of the thermal expansion coefficient α of the string be?

Jarda gets cold on the 16th floor of dormitories.

A period of mathematical pendulum is

$$P = 2\pi\sqrt{\frac{l}{g}}\,,$$

where g is the acceleration due to gravity and l is the length of the string. However, the acceleration due to gravity decreases with the increasing altitude h, because the gravitational force decreases with increasing distance from the Earth's surface. Neglecting the centrifugal force leads to

$$g = \frac{GM}{r^2} = \frac{g_0 R_\oplus^2}{(R_\oplus + h)^2} \approx g_0\left(1 - \frac{2h}{R_\oplus}\right)\,,$$

where $R_\oplus$ is the Earth's radius and g_0 is the acceleration due to gravity at the Earth's surface.

If the period is to remain constant, while acceleration decreases due to gravity, the length of the string has to shorten as well. However, this already happens due to thermal expansion. The length of the pendulum changes linearly with temperature according to relation

$$l = l_0\left(1 - \alpha(t_0 - t)\right) = l_0\left(1 - \alpha(t_0 - t_0 + kh)\right) = l_0\left(1 - \alpha kh\right)\,,$$

where α is the thermal expansion coefficient we are searching for. Since the period does not change, it holds

$$\frac{l_0}{g_0} = \frac{l}{g} = \frac{l_0(1-\alpha kh)}{g_0\left(1 - \frac{2h}{R_\oplus}\right)}\,,$$

which leads to

$$\alpha = \frac{2}{R_\oplus k} \doteq 4.5\cdot 10^{-5}\,\mathrm{K^{-1}}\,.$$

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Problem 25 ... the hollow Earth has many variants 5 points

How much would the value of the gravitational acceleration at the Earth's surface decrease if our planet was only a spherical shell of the same outer radius the Earth has now and of thickness $D = 100.0\,\mathrm{km}$? Consider the spherical shell to be homogeneous and of the same density as the Earth's average density. The result should be the ratio of this acceleration to the actual gravitational acceleration acting at the Earth's surface.

Karel keeps returning to the topic of spheres.

The mass of the planet is spherically symmetrically distributed. Thus, at its surface, we can assume that the problem is equivalent to the situation where the mass is concentrated in its center of gravitation, from which we are distant $R_\oplus$ (radius of the Earth). With the use of Newton's formula, we obtain force acting on a body of mass m, or the acceleration, respectively

$$F = G\frac{mM_\oplus}{R_\oplus^2}\,, \quad a_\mathrm{g} = G\frac{M_\oplus}{R_\oplus^2}\,.$$

Because the planet's mass is the only subject to change (in a way that both cases share the same density, but the new "planet" will be hollow), then the original and new acceleration should differ only in volume. The volume of a hollow sphere expressed by D and $R_\oplus$ is

$$V_\mathrm{H} = \frac{4}{3}\pi\left(R_\oplus^3 - (R_\oplus - D)^3\right) = \frac{4}{3}\pi\left(3R_\oplus^2 D - 3R_\oplus D^2 + D^3\right)$$

The gravitational acceleration on the surface of the hollow planet would be

$$a_\mathrm{H} = \frac{V_\mathrm{H}}{V_\oplus}a_\mathrm{g} = \frac{3R_\oplus^2 D - 3R_\oplus D^2 + D^3}{R_\oplus^3}a_\mathrm{g} \doteq 0.046\,3\,g\,.$$

The gravitational acceleration on the surface of a hollow planet would be less than five percent of the actual gravitational acceleration at the Earth's surface. The thinner the shell, the more negligible the second and third terms in the sum become. If the second and the third terms are neglected, the result will differ from the second valid digit. Neglecting the third term alone will be reflected only in the fifth valid digit.

According to the assignment, we considered only gravitational acceleration (not the net acceleration usually denoted as gravity), as we neglected the centrifugal force from the Earth's rotation. If we wanted to think about the acceleration we would feel on such a planet, we had to know the specific position and speed of rotation. However, we were not provided any information on whether the imaginary planet would rotate in the same or different direction compared to the Earth. One could even deduce that if they wanted the shell-shaped planet to be stable, it should rotate at a lower rate thanks to its less weight.

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Problem 26 ... an oscillating pulley 5 points

Consider a weightless pulley hanging from the ceiling on a spring with stiffness $k = 80\,\mathrm{N/m}$. A weightless rope is passed over the pulley. While one end of the rope is attached to the ground, there is a body of mass $m = 1\,\mathrm{kg}$ attached to the other end. If we pull the body slightly downwards and then release it, what will the period of small oscillations of the system be?

Lego loves oscillations and pulleys, so he finally put them together.

If we denote the distance, along which we have to pull the body down to the ground, as x, the pulley lowers its position just by $x/2$. This comes from the idea that if we pulled both ends of the rope simultaneously by x, then the pulley would also change its position by x. However, we pulled only one end, while the other is fixed; therefore, the pulley lowers by hlaf the distance.

Thus, the spring prolongs by $x/2$ and the force, by which the spring pulls the pulley up, increases by $kx/2$. The question stands, how does the force by which the rope acts on the body increases. The pulley is weightless therefore the force acting on it is always zero. The same stands for the rope, thus the tension has to be the same at each point of the rope. It implies that both parts of the rope (hanging from the pulley) pull the pulley down by the same force. As we already know, the total force (acting on the pulley) increased by $kx/2$; thus the tension in the rope has to increase by half of this value, i.e. $kx/4$.

Finally, if we pull the body down by distance x, the acting force, by which the rope pulls it up, increases by $kx/4$. Thus the stiffness that the body feels is $(kx/4)/x = k/4$. We can substitute the result to the formula for the period of the linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k/4}} = 4\pi\sqrt{\frac{m}{k}} \approx 1.4\,\mathrm{s}\,.$$

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Problem 27 ... one increased, the other increased too — 5 points

Lego was captured by cannibals. They imprisoned him in a hut, where he had an ideal DC voltage source, many perfect conductors and several adjustable resistors (rheostats). To stay alive, Lego must create a specific electric circuit. In such a circuit, there has to be at least one rheostat such that increasing its resistance causes the current flowing through some other rheostat (in the same circuit) to increase. What is the minimum number of rheostats this circuit needs to contain? If such a circuit does not exist, the answer should be 0.

Lego has no clue whether anyone could fill themselves up by him.

We will start by assuming the smallest possible number of rheostats (theoretically). For one rheostat, we can quickly realize that it won't be possible to meet the given conditions because if we increase the resistance on it, the current flowing through it will decrease (and there is no other rheostat in the circuit that would meet the second condition).

We can look at the case of 2 rheostats. If we connect them in series, we reduce the current flowing through both rheostats by increasing the resistance on either of them, so again, we do not meet the second condition. If we connect two rheostats in parallel, and we increase the resistance on one of them, the current flowing through it decreases, and the current flowing through the other rheostat remains unchanged. Even though we still don't have the right solution, you may intuitively feel that we are getting closer.

Let's move on to the case of 3 rheostats. We realize that exclusively serial or exclusively parallel connections will behave analogously to the case of two rheostats. If we connect two rheostats in series and the third one in parallel to them, we again get a situation identical to two rheostats connected in parallel. So we will evaluate another possible connection - two rheostats (let's note them R_1 and R_2), connected in parallel, and the third (R_3) will be connected in series. As the resistance of R_3 increases, the current flowing through the remaining two rheostats decreases, and the other way around, as the resistances of R_1 and R_2 decrease, the current through R_3 decreases (we do not consider the case when one of the pair R_1 and R_2 is zero, as the current would not change again, which is not the case we are looking for). However, let's think about what will happen to the current flowing through R_2, if we increase the resistance of R_1.

First, we need to express the current flowing through R_2 (let's denote it as I_2, and the other currents by analogy). The total current flowing through the circuit is then also equal to I_3. When the circuit splits into branches, the total current I_3 must divide between the two branches, and in the inverse ratio of resistances (a larger current will flow through the smaller resistance). The current $I_2 = I_3R_1/(R_1+R_2)$ will go through R_2. We still need to calculate the total current I_3. We can determine it as the ratio of the voltage U on the source and the total resistance. The total resistance is the sum of the resistance R_3 and the parallel connection, which has the resistance $R_1R_2/(R_1+R_2)$. Thus, the total current is

$$I_3 = \frac{U}{R_3 + R_1R_2/(R_1+R_2)} = U\frac{R_1+R_2}{R_1R_3+R_2R_3+R_1R_2}\,.$$

From here, we can express the current flowing through R_2 as

$$I_2 = I_3\frac{R_1}{R_1+R_2} = U\frac{R_1}{R_1+R_2}\frac{R_1+R_2}{R_1R_3+R_2R_3+R_1R_2} = U\frac{R_1}{R_1R_3+R_2R_3+R_1R_2}\,.$$

We would get the same result if we firstly calculated how the voltage distributes and then determined the ratio of the voltage on the parallel connection and R_2.

But back to the main question – is there any possibility that by increasing R_1, I_2 will increase, too? It seems we have no other choice but to differentiate the result with respect to R_1

$$I_2 = I_3\frac{R_1}{R_1+R_2} = U\frac{R_1}{R_1+R_2}\frac{R_1+R_2}{R_1R_3+R_2R_3+R_1R_2} = U\frac{R_1}{R_1R_3+R_2R_3+R_1R_2}\,.$$

We obtain that the derivative I_2 with respect to R_1 is (for positive resistances) always positive, i.e., the current flowing through the resistance R_2 increases with the resistance R_1. So we have found the circuit that the cannibals were demanding from Lego, and we needed just three rheostats to build this circuit. At the same time, we verified (in the beginning of the solution) that it would not be possible to meet the required conditions for a lower number of rheostats. Hence, the correct answer to the problem is that the circuit must contain at least three rheostats.

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Problem 28 ... Doppler on his way home — 6 points

Christian Doppler was on his way home. After a while, he noticed that people in both directions (the same as his and the opposite) walk at a speed of $v = 1.0\,\mathrm{m\cdot s^{-1}}$ and have distances of $l = 4.0\,\mathrm{m}$ between them. Doppler decided to take the chance to find the speed at which he should walk in order to meet as few people as possible on his way home. What is the minimum number of people that Doppler will meet on his way home if he's $d = 5.2\,\mathrm{km}$ away from his home? You may assume that $d \gg l$, do not consider relativistic effects. *Legolas is kind of autistic.*

Let v_D be Doppler's velocity. The time that passes between meeting two people walking in the opposite direction is

$$t_\mathrm{p} = \frac{l}{v+v_\mathrm{D}}\,.$$

Therefore, the frequency of meeting people walking in the opposite direction is

$$f_\mathrm{p} = \frac{1}{t_\mathrm{p}} = \frac{v+v_\mathrm{D}}{l}\,.$$

Similarly, we can express the frequency of meeting people walking in the same direction as

$$f_\mathrm{r} = \frac{1}{t_\mathrm{r}} = \frac{|v - v_\mathrm{D}|}{l}\,.$$

The number of people that Doppler has met is given by the product of the sum of these meeting frequencies and the total time of home travel $t = \frac{d}{v_\mathrm{D}}$. Precisely, this way of reasoning is possible only in given limit $d \gg l$. The solution is now divided into two cases according to the sign of the argument of the absolute value.

Case when people overtake Doppler ($v_\mathrm{D} \leq v$)

Final frequency takes form

$$f_\mathrm{v} = \frac{v + v_\mathrm{D}}{l} + \frac{v - v_\mathrm{D}}{l} = 2\frac{v}{l}\,,$$

and is independent on Doppler's speed. The number of people encountered is therefore

$$N = f_\mathrm{v} t = 2\frac{v}{l}\frac{d}{v_\mathrm{D}}$$

with the minimal value if, and only if the speed is the maximum possible of the case $v_\mathrm{D} = v$. He totaly meets $N_\mathrm{min} = 2d/l$ people.

Case when Doppler overtakes people ($v_\mathrm{D} \geq v$)

We have the frequency

$$f_\mathrm{v} = \frac{v + v_\mathrm{D}}{l} + \frac{v_\mathrm{D} - v}{l} = 2\frac{v_\mathrm{D}}{l}\,.$$

Total number of the people he meets is given as

$$N = f_\mathrm{v} t = 2\frac{v_\mathrm{D}}{l}\frac{d}{v_\mathrm{D}} = 2\frac{d}{l} = N_\mathrm{min}\,,$$

so if he travels faster than v he encounters N_min people independent of his velocity. The answer to the task is therefore $N_\mathrm{min} = 2d/l = 2\,600$.

Note that interpreting the bywalkers as a wave with wavelength l and propagating at a speed v provides us with the frequency

$$f_\mathrm{p} = \frac{v + v_\mathrm{D}}{l}\frac{v}{v} = \frac{v + v_\mathrm{D}}{v} f_0\,,$$

where $f_0 = \frac{v}{l}$. Thus, this is just the Doppler effect for the moving observer.

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Problem 29 ... psoon 5 points

Jarda took a spoon out of a dishwasher and hung it on a cutlery stand. Obviously, Jarda does not hang cutlery perfectly, so the spoon is now swinging from side to side. Determine the period of its oscillations if we approximate its shape to be planar, composed of a circle of radius $R = 1.5\,\mathrm{cm}$ connected to a rectangle of width $a = 0.7\,\mathrm{cm}$ and length $b = 9\,\mathrm{cm}$. The spoon is $1\,\mathrm{mm}$ thick and is made of a material with density $\rho = 8000\,\mathrm{kg\cdot m^{-3}}$. The spoon is hanging by a small hole which is located $s = 1\,\mathrm{cm}$ from the end of the spoon and in the middle of its width.

Jarda lacks a dishwasher in the dorm.

We will use the relation for the period of a physical pendulum, which is

$$T = 2\pi\sqrt{\frac{J}{mgd}}\,,$$

where J is the moment of inertia of the spoon with respect to the axis of rotation, m is its mass, g is the gravitational acceleration, and d is the distance of the center of gravity from the rotational axis. The center of gravity is located on the axis of symmetry of the spoon, so we only need to calculate one coordinate. We will measure the distance from the upper (rectangular) end of the spoon. The center of gravity of the rectangle is at $x_1 = \frac{b}{2}$, and the one of the circle is at $x_2 = b + R$. To find the position of the center of gravity of the entire object, we must use the relation

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\,.$$

Because the thickness of the spoon is uniform, the mass of each part is proportional to its surface area. Thus, the position of the centre of gravity can be found as

$$x = \frac{ab\frac{b}{2} + \pi R^2 (b + R)}{ab + \pi R^2} = 7.67\,\mathrm{cm}\,,$$

therefore, the distance of the centre of gravity from the axis of rotation is $d = x - s = 6.67\,\mathrm{cm}$. We still have to calculate the moment of inertia, which will be the sum of the moment of inertia of the circle and the rectangle with respect to the rotational axis. The moment of inertia of the circle with respect to the axis passing through its center is $\frac{1}{2} m_\mathrm{c} R_\mathrm{c}^2 = \frac{1}{2}\pi\rho t R_\mathrm{c}^4$, where m_c is the mass of the circle and R_c its radius. Using Steiner's parallel axis theorem, we find the moment of inertia of this part of the spoon with respect to the axis of rotation as

$$J_\mathrm{c} = \frac{1}{2} m_\mathrm{c} R_\mathrm{c}^2 + m_\mathrm{c} (b + R - s)^2 = \pi R^2 t\rho \left(\frac{R^2}{2} + (b + R - s)^2\right)\,,$$
$$J_\mathrm{c} \doteq 5.167\cdot 10^{-5}\,\mathrm{kg\cdot m^2}\,.$$

Similarly, we can find the moment of inertia of the rectangle with respect to the rotation axis (with respect to the center, it would be $\frac{1}{12} m_\mathrm{r}\left(a^2 + b^2\right)$) as

$$J_\mathrm{r} = \frac{1}{12} m_\mathrm{r}\left(a^2 + b^2\right) + m_\mathrm{r}\left(\frac{b}{2} - s\right)^2 = abt\rho\left(\frac{a^2 + b^2}{12} + \left(\frac{b}{2} - s\right)^2\right)\,,$$
$$J_\mathrm{r} \doteq 9.597\cdot 10^{-6}\,\mathrm{kg\cdot m^2}\,.$$

The mass of the entire spoon is

$$m = m_\mathrm{r} + m_\mathrm{c} = t\rho\left(ab + \pi R^2\right) ,$$
$$m \doteq 10.69\,\mathrm{g} .$$

The final equation representing the period is then

$$T = 2\pi\sqrt{\frac{\pi R^2\left(\frac{R^2}{2} + (b+R-s)^2\right) + ab\left(\frac{a^2+b^2}{12} + \left(\frac{b}{2}-s\right)^2\right)}{g\left(ab\frac{b}{2} + \pi R^2\left(b+R\right) - s\left(ab+\pi R^2\right)\right)}} ,$$
$$T \doteq 0.59\,\mathrm{s} .$$

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Problem 30 ... on a beach — 6 points

Jarda is playing ball games with his friends on the beach. In water, Jarda can move with a speed of $0.7\,\mathrm{m\cdot s^{-1}}$*, while on land, he can move with a speed of* $1.0\,\mathrm{m\cdot s^{-1}}$*. Jarda is standing exactly on the straight line which forms the border between water and land and knows that the ball will fall near him in exactly* 3 *seconds. What is the area of the region (on land or water) where the ball might fall in such a way that Jarda will be able to catch it before it hits either land or water?* *Jarda spent some time on a beach in Greece instead of creating problems.*

Jarda has three options go. The first is to move on the beach, where in time $t = 3\,\mathrm{s}$ he can get to the distance $s_\mathrm{s} = v_\mathrm{s}t$ in any direction. On land, he can therefore cover an area of size

$$S_\mathrm{s} = \frac{\pi s_\mathrm{s}^2}{2} \doteq 14.14\,\mathrm{m}^2 .$$

The second option is to run in the water. Here the maximum distance is $s_\mathrm{v} = v_\mathrm{v}t$. However, there is a third, combined option. He can run on the beach, exactly at the borderline, and at some point begin to move through the water.

To describe this motion, we will use an analogy with light refraction. Light follows the curves with the shortest time; in other words, it travels the maximum possible distance in a given time. This is what we need. We will describe Jarda's motion using Snell's law, where the interface is the water-land borderline. However, Snell's law has some refractive indices that do not sunbathe at this beach, so we must change the law a little. By definition, the refractive index is the ratio of the speed of light in a vacuum c and the speed of light in a given medium v. We can then rewrite Snell's law as

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} .$$

In our scenario, Jarda is moving on the water-beach borderline, so α_1 is a right angle, and the angle at which he runs into the water (measured from the perpendicular to the shore) is equal to

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} .$$

No matter where Jarda decides to go into the water, it will always be best for him to head at this angle. Let's denote x the distance from the Jarda's initial position to where he decides to enter the water. The distance he can travel in water can be expressed as

$$s_x = v_\mathrm{v} \left(t - \frac{x}{v_s} \right) .$$

The boundary he can reach by this way is a straight line, with one end on the water-land borderline, distant 3.0 m from the initial position; and with the other end in the water, distant 2.1 m from the initial position, at the angle α_2 (measured from the perpendicular). The area that Jarda can cover in this way is, therefore, a right-angled triangle of area

$$S_\mathrm{k} = \frac{1}{2} v_\mathrm{v} t^2 \sqrt{v_\mathrm{s}^2 - v_\mathrm{v}^2} \doteq 2.25\,\mathrm{m}^2 .$$

Regarding the motion in the water, only a circle sector of the angle α_2 remains. It has an area of

$$S_\mathrm{v} = \frac{\alpha_2}{2\pi} \pi s_\mathrm{v}^2 \doteq 1.70\,\mathrm{m}^2 .$$

The last two areas we mentioned have to be considered twice due to the symmetry. In total, Jarda can cover the area of

$$S = S_\mathrm{s} + 2\left(S_\mathrm{k} + S_\mathrm{v}\right) = \frac{\pi \left(v_\mathrm{s} t\right)^2}{2} + v_\mathrm{v} t^2 \sqrt{v_\mathrm{s}^2 - v_\mathrm{v}^2} + \arcsin \frac{v_\mathrm{v}}{v_\mathrm{s}} \left(v_\mathrm{v} t\right)^2 \doteq 22.1\,\mathrm{m}^2 .$$

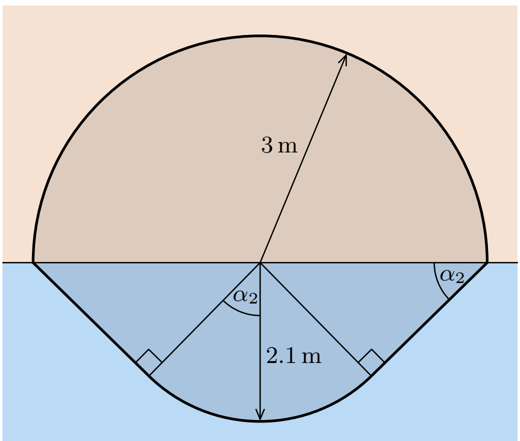

Fig. 4: Jarda's cross section.

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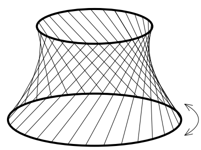
Problem 31 ... an oscillating hoop reloaded 6 points



We have two hoops. The upper one has a radius $r = 1.0\,\mathrm{m}$ and is fixed in a horizontal plane, and the bottom hoop has a radius $R = 1.5\,\mathrm{m}$. The bottom one is attached to the upper hoop by several massless cords, each of which has the same length, and if we look at the hoops from the top, we see the cords pointing radially. Therefore, the bottom hoop "lies" in the horizontal plane, too. The distance between the plane of the upper hoop and that of the bottom hoop is $h_0 = 2.0\,\mathrm{m}$. Let the mass of the bottom hoop be $m = 1.0\,\mathrm{kg}$. If we rotate the bottom hoop a bit around its vertical axis and release it, what is the period of its small oscillations? *Lego adjusted his problem from last year.*

As is the common theme in physics, even the period of small oscillations can be calculated by multiple approaches. Here, we employ a perhaps less well known (you could have seen this approach in the solution to a similar problem in Physics Brawl Online 2020), but very efficient approach to the solution.

The approach is based on the law of conservation of energy. An appropriate coordinate for the description of the problem is the angle of rotation φ of the lower hoop relative to equilibrium.

Kinetic energy of the rotating thin loop is

$$E_\mathrm{k} = \frac{1}{2} I\omega^2 = \frac{1}{2} mR^2\dot{\varphi}^2 \,.$$

Description of potential energy is somewhat more involved. Lets focus now on a single cord and denote its length as l. Denoting the horizontal distance of points of attachement of the cord to the hoops as d and vertical distance of the said points as h, Pythagoras' theorem leads to $l^2 = h^2 + d^2$. In equilibrium, $d = R - r$ and $h = h_0$, and hence $l^2 = h_0^2 + (R - r)^2$. For the evaluation of the potential energy, the change of the distance between the planes the hoops lie in is critical.

We need to determine the distance d after rotating by angle φ. Since d is the horizontal distance, we can use a planar picture. Looking from above, the points of attachement of the cords to the hoops are colinear with the centre of the hoops in the equilibrium ($\varphi = 0$). After rotating by φ, the centre of the hoops, and the two points of attachement form a triangle with sides of lengths R, r, d, where sides R, r meet at angle φ. The distance d can be calculated using the law of cosines

$$d^2 = R^2 + r^2 - 2Rr\cos\varphi \,.$$

Pythagoras' theorem therefore dictates for the distance between the hoops

$$h^2 = l^2 - R^2 - r^2 + 2Rr\cos\varphi \,.$$

Substituting for l^2 and using that for $\varphi \ll 1$ we can approximate $\cos\varphi \approx 1 - \varphi^2/2$, we determine that

$$h^2 = h_0^2 + (R - r)^2 - R^2 - r^2 + 2Rr\left(1 - \frac{\varphi^2}{2}\right) = h_0^2 - Rr\varphi^2 \,.$$

We are however interested in the change of the distance of hoops compared to the equilibrium case. For small x, the approximation $(1 + x)^a \approx 1 + ax$ holds. We therefore get

$$\Delta h = h_0\sqrt{1 - \frac{Rr\varphi^2}{h_0^2}} - h_0 \approx -\frac{Rr\varphi^2}{2h_0} \,.$$

Relative to the equilibrium position, the lower hoop was lifted by $-\Delta h$. The change in potential energy is then

$$E_\mathrm{p} = -mg\Delta h = \frac{1}{2}\frac{mgRr}{h_0}\varphi^2\,.$$

Remember that for a linear harmonic oscillator (point mass on a spring) the kinetic and potential energy are given as

$$E_\mathrm{k} = \frac{1}{2}m\dot{x}^2\,,$$
$$E_\mathrm{p} = \frac{1}{2}kx^2\,.$$

The period of oscillations is then $T = 2\pi\sqrt{m/k}$. We now only need to recognize that the relations for the energy of our system can be transformed to equations of LHO by defining the effective mass $m_\mathrm{ef} = mR^2$ and stiffness $k_\mathrm{ef} = \frac{mgRr}{h_0}$, since the period is independent of the choice of coordinates for the oscillator. The final result is therefore

$$T = 2\pi\sqrt{\frac{m_\mathrm{ef}}{k_\mathrm{ef}}} = 2\pi\sqrt{\frac{mR^2}{\frac{mgRr}{h_0}}} = 2\pi\sqrt{\frac{Rh_0}{rg}} \doteq 3.5\,\mathrm{s}\,.$$

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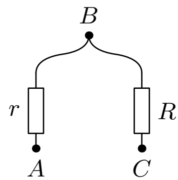
Problem 32 ... finite circuit 6 points

In the figure, you can see a fragment of an electrical circuit with resistors $r = 2.35\,\Omega$ and $R = 271.2\,\Omega$. Any number of fragments can be connected together in a series according to several rules:

- *Each connector A (except for the first fragment) must be connected to exactly one B.*
- *Similarly, each B (except for the last fragment) must be connected to one A.*
- *All connectors C must be connected at one point.*
- *No other connections are allowed.*

Then we connect the A and C of the first fragment with one terminal of a multimeter, and we connect the B of the last fragment to the second terminal. What is the minimum number of fragments we need to use in order to measure a resistance greater than $R_x = 23.7\,\Omega$?



Jáchym wanted to come up with a problem about springs, but somehow it didn't work out.

Let's denote by R_n the resistance of n fragments. From the figure 5, we can derive

$$R_{n+1} = \frac{(r + R_n)\,R}{r + R_n + R}\,,$$

where R_0 is defined as $R_0 = 0$. We are looking for such a natural number n, for which $R_n \geq R_x$. We can write a simple script for this, but even a spreadsheet is sufficient. The result is $n = 26$.

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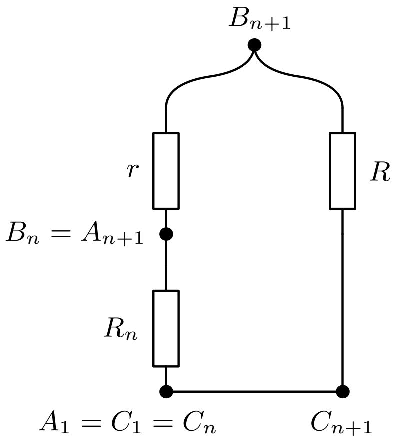

Fig. 5: Circuit diagram of the n+1-th fragment.

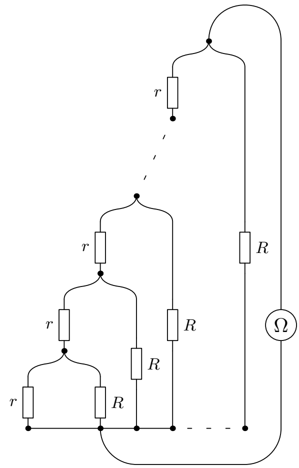

Fig. 6: Circuit diagram of the whole circuit.

Problem 33 ... an inclined roof 6 points

It is raining so much that 20 mm *of precipitation falls in an hour. The rectangular roof of an older building with dimensions* $a = 20\,\mathrm{m}$ *and* $b = 10\,\mathrm{m}$ *has a slope which is very small, but sufficient for all the water from the roof to flow into one place and fall to the ground from there. Determine the force which the stream of falling water exerts on the ground if it bounces upwards after impact with speed* $k = 5$*-times smaller than it was just before impact. Water drains from the roof so fast that a constant amount of water is maintained on it. The roof is at a height of* $h = 3\,\mathrm{m}$ *above the ground. Assume that the water draining from the roof has zero initial vertical velocity.*

Danka was walking in the rain around a building with an inclined roof.

The force by which water (falling from the roof and bouncing off the ground) acts on the ground can be determined using Newton's 2nd law as

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta m \Delta v}{\Delta t}\,.$$

Note that $\frac{\Delta m}{\Delta t}$ is the mass flow Q_m given by the mass of water that falls from the roof at any given moment, thus

$$Q_m = Rab\rho\,,$$

where $R = 20\,\mathrm{mm{\cdot}h^{-1}} = 5.55\,\mathrm{\mu m{\cdot}s^{-1}}$ and $\rho = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water. For the change of the water speed (the bouncing off case), the following holds

$$\Delta v = v_0 + \frac{v_0}{k} = v_0\left(1 + \frac{1}{k}\right)\,,$$

where the speed of a water flow v_0 is calculated from its free-fall as $v_0 = \sqrt{2gh}$, where we assume that the potential energy of water at height h has transformed into kinetic energy. This altogether yields

$$F = Rab\rho\sqrt{2gh}\left(1 + \frac{1}{k}\right) .$$

After substituting the given values, we find that the water flow acts on the ground with a force of $F \doteq 10.2\,\mathrm{N}$.

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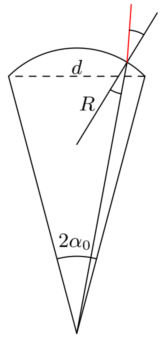
Problem 34 ... ommatidium

6 points

The eyes of insects are composed of many individual small units, the so-called ommatidia. We can consider each ommatidium to be a solid cone with vertex angle $2\alpha_0 = 14^\circ$. While photoreceptors are located at the vertex, the other side of the cone is enclosed by a spherical surface of radius $R = 17\,\mathrm{\mu m}$, which is coaxial with the cone. The diameter of the base of the cone is $d = 25\,\mathrm{\mu m}$. Determine the solid angle that one ommatidium can perceive if its interior is filled with a material of refractive index $n = 1.3$.

Jarda tried a school student project.



Let's begin in reverse order - consider the beam to be emitted from the photoreceptors to the space of the ommatidia. Then it refracts on a spherical surface towards the surrounding world. Let such a ray emanates at an angle α from the cone axis (the one which is perpendicular to bases). Then it impacts the spherical surface of an angle γ from the perpendicular (i.e., from the line connecting the center of curvature and the refractive point of the ray). From the sine theorem, we can determine the sine of the angle γ as

$$\sin\gamma = \frac{l}{R}\sin\alpha\,,$$

where l is the distance from the center of curvature (of ommatidium surface) to the apex of the cone, which can be determined as

$$l = \frac{d}{2\tan\alpha_0} - \sqrt{R^2 - \frac{d^2}{4}} = 90.3\,\mathrm{\mu m}\,.$$

According to Snell's law, we find the exit angle φ from the perpendicular to the spherical surface as

$$\sin\varphi = n\sin\gamma = n\frac{l}{R}\sin\alpha\,.$$

If we subtract angle φ from the angle between line (connecting the point of refraction at the spherical surface and the center of curvature) and the axis of the cone, we get the direction of the ray with respect to the axis of the cone. The angle of this direction can be expressed as

$$\beta = 180 - (180 - \gamma - \alpha) - \varphi = \gamma + \alpha - \varphi = \arcsin\left(\frac{l}{R}\sin\alpha\right) + \alpha - \arcsin\left(n\frac{l}{R}\sin\alpha\right).$$

For example, we can graphically verify that this function is negative and monotonically decreasing. It thus acquires extreme values at the edge of the ommatidia, that is when $\alpha = \alpha_0 = 7^\circ$. For this value, the rays at angle $|\beta| = 10.0^\circ$ from the ommatidia axis also get to the top of the cone. Because $|\beta| > \alpha_0$, the angle of view of the ommatidium is larger than its angular diameter. Note that due to the value of angle β being negative, the ommatidium sees everything in reverse to its axis. We determine the solid angle as the ratio of the area of the spherical cap with apex angle 2β, and the square of the radius of the sphere, i.e.

$$\Omega = \frac{2\pi R v}{R^2} = \frac{2\pi R^2 (1 - \cos\beta)}{R^2} = 2\pi (1 - \cos\beta) = 0.095\,\mathrm{sr}\,.$$

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Problem 35 ... thermodynamic cannon 6 points

We have a horizontally placed cylinder with a base area $S = 2.5\,\mathrm{dm}^2$ and a length $l = 2\,\mathrm{m}$. One end of the cylinder is closed and the other end is not. In the cylinder, there is a piston with a weight of $m = 24\,\mathrm{kg}$, which can move without friction and seals perfectly. The volume of the gas between the piston and the closed end is $V_0 = 3.5\,\ell$. First, everything is in equilibrium – the pressure outside the cylinder and also between the piston and the closed end is equal to normal atmospheric pressure and the temperature is T_0 everywhere. However, suddenly an explosion occurs – the temperature of the enclosed part of the gas changes to $10T_0$, and the amount of substance there becomes twenty times greater. At what speed does the piston leave the cylinder? Assume that classical equilibrium thermodynamics apply and the heat capacity ratio of the gas is $\kappa = 1.4$. *Lego wanted to play a joke on someone.*

Let's call the enclosed part of the gas (between the piston and the closed end of the cylinder) "the inside" and refer to all other gas as "the outside".

What will happen with the gas outside? We can assume the gas outside to be of infinite volume – if we compress it a little (by piston coming off the cylinder), the outside pressure should not change. Consequently, the piston will always be pressed back to the cylinder by pressure p_0.

What will happen with the gas inside? The explosion happened very quickly, therefore there is no time for heat exchange – we have to think of the explosion as an adiabatic process. The adiabatic process keeps the product pV^κ constant. Now we only need to compute the initial value of the pressure. The inside pressure for any position of the piston can be then calculated as

$$p_{\mathrm{in}} = p_{\mathrm{in}_0} \left(\frac{V_0}{V}\right)^\kappa ,$$

where particular volume V is calculated as the product of the area of the cylinder base and the distance between the edge of the piston and the closed base of the cylinder x. The initial value of this distance is $x_0 = V_0/S = 14\,\mathrm{cm}$. We can obtain the pressure just after the explosion using an equation of state $pV = nRT$. The temperature rises by a factor of ten, while the amount of the substance is twenty times its initial value, therefore the right-hand side increases by a factor of 200. The volume on the left-hand side is at the moment of explosion at its initial value, therefore the pressure has to increase by a factor of 200 $p_{\mathrm{in}_0} = 200 p_0$.

The resultant force acting on the piston is $F = S\left(p_{\mathrm{in}} - p_0\right)$. Someone may want to solve a differential equation; it is not necessary here – it is sufficient to integrate this force along the whole path to get the total work the gas did on the piston.

$$W = \int_{x_0}^{l} S\left(200p_0\left(\frac{V_0}{Sx}\right)^{\kappa} - p_0\right)\mathrm{d}x = \left[Sp_0\left(200\left(\frac{V_0}{S}\right)^{\kappa}\frac{1}{1-\kappa}x^{1-\kappa} - x\right)\right]_{x_0}^{l}$$
$$= Sp_0\left(200\left(\frac{V_0}{S}\right)^{\kappa}\frac{1}{1-\kappa}\left(l^{1-\kappa} - x_0^{1-\kappa}\right) - (l - x_0)\right).$$

Now we only need to substitute for x_0. This work has to be equal to the kinetic energy of the piston

$$v = \sqrt{\frac{2}{m}W} = \sqrt{\frac{2}{m}Sp_0\left(200\left(\frac{V_0}{S}\right)^{\kappa}\frac{1}{1-\kappa}\left(l^{1-\kappa} - \left(\frac{V_0}{S}\right)^{1-\kappa}\right) - \left(l - \frac{V_0}{S}\right)\right)} \doteq 96\,\mathrm{m{\cdot}s^{-1}}\,.$$

If someone wanted to avoid the integration, one could obtain the exact solution by using the formula for the work of the adiabatic process (one can notice that even our solution is in the form of "the work done on the piston is equal to the difference of the work done by gas inside and outside").

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Problem 36 ... center of gravity of a snail — 6 points

How far away from the center of a spiral is its center of gravity? Consider a spiral with constant linear density λ, given in polar coordinates as $r = a\varphi$, where $a = 0.1\,\mathrm{m{\cdot}rad^{-1}}$ and φ is the polar angle in radians. The total angle subtended by the spiral is 10π.

Jarda fights snails in his flowerbed.

We set up the Cartesian coordinates in the plane of the spiral. Let axis x have the direction of the angle $\varphi = 0$, and let the ray $\varphi = 90^\circ$ be the positive direction of axis y. Each point of the spiral can be determined by coordinates

$$x = a\varphi\cos\varphi\,,$$
$$y = a\varphi\sin\varphi\,.$$

We can express the length of the element of the spiral arc $\mathrm{d}l$ dependent on angle φ as

$$\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = a\sqrt{\left(\cos\varphi - \varphi\sin\varphi\right)^2 + \left(\sin\varphi + \varphi\cos\varphi\right)^2}\,\mathrm{d}\varphi = a\sqrt{1+\varphi^2}\,\mathrm{d}\varphi\,,$$

where $\mathrm{d}x$ and $\mathrm{d}y$ are the differentials of the equations above. Now we can compute the length of the spiral by expression

$$L = a\int_0^{10\pi}\sqrt{1+\varphi^2}\,\mathrm{d}\varphi\,.$$

Let us firstly compute the x component of the location of the center of gravity. By the definition we get

$$x_C = \frac{\int_0^L x\lambda\,\mathrm{d}l}{\int_0^L \lambda\,\mathrm{d}l} = a\frac{\int_0^{10\pi}\varphi\cos\varphi\sqrt{1+\varphi^2}\,\mathrm{d}\varphi}{\int_0^{10\pi}\sqrt{1+\varphi^2}\,\mathrm{d}\varphi}\,.$$

Both integrals can be computed numerically, e.g., by using WolframAlpha (despite the existence of the analytical solution for the integral in the denominator). We get the first coordinate of the center of gravity

$$x_C \doteq \frac{6.264\,18\,\mathrm{m}}{495.80} \doteq 0.012\,634\,\mathrm{m}\,.$$

We compute the y coordinate of the center of gravity analogically as

$$y_C = a\frac{\int_0^{10\pi} \varphi \sin\varphi \sqrt{1+\varphi^2}\,\mathrm{d}\varphi}{\int_0^{10\pi} \sqrt{1+\varphi^2}\,\mathrm{d}\varphi} \doteq \frac{-98.708\,5\,\mathrm{m}}{495.80} \doteq -0.199\,09\,\mathrm{m}\,.$$

The center of gravity of this part of the spiral is distant

$$d = \sqrt{x_C^2 + y_C^2} = 0.199\,\mathrm{m}$$

from the origin. Regarding the size of the spiral (some points of it are as distant as 3 m from the origin), the location of the center of gravity is relatively close to the center of the spiral. Therefore the spiral is a pretty symmetrical object.

Because the spiral covers angle of 10π and most of its mass is located where φ acquires larger values, we can use approximation $\sqrt{1+\varphi^2} \doteq \varphi$. Integrals for the x coordinate of the center of gravity simplify to

$$x_C = a\frac{\int_0^{10\pi} \varphi^2 \cos\varphi d\varphi}{\int_0^{10\pi} \varphi\,\mathrm{d}\varphi} \doteq \frac{6.283\,\mathrm{m}}{493.5} \doteq 0.012\,7\,\mathrm{m}\,.$$

The other coordinate of the center of gravity returns $-0.199\,9\,\mathrm{m}$, which implies that the center of spiral and its center of gravity are distant 0.200 m. Note that the result obtained by using approximation is very close to the exact one. However, the advantage is that the integrals have an analytical solution, and we do not have to compute them numerically.

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Problem 37 ... very thirsty 6 points

While wandering through a desert, Jarda came across a canister of water. The canister has a cuboid shape. It has width $c = 20\,\mathrm{cm}$, length $b = 40\,\mathrm{cm}$ and height $a = 60\,\mathrm{cm}$. The canister does not have a lid, but it has a tap through which water can flow out, in the middle of one of the bottom edges of length c. The container is 90 percent full. Jarda is so thirsty that he wants the water to flow from the tap as fast as possible. What is the angle (with respect to the vertical) by which he needs to tilt the canister around the edge c? Jarda first tilts the canister and then opens the tap. *Jarda was working at the steppes of South Moravia.*

The water flow rate is proportional to the square root of the height of the water column above the tap, so it is necessary to tilt the container so that the water level above the lower edge is as high as possible.

We find the dependence of the water level on the angle φ. Let h be the height of the water level above the tap, h_1 is the distance of the edge from the tap to the water level and h_2 is the distance from the edge of the base b to the water level. Obviously, we have to tilt the canister to the tap side, so $h_1 \geq h_2$.

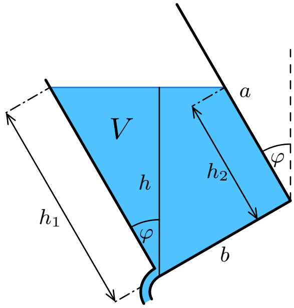

Fig. 7: The canister with water.

Assume that no water has leaked from the top of the canister. Then the volume of water is constant and equals $V = 0.9abc$. From a side view, this volume is an area S (bounded by the water surface and the sides of the container) multiplied by the length c. This area is $S = b\frac{h_1+h_2}{2}$. Notice that $h_1 \cos\varphi = h$ and $b\tan\varphi = h_1 - h_2$. Thus, we can write

$$h_1 = \frac{V}{bc} + \frac{b\tan\varphi}{2}$$

and

$$h = h_1\cos\varphi = \frac{V\cos\varphi}{bc} + \frac{b\sin\varphi}{2}\,.$$

Plugging the volume V into this equation provides

$$h = 0.9a\cos\varphi + \frac{b\sin\varphi}{2}\,.$$

We use the derivative to find the angle for which the function is at its maximum; this angle is $\varphi_{\max} = 20.3^\circ$. Because the canister has no lid, water starts leaking before the canister can be tilted by this angle. If some water leaks from the canister, the water level cannot be at its maximum anymore. The previous function is increasing on the interval from $\varphi = 0$ to $\varphi_{\max}$, so the height of the water level is maximal just when the level touches the upper edge of the canister. This occurs when $h_1 = a$. Therefore

$$\tan\varphi = \frac{2a - \frac{2V}{bc}}{b} = 2a\frac{0.1}{b}$$

and $\varphi = 16.7^\circ$.

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Problem 38 ... rotary capacitor

6 points

A capacitor consists of two rotary semicircular plates with the same radii $R = 2\,\mathrm{cm}$ at a mutual distance of $d = 0.1\,\mathrm{cm}$, separated by air. In the initial state, the plates are exactly on top of each other, the capacitor is charged to a voltage $U_0 = 20\,\mathrm{V}$ and then the voltage source is *disconnected. When rotating one plate away from the other (around the axis passing through the midpoint of the straight side of both plates), we act with a torque. What is the initial magnitude of this torque?* *Jarda couldn't loosen a screw.*

On the plates, the charge Q remains. According to the problem assignment, it is (air permittivity is approximately the same as for vacuum)

$$Q = CU_0 = \frac{\varepsilon S U_0}{d} = \frac{\varepsilon \pi R^2 U_0}{2d}\,.$$

When rotating the plates against each other, the capacitance of the capacitor changes and so does its energy $\frac{Q^2}{2C}$. At the beginning, the area of the capacitor is $\frac{\pi R^2}{2}$, when rotated by an angle φ the area of the capacitor is $\frac{(\pi-\varphi)R^2}{2}$. We substitute the formula for the capacitance $\frac{S\varepsilon}{d}$ into the expression for energy and differentiate with respect to angle, which gives us the torque (so-called virtual work principle)

$$\tau = \frac{\mathrm{d}E}{\mathrm{d}\varphi} = \frac{\partial E}{\partial C}\frac{\mathrm{d}C}{\mathrm{d}\varphi} = \left(-\frac{Q^2}{2C^2}\right)\left(-\frac{\varepsilon R^2}{2d}\right)\,.$$

The magnitude of the torque in the initial position $\varphi = 0$, when $C = \frac{\varepsilon \pi R^2}{2d}$, is

$$\tau = \frac{\varepsilon R^2 U_0^2}{4d} = 3.5 \cdot 10^{-10}\ \mathrm{N{\cdot}m}\,.$$

In the initial position, all marginal phenomena are negligible.

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Problem 39 ... inevitable fall 7 points

Imagine a body of a weight $M = 72\,\mathrm{kg}$, which we release from a height of $h = 12\,\mathrm{m}$ above the ground. At that moment, we start firing at it with a machine gun with a rate of fire $f = 2\,000\,\mathrm{min}^{-1}$ from the point where it would soon fall (the first projectile is shot at $1/f$ after releasing). The projectiles have a mass of $m = 11.3\,\mathrm{g}$, move at a speed of $v = 790\,\mathrm{m{\cdot}s^{-1}}$ and after collisions, they stay inside the body. How long would it take for the body to fall to the ground? *Jáchym knew it would eventually fall, but he wanted to know when exactly.*

Let's call the body's velocity immediately before i-th collision v_i and immediately after i-th collision v_i', both pointing down. The body is falling in a gravitational field, which means, that after time $T = f^{-1}$ it gains velocity $v_{i+1} - v_i' = gT$. Its mass will analogously be $M_i = M + (i-1)\,m$, $M_i' = M + im = M_{i+1}$. Now we write down the equation for perfectly inelastic collision, where only the momentum is conserved,

$$-mv + M_i v_i = (m + M_i)\, v_i' = M_{i+1} v_i'\,,$$

where we've neglected all the small changes in projectiles' velocity due to gravity, for those changes would not (considering high velocity of projectiles and small value of T) have a chance to take effect. We as well neglect the fact, that the projectiles are in fact hitting the body with slightly different period than T, because the body's moving towards them. From the previous equation we get

$$\begin{aligned} v_i' &= \frac{-mv + M_i v_i}{M_{i+1}} = \frac{-mv + (M + (i-1)\,m)\,v_i}{M + im}\,, \\ v_{i+1} &= v_i' + gT = \frac{-mv + (M + (i-1)\,m)\,v_i + (M + im)\,gT}{M + im} = \\ &= \frac{-v + (k + i - 1)\,v_i + (k + i)\,gT}{k + i} \approx -\frac{v}{k+i} + v_i + gT\,, \end{aligned}$$

where $k = \frac{M}{m}$. The approximation at the end is reasonable, because $k \gg 1$ and so $(k + i - 1)\,/\,(k + i) \approx 1$. From here apparently

$$v_i \approx igT - v \sum_{j=2}^{i} \frac{1}{k+j}\,.$$

Notice, that we're computing the sum from $j = 2$, because $v_1 = gT$ has to hold – we're considering the first collision in the time T, so up until that moment, the body's falling "as usual".[3] It would now be easier for us to work with the velocity at the beginning of the interval after collision, so we express it by substituting for v_{i+1} into the original equation

$$v_i' = v_{i+1} - gT \approx (i+1)\,gT - v \sum_{j=2}^{i+1} \left(\frac{1}{k+j}\right) - gT = igT - v \sum_{j=1}^{i} \left(\frac{1}{k+j+1}\right)\,.$$

Total distance traveled until the moment before n-th collision will thus be

$$\begin{aligned} x_n &= \sum_{i=0}^{n-1} \left(v_i' T + \frac{1}{2} g T^2\right) = gT^2 \sum_{i=0}^{n-1} i - vT \sum_{i=0}^{n-1} \sum_{j=1}^{i} \frac{1}{k+j+1} + \frac{n}{2} g T^2 = \\ &= \frac{n^2}{2} g T^2 - vT \sum_{i=0}^{n-1} \sum_{j=1}^{i} \frac{1}{k+j+1} = \frac{n^2}{2} g T^2 - vT\sigma(n)\,. \end{aligned} \tag{1}$$

Now we want to figure out, after how many steps will $x_n > h$ hold. To that end, we can use can use a spreadsheet software like MS Excel or some kind of script. The second option is to

[3] We're neglecting the time it takes the bullet to reach the body, so we assume, that the collision happens exactly at the same time as the shooting. The period $T \doteq 30\,\mathrm{ms}$ is of the same order as flight time of the first bullet (approximately $15\,\mathrm{ms}$), so it might seem this assumption is unreasonable. However, the inaccuracy in the final result will truly be negligible.

approximate the last sum as

$$\begin{aligned}\sigma(n) &= \frac{1}{k}\sum_{i=0}^{n-1}\sum_{j=1}^{i}\left(1+\frac{j}{k}+\frac{1}{k}\right)^{-1} \approx \frac{1}{k}\sum_{i=0}^{n-1}\sum_{j=1}^{i}\left(1-\frac{j}{k}-\frac{1}{k}\right) = \\ &= \frac{1}{k^2}\sum_{i=0}^{n-1}\sum_{j=1}^{i}\left((k-1)-j\right) \quad = \frac{1}{k^2}\sum_{i=0}^{n-1}\left((k-1)\,i-\frac{i\,(i+1)}{2}\right) = \\ &= \frac{1}{2k^2}\sum_{i=0}^{n-1}\left((2k-3)\,i-i^2\right) \quad = \frac{1}{2k^2}\left((2k-3)\,\frac{n\,(n-1)}{2}-\frac{n\,(n-1)\,(2n-1)}{6}\right) = \\ &= \frac{n\,(n-1)\,(3k-4-n)}{6k^2}\,,\end{aligned}$$

which could be done, if $k \gg n$. This holds more than enough for given values values, which means that the function we've found approximates the sum almost with no inaccuracies. At the same time, it is in a way more interesting result than previously derived x_n, which describes the distance traveled by the body only at discrete times $t_n = nT$. Instead, we could now consider the continuous time t and define n outside of the integers as $n(t) = t/T$. By substituting into the expression (1) and by approximating the sum according to the equation above we get

$$X(t) = \frac{1}{2}gt^2 - \frac{vt}{6k^2}\left(\frac{t}{T}-1\right)\left(3k-4-\frac{t}{T}\right)\,.$$

We've already shown, that for integer values of n, $X(nT) \approx x_n$ holds. Now though, we'll go even further and without further proof we say, that $X(t)$ approximates the actual distance traveled by the body $x(t)$ not only in the "integer" times $t = nT$, but also for every real t.[4]

Some kind of physical intuition would be, that we're finding a 3rd polynomial to fit values x_n. And as we know, if the two polynomials of at most N-th degree are equal at at least $N+1$ points, they are necessarily equal at all the other points. And even though in this case functions x and X equal at the points x_n only approximately (and x apparently can't be a polynomial thanks to discontinuous first derivation), the approximation is correct enough with the required precision.

Now it only remains to let $X(t) = h$ and to find the time t at which body hits the ground. Situation is slightly complicated by the fact, that we're solving a general cubic in the form

$$0 = vt^3 + 3\left(gk^2T^2 - vT\,(k-1)\right)t^2 + vT^2\,(3k-4)\,t - 6hk^2T^2\,.$$

We can however easily numerically compute the only positive solution $t = 2.04\,\mathrm{s}$.

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[4] Obviously only for sufficiently small values – after shooting enough projectiles, the condition $k \gg n$ would cease to hold. Fortunately, the body would have fallen on the ground at this point.

Problem 40 ... life is short
7 points

Matěj found a piece of pure radium 225. After a while, he discovered that it decayed by β^- decay to actinium with a half-life of 15 days. However, the story does not end here. Actinium is subject to α decay to francium with a half-life of 10 days. In how many days (from the moment he found the radium) would Matěj have had the maximum amount of actinium?

Matěj stole this from a lecture on Computer Methods.

Let us denote the amount of radium that Matěj posesses at time t by $R(t)$, and the amount of actinium $A(t)$, respectively. Let λ_R and λ_A be the appropriate exponential decay constant. The setup can be described as follows

$$R(t) \xrightarrow{\lambda_R} A(t) \xrightarrow{\lambda_A} F(t)\,.$$

For $R(t)$ and $A(t)$, we can compile the following differential equations

$$\begin{aligned}\dot{R} &= -\lambda_R R(t)\ ,\\ \dot{A} &= -\lambda_A A(t) + \lambda_R R(t)\ .\end{aligned} \tag{2}$$

The solution of the first equation is clearly

$$R(t) = R_0 \mathrm{e}^{-\lambda_R t}\,,$$

where R_0 is the initial amount of Matěj's radium. We plug this solution into (2)

$$\dot{A} = -\lambda_A A(t) + R_0 \lambda_R \mathrm{e}^{-\lambda_R t}\,.$$

The solution of this non-homogeneous differential equation is a bit more complicated. By the initial condition $A(0) = 0$, we get the expression

$$A(t) = \frac{\lambda_R R_0}{\lambda_R - \lambda_A}\left(\mathrm{e}^{-\lambda_A t} - \mathrm{e}^{-\lambda_R t}\right)\,.$$

The amount of actinium is at its maximum if and only if the expression in brackets gets the extremal value. Now, from the equation $\dot{A}(t_{\mathrm{max}}) = 0$ we get

$$\lambda_A \mathrm{e}^{-\lambda_A t_{\mathrm{max}}} - \lambda_R \mathrm{e}^{-\lambda_R t_{\mathrm{max}}} = 0 \quad \Rightarrow \quad t_{\mathrm{max}} = \frac{t_R t_A}{t_A - t_R}\frac{\ln\left(\frac{t_A}{t_R}\right)}{\ln(2)} \doteq 17.5\,\mathrm{days}\,,$$

where $t_{R,A}$ represents half-lifes, we also used formula $\lambda_{R,A} = \frac{\ln(2)}{t_{R,A}}$.

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Problem 41 ... radiant
5 points

An ideal black body radiates with the maximum intensity I_λ at a wavelength of $\lambda_{\mathrm{max}} = 598.34 \cdot 10^{-9}\,\mathrm{m}$. At what wavelength does the black body radiate with half of the maximum intensity? Submit the largest correct answer.

Danka thought she wouldn't like to solve this problem.

The black-body radiation is described by the Planck's law. The law can be written in two forms

$$\mathrm{d}I_\nu = B_\nu \,\mathrm{d}\nu\,, \qquad \mathrm{d}I_\lambda = B_\lambda \,\mathrm{d}\lambda\,,$$

where quantities with the index ν differ from quantities marked with the index λ. They are coupled together, and we can obtain one from another respecting the condition of total intensity equality

$$I = \int_0^\infty B_\nu \,\mathrm{d}\nu = \int_0^\infty B_\lambda \,\mathrm{d}\lambda\,.$$

To calculate it, we need the relation between the wavelength λ and the frequency ν

$$\nu\lambda = c \rightarrow \mathrm{d}\nu = -\frac{c}{\lambda^2}\,\mathrm{d}\lambda\,.$$

A well-known form of the Planck's law for B_ν gives us

$$B_\lambda = B_\nu(\nu = \frac{c}{\lambda}) \cdot \frac{c}{\lambda^2} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}\,.$$

We can obtain the function's maximum by differentiating it; however, it does not have a nice analytic solution (it contains the Lambert W function). The final result is called Wien's displacement law, and it holds as

$$\lambda_{\mathrm{max}} T = b = 2.897771955 \cdot 10^{-3}\,\mathrm{m{\cdot}K}\,.$$

Using the given maximum wavelength value, we get the temperature of the black body $T = 4843.0\,\mathrm{K}$. The next step is to determine the value of the radiation intensity in the maximum. By plugging into we get $B_\lambda^{\mathrm{max}} = 1.09122 \cdot 10^7\,\mathrm{W{\cdot}m^{-2}{\cdot}sr^{-1}{\cdot}\mu m^{-1}}$.

The most challenging part of the problem is to determine when

$$B_\lambda(\lambda) = \frac{1}{2}B_\lambda^{\mathrm{max}} = 5.4561 \cdot 10^6\,\mathrm{W{\cdot}m^{-2}{\cdot}sr^{-1}{\cdot}\mu m^{-1}}\,.$$

We already know all necessary quantities so that we can solve the equation numerically. A different approach is to plot Planck's function for a given temperature and determine the solution from the graph. There are several helpful web tools for dealing with Planck's law, such as `https://www.spectralcalc.com/blackbody_calculator/blackbody.php`, which plots the dependence on the wavelength interval. By the graph of the whole function, we can evaluate that the half value for greater wavelengths is located somewhere in the $1000-1200\,\mathrm{nm}$ interval. After plotting the interval, we can narrow it down to $1080-1090\,\mathrm{nm}$ and then to $1087-1088\,\mathrm{nm}$. Finally, it can be estimated that $\lambda_{1/2} = 1087.2\,\mathrm{nm}$.

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Problem 42 ... anti-reflective glasses — 7 points

Patrik has a thin anti-reflective layer on his glasses of a thickness $d = 250\,\mathrm{nm}$ and of a refractive index $n_2 = 1.4$. Which part of the visible light is subject to destructive interference? The refractive index of air is $n_1 = 1$, for glass it is $n_3 = 1.55$ and light rays are perpendicular to the glasses.

Patrik was about to buy sunglasses.

Since $n_1 < n_2 < n_3$ applies to individual layers, the light always changes phase by $\frac{\lambda}{2}$ if reflected on a more dense layer, and because it happens twice, the chages cancel out each other. The condition for interference the minimum is $\Delta l = (2k+1)\frac{\lambda}{2}$, while the difference of the optical path when passing through the reflective layer is $\Delta l = 2n_2 d$. These two paths have be equal

$$2n_2 d = (2k+1)\frac{\lambda}{2},$$
$$\lambda = \frac{4n_2 d}{2k+1}.$$

We are looking for a multiple of k, for which the part of the visible spectrum $(400\,\mathrm{nm}, 700\,\mathrm{nm})$ is canceled out

$$\lambda(k=0) = \frac{4n_2 d}{1} = \frac{4 \cdot 1.4 \cdot 250}{1}\,\mathrm{nm} = 1\,400\,\mathrm{nm},$$
$$\lambda(k=1) = \frac{4n_2 d}{3} = \frac{4 \cdot 1.4 \cdot 250}{3}\,\mathrm{nm} = 467\,\mathrm{nm},$$
$$\lambda(k=2) = \frac{4n_2 d}{5} = \frac{4 \cdot 1.4 \cdot 250}{5}\,\mathrm{nm} = 280\,\mathrm{nm}.$$

Hence, the light of a wavelength of 467 nm, which corresponds to the blue part of the spectrum, is subject to destructive interference.

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Problem 43 ... Dano is flying to an exoplanet — 7 points

For exoplanet research, Dano would like to fly to Proxima Centauri, at a distance of 4.2 light-years. Calculate the Lorentz factor if Dano wants to get exactly half a year older on his way there and back. Do not consider acceleration of the spaceship.

Vašek was jealous of Dano's research on exoplanets.

Due to a time dilatation, Dano will age a total of

$$\Delta\tau = \frac{\Delta t}{\gamma}$$

where Δt is the time elapsed on Earth and γ is the Lorentz factor. According to the values from the assignment, it is clear that Dano will have to travel almost at the speed of light c; therefore, we can estimate

$$\Delta t \approx \frac{2l}{c}\,,$$

where $2l = 8.4\,\mathrm{ly}$. By combining the two equations above we obtain

$$\gamma \approx \frac{2l}{c\Delta\tau} \doteq 16.8\,.$$

In a more accurate calculation, we would assume that Dano moves at a speed of v with respect to the Solar System. Then for observers on Earth, Dano's journey will take time

$$\Delta t = \frac{2l}{v}\,.$$

Again, by combining the two equations we exculde the time Δt,

$$\frac{\Delta\tau}{2l} = \frac{1}{\gamma v} = \frac{1}{v}\sqrt{1 - \frac{v^2}{c^2}}$$

From here, by simple algerbraic adjustements, we can express the factor γ as

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{1 + \left(\frac{2l}{c\Delta\tau}\right)^2} \doteq 16.8\,.$$

Within the accuracy of the problem, we received the same result as above.

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Problem 44 ... platform 9 and 3/4 6 points

We are standing on a train platform 2 m *away from the track. Next to us, there is a very long train which is slowly braking. We are located roughly near the middle of the train. We hear a really unpleasant squeal of brakes with loudness of* 110 dB *coming from the entirety of the train. Moving away from the track might help us a bit. How many decibels do we measure at the other side of the platform,* 7 m *away from the track?*

Jarda believes in Harry Potter and loves Hermione.

Firstly, we need to know how does the sound intensity I changes with a distance. The intensity of the sound is defined as an acoustic power carried by sound waves through a specific area. Because the sound comes out of the whole very long train, we can consider this area to be a lateral surface of a cylinder with the axis being in the center of the train. The surface area grows linearly with distance from the train; the intensity is thus inversely proportional to the distance – this fact is crucial to understand the problem; unlike point source, where $I \propto 1/r^2$, in this case, it holds $I \propto 1/r$.

The sound loudness L (in decibels) is given by

$$L = 10\log_{10}\frac{I}{I_\mathrm{p}}\,,$$

where I_p is the intensity of the threshold of perception.

Let's try to subtract the loudnesses at distances $r_1 = 2\,\mathrm{m}$ and $r_2 = 7\,\mathrm{m}$ from the train

$$L_1 - L_2 = 10\left(\log_{10}\frac{I_0}{r_1 I_\mathrm{p}} - \log_{10}\frac{I_0}{r_2 I_\mathrm{p}}\right) = 10\log_{10}\frac{r_2}{r_1}\,.$$

Now we can easily express the loudness L_2 as

$$L_2 = L_1 - 10\log_{10}\frac{r_2}{r_1} = 104.6\,\mathrm{dB}\,.$$

The intensity has changed quite a bit; however, our perception did not change that much.

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Problem 45 ... a ride on the exponential 8 points

The exponential is an essential function in physics, which is why Jarda built a roller coaster in the shape of this function in his new amusement park. The ride starts at a height of $h = 10\,\mathrm{m}$. The cart then goes down a path which, if viewed from the side, has a profile given by the function $y = h\mathrm{e}^{-\frac{x}{h}}$, where x is the distance in the horizontal direction from the base of the entrance tower at the beginning of the ride. At what height above the ground does the descending cart exert the greatest force on the roller coaster?

Jarda is tired of taking the elevator.

On the descent, the cart with the visitors exerts two forces on the rollercoaster – gravitational and centripetal. The centripetal force acts perpendicularly to the track, and the gravitational force has to be decomposed into a normal and a tangent component. The resulting force will be the sum of the normal gravitational force and the centripetal force. The magnitude of the centripetal force can be found using the speed of the cart and the radius of curvature. First, we will determinate the speed v using the law of conservation of energy

$$v = \sqrt{2g\left(h - y(x)\right)} = \sqrt{2gh(1 - \mathrm{e}^{-\frac{x}{h}})}\,,$$

where g is the gravitational acceleration and $y(x)$ is the height above the ground depending on the horizontal coordinate. The radius of curvature r can be found using the formula

$$r = \frac{(1 + {y'}^2)^{\frac{3}{2}}}{|y''|}\,,$$

where y', y'' are the first and the second derivative of the function y with respect to x. The exponential is relatively easy to differentiate, so we can write the expression for the centripetal force with respect to x right away as

$$F_\mathrm{c} = \frac{mv^2}{r} = mg\frac{2\left(1 - \mathrm{e}^{-\frac{x}{h}}\right)\mathrm{e}^{-\frac{x}{h}}}{\left(1 + \left(-\mathrm{e}^{-\frac{x}{h}}\right)^2\right)^{\frac{3}{2}}}\,.$$

Now we can find the normal component of the gravitational force

$$F_{g_\mathrm{n}} = mg\cos\alpha = mg\frac{1}{\sqrt{1 + \tan\alpha^2}}\,,$$

where α is the angle between the slope of the track and the ground. In a given point, the function $\tan\alpha$ is numerically equal to the derivative of y with respect to x, therefore we can write

$$F_{g_\mathrm{n}} = mg\frac{1}{\sqrt{1 + \mathrm{e}^{-\frac{2x}{h}}}}\,.$$

Both the centripetal and gravitational force act on the track, which is why we can add them up to find the resulting force that the cart exerts on the rollercoaster

$$F = mg\left(\frac{1}{\sqrt{1 + \mathrm{e}^{-\frac{2x}{h}}}} + \frac{2\left(1 - \mathrm{e}^{-\frac{x}{h}}\right)\mathrm{e}^{-\frac{x}{h}}}{\left(1 + \mathrm{e}^{-\frac{2x}{h}}\right)^{\frac{3}{2}}}\right) = mg\frac{1 + 2\mathrm{e}^{-\frac{x}{h}} - \mathrm{e}^{-\frac{2x}{h}}}{\left(1 + \mathrm{e}^{-\frac{2x}{h}}\right)^{\frac{3}{2}}}\,.$$

To find the greatest value of this function, we will differentiate it with respect to x

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{\mathrm{e}^{-\frac{3x}{h}} - 4\mathrm{e}^{-\frac{2x}{h}} - 5\mathrm{e}^{-\frac{x}{h}} + 2}{\left(1 + \mathrm{e}^{-\frac{2x}{h}}\right)^2}\,.$$

To find the extremum, we equalize the derivative to zero

$$\mathrm{e}^{-\frac{3x}{h}} - 4\mathrm{e}^{-\frac{2x}{h}} - 5\mathrm{e}^{-\frac{x}{h}} + 2 = 0$$

and solve the equation numerically. We obtain that the only positive numerical solution is 11.30 m, where the track is 3.23 m above ground. Thus, at this height, the descending cart will exert the greatest force on Jarda's rollercoaster.

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Problem 46 ... carousel ride — 7 points

In addition to the roller coaster, Jarda also bought a chain carousel for his new amusement park. A horizontal disc with a diameter of $d = 8\,\mathrm{m}$ rotates with a period of $T = 8\,\mathrm{s}$, and the seats are attached to the edge of the disc using ropes of length $l = 8\,\mathrm{m}$. Determine the force exerted by a child weighing $m = 25\,\mathrm{kg}$ on a seat when the carousel is in a steady state.

Jarda works around the clock.

In a rotating coordinate system, a child is subject to three forces - tensile force of the rope, the force of gravity, and centrifugal force. In a steady state, the net force and torque are zero. Let us denote the angle between the rope and perpendicular to the ground as α. Then the seat is distant $x = \frac{d}{2} + l\sin\alpha$ from the rotational axis, and the magnitude of centrifugal force is

$$F_\mathrm{c} = m\omega^2 x = m\frac{4\pi^2}{T^2}\left(\frac{d}{2} + l\sin\alpha\right)\,.$$

The resultant force that the child is acting on the seat is the sum of gravity and centrifugal force. Thus, the magnitude of the resultant force is

$$F = \sqrt{F_g^2 + F_\mathrm{c}^2}\,.$$

From the equality of the moments of the two forces, it holds

$$\tan\alpha = \frac{F_\mathrm{c}}{F_g}\,.$$

Now we substitute from the last two equations into the first one and get

$$F_\mathrm{c} = \frac{4\pi^2}{T^2}m\left(\frac{d}{2} + l\frac{F_\mathrm{c}}{F}\right)\,,$$

$$F_\mathrm{c}\left(1 - \frac{4\pi^2 ml}{FT^2}\right) = \frac{4\pi^2}{2T^2}md\,,$$

$$\left(F^2 - m^2g^2\right)\left(F - \frac{4\pi^2 ml}{T^2}\right)^2 = \frac{4\pi^4 m^2 d^2}{T^4}F^2\,.$$

This is the equation of degree four in variable F; therefore, we must solve it numerically. The only positive root is the sought value $F = 270\,\mathrm{N}$.

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Problem 47 ... catapult

7 points

We thread a small bead on a rod of a length $l = 30\,\mathrm{cm}$, which lies on a horizontal pad and can only rotate vertically around one of its ends, which is fixed on the pad. We start to rotate the rod upwards at a constant angular speed. What is the maximum distance from the center of rotation at which we can place the bead such that it leaves the rod just at the moment when it reaches the vertical position? The bead moves along the rod without friction.

Jarda wanted to hit the professor while in a lecture on Theoretical Mechanics.

Let's solve the problem in the reference frame associated with the rotating rod. In such a frame, gravitational, centrifugal, Coriolis and rod reaction forces will act on the bead. However, since the bead can only move radially relative to the rod, the Coriolis force will have no effect on the motion, because it will cancel with the reaction force component of the rod. Let us describe the position of the bead by the coordinate x, which indicates its distance from the centre of rotation. The centrifugal force has a magnitude of

$$F_{\mathrm{ods}} = m\omega^2 x\,,$$

where m is mass of bead and ω is the angular velocity of rotation of the rod. This force acts in the direction away from the centre of rotation. Another important force is the gravitational force. Let's divide it into two components, one in the direction of the rod and the other perpendicular to it. We can calculate the components in the direction of the rod as

$$F_{\mathrm{gt}} = mg\sin\varphi = mg\sin\omega t\,,$$

where g is gravitational acceleration and $\varphi = \omega t$ is angle of inclination of the rod with respect to the horizontal plane. As the angle of inclination increases with time, this component of the gravitational force acts towards the centre of rotation. As with Coriolis force, the normal component of the gravitational force will be compensated by the reaction of the rod, because the bead only moves radially in our reference frame. The equation of motion of the bead on the rod is

$$ma = m\omega^2 x - mg\sin\omega t\,.$$

This is a second order differential equation. However, we can try to guess its solution. We divide the function x into two parts, x_1 and x_2. We are looking for a function x_1, which, if we derive twice, we get the same function multiplied by the factor ω^2. The function satisfying part of the condition is the sum of two exponentials, i.e.

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 Ae^{Bt} + Ce^{Dt}}{\mathrm{d}t^2} = B^2Ae^{Bt} + D^2Ce^{Dt}\,.$$

Now we apply the condition $\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = \omega^2 x$ and we obtain the relation

$$B^2Ae^{Bt} + D^2Ce^{Dt} = \omega^2\left(Ae^{Bt} + Ce^{Dt}\right)\,.$$

For the two sides to be equal, the constants B and D must be equal to $B = \omega$ and $D = -\omega$ (or the opposite, if they were both the same, then it would be just one exponential, and it can't be any other numbers because of the ω^2 factor). We derive the factors A and C from the initial conditions.

In a similar way we can find the second part of the function x, x_2. This is the part whose second derivative is equal to this function multiplied by the ω^2 factor minus the sine function. So let's try to put $x_2 = E\sin\omega t$. Then

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 E\sin\omega t}{\mathrm{d}t^2} = -E\omega^2\sin\omega t\,.$$

We know that

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -E\omega^2\sin\omega t = \omega^2 x_2 - g\sin\omega t = \omega^2 E\sin\omega t - g\sin\omega t\,.$$

In order for this equation to be true, it is necessary that

$$E = \frac{g}{2\omega^2}\,.$$

Thus

$$x = Ae^{\omega t} + Ce^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t\,.$$

The first initial condition is zero velocity at time $t = 0$. Let's derive the function x to find the velocity of the bead v and after putting $t = 0$ we get the equation

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega e^{\omega t} - C\omega e^{-\omega t} + \frac{g}{2\omega}\cos\omega t = 0\,,$$

from which we get

$$A - C = -\frac{g}{2\omega^2}\,.$$

According to the task, at an angle of $\varphi = \omega t = \frac{\pi}{2}$ the bead should fly out of the rod, so it must be true that

$$l = Ae^{\frac{\pi}{2}} + Ce^{-\frac{\pi}{2}} + \frac{g}{2\omega^2}\,,$$

from which, using the first condition, we can derive

$$A = \frac{l - \frac{g}{2\omega^2}\left(1 + e^{-\frac{\pi}{2}}\right)}{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}} = \frac{l - \frac{g}{2\omega^2}\left(1 + e^{-\frac{\pi}{2}}\right)}{2\cosh\frac{\pi}{2}}\,.$$

Therefore

$$C = A + \frac{g}{2\omega^2} = \frac{l - \frac{g}{2\omega^2}\left(1 - e^{\frac{\pi}{2}}\right)}{2\cosh\frac{\pi}{2}}\,.$$

At time $t = 0$, the position of the bead is

$$x_0 = A + C = \frac{l - \frac{g}{2\omega^2} - \frac{g}{4\omega^2}\left(e^{-\frac{\pi}{2}} - e^{\frac{\pi}{2}}\right)}{\cosh\frac{\pi}{2}}\,.$$

The greater the angular velocity, the higher the initial position. For $\omega \to \infty$ the value of the initial distance will be

$$x_0 = \frac{l}{\cosh\frac{\pi}{2}} = 12.0\,\mathrm{cm}\,.$$

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Problem 48 ... too many cubes

5 points

Consider an infinite cube net consisting of identical resistors $R = 24\,\Omega$, which are located on the edges of all cubes. What resistance do we measure between two adjacent vertices?

Jarda wanted a challenging problem with a short solution.

Let us bring the current I to vertex A. Considering the symmetry of the net, the current flowing to each of the six adjacent vertices is $\frac{I}{6}$. From the adjacent vertex B, the current of the same magnitude I has to flow away and from the symmetry again, each current flowing into the B is equal to $\frac{I}{6}$. From the superposition principle, we get that the current flowing between vertices A and B is

$$\frac{I}{6} + \frac{I}{6} = \frac{I}{3}\,,$$

thus, the voltage on the resistor between A and B is $U = R\frac{I}{3}$. This voltage is equal to the product of the current I and the resulting resistance R_c of the cube net between vertices A and B. Therefore, $U = R_\mathrm{c}I$, and thus,

$$R_\mathrm{c} = \frac{U}{I} = \frac{R\frac{I}{3}}{I} = \frac{R}{3} = 8\,\Omega\,.$$

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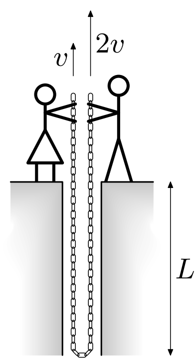
Problem 49 ... we work by pulling

8 points

Alice and Bob are pulling a non-elastic chain of linear density $\lambda = 0.4\,\mathrm{kg{\cdot}m^{-1}}$ from a pit. Alice is pulling the chain out at a constant speed $v = 0.6\,\mathrm{m{\cdot}s^{-1}}$ and Bob at twice the speed. Consider the work done between the point in time when the lowest point of the chain is at a depth of $L = 5\,\mathrm{m}$ and when it is fully pulled out. How many times larger will Bob's work be than Alice's? Assume that they have already been pulling for some time at the starting point.

Matěj is fascinated by the Mould effect.



At a first glance, we might think that Bob needs to do double the work, because both Bob and Alice pull by the same force and Bob pulls on double the distance. However, he needs to pull by a larger force, because as he is pulling, he accelerates the chain from the Alice's velocity $v_\mathrm{A} = v$ to $v_\mathrm{B} = 2v_\mathrm{A}$. We can assume that Alice and Bob are close to each other relative to the chain's length, so the lower bent part of the chain is negligibly small.

Let a depth of the lowest chain's point be l, in the beginning $l = L$ and in the end $l = 0$. Alice exerts a force on the chain, which is just as large as gravity acting on Alice's half of the chain,

$$F_\mathrm{A} = F_g = \lambda l g\,.$$

We need to use a general Newton's motion law to evaluate Bob's force. A resultant force acting on his part of the chain must be equal to the time derivative of a momentum

$$F_\mathrm{B} - F_g = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}\,(mv)}{\mathrm{d}t} = v\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}v}{\mathrm{d}t}\,.$$

Bob's part of the chain does not accelerate as a whole, therefore the last term equals zero. The first term tells us how large a force needs to be to accelerate a mass $\mathrm{d}m$ during the time $\mathrm{d}t$ by a velocity difference Δv. In this case $\Delta v = v_\mathrm{B} - v_\mathrm{A}$.

During the period $\mathrm{d}t$ there will be pulled out $\mathrm{d}x = (v_\mathrm{A} + v_\mathrm{B})\,\mathrm{d}t$ of the chain. Just before this moment, one half of this length has been on Alice's side and the other half on Bob's side. On the other hand, Bob pulled out a different chain's length $v_\mathrm{B}\,\mathrm{d}t$ and therefore a segment "flowed" to his part with a length

$$\mathrm{d}l_\mathrm{B} = v_\mathrm{B}\,\mathrm{d}t - \frac{\mathrm{d}x}{2} = \frac{v_\mathrm{B} - v_\mathrm{A}}{2}\,\mathrm{d}t\,.$$

A weight of this segment is $\mathrm{d}m = \lambda\,\mathrm{d}l_\mathrm{B}$ and we obtain the Bob's total force by plugging values in

$$F_\mathrm{B} = F_g + v\frac{\mathrm{d}m}{\mathrm{d}t} = F_g + \frac{\lambda}{2}\left(v_\mathrm{B} - v_\mathrm{A}\right)^2\,.$$

Why is this additional force exerted only by Bob and not both as with gravity? The answer is that only Bob causes the acceleration. Alice's force effects are equivalent to a situation where she would pull her part with a length l, from whose end the chain's link were falling off. It would shorten itself, but no force is created. In fact, these links will be pulled to Bob, so he needs some extra force.

We find the lowest point of the chain at the time t as

$$l = L - \frac{v_\mathrm{A} + v_\mathrm{B}}{2}t\,,$$

The total pulling time will be $T = \frac{2L}{v_\mathrm{A}+v_\mathrm{B}}$. Now we have one remaining job: solve for work. For Alice's work, we get

$$W_\mathrm{A} = \int_0^T F_\mathrm{A} v_\mathrm{A}\,\mathrm{d}t = \lambda g v_\mathrm{A}\int_0^T \left(L - \frac{v_\mathrm{A}+v_\mathrm{B}}{2}t\right)\mathrm{d}t = \lambda g v_\mathrm{A}\left[Lt - \frac{v_\mathrm{A}+v_\mathrm{B}}{4}t^2\right]_0^T = \\ = \frac{\lambda g v_\mathrm{A} L^2}{v_\mathrm{A}+v_\mathrm{B}} = \frac{\lambda g L^2}{3}\,.$$

This applies for Bob as well. The contribution of gravity will be almost the same, just twice as large due to twice the speed. So let's focus on the second force

$$W_\mathrm{B} = \int_0^T F_\mathrm{B} v_\mathrm{B}\,\mathrm{d}t = \frac{v_\mathrm{B}}{v_\mathrm{A}}W_\mathrm{A} + \int_0^T \frac{\lambda}{2}\left(v_\mathrm{B}-v_\mathrm{A}\right)^2 v_\mathrm{B}\,\mathrm{d}t = 2W_\mathrm{A} + \lambda L\frac{\left(v_\mathrm{B}-v_\mathrm{A}\right)^2}{v_\mathrm{B}+v_\mathrm{A}}v_\mathrm{B} = \\ = 2W_\mathrm{A} + \frac{2\lambda L v_\mathrm{A}^2}{3}\,.$$

To finish, we have the works' ratio

$$\frac{W_\mathrm{B}}{W_\mathrm{A}} = 2 + \frac{2\lambda L v_\mathrm{A}^2}{3W_\mathrm{A}} = 2 + \frac{2v_\mathrm{A}^2}{gL} = 2.015\,.$$

You may find interesting that the ratio doesn't depend on chain's length density, but does depend on gravity, the initial depth and the pulling speed.

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Problem 50 ... snowball to a window

8 points

Jarda wanted to distract his quarantined friend, so he decided to throw a snowball at his window. The lower frame of the window is at a height of $h_1 = 3.5\,\mathrm{m}$ and the upper frame at $h_2 = 4.8\,\mathrm{m}$. The window is $d = 2\,\mathrm{m}$ wide. Jarda was standing four meters from the base of the house, right in front of the center of the window. What part of the area of the window (in percent) could he hit if he threw the ball at a speed of $v = 8\,\mathrm{m{\cdot}s^{-1}}$, from $H = 1.8\,\mathrm{m}$ above ground?

Jarda aims high.

We introduce the Cartesian coordinate system with its origin at the place where Jarda is standing so that the plane xy represents the ground and the y-axis points to the horizontal center of the window. Jarda can throw the snowball only at a particular area of the space, which is delimited by what is called a parabola of safety (in our three-dimensional space, a rotational paraboloid). We will not derive the equation of this paraboloid here, but the height z of the protective paraboloid with respect to the distance r from Jarda's feet is

$$z = H + \frac{v^2}{2g} - \frac{r^2 g}{2v^2}\,.$$

Points, where the paraboloid intersects the window determine the border of all points Jarda can hit. These points lie at a ground distance of

$$r = \sqrt{D^2 + x^2}$$

from the place where Jarda is standing. The distance between Jarda and the window is $D = 4\,\mathrm{m}$, and x is the position of a point from the center of the window. Thus, the safety paraboloid intersects the window at points

$$z = H + \frac{v^2}{2g} - \frac{D^2 g}{2v^2} - \frac{x^2 g}{2v^2}\,. \tag{3}$$

The area under these points can be easily calculatad with the integral

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \left(H + \frac{v^2}{2g} - \frac{D^2 g}{2v^2} - \frac{(x^2) g}{2v^2}\right) \mathrm{d}x = \left[Hx + \frac{v^2}{2g}x - \frac{D^2 g}{2v^2}x - \frac{x^3 g}{6v^2}\right]_{-\frac{d}{2}}^{\frac{d}{2}} = \\ = \left(H + \frac{v^2}{2g} - \frac{D^2 g}{2v^2}\right) d - \frac{d^3 g}{24v^2}\,.$$

We will now subtract the area of the wall under the window - that is $h_1 d$, and in order to obtain the searched ratio, we devide this value by the total area of the window $(h_2 - h_1)\,d$. Thus, the solution is

$$\frac{H - h_1 + \frac{v^2}{2g} - \frac{D^2 g}{2v^2} - \frac{d^2 g}{24v^2}}{(h_2 - h_1)} = 23.9\,\%\,.$$

It is also important to verify that our area does not extend above the window. We determine the maximum height from the equation (3) by substituting $x = 0$. After numerical substitution we get $z \doteq 3.8\,\mathrm{m}$, which is less than h_1, so we can't hit the area above the top edge of the window.

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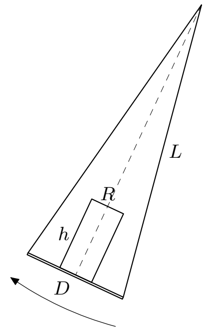
Problem 51 ... a bottle on a swing

8 points

Jarda is swinging harmoniously on a swing with a frequency $f = 0.3\,\mathrm{Hz}$. He has a full cylinder-shaped bottle with him, placed exactly in the middle of the width of the swing. The bottle has a height $h = 30\,\mathrm{cm}$ and it stands on its base of a diameter $R = 3.0\,\mathrm{cm}$. The swing is $D = 40\,\mathrm{cm}$ wide and hangs on ropes that are $L = 1.4\,\mathrm{m}$ long. The friction between the swing and the bottle is enormous, so the bottle stands in its place. Determine the smallest amplitude of the swing's inclination angle upon which the bottle starts to wobble. *Jarda keeps dropping things.*



Let us consider a system connected to a swing. In this system, the forces acting upon the bottle are gravitational, a reaction of the swing, inertial, friction and centrifugal. We will decompose the forces in directions perpendicular and parallel to the swing and we will examine their torques. Friction is acting upon the bottle in the direction that is tangential to the swing and its point of action lies in the plane of the lower base. Because of that, the friction will not exert any torque with respect to the point which would lie on the rotational axis, if the bottle were to wobble.

The bottle on the sway will begin to wobble if the torque of the forces acting parallel to the plane of the swing is greater than the torque of forces acting in the axis of the bottle. Let us denote φ an angle of deflection of the swing from vertical. Since Jarda swings harmonically the time dependence of the size of the angle is

$$\varphi = \varphi_0 \sin(\omega t)\ ,$$

where φ_0 is the maximum deflection and $\omega = 2\pi f$. Angular velocity and angular acceleration are

$$\dot{\varphi} = \omega\varphi_0 \cos(\omega t) = \omega\sqrt{\varphi_0^2 - \varphi^2}\,,$$
$$\ddot{\varphi} = -\omega^2\varphi_0 \sin(\omega t) = -\omega^2\varphi\,.$$

Let us begin with the gravitational force. We will decompose it into two components with perpendicular directions, analogous to how we would do it in an inclined plane problem. The component of the gravitational force with direction of the normal to the plane of the swing is

$$F_g^{\mathrm{n}} = mg\cos\varphi\,,$$

where m is the mass of the bottle. In the tangential direction to the plane acts component of the gravitational force with magnitude

$$F_g^{\mathrm{t}} = mg\sin\varphi\,.$$

Both components of the gravitational force act in the centre of the mass of the bottle. We calculate the moment arm of torques with respect to the lower edge of the bottle around which the bottle could theoretically rotate. For calculating magnitudes of torques we just need to take the perpendicular distance with respect to the direction of the force to the rotational axis, we get

$$M_g^{\mathrm{n}} = Rmg\cos\varphi\,,$$
$$M_g^{\mathrm{t}} = \frac{h}{2}mg\sin\varphi\,.$$

Both torques are acting against each other.

We will now consider the inertial forces which depend on the distance from the axis of rotation. For the centre of the lower base of the bottle it is

$$l = \sqrt{L^2 - \left(\frac{D}{2}\right)^2} \doteq 1.386\,\mathrm{m}\,.$$

Let us denote x the perpendicular distance from the plane of the swing and let us consider an element of the bottle in the shape of an annulus with a height $\mathrm{d}x$, a with $\mathrm{d}r$ in the distance r from the axis of the cylinder and x from the plane of the swing. We will examine the effect of individual inertial forces on this small volume.

We will begin with the centrifugal force. The angular velocity is the same for all points but the distance from the axis of rotation and the direction of force changes. We will divide the annulus into angular elements $\mathrm{d}\alpha$, where α denotes the angular distance of such an element from a line that is parallel with the axis of swinging and lies in the plane of the chosen annulus. Let us denote β the angle which is formed by a connecting line of an element $\mathrm{d}\alpha$ and the swinging axis with the axis perpendicular to the plane of the swing. The magnitude of centrifugal force is then

$$\begin{aligned}\mathrm{d}F_\mathrm{o} &= \dot{\varphi}^2 \frac{l-x}{\cos\beta}\,\mathrm{d}m\,,\\ \mathrm{d}m &= \rho\,\mathrm{d}V \;= \rho r\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}\alpha\,.\end{aligned}$$

Let us decompose this force into a component that is perpendicular to the plane of the swing, pointing “into” it, and a component that is parallel with the plane. Due to the circular symmetry is the resultant of these parallel components zero. Components acting into the swing are

$$\mathrm{d}F_\mathrm{o}^\mathrm{n} = \mathrm{d}F_\mathrm{o}\cos\beta = \dot{\varphi}^2\left(l-x\right)\rho r\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}\alpha\,.$$

We integrate with respect to the angle α, then to r and finally to x

$$\begin{aligned}F_\mathrm{o} &= \int_0^h \int_0^R \int_0^{2\pi} \dot{\varphi}^2\left(l-x\right)\rho r\,\mathrm{d}\alpha\,\mathrm{d}r\,\mathrm{d}x = \dot{\varphi}^2\rho \int_0^h \int_0^R 2\pi\left(l-x\right) r\,\mathrm{d}r\,\mathrm{d}x =\\ &= 2\pi\dot{\varphi}^2\rho \int_0^h \frac{R^2}{2}\left(l-x\right)\mathrm{d}x = \pi R^2\dot{\varphi}^2\rho\left(lh - \frac{h^2}{2}\right)\,.\end{aligned}$$

We found the magnitude of the centrifugal force, but we are more interested in the torque. Moment arm is $R + r\sin\alpha$, so for the element of torque we get

$$\mathrm{d}M_\mathrm{o} = \left(R + r\sin\alpha\right)\mathrm{d}F_\mathrm{o}\,.$$

We integrate this expression again over the whole volume of the cylinder. We can notice that right after the integration over the angle α we will get the same expression as what we get when calculating a magnitude of force multiplied by R. The resulting torque is

$$M_\mathrm{o} = RF_\mathrm{o} = \pi R^3\dot{\varphi}^2\rho h\left(l - \frac{h}{2}\right) = mR\dot{\varphi}^2\left(l - \frac{h}{2}\right)\,.$$

We can see that the torque of the centrifugal force is the same as if we replaced all of the mass of the object by a mass point in the centre of mass. This torque returns the bottle into a standing position. We will calculate the torque of inertial force in the same manner. We will come to conclusion that there is inertial force acting upon individual elements of cylinder with height $\mathrm{d}x$

$$\mathrm{d}F_\mathrm{s} = \ddot{\varphi}\,(l-x)\,\mathrm{d}m = \pi\ddot{\varphi}\,(l-x)\,\rho R^2\,\mathrm{d}x\,.$$

These forces also act in the centres of those thin cylinders and their direction is parallel with the plane of the swing. We calculate their moment arm also from the lower edge of the bottle and it has a length x. Torque of inertial force is

$$M_\mathrm{s} = \int_0^h \pi\ddot{\varphi}\,(l-x)\,x\rho R^2\,\mathrm{d}x = \pi\ddot{\varphi}\rho R^2\left(\frac{lh^2}{2}-\frac{h^3}{3}\right) = \ddot{\varphi}mh\left(\frac{l}{2}-\frac{h}{3}\right)\,.$$

This torque has the same direction as the torque of the centrifugal force, it acts against the wobbling. The resulting torque acting upon the bottle is

$$M = M_g^\mathrm{t} - M_\mathrm{s} - M_\mathrm{o} - M_g^\mathrm{n}\,,$$

while if $M > 0$, the bottle will begin to wobble. When we substitute into the equation and simplify, we get

$$M = \frac{h}{2}g\sin\varphi - \frac{h}{2}\omega^2\varphi\left(l-\frac{2}{3}h\right) - Rg\cos\varphi - R\omega^2\left(\varphi_0^2-\varphi^2\right)\left(l-\frac{h}{2}\right)\,.$$

It is obvious that when $\varphi = 0$ then $M < 0$ and the bottle is stable. Solution of this problem is to find the minimal angle φ_0 for which exists an angle φ such that $|\varphi| < \varphi_0$, while satisfying $M(\varphi,\varphi_0) = 0$.

Instead of verifying all possible combinations of φ and φ_0 we can calculate for all φ_0 in which φ has M maximum and verify if in given φ and φ_0 the torque is positive of negative. For every fixed φ_0 the torque M is a function of variable φ, so for finding the extreme we differentiate

$$\frac{\mathrm{d}M}{\mathrm{d}\varphi} = \frac{h}{2}g\cos\varphi - \frac{h}{2}\omega^2\left(l-\frac{2}{3}h\right) + Rg\sin\varphi + 2R\omega^2\varphi\left(l-\frac{h}{2}\right)\,.$$

Let us put this expression equal to zero and let us denote the solution of the equation φ_m. Notice that φ_m does not depend on φ_0, which means that the maximum will be the same for all φ_0. To be more precise, it would be the same if all of the values of φ were allowed for given φ_0. Since $|\varphi| < \varphi_0$ holds for $|\varphi_0| < |\varphi_\mathrm{m}|$, there is not a point with derivative equal to zero in allowed interval, so we have to look for maximum at the boundary, i.e. in points $\pm\varphi_0$.

The value of φ_m has to be calculated numerically. The only one satisfying $|\varphi_\mathrm{m}| \leq \pi/2$ is $\varphi_\mathrm{m} \doteq -0.781$, but in this point the $M(\varphi = \varphi_0 = \varphi_\mathrm{m})$ is even smaller than M in zero. Therefore it is not a maximum but a minimum. That means that for positive values of φ the torque should be increasing to the next root of the last equation, which is at the point 1.633. Maxima of torques for $0 < \varphi_0 < \pi/2$, therefore, must lie at the boundary which corresponds to the turning point.

Now we need to find the minimal positive φ_0 which satisfies $M(\varphi = \varphi_0) = 0$

$$0 = M(\varphi=\varphi_0) = \frac{h}{2}g\sin\varphi_0 - \frac{h}{2}\omega^2\varphi_0\left(l-\frac{2}{3}h\right) - Rg\cos\varphi_0\,.$$

The result of this straightforward numerical calculation is $\varphi_0 \doteq 0.342$. Notice that for $\varphi = \varphi_0$ the part which contains centrifugal force equals zero because at that moment the angular velocity of the bottle equals zero.

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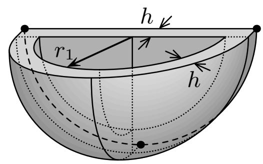
Problem 52 ... washbasin

8 points



A washbasin has two parts: the front part is shaped like a quarter of a sphere, with an inner radius of $r_1 = 30\,\mathrm{cm}$, and it is attached to the back part in the shape of a half-cylinder. The thickness of all its walls is $h = 2.0\,\mathrm{cm}$ and the whole washbasin is made of a material with a density of $\rho = 3\,100\,\mathrm{kg{\cdot}m^{-3}}$. It is attached to a wall at two points located at its only two vertices (corners). Other than that, its bottom edge leans freely on the wall. What is the minimum force which the hanging mechanism at each corner must withstand when the basin is completely filled with water? Assume that the bottom edge acts on the wall only at its lowest point and only in the direction perpendicular to the wall. *Dodo was repairing a bathroom tap.*

Let us first introduce the coordinate system in which we are going to solve the problem. Let its origin be in the middle of the top edge of the closer side to the wall. Let the x-axis be perpendicular to the wall, the y-axis points downwards, and let the z-axis point along the edge of the washbasin. In a static situation, the resultant force must be zero, and also the resultant torque must be zero.

Since the plane xy is the plane of symmetry of the washbasin, a gravitational force of magnitude $F_g = mg$ acts upon the washbasin at its center of gravity with coordinates $(x_T, y_T, 0)$, pointing straight downward. Next, at the lower edge (at the point $(0, r_2, 0)$, where $r_2 = r_1 + h$) a force of unknown magnitude $\mathbf{F}_1 = (F_1, 0, 0)$ acts upon the washbasin in a direction perpendicular to the wall. Finally, at the hinge points (i.e. $(0, 0, \pm r_2)$) a force $\mathbf{F}$ acts upon the washbasin in an unknown direction. Since the magnitude of this force has to be as small as possible, its components must be $\mathbf{F} = (F_x, F_y, 0)$.

The balance of forces gives us the equations

$$F_1 + 2F_x = 0\,,$$
$$F_g + 2F_y = 0\,.$$

From the balance of torques, we have one non-trivial equation.

$$-r_2 F_1 + x_T F_g = 0\,.$$

In total, we have three equations for the three unknowns, while we are looking for the magnitude of a force

$$F = \sqrt{F_x^2 + F_y^2} = \frac{mg}{2}\sqrt{1 + \left(\frac{x_T}{r_2}\right)^2}\,. \tag{4}$$

Therefore we have to calculate the mass of a full washbasin and a x coordinate of its center of gravity. First, let us deal with an object of the shape of a quarter sphere, with a radius R, and homogeneous density distribution. Its volume is apparently

$$V(R) = \frac{\pi}{3} R^3\,,$$

however, determining the position of the centre of gravity is more demanding. If we place the object in the aforementioned coordinate system, from the mirroring the xy plane we have $z_T = 0$, and from the symmetry with respect to the plane $x = y$ we obtain $x_T = y_T$. Thus, the centre of gravity has to be somewhere on this line. The position of a centre of gravity of a homogeneous object is by definition

$$\mathbf{x}_T = \frac{1}{M}\int_V \rho(\mathbf{x})\,\mathbf{x}\,\mathrm{d}V = \frac{1}{V}\int_V \mathbf{x}\,\mathrm{d}V\,.$$

We are interested only in the x coordinate of the centre of gravity which we denote as X_T. We transfer to spherical coordinates (r,θ,φ), where $r \in \langle 0, R\rangle$ is the distance from the origin, $\theta \in \langle 0, \pi\rangle$ is the deflection from the z axis, and $\varphi \in \left\langle 0, \frac{\pi}{2}\right\rangle$ is the angle between the projection of the position vector onto the xy plane, and the x axis. The transformation relation is $x = r\sin\theta\cos\varphi$, the volume differential changes is then $\mathrm{d}V = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

From the definition above, we have

$$\begin{aligned} X_T &= \frac{1}{V}\int_V x\,\mathrm{d}V = \frac{1}{V}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{R} r^3\sin^2\theta\cos\varphi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = \\ &= \frac{1}{V}\left[\frac{r^4}{4}\right]_0^R \int_0^{\frac{\pi}{2}}\int_0^{\pi}\sin^2\theta\cos\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{R^4}{4V}\left[\sin\varphi\right]_0^{\frac{\pi}{2}}\int_0^{\pi}\sin^2\theta\,\mathrm{d}\theta = \\ &= \frac{R^4}{4V}\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta = \frac{R^4}{4V}\left[\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi} = \frac{\pi R^4}{8V} = \frac{3}{8}R\,. \end{aligned}$$

Now, we can put the acquired knowledge together. We consider that the washbasin consists of a quarter sphere of radius r_2 with density ρ centered at the point $(h,0,0)$, another quarter sphere of radius r_1 with density $\rho_\mathrm{v} - \rho < 0$ centered at the same point, and a half cylinder forming the back part of the washbasin of radius r_2 and heigth h. The quantity ρ_v denotes the density of water. For the total mass we have

$$m = \frac{\pi}{3}r_2^3\rho + \frac{\pi}{3}r_1^3\left(\rho_v - \rho\right) + \frac{\pi}{2}r_2^2 h\rho \doteq 56.97\,\mathrm{kg}\,.$$

By taking the weighted arithmetic mean for the positon of the centre of gravity in the x direction, we get

$$\begin{aligned} x_T &= \frac{1}{m}\left[\frac{\pi}{3}r_2^3\rho\left(h + \frac{3}{8}r_2\right) + \frac{\pi}{3}r_1^3\left(\rho_v - \rho\right)\left(h + \frac{3}{8}r_1\right) + \frac{\pi}{2}r_2^2 h\rho\frac{h}{2}\right]\,, \\ x_T &\doteq 12.5\,\mathrm{cm}\,. \end{aligned}$$

When we put this into the equation (4), we obtain the magnitude of the force $F \doteq 300\,\mathrm{N}$ that each of the hanging mechanisms has to bear.

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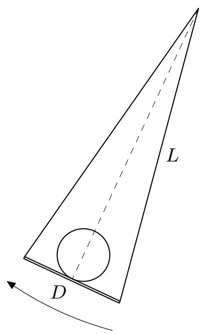
Problem 53 ... bottle on a swing reloaded

8 points



Jarda is swinging harmoniously on a swing with a frequency $f = 0.5\,\mathrm{Hz}$ and a maximum inclination angle 5°. Next to him, there is an empty, relatively narrow, and long cylinder-shaped bottle, exactly in the middle of the width of the swing. The bottle is placed horizontally, and its axis is perpendicular to the direction of motion. The swing is $D = 40\,\mathrm{cm}$ wide and hangs on ropes that are $L = 3\,\mathrm{m}$ long. At first, Jarda is holding the bottle, but he releases it when he passes through the equilibrium position. When does the bottle fall off the swing? Assume that the bottle is rolling without slipping.

Jarda drank a bottle of wine and started thinking about problems.

Let's work in a swing-fixed frame. The forces acting on the bottle are gravity, normal force from the swing seat, inertial force, and centrifugal force. These forces can be decomposed into parallel and perpendicular components with respect to the swing plane.

The Coriolis force acts only perpendicular to the velocity vector. The bottle moves only on the desk, so this force acts only into the desk or in the opposite direction. In this case, the Coriolis force might be greater than the gravity force, but it doesn't occur in this problem (see below).

The centrifugal force is not constant throughout the bottle; however, the distance from the axis to the center of the swing is much greater than the diameter of the bottle, so we can neglect this, and we can assume the bottle to be a point mass in its center of gravity. The resultant of this force points in the "axis of the rotation – the center of gravity of the bottle" line, and is not perpendicular to the pad. It is of magnitude

$$F_\mathrm{c} = m\dot{\varphi}^2 \frac{L}{\cos\alpha}\,,$$

where m is the mass of the bottle, $\dot{\varphi}$ is the angular velocity of the swing, and α is an angle adjacent to the rotation axis, covering the bottle and the center of the swing. The distance between the rotational axis and center of the swing is not exactly L, but regarding the given, this approximation is adequate. Since the bottle is slim, we do not take into account the distance between the center of gravity of the bottle and the center of the swing. Thus, we split the centrifugal force into two components. Since the bottle is moving in the plane, the perpendicular force doesn't affect its motion. The component parallel to the desk is of magnitude

$$F_\mathrm{c}^\mathrm{t} = F_\mathrm{c}\sin\alpha = m\dot{\varphi}^2 L\tan\alpha = m\dot{\varphi}^2 x\,,$$

where x is distance between the bottle and the center of the swing.

The parallel component of the gravity force is

$$F_g^\mathrm{t} = mg\sin\varphi\,,$$

where m is mass of the bottle, $\varphi = \varphi_0\sin(\omega t)$ is the instantaneous angle of the swing, $\varphi_0 = 5^\circ$ is the swing amplitude, $\omega = 2\pi f$ is the angular frequency of swinging and t is time since the release of the bottle. Because the amplitude is reasonably small, we can use the approximation $\sin x \approx x$. The parallel component of the gravity force then becomes $F_g^\mathrm{t} \approx mg\varphi$.

The other component of the gravity force is aproximately mg (because φ_0 is small). Maximal magnitude of Coriolis force is $2\omega m\varphi_0 v \doteq 0.55mv$. The velocity of the bottle must be approximately $18\,\mathrm{m\cdot s^{-1}}$ to make the Coriolis force greater than the gravity force. This doesn't occur in this problem, so the Coriolis force is not important in our solution.

Then there is also the inertial force emerging from the acceleration and deceleration of the swing. Due to the negligible radius of the bottle compared to the rope length L, the acceleration caused by inertial force is

$$a_\mathrm{s} \approx \ddot{\varphi} = -\varphi_0\omega^2 \sin(\omega t)\,L = -\omega^2 L\varphi\,.$$

The inertial force magnitude is $F_\mathrm{s} = ma_\mathrm{s}$, and it points opposite to the tangential acceleration of the swing, i.e., opposite to the parallel component of the gravity force, which we calculated in the previous paragraph. The magnitude of the parallel component of inertial force is independent of the position of the bottle relative to the swing. When the bottle moves off the swing axis, the inertial force changes both magnitude and direction; nevertheless, its parallel component magnitude stays constant.

Now we can see the magnitude of the tangential component of the centrifugal force F_c^t is negligible to the inertial and gravity force because there is a small angle φ_0 squared, but in the inertial and gravity force, it occurs only in the first power.

The resultant force acting on the bottle in direction prallel to the swing plane is

$$F = F_g^\mathrm{t} - F_\mathrm{s} = m\varphi\left(g - \omega^2 L\right)\,.$$

Notice that if the swing oscillated at an angular frequency of a mathematical pendulum, the angular frequency would be $\omega^2 = \frac{g}{L}$ and the two forces would exactly cancel out each other. However, the swing oscillates with frequency $f = 0.5\,\mathrm{Hz}$ which is not a frequency of mathematical pendulum with length L.

We can approximate that the force acts in the center of the bottle (because the length of the rope is large compared to the bottle radius). The torque with respect to the point of contact between the bottle and swing is $M = RF$, where R is bottle radius. The moment of inertia of the bottle for rotation around this point is from the parallel axis theorem

$$J = J_\mathrm{s} + mR^2 = mR^2 + mR^2 = 2mR^2\,,$$

where $J_\mathrm{s} = mR^2$ is the moment of inertia of a cylindrical shell (we neglect the bottom and the neck of the bottle). Let x be the distance of the bottle center from the swing axis. We derive the equation

$$\ddot{x} = \varepsilon R = \frac{MR}{J} = \frac{\varphi\left(g - \omega^2 L\right)}{2} = \frac{\varphi_0\left(g - \omega^2 L\right)}{2}\sin(\omega t)\,,$$

where ε is the angular acceleration of the bottle. We integrate the equation and obtain

$$\dot{x} = -\frac{\varphi_0\left(g - \omega^2 L\right)}{2\omega}\cos(\omega t) + C\,.$$

Boundary condition is $\dot{x}(0) = 0$, thus

$$C = \frac{\varphi_0\left(g - \omega^2 L\right)}{2\omega} \quad \Rightarrow \quad \dot{x} = \frac{\varphi_0\left(g - \omega^2 L\right)}{2\omega}\left(1 - \cos(\omega t)\right)\,.$$

By integrating the equation for speed $\dot{x}$, we get the position dependence on time

$$x = \frac{\varphi_0\left(g-\omega^2 L\right)}{2\omega}\left(t-\frac{1}{\omega}\sin(\omega t)\right) = \frac{\varphi_0}{2}\left(\frac{g}{\omega^2}-L\right)(\omega t-\sin(\omega t)) \;.$$

where we used the boundary condition $x(0) = 0$. The bottle will fall off when the position $|x|$ is larger than $\frac{D}{2}$. The time can be calculated from equation

$$D = \left|\varphi_0\left(\frac{g}{\omega^2}-L\right)(\omega t-\sin(\omega t))\right| \;.$$

This equation has to be solved numerically. The bottle will fall at time $t \doteq 0.86\,\mathrm{s}$.

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Problem 54 ... minimisation of acceleration 7 points

We are driving a car, which we consider to be a point mass, at a speed of $v_0 = 20.0\,\mathrm{m\cdot s^{-1}}$. At some point, we notice that we are approaching a curve, at the end of which we'll need to stop at traffic lights. Due to our insurance company, we want the maximum absolute value of the acceleration acting on the car (until it stops at the traffic lights) to be as small as possible. If we start braking $d = 30.0\,\mathrm{m}$ before the beginning of the curve, the radius of the curve is $R = 10.0\,\mathrm{m}$ and we turn 22.5° on the curve, what time does it take to reach the end of the curve from the point where we start braking? *Karel's insurance company insists on minimum acceleration.*

Let us denote the acceleration of braking as a. At first, we will drive the distance d as uniformly decelerated motion. By expressing the time from the equation for distance of such a motion, we get

$$d = v_0 t_1 - \frac{1}{2}a t_1^2 \,,$$
$$t_1 = \frac{v_0 - \sqrt{v_0^2 - 2ad}}{a} \,,$$

where we do not consider the solution with a positive sign (+) before the square root as it represents the situation when the car does not stop accelerating even after it stops – it would return back. Hence, the car will move at speed $v_1 = v_0 - at_1 = \sqrt{v_0^2 - 2ad}$ at the beginning of the curve.

Now we get to the point where the situation becomes more complex because the car will also turn in addition to braking. It means that centripetal acceleration (which is always perpendicular to the braking) will act on the car, too. We can obtain the magnitude of centripetal acceleration by expression $a_\mathrm{c} = v^2/R$. Then, if in the sum, we do not want to exceed the acceleration a, it remains $a_\mathrm{b} = \sqrt{a^2 - v^4/R^2}$ on braking. If we denote the speed in time t as $v(t)$, we get the following differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \sqrt{a^2 - v^4/R^2} \,.$$

This is separable; nevertheless, we would get hypergeometric functions from the integral. Furthermore, we need to determine the distance that car would travel somehow to find out whether it would stop before the traffic lights. So if we do not have a smarter idea of how to avoid it, we have to solve the problem numerically.

The outline of the algorithm: We choose the acceleration and compute the car's speed at the beginning of the curve, and the time it needs to get there analytically. Then we proceed by tiny steps and compute the change in distance (and add it to total distance, which it traveled in the curve), and by how much it slows down (and subtract it from the speed) in each of the steps. When speed becomes less than 0 (the car stops), we check the distance that the car already traveled and compare it to the length of the curve, which is $\pi R/8$. If it traveled more, it means we need to increase the acceleration and vice versa. Then we repeat the process with such adjusted acceleration.

Regarding adjusting the acceleration, we can use the bisection method, where the initial interval could be from $a_{\min} = v_0^2/(2d + R)$ (at this acceleration, the car gets to the beginning of the curve at speed, when the centripetal acceleration is equal to $a_{\min}$, thus, for any less acceleration it could not follow the curve) to $a_{\max} = v_0^2/(2d)$ (at this acceleration, the car stops exactly at the beginning of the curve). Hence, the sought acceleration has to be in this interval, and by selecting it, we avoided all potential square roots of negative numbers. When do we stop bisecting the interval? The question in the problem assignment asks about the time duration of the braking. Therefore, in addition to acceleration, we have to remember the particular time at the boundaries of the interval. Then we will bisect until the difference of these 2 times becomes less than the accuracy required by the task.

```
import numpy as np
v0 = 20
d = 30
R = 10
l = np.pi * R / 8
amin = v0 * v0 / (2 * d + R)
amax = v0 * v0 / (2 * d)
tmin = 0.
tmax = 10000.
dt = 0.0001
while abs(tmin - tmax) > 10 * dt:
   a = (amin + amax) / 2
   v = np.sqrt(v0 * v0 - 2 * a * d)
   t1 = (v0 - v) / a
   s = 0.0
   i = 0
   while v > 0:
     i += 1
     s += v * dt
     at = np.sqrt(a * a - v * v * v * v / R / R)
     v -= at * dt
   if s < l:
     amax = a
     tmin = t1 + i * dt
   else:
```

```
        amin = a
        tmax = t1 + i * dt
print(tmin)
print(tmax)
print(a)
```

After the convergence of the script, it will tell us that the optimal acceleration is approximately $6\,\mathrm{m\cdot s^{-2}}$ with the braking time of $3.42\,\mathrm{s}$.

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Problem M.1 ... the boat 3 points

Lego wants to cross a river of a width $l = 5.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m\cdot s^{-1}}$ and water flows at speed $v_\mathrm{F} = 0.50\,\mathrm{m\cdot s^{-1}}$.

What is the fastest time in which Lego can cross the river if he does not care how far along the shore the current takes him? *Lego just got carried away.*

If we do not mind, how far will the current take us; we shall paddle perpendicular to the river. The perpendicular (to the river) velocity component is then v_B (while the velocity component in the flow direction is v_F). Therefore, Lego will cross the river in time

$$t = \frac{l}{v_\mathrm{B}} = 5.0\,\mathrm{s}\,.$$

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Problem M.2 ... the boat reloaded 3 points

Lego wants to cross a river of a width $l = 5.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m\cdot s^{-1}}$ and water flows at speed $v_\mathrm{F} = 0.50\,\mathrm{m\cdot s^{-1}}$ in the whole river.

What is the fastest time in which Lego can cross the river if he wants to move in the direction perpendicular to the water flow at all times?

Lego found out that the shortest path is not always the fastest one.

To move perpendicular to the river at any time, we need the paddle velocity component in the flow direction to cancel out with the velocity of the river flow i.e. the magnitude of this component has to be equal to v_F. Therefore; the component perpendicular to the river will be

$$v_\mathrm{n} = \sqrt{v_\mathrm{B}^2 - v_\mathrm{F}^2} \doteq 0.87\,\mathrm{m\cdot s^{-1}}\,.$$

Thus, we will cross the river in time

$$t = \frac{l}{v_\mathrm{n}} \doteq 5.8\,\mathrm{s}\,.$$

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Problem M.3 ... the boat revolutions 3 points

Lego wants to cross a river of a width $l = 5.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m\cdot s^{-1}}$. He wants to move in the direction perpendicular to the river again; however, it has been raining and the water seems to flow faster. What is the slowest speed of the flow for which he cannot cross the river?

Legolas needed one more problem for the series.

In the previous problem, we found out that if we want to move perpendicular to the river, the magnitude of the final velocity will be

$$v_\mathrm{n} = \sqrt{v_\mathrm{B}^2 - v_\mathrm{F}^2}\,,$$

where v_F is the speed of the water flow. This equation returns positive final speed for all $v_\mathrm{F} < v_\mathrm{B}$; therefore for the speed of water flow slower than our maximum speed, Lego can cross the river.

On the other hand, the equation for $v_\mathrm{F} > v_\mathrm{B}$ does not make sense; therefore for such velocities, Lego is not able to cross the river. In the case when $v_\mathrm{F} = v_\mathrm{B}$, the final velocity is zero i.e. Lego will paddle at the same spot - the river will not take him away; however, he will not be moving forward either.

Thus, for the case when $v_\mathrm{F} = v_\mathrm{B} = 1.0\,\mathrm{m\cdot s^{-1}}$, Lego already cannot cross the river.

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Problem M.4 ... the boat resurrections 4 points

Lego wants to cross a river of a width $l = 6.0\,\mathrm{m}$ on a small boat. The maximum speed at which he can paddle is $v_\mathrm{B} = 1.0\,\mathrm{m\cdot s^{-1}}$. The water flows at a speed $v_\mathrm{F} = 1.5\,\mathrm{m\cdot s^{-1}}$. From the previous problem, Lego already knows that in this situation, it is impossible to move in the direction perpendicular to the river. He would like to know the minimum distance by which the current would take him away (from the point on the other shore of the river directly opposite to his starting point) while crossing the river. *Lego optimizes.*

One way to find the solution is to express the distance (that the current will take him away) for a general direction of paddling and then find the minimum (e.g. by derivative).

Another way to find the solution is by geometric thought. To express the questioned distance, we need only the direction of the final velocity. This distance will be minimum when the angle between the direction of the final velocity and the direction of the river flow will be maximum.

This occurs when the direction of the final velocity is perpendicular to the direction of the river flow (the triangle consisting of the velocity of the river flow, the paddling velocity, and the final velocity will be a right triangle, while the hypotenuse will be the river flow velocity).

The maximum angle between the final velocity and the riverbank can be computed as

$$\varphi = \arcsin\frac{v_\mathrm{B}}{v_\mathrm{F}} \doteq 42\,^\circ\,.$$

The distance that the current will take Lego away is

$$d = \frac{l}{\tan\varphi} \doteq 6.7\,\mathrm{m}\,.$$

Note that the width of the river is greater (by one meter) than in previous problems.

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Problem E.1 ... discharging a capacitor 3 points

We are discharging a capacitor of a capacitance $C = 42.0\,\mathrm{mF}$ through a resistor with a resistance of $R = 9.81\,\mathrm{k\Omega}$. We know that the equation $u(t) = U_0 \mathrm{e}^{-\frac{t}{RC}}$ holds for the evolution of voltage in time. In what time will the capacitor's electric field have half the energy of the fully charged state? *Karel wanted the participants to get familiar with discharging.*

The energy of a capacitor is

$$E = \frac{1}{2} C U^2 \,.$$

Thus, the problem can be described by the following equation

$$\frac{1}{4} C U_0^2 = \frac{1}{2} C U_0^2 \mathrm{e}^{-2\frac{t}{RC}} \quad \Rightarrow \quad \frac{1}{2} = \mathrm{e}^{-2\frac{t}{RC}} \,,$$

which we solve for t.

Now, we have two options how to get the solution. We can substitute the equation directly into some computational software, which will solve it for us. An example is Wolfram Alpha.[5] By this approach, we directly get the solution $t \doteq 143\,\mathrm{s}$.

If we are familiar with the logarithmic functions, we can choose an analytical path that is not so demanding, too. We take advantage of the fact that $x = \mathrm{e}^{\ln x}$, and adjust the equation to

$$\frac{1}{2} = \mathrm{e}^{\ln\frac{1}{2}} = \mathrm{e}^{-\ln 2} \quad \Rightarrow \quad -\ln 2 = -2\frac{t}{RC} \,,$$

$$t = \frac{RC}{2} \ln 2 \doteq 143\,\mathrm{s} \,.$$

The energy on the capacitor drops by half in 143 s.

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Problem E.2 ... conductive chewing gum 4 points

Rado is chewing a chewing gum with a specific electrical resistance $\rho = 1\,\mathrm{m\Omega{\cdot}mm}$ and volume $V = 1\,\mathrm{cm}^3$. He splits it in half and uses each half as a separate conductor. In his backpack, he has an electrical appliance of an internal resistance $R = 1\,\Omega$ that can only be powered by voltages up to $U_{\max} = 5\,\mathrm{V}$, and a DC voltage source $U = 12\,\mathrm{V}$. What is the minimum distance l between the electrical appliance and the power supply if he wants to connect them with the two conductors made of chewing gum? The conductors should have the same constant cross-sections. *Rado had chewing gum after a long time.*

The volume of both gums is

$$V_{\mathrm{gum}} = \frac{V}{2} = Sl \,,$$

[5] https://www.wolframalpha.com/input/?i=exp%5B-2*t%2F%289.81*10%5E3*42*10%5E%28-3%29%29%5D+%3D+1%2F2

where S is the area of the cylinder base and l is its length (the distance between the appliance and the source). The resistance of one chewing gum is calculated by the specific electrical resistance as follows

$$R_{\mathrm{gum}} = \rho\frac{l}{S} = \rho\frac{l^2}{V_{\mathrm{gum}}} = 2\rho\frac{l^2}{V}\,.$$

The total resistance of the circuit is $R_{\mathrm{c}} = 2R_{\mathrm{gum}} + R$. From Ohm's law, we obtain

$$U = R_{\mathrm{c}}I = (2R_{\mathrm{gum}} + R)I = \left(4\rho\frac{l^2}{V} + R\right)I\,,$$

where $I = \frac{U_{\max}}{R}$, and thus

$$l = \sqrt{\frac{VR\left(\frac{U}{U_{\max}} - 1\right)}{4\rho}} \doteq 0.59\,\mathrm{m}\,.$$

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Problem E.3 ... cheap voltmeter 3 points

An AC voltmeter works as follows. At first, the DC component is filtered out of the input voltage using a high-pass filter. Subsequently, the obtained voltage is rectified and smoothed by a low-pass filter. The resulting DC voltage is measured and converted to the displayed value by multiplying it by a numerical factor. This factor is chosen in such a way that for a sine wave, the displayed value is the effective value of voltage. What value is being displayed if the input AC voltage has a sawtooth waveform oscillating between $U_1 = 0.00\,\mathrm{V}$ and $U_2 = 1.00\,\mathrm{V}$?

Dodo steals ideas for problems from the Department of Physics Education.

Firstly, we determine the numerical factor used by the device. After the rectification of harmonic voltage given by the relation $U = U_0\sin(\omega t)$, which does not have a DC component (its time mean is zero), we get the timecourse of the voltage $U = U_0\left|\sin(\omega t)\right|$.

Subsequent smoothing makes a mean value of it. The rectified signal is π-periodic, so we only need to calculate the mean of the following expression

$$U_{\mathrm{m}} = \frac{1}{\pi}\int_0^{\pi} U_0\sin\varphi\,\mathrm{d}\varphi = \frac{2}{\pi}U_0\,.$$

To obtain the RMS value given by the mean square of the voltage, we have to multiply the measured voltage by the conversion factor a.

$$U_{\mathrm{eff}} = \sqrt{\frac{1}{\pi}\int_0^{\pi} U_0^2\sin^2\varphi\,\mathrm{d}\varphi} = \frac{1}{\sqrt{2}}U_0\,.$$

The conversion factor therefore has a value of

$$a = \frac{U_{\mathrm{eff}}}{U_{\mathrm{m}}} = \frac{\pi}{2\sqrt{2}}\,.$$

In the case of a sawtooth waveform voltage, we have to firstly remove the DC component – the long-term time mean is equal to $U = 0.50\,\mathrm{V}$, so the resulting sawtooth waveform has values between $-0.50\,\mathrm{V}$ and $0.50\,\mathrm{V}$. After the rectification, we get consecutive triangles all with a base on the timeline and a height of $0.50\,\mathrm{V}$.

After smoothing, we get the mean value of voltage $U'_\mathrm{m} = 0.25\,\mathrm{V}$ and after conversion in the device to "effective value", the displayed value is $U_\mathrm{d} = aU'_\mathrm{m} \doteq 0.28\,\mathrm{V}$. By the way, the actual value of the effective value is $U'_\mathrm{eff} = \frac{1}{\sqrt{3}}\,\mathrm{V} \doteq 0.58\,\mathrm{V}$ i.e. almost twice as high.

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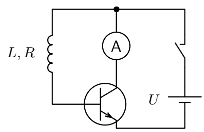
Problem E.4 ... coil on a base — 4 points

Consider a circuit with a DC source ($U > 50\,\mathrm{V}$), an inductor with inductance $L = 5\,\mathrm{H}$ and resistance $R = 10\,\mathrm{k\Omega}$, an NPN transistor, an ideal ammeter, and a switch. Some time after closing the switch, the value on the ammeter will settle on $I = 1\,\mathrm{A}$. At what time after closing the switch does the ammeter display the value $I' = 500\,\mathrm{mA}$?



Vojta was not building a bomb.

Notice, that we can omit the BE voltage thanks to $U > 50\,\mathrm{V}$. Let β be the current gain of the transistor. We can express the currents flowing through the inductor in terms of currents flowing through the ammeter. In the time ammeter was displaying value I, the current $I_\mathrm{max} = I/\beta$ was flowing through the inductor, and analogously when ammeter was showing I', $I_L = I'/\beta$ flowed through the inductor.

Now we proceed by using the following relation, which can be easily derived by solving a differential equation for self-induction of a coil,

$$I_L = I_{max}(1 - e^{-\frac{R}{L}t})\,,$$

from where it is straightforward to substitute for the currents and rearrange the terms to get

$$t = \frac{L}{R}\ln\left(\frac{I}{I - I'}\right) \doteq 0.35\,\mathrm{ms}\,.$$

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Problem X.1 ... I'll pull her down with me — 3 points

Let us assume we have a ball of air of radius $R = 2.1\,\mathrm{cm}$, which is wrapped in a layer of polystyrene of thickness $t = 2.1\,\mathrm{cm}$ and density $\rho = 33\,\mathrm{kg{\cdot}m^{-3}}$. We attach the ball to a force gauge by a rigid, weightless string and then pull it down so that half of the ball's volume is under the water surface. What is the value we see on the force gauge?

Karel had balloons on his mind.

The force gauge will display the difference of the buoyancy force (that will keep the ball floating) and the gravity acting on the ball (where we neglect the mass of the air). The volume of the polystyrene is the difference of the volume of the whole ball V_{ball} and the volume of the air V_{a}. Let us denote the density of the water as ρ_{w}. Then

$$F = F_{\mathrm{b}} - F_g = \frac{1}{2} V_{\mathrm{ball}}\, \rho_{\mathrm{w}} g - \left(V_{\mathrm{ball}} - V_{\mathrm{a}}\right) \rho g$$
$$= \frac{2}{3}\pi \left(R+t\right)^3 \rho_{\mathrm{w}} g - \left(\frac{4}{3}\pi\left(R+t\right)^3 - \frac{4}{3}\pi R^3\right) \rho g \doteq 1.43\,\mathrm{N}\,.$$

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Problem X.2 ... to the bottom of the dam — 4 points

Consider a piece of polystyrene with weight $m_0 = 1.25\,\mathrm{kg}$ and density $\rho_0 = 27\,\mathrm{kg{\cdot}m^{-3}}$ that we want to completely submerge to the bottom of a water dam. We have plenty of stones of various sizes and the same density $\rho = 2\,650\,\mathrm{kg{\cdot}m^{-3}}$. What is the minimum weight of a stone which we should attach to the polystyrene? *Karel was thinking about Archimedes.*

Let's denote the weight of the stone by m. After submerging the polystyrene-stone system in water, it is subjected to buoyancy $F_{\mathrm{b}} = V\rho g$ and gravity $F = Mg$ force, where $V = \frac{m_0}{\rho_0} + \frac{m}{\rho}$ is the total volume and $M = m_0 + m$ is the total weight.

In the extreme case, the magnitude of the gravitational force will be the same as the magnitude of the buoyancy force

$$(m_0 + m)g = \left(\frac{m_0}{\rho_0} + \frac{m}{\rho}\right)\rho_{\mathrm{w}} g\,,$$

where $\rho_{\mathrm{w}} = 998\,\mathrm{kg{\cdot}m^{-3}}$ is the density of water.

By solving this equation we get

$$m = m_0 \frac{\frac{\rho_{\mathrm{w}}}{\rho_0} - 1}{1 - \frac{\rho_{\mathrm{w}}}{\rho}} \doteq 72.1\,\mathrm{kg}\,.$$

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Problem X.3 ... an unknown ball in an unknown fluid — 4 points

We are given a ball of unknown radius, submerged in a fluid of unknown (but constant) density. We know that the difference between hydrostatic pressures at the top and the bottom point of the ball is $\Delta p = 850\,\mathrm{Pa}$, and the total buoyant force acting on the ball is $F_{\mathrm{b}} = 150\,\mathrm{N}$. What is the density of the fluid? *Lego was inventing buoyancy problems.*

The hydrostatic pressure difference can be expressed as $\Delta p = \Delta h \rho g$. In this problem, the height difference between the top and the bottom point of the ball is the diameter of the ball i.e. double the radius ($\Delta h = 2r$). Therefore, we can express the radius of the ball as

$$r = \frac{\Delta p}{2\rho g}\,.$$

Because the volume of the ball is $V = 4/3\pi r^3$, we only need to substitute into the formula for buoyancy force

$$F_\mathrm{b} = V\rho g = \frac{4}{3}\pi\frac{\Delta p^3}{8\rho^3 g^3}\rho g\,,$$

now, we can express the density of the unknown fluid

$$\rho = \sqrt{\frac{\pi\Delta p^3}{6F_\mathrm{b}g^2}} \doteq 150\,\mathrm{kg{\cdot}m^{-3}}\,.$$

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Problem X.4 ... attractive 3 points

Danka has two brass balls with diameters of 2.00 cm *carrying the same charges. One of the balls is fixed and the other one hovers above it at a distance of* $d_1 = 20.0\,\mathrm{cm}$ *(we assume that the ball can only move in the vertical direction). We now immerse both of the balls in oil of density* $\rho_0 = 910\,\mathrm{kg{\cdot}m^{-3}}$*, where a new equilibrium is established. The balls are now* $d_2 = 14.0\,\mathrm{cm}$ *away from each other. What is the relative permittivity of the oil we used? Consider the density of brass to be* $\rho = 8\,400\,\mathrm{kg{\cdot}m^{-3}}$*. Distance between the two balls means the distance of their centers.* *Danka was playing with balls.*

To solve the problem, we need to examine the forces acting on the upper ball. In the air, it is subjected to the electrostatic force given by Coulomb's law which compensates for the gravitational force. Therefore, we have

$$V\rho g = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{d_1^2}\,,$$

where $V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$ is the volume of one ball, ε_0 is the permittivity of vacuum and Q is the magnitude of electric charge of one ball. When immersed in oil, in addition to these two forces, a significant hydrostatic buoyancy force acts on the upper ball. This situation has the equilibrium described by a similar relation

$$V\rho g = \frac{1}{4\pi\varepsilon_0\varepsilon_\mathrm{r}}\frac{Q^2}{d_2^2} + V\rho_0 g\,,$$

where ε_r is the relative permittivity of oil, we're searching for. The first and the second equilibrium relations give us two equations of two unknowns. To simplify the calculation, we express the fraction from the first relation as

$$\frac{Q^2}{4\pi\varepsilon_0}\,,$$

and plug it into the second one. From here, we can express the relative permittivity of oil as

$$\varepsilon_\mathrm{r} = \left(\frac{d_1}{d_2}\right)^2\frac{\rho}{\rho - \rho_0}\,.$$

After substituting with numerical values, we get that the relative permittivity of the oil we used is $\varepsilon_{\mathrm{r}} \doteq 2.3$.

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Solutions of Physics Brawl Online 2022

Problem 1 ... my hand is slipping off 3 points

According to information from Wikipedia, the longest escalators in Prague are at Náměstí Míru. They travel a distance $l = 87.1\,\mathrm{m}$ in $t = 140\,\mathrm{s}$. It is also said that the handrails on the escalators move $\delta v = 2.0\,\%$ faster than the stairs in order to ensure alertness. How much would your arm move relative to the rest of your body (if it was long enough) during the entire escalator ride, if you stood on the same stair while holding one spot the whole time?

Karel was wondering about escalators again.

The handrails are moving at a speed

$$u = v \cdot (1 + \delta v) ,$$

so the hand will get to the end of the escalator ride in time

$$\tau = \frac{l}{u} = \frac{l}{v} \cdot \frac{1}{1 + \delta v} .$$

Meanwhile, the rest of the body travels the distance

$$l_0 = v \cdot \tau = l_0 \cdot \frac{1}{1 + \delta v} ,$$

from which we find that the hand has moved relative to the rest of the body by

$$\Delta l = l - l_0 = \frac{\delta v}{1 + \delta v} \cdot l ,$$

$$\Delta l = 1.71\,\mathrm{m} .$$

The hand would move by 1.71 m with respect to the legs. Therefore, it is not realistic for most people to hold themselves in one place all the time. It depends on how exactly the escalators are set up. However, the fact that you sometimes have to change the position of your hand (and you probably do so subconsciously) is usually not a fault of the escalators, but instead a desired feature.

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Problem 2 ... comparison of slings 3 points

Lego got bored of bows, so he went to buy a sling. We approximate it as a spring with constant stiffness and a zero rest length. Lego can stretch the sling with force $F = 31.4\,\mathrm{N}$, and he wants it to store as much energy as possible while it is stretched. There were two slings at the shop, with stiffnesses $k_1 = 159\,\mathrm{N/m}$ and $k_2 = 265\,\mathrm{N/m}$. What will be the difference between the potential energies of those slings stretched with force F? Give a positive result if the sling with k_1 has more energy and a negative one if the opposite is true.

Lego got bored of bows.

The potential energy of a spring is given by $E_p = \frac{1}{2}ky^2$, where k is a stiffness of a spring, and y is its elongation. If we stretch the spring with the stiffness k_1 using the force F, its length will be increased by $y_1 = \frac{F}{k_1}$. Then for the potential energy of the first spring, we get

$$E_{p1} = \frac{1}{2} k_1 y_1^2 = \frac{F^2}{2k_1} .$$

Analogically we can express the potential energy of the second spring with the same elongation. Then, we calculate the difference between those two energies, which will give us

$$E_{p1} - E_{p2} = \frac{F^2}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) = 1.24\,\mathrm{J}\,,$$

which is our solution.

As we can see, if we fixate the force with which we are stretching the string, we will end up doing more work when the spring has smaller stiffness. Why don't we make all slings and bows with the least possible stiffness? The short answer would be that even if we don't consider the maximum force we can use, our arms are limited by the maximum length they can be apart. (More complex answer includes that force, which we can apply, isn't independent of the relative position of our hands...)

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Problem 3 ... mountain biking 3 points

Matěj is riding on a mountain bike trail, passing oncoming cyclists. At the top of the hill, exhausted riders are moving at an average speed of $5\,\mathrm{km{\cdot}h^{-1}}$ in both directions and Matěj meets them here at an average rate of 0.02 per second. On the other hand, at the lowest point of the trail, the cyclists are revved up (from both directions) and moving at an average speed of $50\,\mathrm{km{\cdot}h^{-1}}$. How many cyclists does Matěj meet here on average? The bike trail has no turnoffs, and Matěj always moves at the speed of an average cyclist. *Matěj mountain biked.*

The problem is solvable using a trick and thus has a trivial solution. Assuming that no one has crashed, to conserve the number of cyclists, the same number of riders must pass each point on the trail per second. Due to Matěj moving at the same speed as the oncoming cyclists, the number of riders encountered per second is double (relative to a static observer) but remains constant. The solution is therefore 0.02 cyclists per second.

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Problem 4 ... hunter, shoot yourself 3 points

What is the largest radius a planet can have for a person to be able to shoot oneself on it? Hunter shoots just above its surface, parallel to it, and the planet will have no atmosphere. The velocity of a flying bullet is $v = 380\,\mathrm{m{\cdot}s^{-1}}$, and the density of the planet is $\rho = 3\,200\,\mathrm{kg{\cdot}m^{-3}}$. Furthermore, assume the planet to be spherical and homogeneous.

Karel wanted planetary shooting.

The problem statement basically asks us what the radius of the planet must be, for the bullet to move in a circular path just above its surface (it would move in an elliptical orbit with greater velocity). Denoting m the mass of the bullet, M the mass of the planet, and R its radius, we are interested with the equality of forces

$$m\frac{v^2}{R} = G\frac{mM}{R^2}\,,$$

where the mass M can be further expressed as

$$M = \rho V = \frac{4}{3}\pi\rho R^3 \,.$$

By addition and rearranging the equation, we get

$$R = v\sqrt{\frac{3}{4\pi\rho G}} \doteq 402\,\mathrm{km}\,.$$

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Problem 5 ... falling icicles 3 points

A small icicle broke loose from the edge of the roof and started its free fall along the vertical wall of the house. The icicle flew past a window of height $h = 1.50\,\mathrm{m}$ during a time $t = 0.10\,\mathrm{s}$. What is the distance between the edge of the roof and the top of the window?

Kuba was once almost hit by an icicle.

We know that the height of the window is h and that the icicle was falling past it during the time t, so it has traveled the path h in the time t. Clearly, the icicle already had some non-zero velocity at the top edge of the window. We shall therefore use the formula for the path of a uniformly accelerated rectilinear motion with a non-zero initial speed

$$h = v_0 t + \frac{1}{2}gt^2 \,.$$

The only quantity in this relation that we do not know is the speed at the upper edge of the window v_0. We can therefore express it as

$$v_0 = \frac{h - \frac{1}{2}gt^2}{t} \,.$$

Finally, it remains to realize that if the icicle started its fall from rest and it gained the speed v_0 in a time $t_0 = v_0/g$. Thus, from the moment it started falling to the moment it was at the top edge of the window, it fell by

$$h_0 = \frac{1}{2}gt_0^2 = \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{\left(h - \frac{1}{2}gt^2\right)^2}{2gt^2} \doteq 10.7\,\mathrm{m}\,.$$

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Problem 6 ... worn tire

3 points

The tires on our new car have a diameter of $D = 634.5\,\mathrm{mm}$ *and a tread depth of* $d_0 = 8.0\,\mathrm{mm}$. *The technician set the car's speedometer to measure accurately for this original wheel size. However, we drive frequently, so after a while, the tire tread came down to the minimum depth for summer use,* $d_{\mathrm{min}} = 1.6\,\mathrm{mm}$. *How does the actual speed now differ from the speedometer reading? Give the answer as the ratio of the speed on the speedometer to the actual speed.*

Karel was refueling.

We consider the wheel to be perfectly circular; then its circumference is $o_0 = \pi D$. After the tire is worn out, we reach the diameter $D_1 = D - 2(d_0 + d_{min})$ and the new circumference of the wheel is $o_1 = \pi\,(D - 2d_0 + 2d_{\mathrm{min}})$. When the car travels the circumference of the new wheel, the "speedometer thinks" it has traveled a little more, namely the length of the original circumference. The ratio of speeds shown by the speedometer will correspond to the ratio of the circumferences, i.e.

$$k = \frac{o_0}{o_1} = \frac{\pi D}{\pi\,(D - 2d_0 + 2d_{\mathrm{min}})} = \frac{D}{D - 2d_0 + 2d_{\mathrm{min}}} \doteq 1.02\,.$$

The speedometer then shows $102\,\%$ of the actual speed.

In fact, the speedometer should always be set to show a little more than the actual speed, even with unworn tread. The real situation is also complicated by the fact, that the tires change shape during the motion; we're not driving just on a flat road, etc. However, these effects are relatively small.

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Problem 7 ... Drop Da Bomb

3 points

At the end of The Simpsons Movie, Homer and Bart are riding a motorcycle on the inside of a glass dome. Assume the dome has a radius of $1.00\,\mathrm{km}$. *What minimum speed does the motorcycle have to travel at to ensure that Homer and Bart do not fall off even at the highest point of the dome?* *Jindra was testing his new motorcycle at The Ondřejov Observatory.*

We will be using a non-inertial system coupled to the motorcycle. Two forces act on the motorcycle at the top of the dome: a downward gravitational force and an upward centrifugal force. The limiting case occurs when the two forces are equal

$$mg = \frac{mv^2}{r}.$$

We have denoted the radius of the sphere by $r = 1.00\,\mathrm{km}$, m is the mass of the motorcycle with riders, $g = 9.81\,\mathrm{m\cdot s^{-2}}$ is the acceleration due to gravity, and v is the minimum speed of the motorcycle. Expressed

$$v = \sqrt{gr} = 99.0\,\mathrm{m\cdot s^{-1}} = 357\,\mathrm{km\cdot h^{-1}}.$$

Homer and Bart would have to travel at $357\,\mathrm{km\cdot h^{-1}}$.

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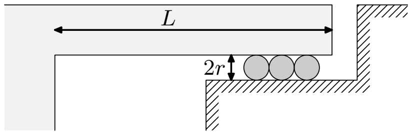
Problem 8 ... moving bridge

3 points



A road bridge of length $L = 8\,\mathrm{m}$ is firmly anchored on one side. The other side is placed on sturdy cylinders of radius $r = 6\,\mathrm{cm}$. The cylinders lay on a rigid flat surface on which they can move freely in a horizontal direction. By what angle will the cylinders rotate between winter, when the temperature is $t_z = -10\,\mathrm{^\circ C}$, and summer, when the temperature is $t_l = 30\,\mathrm{^\circ C}$? The coefficient of linear thermal expansion of the bridge is $\alpha = 1.0 \cdot 10^{-5}\,\mathrm{K^{-1}}$. Note that the cylinders do not slip and that there is a gap between the road and the moving end of the bridge.

Jarda can feel the ground moving beneath his feet.

The length of the bridge changes by $\Delta l = l\alpha\Delta T$, where l is its original length, α is the coefficient of linear thermal expansion, and $\Delta T = 40\,\mathrm{^\circ C}$ is the temperature difference between winter and summer.

The moving end of the bridge is therefore displaced by $\Delta l = 3.2\,\mathrm{mm}$. As the end of the bridge moves, the position of the cylinders changes. Since the cylinders do not slip, the distances on both flat and curved surfaces are the same. When the end of the bridge moves, the cylinder moves half the distance, because it does not slip at the bottom or the top - the center of the cylinder moves forward at a certain velocity, but the top of the cylinder moves faster by a factor of the angular velocity times the radius of the cylinder.

Thus the angle in degrees is calculated as

$$\varphi = \frac{l\alpha\Delta T}{2}\frac{360\,^\circ}{2\pi r} = 1.5^\circ\,,$$

where r is the radius of the cylinder.

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Problem 9 ... two rectangular cuboids on a hill

4 points

Let us have "a hill" made of two inclined planes with inclinations α_1 and α_2 with respect to the horizontal plane. We lay a rectangular cuboid with the mass $m_1 = 10\,\mathrm{kg}$ on the first plane with inclination $\alpha_1 = 25^\circ$. On the second plane with inclination $\alpha_2 = 35^\circ$, we place a different rectangular cuboid with the mass $m_2 = 15\,\mathrm{kg}$. Then we connect the cuboids with a massless rope that is led through a massless pulley at the top of the hill. The rope is always stretched parallel to the given plane, the coefficient of dynamic friction between the cuboids and inclined planes is $f = 0.15$, and the coefficient of static friction is small enough for cuboids to start moving. What is the acceleration of the cuboid m_1 if both cuboids are at rest at the beginning? The positive sign will indicate acceleration downhill, and a negative sign will indicate acceleration uphill.

Lego was hiking.

The attentive solver will immediately notice that the rectangular cuboid m_2 is heavier and on the inclined plane with a greater slope. So, intuitively, the cuboid m_2 will accelerate downhill and m_1 will accelerate uphill. We will verify this by calculation. It is important to note, that if we made this calculation with the assumption that m_1 is accelerating downhill, we would get negative acceleration. However, this acceleration is not our solution, because it was calculated with the wrong assumption, that the cuboids are moving with the opposite acceleration to what they truly are. Hence, the friction forces have the opposite orientation (this incorrect solution is therefore larger in absolute value than the correct one).

Enough talking about how not to solve this problem; let's see how to solve it correctly. The cuboid m_2 is pulled down by a component of its weight parallel with the inclined plane, its magnitude, therefore, is $F_{\mathrm{k}2} = m_2 g \sin\alpha_2$. It will be slowed down by the component of its weight perpendicular to the base of the cuboid, i.e. with magnitude $F_{\mathrm{t}2} = f m_2 g \cos\alpha_2$. Moreover, it will be pulled upwards by the rope with a force whose magnitude T we do not know. The equation of motion for this cuboid will be $m_2 a_2 = F_{\mathrm{k}2} - F_{\mathrm{t}2} - T$. The forces for the cuboid m_1 will look similar, except that the rope will pull it up in the direction of its acceleration, while its weight will pull it down, i.e. in the opposite direction of its acceleration. The equation of motion for it will therefore be $m_1 a_1 = -F_{\mathrm{k}1} - F_{\mathrm{t}1} + T$.

Since the rope and the pulley are intangible, the rope pulls both cuboids with a force of the same magnitude T. At the same time, the acceleration of the cuboids must be equally large, so in our equations, $a_1 = a_2$. What remains is to sum up the equations to get rid of the unknown T and express a_2:

$$\begin{aligned}(m_2 + m_1) a_2 &= F_{\mathrm{k}2} - F_{\mathrm{t}2} - F_{\mathrm{k}1} - F_{\mathrm{t}1}\,,\\ a_2 &= g\frac{m_2 \sin\alpha_2 - f m_2 \cos\alpha_2 - m_1 \sin\alpha_1 - f m_1 \cos\alpha_1}{m_2 + m_1}\,,\\ a_2 &= 0.46\,\mathrm{m{\cdot}s^{-2}}\,.\end{aligned}$$

Let us note that the problem statement asked for a negative acceleration if the cuboid m_1 accelerates uphill, so the correct answer is $-0.46\,\mathrm{m{\cdot}s^{-2}}$.

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Problem 10 ... energy-saving electric kettle 4 points

When prices rise, you must save. What percentage of our finances will we save if we heat less water for tea and more economically at the same time? The original method was to heat $V_1 = 1.00\,\mathrm{l}$ of water from the original temperature $t_0 = 18.5\,\mathrm{^\circ C}$ (both the kettle and water temperature) to the boiling point $t_\mathrm{v} = 98.5\,\mathrm{^\circ C}$. We use a kettle with heat capacity $C = 442\,\mathrm{J{\cdot}K^{-1}}$. The new method will be different in a smaller volume of heated water $V_2 = 0.800\,\mathrm{l}$ and in a lower reached temperature of $t_2 = 83\,\mathrm{^\circ C}$. Regardless of the current price development, consider the price of electricity for consumers as constant $p = 4.45\,\mathrm{Kč{\cdot}kWh^{-1}}$. Assume that the efficiency of the kettle is constant $95\,\%$ and that the temperature is the same in the whole volume at any given moment. *Karel was looking at the rising inflation.*

This is a relatively straightforward application of the calorimetric equation, which is only made more complicated by the fact that we are comparing two situations and at the end, we will have to calculate their ratio. The price given in the problem statement isn't necessary to solve the problem. The question is aimed at the percentage saving, which does not depend on the price of electricity if it is constant. However, if we heated in the first way before the price increase and in the second way after the price increase, we would need to know the current prices. Similarly, the information about efficiency is superfluous as well, because it only changes the absolute values of energy/heat, but the relative savings remain the same.

The heat received in the first case is the sum of the heat received by the water and by the kettle. Thus, the total

$$Q_1 = m_1 c \Delta t_1 + C \Delta t_1 = (\rho V_1 c + C)(t_v - t_0) \ ,$$

where $m_1 = \rho V_1$ is the mass of water, ρ is the density of water, c is the specific heat capacity of water. We assume that the material constants for water are sufficiently accurate to be constant for the given temperature range. Similarly, we can write a relation for the more economical heat

$$Q_2 = m_2 c \Delta t_2 + C \Delta t_2 = (\rho V_2 c + C)(t_2 - t_0) \ ,$$

where the variables are denoted analogously.

We want to compare the percentage savings with the original consumption. Let's denote the savings ratio as k. As mentioned earlier, in our case, the savings ratio is the same as the heat ratio

$$k = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\rho V_2 c + C}{\rho V_1 c + C} \frac{t_2 - t_0}{t_\mathrm{v} - t_0} \doteq 34.0\,\%$$

From the final expression, we see that if we were satisfied with the result neglecting the heat capacity of the kettle, we would not even need to know the density of water nor the specific heat capacity of water. However, in that case, we would obtain the result 35.5 %, which is noticeably different within the specified number of significant digits, so we did not recognize the result with the neglected heat capacity of the kettle as valid.

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Problem 11 ... induced voltage reloaded — 3 points

Let us have an area of a homogenous magnetic field. The area has a rectangular cross-section with sides $a = 3.0\,\mathrm{m}$, $b = 2.0\,\mathrm{m}$, and the magnetic field vector $B = 1.0 \cdot 10^{-3}\,\mathrm{T}$ is perpendicular to it. We place a sufficiently long wire parallel to the diagonal of the rectangle so that it all lies entirely outside of the area with the magnetic field. We connect the ends of the wire to a voltmeter.

Now we start moving the wire in a rectilinear motion at a speed of $v = 0.20\,\mathrm{m{\cdot}s^{-1}}$ perpendicularly to the direction of the magnetic field vector and the diagonal to which it is parallel. What is the peak magnitude of voltage displayed by the voltmeter during the movement of the wire? *Vojta was proofreading.*

We will calculate the voltage induced on a conductor of length l moving with a speed v perpendicularly to the magnetic field of induction B using

$$U = Bvl \,.$$

Since we are interested in the highest value of the voltage, the length of the part of the conductor inside the magnetic field must be maximal, as all other parameters are constant. The length can be at most $\sqrt{a^2 + b^2}$, so we get

$$U_\mathrm{max} = Bv\sqrt{a^2 + b^2} \doteq 0.72\,\mathrm{mV} \,.$$

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Problem 12 ... Archimedeskugeln 5 points

Consider two identical containers of mass $M_\mathrm{n} = 340.0\,\mathrm{g}$ *with the identical amount of water of mass* $M_\mathrm{w} = 150.0\,\mathrm{g}$*. We place each of the containers on a scale separately. Into the first container, we immerse a steel ball with density* $\rho_\mathrm{o} = 7\,850\,\mathrm{kg{\cdot}m^{-3}}$*, which is suspended from an external frame that does not stand on the scale. The ball is fully immersed and does not touch the bottom. In the second container, we attach a ball made of polystyrene foam with density* $\rho_\mathrm{p} = 27.0\,\mathrm{kg{\cdot}m^{-3}}$ *and radius* $r_\mathrm{p} = 1.5\,\mathrm{cm}$ *to the bottom with a thin string. The ball is fully submerged, not breaking the surface. What must be the radius of the steel ball so the two scales will display the same weight?*

Jindra has been learning German to hide the balls in the title.

In the first case, the force of gravity $\rho_\mathrm{o} V_\mathrm{o} g$ pointing downwards acts on a steel ball of volume V_o. In the upward direction, the two forces act – the buoyancy $\rho_\mathrm{w} V_\mathrm{o} g$ and the tensile force of the string T_o. The water in the container is subject to downward force $\rho_\mathrm{w} V_\mathrm{o} g$, which is a reaction to the buoyant force acting on the ball. Thus, the first container pushes on the scale with a total force $M_\mathrm{n} g + M_\mathrm{w} g + \rho_\mathrm{w} V_\mathrm{o} g$, where the first two terms express the gravitational forces of the container and the water.

In the second case, the upward buoyant force of the water $\rho_\mathrm{w} V_\mathrm{p} g$ acts on the foam ball of volume V_p. In the downward direction, the gravitational force $\rho_\mathrm{p} V_\mathrm{p} g$ acts. From the equilibrium of forces acting on this ball, we determine the tensile force of the string as

$$T_\mathrm{p} = (\rho_\mathrm{w} - \rho_\mathrm{p})\, V_\mathrm{p} g\,,$$

which acts downwards on the foam ball.

The reaction force from the ball $\rho_\mathrm{w} V_\mathrm{p} g$ acts downwards on the water. However, the string also pulls on the bottom with a force of magnitude T_p upwards, which lifts the container. The total force acting on the scale is, therefore

$$M_\mathrm{n} g + M_\mathrm{w} g + \rho_\mathrm{w} V_\mathrm{p} g - (\rho_\mathrm{w} - \rho_\mathrm{p})\, V_\mathrm{p} g = M_\mathrm{n} g + M_\mathrm{w} g + \rho_\mathrm{p} V_\mathrm{p} g\,.$$

We could have derived the same result a little easier if we realized that the container, the water, and the polystyrene-foam ball form one system with mass $M_\mathrm{n} + M_\mathrm{w} + \rho_\mathrm{p} V_\mathrm{p}$, on which the gravitational force $(M_\mathrm{n} + M_\mathrm{w} + \rho_\mathrm{p} V_\mathrm{p})\, g$ acts.

Both scales display the same value if $\rho_\mathrm{w} V_\mathrm{o} = \rho_\mathrm{p} V_\mathrm{p}$ holds (the masses of water and container cancel out), from which we obtain

$$r_\mathrm{o} = r_\mathrm{p} \sqrt[3]{\frac{\rho_\mathrm{p}}{\rho_\mathrm{w}}} = 0.45\,\mathrm{cm}\,.$$

Interestingly, the desired radius does not depend on the density of the steel.

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Problem 13 ... neglecting the Earth's movement 4 points

Have you ever considered that when we calculate the free fall of a body on Earth, we neglect the Earth's motion towards the body? This calculation corresponds to an Earth with infinite mass. However, is it possible to measure the Earth's motion, at least in theory? According to *today's physics, we generally cannot measure distances smaller than Planck's length, which is* $l_\mathrm{P} = \sqrt{\hbar G/c^3} \doteq 1.616 \cdot 10^{-35}\,\mathrm{m}$*. What is the minimum mass that a body needs to have for the Earth to move from its initial position by at least* $n = 10^4$ *Planck's lengths from the point when this body is released until the point when it impacts the Earth? The body is released at the height* $h = 1.00\,\mathrm{m}$ *above the Earth's surface. Neglect the Earth's rotation, atmosphere, and any other motion of the Earth.* *Karel was thinking about approximations and Planck units.*

Solution using free fall

For simplicity, let's assume we're dealing with simple free fall. During some time, the body moves by h and meanwhile, the Earth moves by nl_P. We could also consider that the displacement of the body during free fall is actually smaller because the Earth moved towards it, but this difference is so small that we can neglect it.

Since we're dealing with motion from rest with constant acceleration in our reference frame,

$$h = \frac{1}{2}a\,t^2\,, \quad s_\oplus = \frac{1}{2}a_\oplus t^2\,,$$

where a is the acceleration of the body, t is the duration of the fall, $s_\oplus$ is the displacement of the Earth and $a_\oplus$ is the acceleration of the Earth. Both accelerations are gravitational, so

$$a = \frac{GM_\oplus}{R_\oplus^2}\,, \quad a_\oplus = \frac{Gm}{R_\oplus^2}\,,$$

where $R_\oplus$ is the radius of the Earth, and we can again neglect both h and nl_P compared to this radius and consider the acceleration during the fall to be constant.

The durations of the "falls" are equal, so

$$\frac{2h}{a} = \frac{2s_\oplus}{a_\oplus} \quad \Rightarrow \quad s_\oplus = \frac{a_\oplus}{a}h = \frac{m}{M_\oplus}h\,.$$

We need the displacement to be greater than nl_P, so

$$s_\oplus > nl_\mathrm{P} \quad \Rightarrow \quad nl_\mathrm{P} < \frac{m}{M_\oplus}h \quad \Rightarrow \quad m > nl_\mathrm{P}\frac{M_\oplus}{h} \doteq 9.65 \cdot 10^{-7}\,\mathrm{kg} = 0.965\,\mathrm{mg}\,.$$

The mass of the body would need to be greater than $0.965\,\mathrm{mg}$.

Solution using center of mass

Since the problem statement says we're dealing with an isolated system of the Earth and the body (we don't consider other motions of the Earth), their common center of mass must perform uniform linear motion according to Newton's 1st law. If we're looking at the problem in the reference frame of this center of mass, we know that it does not move. If we denote the distances of the center of the Earth and the body from the common center of mass by $d_\oplus$ and d respectively, a well-known formula says that

$$d_\oplus M_\oplus = dm\,, \tag{1}$$

where $M_\oplus$ is the mass of the Earth. The height above the Earth's surface is $h + R_\oplus = d_\oplus + d$, where $R_\oplus$ is the radius of the Earth.[1] The formula for the distances of the centers of mass 1 holds all the time, during the fall and even after the body's impact on the surface. At the moment of impact, the distances change to $d'_\oplus$ and d', which satisfy $R_\oplus = d'_\oplus + d'$ and $d'_\oplus M_\oplus = d'm$. We're interested in displacements, not the distances themselves. Therefore, let's subtract these equations

$$\Delta d_\oplus M_\oplus = \Delta d m\,, \qquad h = \Delta d_\oplus + \Delta d\,.$$

Solving this system of two equations with two unknowns, we find the formula for displacement

$$\Delta d_\oplus = \frac{mh}{m + M_\oplus} \approx \frac{mh}{M_\oplus}\,.$$

We're looking for the smallest m which satisfies $\Delta d_\oplus > n l_\mathrm{P}$

$$m > \frac{M_\oplus n l_\mathrm{P}}{h} \doteq 9.65 \cdot 10^{-7}\,\mathrm{kg} = 0.965\,\mathrm{mg}\,.$$

It turns out that the mass of the body would need to be greater than approximately a milligram.

Finally, note that even 10^4 Planck lengths isn't a displacement which today's experimental physics would be able to measure for something as large as the Earth, so this problem is really just a theoretical plaything. Another alternative solution could use conservation of momentum.

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Problem 14 ... hot light bulb

4 points

Danka has a lamp that has an old light bulb with an input power of $P = 60\,\mathrm{W}$ on her desk. At a distance $h = 40\,\mathrm{cm}$ below the bulb, there is a computer mouse with an elliptical cross-section. The semi-major axis of the ellipse is $a = 60\,\mathrm{mm}$ and the semi-minor $b = 30\,\mathrm{mm}$. When the light bulb was turned on, the mouse had a room temperature $T_1 = 23.0\,\mathrm{^\circ C}$. What will be the temperature of the mouse after 90 minutes of illumination? Assume that the bulb radiates isotropically into the entire space, $\eta = 83\%$ of energy hitting the mouses cross-section is converted into heat, while mouse is not losing heat itself. The heat capacity of the mouse is $C = 200\,\mathrm{J{\cdot}K^{-1}}$. *Danka's mouse is warming up from the lamp.*

From the calorimetry formula, we know that the heat Q received by the mouse is equal to

$$Q = C(T_2 - T_1)\,,$$

where T_2 is the temperature of the mouse after 90 minutes of illumination. This is equal to the heat it receives from the bulb. The bulb has a heat output power equal to $P\eta$, distributed evenly throughout the space. At distance h, the heat output power of the bulb per unit area is equal to $\frac{P\eta}{4\pi h^2}$. Multiplying this quantity by the area occupied by the cross-section of the mouse and by the time t gives the total heat that is supplied to the mouse. Since the mouse has non-zero dimensions, when its center is at a distance h from the bulb, its edges are at a slightly greater distance. However, we can calculate that the angle at which the longest dimension of

[1] Here we should correctly use the distance from the center of the Earth to the surface. Next, we silently assume in the whole solution that the body falls along the line connecting its initial position and the center of the Earth. However, this is a good approximation in our problem for a real non-spherical Earth.

the mouse is visible (as viewed from the bulb) is approximately $15\,^\circ$. From this, we can calculate that the difference in the distances between the edge and center of the mouse from the bulb is negligible. We can then assume that the same energy flux is delivered to the entire surface of the mouse. The mouse's cross-section is the area of an ellipse, hence πab. From these observations, we obtain the equation for the equality of the radiated and received heat

$$\frac{P\eta t}{4\pi h^2}\pi ab = C(T_2 - T_1)\,.$$

We rearrange and express the search temperature T_2

$$T_2 = T_1 + \frac{P\eta tab}{4Ch^2}\,.$$

After plugging in the numerical values, we get $T_2 \doteq 27\,^\circ\mathrm{C}$.

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Problem 15 ... bead with spring 4 points

A bead of mass $m = 358\,\mathrm{g}$ is strung on a straight wire (so the bead can only move in one direction). We attach to the bead an ideal spring with zero rest length and spring constant $k = 979\,\mathrm{N{\cdot}m^{-1}}$. We fix the other end of this spring at a distance $l = 323\,\mathrm{mm}$ from the line on which the wire lies. What will be the period of the small oscillations of the bead around its equilibrium position? *Lego hasn't set a problem on small oscillation for a long time.*

Of course, we could solve the problem by calculating the force on the bead after a deflection by a small Δx, but it is quicker and more elegant to calculate how its potential energy increases after such a deflection.

The potential energy of a spring with zero rest length is equal to

$$E_p = \frac{1}{2}ky^2\,,$$

where y is its length. The equilibrium position of the bead is naturally at the point on the wire closest to the spring attachment point (because that is when the spring is shortest). From the problem statement, we know that the attachment point is at a distance l from the wire, so in the equilibrium position, the spring has a direction perpendicular to the wire and a length equal to l.

When we move the bead from its equilibrium position by Δx, we move it only perpendicular to the equilibrium direction of the spring. The new position of the spring will therefore be the hypotenuse of a right triangle, while the equilibrium position of the spring l and the deflection of the bead Δx are the legs. Then we can directly calculate the square of the new length of the spring as $l^2 + \Delta x^2$ from Pythagoras' theorem. The potential energy of the spring will therefore be

$$E_p = \frac{1}{2}k\left(l^2 + \Delta x^2\right)\,.$$

However, we must remember that the potential energy of the spring in the equilibrium position was $kl^2/2$, so by deflecting the bead by Δx, the potential energy increased by $k\Delta x^2/2$. Now we can either observe that this corresponds to a linear harmonic oscillator with stiffness k, or we can differentiate the potential energy by position and obtain that the force that counteracts towards the deflection Δx is $k\Delta x$. Either way, it remains to fit this to the formula for the period of a linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k}} \doteq 0.120\,\mathrm{s}\,.$$

Interestingly, the bead oscillates with this period independently of the magnitude of its initial deflection on the wire; hence, we do not need to restrict to small oscillations.

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Problem 16 ... ballad of a sinful soul — 5 points

Jarda's FYKOS soul will once be relieved of its physical burden and head to heaven to meet the FYKOS bird. However, weighed down by sins, it will slip into hell, where it will be enclosed in a cauldron of $V = 666\,\mathrm{cm^3}$. It will remain preserved there at drastic conditions of $T = 666\,\mathrm{^\circ C}$ and $p = 666 \cdot 10^5\,\mathrm{Pa}$, while it will try to get rid of the sin molecules to rise again. The molar mass of sin is $666\,\mathrm{g{\cdot}mol^{-1}}$. How many sins will Jarda accumulate during his time at FYKOS? The mass of a pure soul is $21\,\mathrm{g}$, and its density at normal conditions is $0.70\,\mathrm{kg{\cdot}m^{-3}}$. Consider the soul and the sins to be an ideal gas. Jarda's proposition was not politically correct.

Jarda's soul boiling in an isolated cauldron satisfies the ideal gas equation of state

$$\frac{pV}{T} = nR\,,$$

where R is the gas constant and $n = n_\mathrm{h} + n_\mathrm{č}$ is the sum of the number of sin and pure-soul moles.

Jarda's soul was originally as pure as any other, but sins found a way to get into it. We calculate the number of moles of the pure soul from its molar mass and $m = 21\,\mathrm{g}$ as

$$n_\mathrm{č} = \frac{m}{M_\mathrm{č}}\,,$$

where $M_\mathrm{č}$ is obtained from another equation of state

$$\frac{p_\mathrm{a}}{T_\mathrm{a}\rho} = \frac{R}{M_\mathrm{č}}\,,$$

which comes from the density information at normal conditions T_a and p_a. If we substitute, we get the number of moles of sins as

$$n_\mathrm{h} = \frac{pV}{TR} - n_\mathrm{č} = \frac{pV}{TR} - \frac{mp_\mathrm{a}}{RT_\mathrm{a}\rho}\,.$$

By multiplying by the molar mass, we get

$$m_\mathrm{h} = \frac{M_\mathrm{h}}{R}\left(\frac{pV}{T} - \frac{mp_\mathrm{a}}{T_\mathrm{a}\rho}\right) = 2.95\,\mathrm{kg}\,.$$

Amen.

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Problem 17 ... is that what you meant, Mr. Planck? 5 points

Determine the Planck's surface tension.

Hint: Planck units are those formed by combining three fundamental physical constants – the gravitational constant G, the reduced Planck constant $\hbar$ and the speed of light c. Just for the record, the standard model of quantum physics fails on Planck scales in the microworld, and the effects of quantum gravity (of which we know nothing yet) must be taken into account.

Jindra came up with a problem that has a shorter statement (without a hint) than the title.

The three physical constants to determine the Planck scales are the gravitational constant $G = 6.673 \cdot 10^{-11}\,\mathrm{kg^{-1}{\cdot}m^3{\cdot}s^{-2}}$, the Planck constant $\hbar = 1.055 \cdot 10^{-34}\,\mathrm{kg{\cdot}m^2{\cdot}s^{-1}}$, and the speed of light $c = 2.998 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$. The surface tension has the unit $\mathrm{N{\cdot}m^{-1}}$ or $\mathrm{kg{\cdot}s^{-2}}$.

We will use dimensional analysis. We assume that the Planck surface tension σ_P is the product of the powers of the three constants

$$\sigma_\mathrm{P} = CG^\alpha \hbar^\beta c^\gamma . \tag{2}$$

C is a dimensionless constant that we cannot determine from dimensional analysis. When deriving Planck units, $C = 1$ is the standard convention. However, when using dimensional analysis in general, it is better to leave the unknown C in the expression to make it clear that the correct relationship may differ by a multiple of the constant.

We plug the units of the physical quantities into the equation (2)

$$\mathrm{kg{\cdot}s^{-2}} = (\mathrm{kg^{-1}{\cdot}m^3{\cdot}s^{-2}})^\alpha (\mathrm{kg{\cdot}m^2{\cdot}s^{-1}})^\beta (\mathrm{m{\cdot}s^{-1}})^\gamma = \mathrm{kg}^{-\alpha+\beta} \cdot \mathrm{m}^{3\alpha+2\beta+\gamma} \cdot \mathrm{s}^{-2\alpha-\beta-\gamma} .$$

Since the number of units on the left and right sides must be the same, we get a system of three linear equations with three unknowns.

$$\begin{aligned} -\alpha + \beta &= 1 \\ 3\alpha + 2\beta + \gamma &= 0 \\ -2\alpha - \beta - \gamma &= -2 \end{aligned}$$

Summing the second and third equations gives $\alpha + \beta = -2$. Combined with the first equation, we get $\alpha = -3/2$, $\beta = -1/2$. This yields $\gamma = 11/2$. The Planck surface tension is

$$\sigma_\mathrm{P} = \sqrt{\frac{c^{11}}{\hbar G^3}} = 7.49 \cdot 10^{78}\,\mathrm{N{\cdot}m^{-1}} .$$

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Problem 18 ... red-hot resistance wire 5 points

Consider a wide resistive wire which at temperature $T_0 = 20\,^\circ\mathrm{C}$ has the length $l_0 = 10\,\mathrm{m}$ and resistance $R_0 = 1.23\,\Omega$. As the current passes through it, it heats up to $T = 100\,^\circ\mathrm{C}$, changing its dimensions and specific resistance. Our wire has a coefficient of linear thermal expansion $\alpha_l = 2.43 \cdot 10^{-3}\,\mathrm{K^{-1}}$ and its temperature coefficient of specific electrical resistance is $\alpha_R = 3.92 \cdot 10^{-3}\,\mathrm{K^{-1}}$. What will be the resistance of the wire at temperature T if it is connected *to a source such that its dimensions can change with temperature?*

Karel was learning to teach and thought about material constants.

Let us denote the specific electrical resistance of a wire at temperature T_0 as ρ_0 and its cross section as S_0. The standard relation that links the length of a body to its temperature through the coefficient of linear thermal expansion has the following form

$$l = l_0 \left(1 + \alpha_l \left(T - T_0\right)\right) .$$

A similar relationship holds for the cross-section, except that it is scaled with linear dimension as its square. Therefore,

$$S = S_0 \left(1 + \alpha_l \left(T - T_0\right)\right)^2 .$$

As for the resistance, we know that for constant dimensions of the wire, it should increase as

$$R = R_0 \left(1 + \alpha_R \left(T - T_0\right)\right) .$$

Since the only variable that can change if the wire dimensions are constant is the resistivity, and moreover, the resulting resistance is directly proportional to it, it must follow

$$\rho = \rho_0 \left(1 + \alpha_R \left(T - T_0\right)\right) .$$

Now all that remains is to substitute into the known relation for the resistance of the wire

$$\begin{aligned} R &= \rho \frac{l}{S} , \\ R &= \rho_0 \left(1 + \alpha_R \left(T - T_0\right)\right) \frac{l_0 \left(1 + \alpha_l \left(T - T_0\right)\right)}{S_0 \left(1 + \alpha_l \left(T - T_0\right)\right)^2} , \\ R &= R_0 \frac{1 + \alpha_R \left(T - T_0\right)}{1 + \alpha_l \left(T - T_0\right)} \approx 1.35\,\Omega . \end{aligned}$$

In reality, however, the coefficients of thermal expansion are orders of magnitude negligible compared to the temperature coefficients of electrical resistivity. Therefore, in practice, it is not necessary to account for changes in the dimensions of the wire if we only want to determine its resistance.

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Problem 19 ... golf on the hill

5 points

A golfer was confident about hitting the ball across the whole course. However, his hands were shaking during the broadcast, and his drive was not successful at all. He underhanded the ball so that it came out at an angle $\alpha = 80^\circ$ to the horizontal plane at speed $v = 8.0\,\mathrm{m\cdot s^{-1}}$. The launch was made in the direction down the hill, which has a deviation from the horizontal direction of $\beta = 10^\circ$. The disappointed golfer turned back and did not look to see how far from the point of drive the ball landed. Therefore, calculate this figure.

That day, Kuba lost another ball…

Let's introduce a Cartesian coordinate system with the center at the point of the ball drive. In the direction of the x axis, no force acts on the ball, but in the direction of the y axis, the gravitational force gives it a negative acceleration g. Hence, at time t the position of the ball is $x = v_0 t\cos\alpha$ and $y = v_0 t\sin\alpha - \frac{1}{2}gt^2$. From the equation for x, we express the time t and substitute it into the equation for y. By this, we get rid of the parameter t and obtain the equation of the parabola along which the ball is moving

$$t = \frac{x}{v_0\cos\alpha} \Rightarrow y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2(\alpha)}\,.$$

The plane of the hill has the direction $\tan(180^\circ - \beta) = -\tan\beta$, so it is described by the equation $y = -\,x\tan\beta$. By comparing the equations of the parabola and the hill plane, we get their intersections, i.e. the point of drive $x = 0$ and the point of impact

$$x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2(\alpha)} = -x\tan\beta\,.$$

We want to get the horizontal distance of the point of impact where $x \neq 0$ (the case $x = 0$ would correspond to the point of drive), so we divide the equation by x

$$\frac{gx}{2v_0^2\cos^2(\alpha)} = \tan\alpha + \tan\beta$$

$$x = \frac{2v_0^2\cos^2(\alpha)\,(\tan\alpha + \tan\beta)}{g}\,.$$

Now all that is left is to express the distance d we're looking for using the horizontal distance x. Notice that the following holds

$$\cos\beta = \frac{x}{d}\,.$$

The resulting distance corresponds to

$$d = \frac{x}{\cos\beta} = \frac{2v_0^2\cos^2(\alpha)\,(\tan\alpha + \tan\beta)}{g\cos\beta} \doteq 2.3\,\mathrm{m}\,.$$

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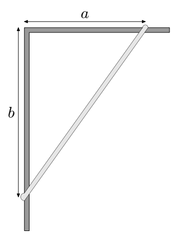
Problem 20 ... rubberband around the corner 4 points

We put a massless rubber band around the corner of the L-shaped rigid body. What is the smallest coefficient of friction between the rubber band and the solid that will prevent it from contracting it? The rubber band bends over the edges of the body at points $a = 5.0\,\mathrm{cm}$ and $b = 7.0\,\mathrm{cm}$ from the corner.

They're gonna put Jarda on a rack.



We can think of a rubber band as of a set of two springs (each on one side of the body), both of which start and end at the points where the rubber band touches the object. Let us denote these points as A and B. The two springs together exert a force F at point A towards point B (and vice versa).

To prevent the rubber band from contracting, the frictional force at both points must be greater than the elastic force. The frictional force at point A is $F_{\mathrm{TA}} = fF\cos\alpha$, while at point B, it is $F_{\mathrm{TB}} = fF\sin\alpha$. For angle α holds $\tan\alpha = a/b$.

The force trying to pull the rubber band down equals $F_\mathrm{A} = F\sin\alpha$ at point A, and $F_\mathrm{B} = F\cos\alpha$ at point B. For the rubber band to stay still, the following must hold

$$fF\cos\alpha \geq F\sin\alpha\,,$$
$$fF\sin\alpha \geq F\cos\alpha\,.$$

If we multiply the first inequality by f and substitute it into the second one, we get the condition $f \geq 1$.

In our case, $\alpha < 45^\circ$. Then the first inequality is satisfied for all $f \geq 1$. However, the second inequality implies

$$f \geq \frac{\cos\alpha}{\sin\alpha} = \frac{b}{a} = 1.4\,.$$

So the coefficient of friction must be at least 1.4.

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Problem 21 ... level sensor 6 points

Consider a vessel with a non-conductive liquid of relative permittivity $\varepsilon_\mathrm{r} = 1.80$, in which two vertical partially immersed parallel plates form a plate capacitor. The capacitor is connected to a coil so that together they are part of the LC oscillator circuit. How many times does the natural frequency of this oscillator increase when the plates of the capacitor are quarterly submerged, compared to the case when the vessel is empty? *I bet that, just like Vašek, you have already thought about a way of measuring the level in a non-mechanical way.*

The frequency of the LC oscillator can be determined by the relationship

$$f = \frac{1}{2\pi\sqrt{LC}}\,,$$

where the capacitance of a plate capacitor with a plate area S spaced by a distance d in the air is given by

$$C = \varepsilon_0\frac{S}{d}\,.$$

When we immerse the plates into the liquid, we can view the device as two capacitors in parallel – one with a plate area of $3S/4$, the other with a plate area of $S/4$. Their capacitance will then be given by the sum of their individual capacitances due to the parallel connection. It will thus apply

$$C' = \varepsilon_0\frac{3S}{4d} + \varepsilon_0\varepsilon_\mathrm{r}\frac{S}{4d} = \varepsilon_0\frac{S}{d}\frac{3+\varepsilon_\mathrm{r}}{4}\,.$$

Now we can determine the ratio of the natural frequencies as

$$\frac{f'}{f} = \sqrt{\frac{C}{C'}} = \sqrt{\frac{4}{3+\varepsilon_\mathrm{r}}} \doteq 0.913\,.$$

So the natural frequency is reduced. Let's mention that it's actually *a capacitive liquid level sensor*.

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Problem 22 ... mountain biking reloaded
4 points

Matěj is riding on a mountain bike trail, passing oncoming cyclists. At the top of the hill, exhausted cyclists are moving at an average speed of $5\,\mathrm{km{\cdot}h^{-1}}$ *in both directions, and Matěj meets there* 0.02 *oncoming bikes per second on average. In contrast, at the lowest point of the bike trail, cyclists are revved up (from both directions) and ride at an average speed of* $50\,\mathrm{km{\cdot}h^{-1}}$. *How many oncoming bikes do Matěj encounter here on average? The bike trail has no turnoffs, and Matěj always moves at a speed of* $10\,\mathrm{km{\cdot}h^{-1}}$.

Matěj mountain biked while reloading.

Let us denote the frequency f_0 with which cyclists pass a static observer in one direction. If cyclists move at an average speed of v at a given location, then their average linear density on the trail is $\lambda = f_0/v$. Let's denote Matěj's speed by u. Relative to the static observer, Matěj passes $u\lambda$ more cyclists per second. Therefore, Matěj's frequency of passing oncoming cyclists is $f = f_0 + u\lambda = f_0\,(1 + u/v)$.

We have two equations (for the top and bottom of the hill) from which we can exclude the unknown f_0

$$f_{\mathrm{top}} = f_0\left(1 + \frac{u}{v_{\mathrm{top}}}\right)\,,$$
$$f_{\mathrm{bot}} = f_0\left(1 + \frac{u}{v_{\mathrm{bot}}}\right)\,,$$

which gives us the desired frequency

$$f_{\mathrm{bot}} = f_{\mathrm{top}}\frac{v_{\mathrm{top}}(v_{\mathrm{bot}} + v)}{v_{\mathrm{bot}}(v_{\mathrm{top}} + v)} = 0.008\,.$$

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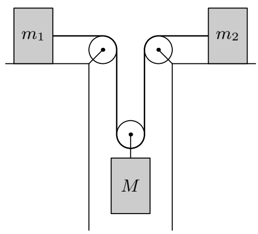
Problem 23 ... a pit with pulleys
5 points



We have a pit in which a block with mass $M = 84.6\,\mathrm{kg}$ *is hanging from a movable pulley. On the horizontal surface to the left of the pit, there is a block with mass* $m_1 = 26.4\,\mathrm{kg}$, *and to the right of the pit, there is another block with mass* $m_2 = 33.8\,\mathrm{kg}$. *To each of these is tied one end of a rope, with a movable pulley hanging in the pit. There are, of course, two pulleys on the edges of the pit so that the rope runs only horizontally and vertically.*

Assume that everything moves without friction; the rope and pulleys are immaterial. What is the magnitude of acceleration the block in the pit will move with?

Lego felt like he does not create enough problems with pulleys.

Since both the rope and the pulleys are immaterial, the total force acting on each element of the rope must be zero (because $F = ma$). Therefore, the rope is stretched along its entire length by the same force; let us denote it by T. Since the gravity of blocks m_1 and m_2 is canceled out by the normal force of the ground, the tensile force from the rope is also the resultant force acting on them. Thus, block m_1 will accelerate with $a_1 = T/m_1$ towards the pit and similarly, the acceleration of m_2 will be $a_2 = T/m_2$.

The block inside the pit is pulled downwards by its gravity of magnitude $F_g = Mg$, and upwards by the tensile force of the rope T on both sides, so in total $2T$. Its resultant acceleration downwards is therefore $a = g - 2T/M$.

How are these accelerations related? If we moved the first block by x_1 and the second block by x_2, both towards the pit, then the block in the pit (because it is on a movable pulley between those two blocks) will move by the average of these two displacements (for intuition, we can easily check this on cases when $x_1 = x_2$ or $x_2 = 0$). Symbolically $x = (x_1 + x_2)/2$. When we derive this equation twice by time, we get the equation for acceleration $a = (a_1 + a_2)/2$ which we are looking for.

We plug all calculated accelerations into this equation and express T

$$\begin{aligned} g - 2\frac{T}{M} &= \frac{\frac{T}{m_1} + \frac{T}{m_2}}{2} \\ g &= T\left(\frac{2}{M} + \frac{1}{2m_2} + \frac{1}{2m_1}\right) \\ T &= \frac{g}{\frac{2}{M} + \frac{1}{2m_2} + \frac{1}{2m_1}} \,. \end{aligned}$$

What remains is to plug T back into the formula for the acceleration of the block inside the pit, and we have the result

$$a = g - 2\frac{T}{M} = g - 2\frac{g}{\frac{2M}{M} + \frac{M}{2m_2} + \frac{M}{2m_1}} = g\left(1 - \frac{1}{1 + \frac{M}{4m_2} + \frac{M}{4m_1}}\right) = 5.77\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem 24 ... effective trapezoid — 5 points

Consider an electrical source that provides a trapezoidal voltage such that for the first third of the period, the voltage increases linearly from $0.00\,\mathrm{V}$ *to* $5.00\,\mathrm{V}$*, then remains constant at* $5.00\,\mathrm{V}$ *for a third of the period, and then decreases linearly back to* $0.00\,\mathrm{V}$ *in the last third of the period. We will connect the source to a resistor. What constant-voltage source could we replace the original source with to have the same average power on the component (i.e., determine the effective voltage of the source)?*

Karel was varying the problem on effective values.

The mean power of the AC source is determined by integrating the instantaneous power over one period (and dividing by this period), i.e.

$$\frac{U_{\mathrm{ef}}^2}{R} = \overline{P} = \frac{1}{T}\int_0^T P\,\mathrm{d}t\,.$$

In our case, we can write instantaneous power as

$$P = \frac{U^2}{R}\,,$$

where the voltage U has the following waveform on the time interval $[0,T]$

$$U = \begin{cases} \frac{3t}{T}\cdot 5\,\mathrm{V} & t\in[0,T/3)\,\mathrm{s} \\ 5\,\mathrm{V} & t\in[T/3,2T/3]\,\mathrm{s} \\ \frac{3(T-t)}{T}\cdot 5\,\mathrm{V} & t\in(2T/3,T]\,\mathrm{s} \end{cases}$$

Thus, we can substitute and break it down into three integrals. Then we get

$$U_{\mathrm{ef}} = \sqrt{\frac{1}{T}\left(\int_0^{T/3}\frac{225t^2}{T^2}\,\mathrm{d}t + \int_{T/3}^{2T/3} 25\,\mathrm{d}t + \int_{2T/3}^{T}\frac{225(T-t)^2}{T^2}\,\mathrm{d}t\right)}\,\mathrm{V} =$$

$$= \sqrt{\frac{450}{T^3}\int_0^{T/3} t^2\,\mathrm{d}t + \frac{25}{3}}\,\mathrm{V} =$$

$$= \frac{5\sqrt{5}}{3}\,\mathrm{V} \doteq 3.73\,\mathrm{V}\,.$$

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Problem 25 ... race against time 4 points

You are sitting in a car that is traveling at a constant velocity of $80\,\mathrm{km\cdot h^{-1}}$*, and you enter a* $5\,\mathrm{km}$ *long tunnel with no cellular signal. At the same time, you are watching a stage of the Tour de France on your cell phone with* $3.5\,\mathrm{km}$ *to go. The cyclists are going at* $40\,\mathrm{km\cdot h^{-1}}$ *and just as your signal drops out, they begin to descend, accelerating steadily until the finish line. What is their largest possible acceleration so that you can see the last* $30\,\mathrm{s}$ *of the stage when you exit the tunnel and get the signal again?* *Dávid was watching cycling in the car.*

When setting up the equation for the motion of cyclists, we first need to realize how long it will take the cyclists to arrive at the finish line under the given conditions. It is clear that this time (t_c) is equal to the sum of the time it takes the car to pass through the tunnel (t_a) and the reserve time (t_r), which expresses how much spare time the cyclists shall have after exiting the tunnel to reach the finish line. The mathematical expression looks as follows

$$t_{\mathrm{c}} = t_{\mathrm{a}} + t_{\mathrm{r}} = \frac{s_{\mathrm{a}}}{v_{\mathrm{a}}} + t_{\mathrm{r}} = 255\,\mathrm{s}\,. \tag{3}$$

Now we can set up an equation for the distance traveled by the cyclists, and then use this equation to express the acceleration a_{max}

$$s_{\mathrm{c}} = v_{\mathrm{c}}t_{\mathrm{c}} + \frac{a_{\mathrm{max}}t_{\mathrm{c}}^2}{2} \quad\Rightarrow\quad a_{\mathrm{max}} = \frac{2\left(s_{\mathrm{c}} - v_{\mathrm{c}}t_{\mathrm{c}}\right)}{t_{\mathrm{c}}^2}\,. \tag{4}$$

For the general expression of the result, we insert the expression we got in (3) into equation (4)

$$a_{\mathrm{max}} = \frac{2\left(s_{\mathrm{c}} - v_{\mathrm{c}}\left(\frac{s_{\mathrm{a}}}{v_{\mathrm{a}}} + t_{\mathrm{r}}\right)\right)}{\left(\frac{s_{\mathrm{a}}}{v_{\mathrm{a}}} + t_{\mathrm{r}}\right)^2} = 0.021\,\mathrm{m\cdot s^{-2}}\,.$$

The result is therefore $a_{\mathrm{max}} = 0.021\,\mathrm{m\cdot s^{-2}}$.

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Problem 26 ... escape across the frozen lake 4 points

A migrant tries to flee the country in winter, and his path leads across a frozen lake. The border guards are on his tail, so he must plan his route as efficiently as possible. There is a direct national border that runs across the lake, beyond which he will be safe. There are thick reeds all around the shore, but suddenly he sees a free area of ice. He runs onto the frozen surface at a speed of $u = 4.3\,\mathrm{m\cdot s^{-1}}$ in a direction parallel to the border from which he is $d = 38\,\mathrm{m}$ away. How long will it take him, at the minimum, to reach safety if the coefficient of friction between the ice and his boots is $f = 0.05$ and no other obstacles stand in his way?

Jarda thinks these are the real problems in Eastern Europe right now.

The maximal force he can act in any horizontal direction is $F = fmg$. From Newton's second law, we get an equation for his maximal acceleration as $a = gf$.

His net acceleration must point perpendicular to the border while he wants to cross the border as fast as possible. So he moves uniformly accelerated, and his distance in that direction is $s = \frac{1}{2}at^2$. After substituting $a = gf$ and $s = d$ we get

$$t = \sqrt{\frac{2d}{gf}} = 12.4\,\mathrm{s}\,.$$

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Problem 27 ... heating the helium 5 points

An unknown volume V_0 of helium gas is stored at atmospheric pressure and at a temperature $T_0 = 297\,\mathrm{K}$. How large is this volume if, by supplying heat $Q = 42\,\mathrm{J}$ at constant pressure, the temperature of the gas is increased by $\Delta T = 2.5\,\mathrm{K}$?

Karel was browsing the textbooks and hadn't seen this one.

In an isobaric process (i.e. process at constant pressure), the equation of state

$$pV = nRT$$

shows that temperature and volume will be directly proportional. If the temperature is increased from T_0 to $T_1 = T_0 + \Delta T$, then the volume is increased from V_0 to

$$V_1 = V_0 \frac{T_1}{T_0} = V_0 \left(1 + \frac{\Delta T}{T_0}\right)\,.$$

We can thus immediately calculate the work done in the process. Since $\mathrm{d}W = p\,\mathrm{d}V$ and the pressure does not change in this process, we only need to multiply the pressure by the change in volume and the work done is

$$W = p_\mathrm{a} \Delta V = p_\mathrm{a} V_0 \frac{\Delta T}{T_0}\,.$$

However, we have to realize that the supplied heat Q is converted into both an increase in the internal energy of the gas and the carried-out work. We know that the internal energy of a gas is proportional to nRT (so it is also proportional to pV). But what is the constant of proportionality? The equipartition theorem says that this constant is $n/2$, where n is the number of degrees of freedom for the motion of the molecules. For a monatomic gas, $n = 3$ as the molecule can move independently in the x, y, z directions. Since helium is a monatomic gas, the internal energy of the gas has increased by

$$\Delta U = U_1 - U_0 = \frac{3}{2}nRT_1 - \frac{3}{2}nRT_0 = \frac{3}{2}p_\mathrm{a}\Delta V\,.$$

By substituting into the first law of thermodynamics, we get

$$Q = W + \Delta U = p_\mathrm{a}V_0\frac{\Delta T}{T_0} + \frac{3}{2}p_\mathrm{a}V_0\frac{\Delta T}{T_0}\,,$$

$$V_0 = \frac{2QT_0}{5p_\mathrm{a}\Delta T} = 1.97\cdot 10^{-2}\,\mathrm{m^3}\,.$$

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Problem 28 ... tropical climate 5 points

Viktor wants his house plants to thrive, so he decided to increase the humidity in the room. Before turning on the humidifier, the humidity in the room was 30 %. The humidifier can evaporate 5 ml of water every minute. At what percentage would the humidity in the room stabilize if Viktor opened the window and every minute, 3 % of air was replaced in the room? Assume that the outside atmosphere has the same humidity as the air inside the room before the device was turned on. Viktor's room has dimensions $4\times 5\times 2.5\,\mathrm{m}$, and the air temperature is 25 °C. *Viktor wanted to experience what it was like to live in the rainforest.*

Let us denote the initial relative humidity in the room Φ_0 and the relative humidity at steady state Φ_e. If the relative humidity of the room is steady state, the total mass of water in the air in the room does not change. Looking at the balance, a humidifier adds water vapor to the air every minute with a mass of

$$m_1 = V_0\rho\,,$$

where $V_0 = 5\,\mathrm{ml}$ is the volume of water evaporated by the humidifier in one minute and $\rho = 997\,\mathrm{g\cdot cm^{-3}}$ is the density of water at 25 °C. At the same time, however, ventilation exchanges $\eta = 3\,\%$ of the more humid air from the room for less humid air from outside. To find out what the mass m_2 of water vapor that leaves the room per minute by this process is, we need to know two equations. The mass of water vapor m in the room air and the absolute humidity θ are linked by the equation

$$\theta = \frac{m}{V}\,,$$

where V is the volume of the room. The relative humidity is then calculated from the absolute humidity as

$$\Phi = \frac{\theta}{\theta_\mathrm{n}}\,,$$

where θ_n is the absolute humidity of the air saturated with water vapor (in that case the relative humidity is 100 %). This value is temperature dependent, at the given 25 °C is $\theta_\mathrm{n} = 23\,\mathrm{g\cdot m^{-3}}$. Using these two equations for absolute and relative humidity, we can write

$$m_2 = \eta abc(\theta_e - \theta_0)\,,$$

$$m_2 = \eta abc\theta_\mathrm{n}(\Phi_e - \Phi_0)\,,$$

where abc is the volume of the room. In equilibrium, $m_1 = m_2$. Thus we get

$$V_0\rho = \eta abc\theta_\mathrm{n}(\Phi_e - \Phi_0)\,.$$

By rearranging, we express the steady-state relative humidity Φ_e as

$$\Phi_e = \Phi_0 + \frac{V_0\rho}{\eta abc\theta_\mathrm{n}}\,.$$

Substituting the numeric values we get $\Phi_e \doteq 44\,\%$.

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Problem 29 ... The Pit and the Pendulum 6 points

In a cylindrical pit that is 30 cm *wide, a small ball is suspended on a* 50 cm *long massless string. The point of suspension lies on the symmetry axis of the pit. We give the ball a horizontal velocity v such that it bounces perfectly elastically off the vertical walls twice before returning to its original position. It will not pass through any point in space twice during this process and will have the same velocity vector at the end as at the beginning. Determine the magnitude of the velocity v.* *Jarda happened to be paying attention in literature class.*

Once the horizontal velocity is given, the ball starts to move in a circle as far as the string to which it is attached allows. After a perfectly elastic bounce from the wall, the vertical component of the velocity is conserved. The horizontal component will change the sign but not the direction. Hence, the ball will move back towards the axis of symmetry of the cylindrical pit. Then a second bounce occurs, and the ball moves back again. Since we require that it does not pass through any point twice during this process, it will move through the air along the parabola after the first bounce. After the second bounce, it will again begin to follow the arc of the circle.

The law of reflection applies for bounces, so the angles of the trajectories (circle and parabola) are the same with respect to the vertical axis.

We calculate the angle of incidence α from the geometry of the problem as

$$\sin\alpha = \frac{d}{2l}\,,$$

where d is the width of the pit and l is the length of the pendulum.

Let us denote the velocity of the ball after reflection as v_o. Then, it will move along a parabola characterized by the following equation

$$y = x\tan\alpha - \frac{g}{2}\frac{x^2}{v_\mathrm{o}^2\cos^2\alpha}\,,$$

where y is the vertical coordinate, x is the horizontal coordinate in the direction from the wall, and g is the acceleration due to gravity. The bounce occurs for $y = 0$ and $x = d$, so from the equation we express v_o as

$$v_\mathrm{o} = \sqrt{\frac{gd}{2\sin\alpha\cos\alpha}}\,.$$

Yet, we still have to determine the velocity of the ball at the lowest point from the law of conservation of energy

$$v = \sqrt{v_0^2 + 2gl\left(1 - \cos\alpha\right)} = \sqrt{\frac{gd}{2\sin\alpha\cos\alpha} + 2gl\left(1 - \cos\alpha\right)}$$

$$= \sqrt{\frac{gl}{\sqrt{1 - \left(\frac{d}{2l}\right)^2}} + 2g\left(l - \sqrt{l^2 - \left(\frac{d}{2}\right)^2}\right)} = 2.4\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem 30 ... magneto-resistor 5 points

Let us assemble an electric circuit. We have a long straight wire of resistance $R = 57\,\mathrm{m\Omega}$. In parallel to the wire, we connect a magneto-resistor, i.e., a component that changes its resistance depending on the external magnetic field. The distance between the magneto-resistor and the wire is $d = 3.5\,\mathrm{cm}$, and its resistance at zero magnetic field is $r_0 = 85\,\Omega$. In parallel to the wire, we connect a power supply with voltage $U = 12\,\mathrm{V}$. A current $I = 140\,\mathrm{mA}$ flows through the magneto-resistor after it has stabilized. Assume that its resistance varies linearly with the magnetic field B over a given range. Determine the coefficient of proportionality.

Jarda got lost in a physicists' warehouse.

The resistance of a magneto-resistor changes as $r = r_0 + \alpha B$, where

$$\alpha = \frac{r - r_0}{B}$$

is the parameter we want to determine. The dependency of the magnetic field B on the electric current flowing through the wire can be represented by the following formula

$$B = \frac{\mu_0 I_R}{2\pi d} = \frac{\mu_0 U}{2\pi d R}\,,$$

where μ_0 is the permeability (of vacuum), and I_R is the current flowing through a resistor.

Resistance r_0 is known from the problem statement, and resistance r is equal to U/I. After substituting the values to the previous equation, we get

$$\alpha = \frac{2\pi}{\mu_0} dR\left(\frac{1}{I} - \frac{r_0}{U}\right) \doteq 590\,\Omega{\cdot}\mathrm{T}^{-1}\,.$$

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Problem 31 ... wheel against the direction of travel

5 points

What fraction of the total travel time does the velocity (component parallel to the ground) of one fixed point of the outermost edge of the railway wheel (the flange) have the opposite direction as the train itself? Consider that the wheel overhangs the track by 35 mm *and its diameter with the flange is* 1 400 mm. *Karel was thinking about trains.*

The time when the flange is moving against the direction of the travel depends only on the diameter of the wheel $d = 1\,400\,\mathrm{mm}$ and the overhang of the flange $l = 35\,\mathrm{mm}$. A point on the outermost edge of the wheel makes a uniform rectilinear motion in the direction of the travel, and a uniform motion along the circle centered in the axis of the wheel. The angle between this point and the horizontal axis is angle α. In order to express both motions using the given values, we introduce ω, which is an angular velocity at which the wheel rotates, and it is constant. Using that, we express the velocity of the rectilinear motion in the direction of travel v_1

$$v_1 = \omega\left(\frac{d}{2} - l\right) .$$

The velocity of the point on the edge of the wheel as it moves along the circle v_2 is

$$v_2 = \omega\frac{d}{2} .$$

The velocity vector v_2 is always perpendicular to the position vector of the point on the edge of the wheel. We decompose it into two vectors, one of which is parallel to the velocity vector v_1, which is the one we are interested in when it has an opposite direction to v_1. We denote it as v_3. If the magnitude of this vector is greater than the magnitude of v_1, it means, that the point on the edge of the wheel is moving against the direction of travel, while its horizontal motion is the sum of vectors v_1 and v_2. Using the goniometric functions we express v_3 from v_2.

$$v_3 = v_2\cos\left(\frac{\pi}{2} - \alpha\right) = \omega\frac{d}{2}\cos\left(\frac{\pi}{2} - \alpha\right) = \omega\frac{d}{2}\sin\alpha .$$

Now we set v_1 to be equal to v_3 and see for what angles α the equality holds.

$$\omega\frac{d}{2}\sin\alpha = \omega\left(\frac{d}{2} - l\right)$$

$$\sin\alpha = \frac{\frac{d}{2} - l}{\frac{d}{2}} .$$

We get $\alpha_1 = 1.253\,\mathrm{rad}$ and $\alpha_2 = 1.888\,\mathrm{rad}$. In the interval between those two angles, the velocity v_3 is greater than the velocity v_2, and the point on the edge of the wheel is moving against the direction of travel. It remains to find out how much of the total time this is.

$$\frac{\alpha_2 - \alpha_1}{2\pi} = 0.10 .$$

From the result, we see that the point at the edge of the wheel moves in the opposite direction for a tenth of the total time.

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Problem 32 ... too high frequency

6 points

Let's have a resistor with resistance $R = 11\,\mathrm{k\Omega}$ and a capacitor with capacitance $C = 2\,\mathrm{\mu F}$ connected in series to the AC voltage source. For what value of frequency in the circuit will the voltage amplitude across the capacitor drop to one tenth of the maximum achievable value?

Jarda doesn't like quick changes.

The impedance of a resistor is simply equal to R, while for a capacitor it is defined as $\frac{-i}{C\omega}$, where i is the imaginary unit and $\omega = 2\pi f$ is the angular frequency of the source. We will solve the problem using complex numbers.

The total impedance in series is $Z = R - \frac{i}{C\omega}$. The complex current in the circuit will thus be $\frac{U}{Z}$. The voltage drop on the resistor os RI and the capacitor is left with

$$U_C = U - RI = U\left(1 - \frac{R}{Z}\right) = U\frac{-\frac{i}{C\omega}}{R - \frac{i}{C\omega}} = U\frac{-i}{RC\omega - i}\,.$$

We are interested in the amplitude of the voltage across the capacitor, which is determined by the absolute value of U_C according to the relation

$$|U_C| = \frac{U}{\sqrt{(RC\omega)^2 + 1}}\,.$$

The maximum voltage across the capacitor clearly occurs at zero frequency (DC), which is $U_{\max} = U$. From the condition in the problem statement, we get the equation

$$\frac{|U_C|}{|U|} = \frac{1}{10} \quad \Rightarrow \quad f = \frac{\sqrt{99}}{2\pi RC} \doteq 72\,\mathrm{Hz}\,.$$

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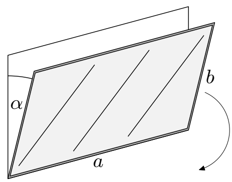
Problem 33 ... attention, the window is falling

7 points

The rectangular window is open on its upper side so that its plane makes an angle of 10° with the vertical direction. However, due to strong wind, the latch holding it in place is released, and the window begins to rotate about its lower horizontal axis before hitting the wall below. At what angular velocity does it hit the wall? The glass alone has an area density $\rho = 15\,\mathrm{kg{\cdot}m^{-2}}$, dimensions $a = 130\,\mathrm{cm}$ by $b = 60\,\mathrm{cm}$, and is embedded in a frame of mass $m_\mathrm{r} = 4\,\mathrm{kg}$ (do not consider its dimensions). Consider the whole window to be planar and homogenous in parts.



It's unfortunate. Jarda has to find a glazier, but at least he has a problem for Physics Brawl Online.

We solve the problem using the law of conservation of energy. The potential energy converts to rotational about the bottom axis according to the formula

$$Mg\Delta h = \frac{1}{2}J\omega^2\,,$$

where M is the total mass of the window, g is gravity, Δh is the difference in positions of the center of gravity at the beginning and the end of the motion, J is the moment of inertia with respect to the rotational axis, and ω is the angular velocity of the window at impact.

The difference in positions is given by

$$\Delta h = \frac{(1+\cos\alpha)}{2} b\,,$$

where $b = 60\,\mathrm{cm}$ is the height of the window and α is the initial angle of opened window. The two in the denominator is there because the center of gravity of the window is in the center of its height.

The total mass of the window is $M = m_\mathrm{r} + m_\mathrm{s} = m_\mathrm{r} + ab\rho$.

The total moment of inertia is given by the sum of the moment of inertia of the glass and the frame. If we look at the window from the side, we see it as a rotating line. The moment of inertia of a rod rotating about one of its ends is $ml^2/3$, where m is the mass of the rod, and l is its length. In our case, it does not matter if a rod or a rectangle is rotating (they look the same from the side). Therefore, the moment of inertia of the glass is $J_\mathrm{s} = ab\rho b^2/3$.

For the frame, the situation is a bit more difficult. It is divided into four segments, one of which is not rotating, the next two look like rods when looking from the side, and the last one (originally at the top) looks like a point. When calculating the moments of inertia of individual segments, we introduce the linear density of the frame as

$$\lambda = m_\mathrm{r}/(2a+2b)\,.$$

The moment of inertia of the original upper part of the frame is then equal to $m_\mathrm{rh} b^2 = \lambda a b^2$, and for the side parts of the frame together

$$2\cdot\frac{1}{3} m_\mathrm{rb} b^2 = \frac{2}{3}\lambda b^3\,.$$

The total moment of inertia of the frame is therefore

$$J_\mathrm{r} = \lambda\left(ab^2 + \frac{2}{3}b^3\right) = m_\mathrm{r} b^2 \frac{a + \frac{2}{3}b}{2a+2b}\,.$$

Now we have all we need to write the final result as

$$\omega = \sqrt{\frac{2Mg\Delta h}{J}} = \sqrt{\frac{3\left(m_\mathrm{r} + ab\rho\right) g\left(1+\cos\alpha\right)}{m_\mathrm{r} b\frac{3a+2b}{2a+2b} + a\ \rho b^2}} = 9.5\,\mathrm{rad\cdot s^{-1}}\,.$$

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Problem 34 ... blocks on top of each other on a plane 5 points

Two blocks of masses 600 g *and* 700 g *are placed on top of each other on an inclined plane which is deviated from the horizontal direction by* 30°. *The heavier of them is on the bottom, and the interface between them is perfectly smooth. The coefficient of static friction between the lower block and the plane is* 0.3, *while the dynamic friction coefficient is* 0.2. *How large is the angle* *between the force exerted by the plane on the block and the inclined plane itself?*

Marek found the block to be lonely on an inclined plane.

Let us denote the mass of the upper block m_1 and the bottom one m_2. We denote the angle of the inclined plane as α. The force exerted by the inclined plane on the block can be decomposed into the direction perpendicular and parallel to the plane. Let us denote the perpendicular force as F_n and the parallel as F_t. The F_t is realized by friction and is, therefore, proportional to F_n.

If the force corresponding to the static friction "holds" the bottom block, $F_\mathrm{t} = 0.3F_\mathrm{n}$ holds; if not, $F_\mathrm{t} = 0.2F_\mathrm{n}$ will hold.

Since the upper block cannot fall through the lower block, and the lower block cannot fall through the plane, the following equation holds

$$F_\mathrm{n} = (m_1 + m_2)g\cos\alpha\,.$$

We denote the force acting on the lower block against the force F_t as F_1. The following then holds

$$F_1 = m_2 g\sin\alpha\,.$$

The condition that the static friction "holds" the lower block corresponds to the inequality

$$\begin{aligned} 0.3F_\mathrm{n} &> F_1\,,\\ 0.3 &> \frac{m_2}{m_1+m_2}\tan\alpha\,,\\ 0.3 &> 0.31\dots\,, \end{aligned}$$

which does not hold so the bottom block moves.

We denote the angle we are looking for as β. For it

$$\tan\beta = \frac{F_\mathrm{N}}{F_\mathrm{t}} = \frac{F_\mathrm{N}}{0.2F_\mathrm{N}} = \frac{1}{0.2}$$

holds, and thus finally

$$\beta = \arctan\frac{1}{0.2} \doteq 78.7^\circ\,.$$

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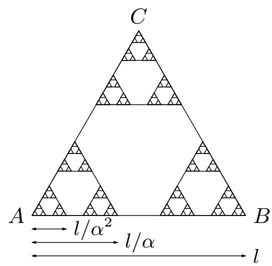
Problem 35 ... long power line

8 points

Consider a wire with constant electrical resistance $r_0 = 1\,\Omega$ per unit length l. Let us construct a resistive network from this wire as shown in the diagram. We start from a large equilateral triangle with side length l. Let us connect the points on the sides of this triangle at a distance l/α from the nearest vertex with a wire (where $\alpha \geq 2$), making an α times smaller triangle at each vertex of the large triangle. We repeat the same procedure for these smaller triangles and so on indefinitely (each step produces triangles α times smaller than in the previous step). Finally, we connect the source terminals to vertices A and B. What will the resistance of this network be if $\alpha = 4$?



Radka couldn't find the end of the extension cord.

Using the triangle-star transformation, any triangle with smaller triangles at its vertices can be transformed into an equivalent simple triangular circuit.

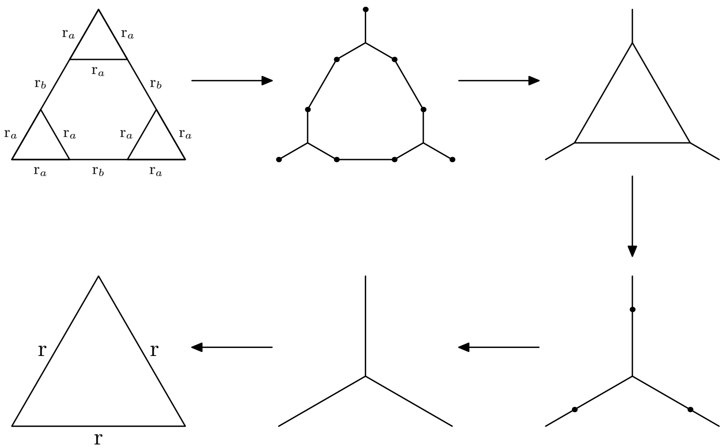


Fig. 1: Transforming circuits.

Specifically, if we use the notation of resistances in the figure (r_a is the resistance of one side of a small triangle and r_b is the resistance of one side of the large triangle minus twice the side of a small triangle), we obtain the formula

$$r = \frac{5}{3} r_a + r_b \,.$$

If we tried constructing the network by gradually adding smaller triangles at the vertices of larger triangles, the equivalent resistance between two vertices of the largest triangle would follow the formula

$$r_{i+1} = \frac{5}{3\alpha} r_i + \left(1 - \frac{2}{\alpha}\right) r_0 \,.$$

where the coefficient $1/\alpha$ in the first term accounts for the way new triangles get smaller than previously added ones. In the limit of infinitely many steps, we may use $r_i = r_{i+1} = r_\mathrm{t}$, where r_t is the resistance of one side of an equilateral triangle which would be equivalent to the whole network. From the equation above, we get

$$r_\mathrm{t} = \frac{3(1 - \frac{2}{\alpha})}{3 - \frac{5}{\alpha}} r_0$$

and for $\alpha = 4$

$$r_\mathrm{t} = \frac{6}{7}\,\Omega \,.$$

The resulting resistance between vertices A and B is

$$R = \frac{2}{3} r_\mathrm{t} = \frac{2(1 - \frac{2}{\alpha})}{3 - \frac{5}{\alpha}} r_0 = \frac{4}{7}\,\Omega \,.$$

Alternative solution

We won't use the triangle-star transformation or limits, but superposition of networks. Let the current flowing through the bottom wire with resistance $r_1 = r_0\left(1-\frac{2}{\alpha}\right)$ be I_1 and the current flowing through the two diagonal wires with resistance r_1 be I_2. Imagine the whole network as a superposition of a network without the bottom wire and another network without the diagonal wires - voltage in each vertex is the sum of voltages in the two circuits. We express the voltage between vertices A and B using equations

$$V = 2\left(\frac{I_1}{2}+I_2\right)\frac{R}{\alpha} + 2I_2 r_1 + I_2\frac{R}{\alpha}\,,$$
$$V = 2\left(I_1+\frac{I_2}{2}\right)\frac{R}{\alpha} + I_1 r_1\,,$$
$$V = (I_1+I_2)\,R\,.$$

Subtracting the first and second equation, we get $(2I_2 - I_1)\frac{r_t}{\alpha} = (I_1 - 2I_2)r_1$. For positive resistances, we must then have $I_1 = 2I_2 = \frac{2V}{3R}$; plugging this back in and dividing by positive V gives the same result

$$R = \frac{2}{3-\frac{5}{\alpha}}r_1 = \frac{2(1-\frac{2}{\alpha})}{3-\frac{5}{\alpha}}r_0\,.$$

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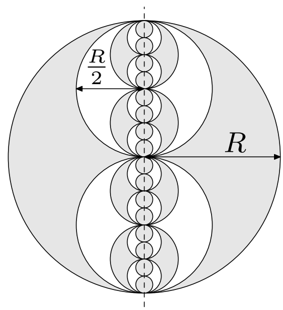
Problem 36 ... a ball for Jáchym 6 points

We will create a special ball for Jáchym in successive steps. We begin with a homogenous solid ball of radius R and material density ρ, and choose one major axis passing through the center of the ball. We remove two smaller balls of radii $R/2$ from the large ball, whose centers lie on the major axis and are at a distance $R/2$ from the center of the large ball so that the resulting holes are adjacent to each other. Then we add 4 balls inside with radii $R/4$. Their centers lie on the major axis, and we place the balls next to each other. We continue in the same way. That is, we make two spherical holes inside each of the small balls, and then put half-radius balls in them again. We continue this way to infinity.



What is the ratio of the total mass of the ball for Jáchym and the mass of the full ball with the radius R and the same density ρ? *Karel wanted Jáchym to get his special ball, too.*

Note that in each step, we add/remove twice as many balls as in the previous step. These balls all have their radius halved. Their total volume (and thus, due to the constant density, also their total mass) will therefore be $2\cdot(1/2)^3$ times the volume of the balls added/removed in the previous step. If we denote by M the mass of the ball we start with, we get the mass of the ball for Jáchym as a sum of an infinite series

$$m = M - \frac{M}{4} + \frac{M}{16} - \frac{M}{64} + \dots = \sum_{i=0}^{\infty} M\left(\frac{-1}{4}\right)^i\,,$$

which we can add up using

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}\,,$$

as

$$m = \frac{4}{5}M\,.$$

Since we want three significant figures in the result, the answer is 0.800.

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Problem 37 ... polarization as far as it goes — 6 points

Karel decided to put five linear polarizers behind each other. He chose the angle between the planes of the first and second polarizer to be α, the next $\alpha - \pi/2$, then α again, and so on. What is the maximum intensity of light that Karel can obtain in this way if he shines on the first polarizer with unpolarized light of intensity I_0? Express as the ratio $I_{\max}/I_0$.

Petr was thinking about stacking polarizing filters.

We assume that $0 \le \alpha \le \pi/2$ because we are only examining the angle between the planes. Also, note that even equality is out of the question while the polarizing filters perpendicular to each other will screen out 100 % of the radiation.

When incident on the first polarizer, the light is linearly polarized, so we observe a 1/2 decrease in intensity. Furthermore, if the polarized light falls on a polarizing filter rotated by an angle α, we can determine the intensity I' of the transmitted radiation as

$$I' = I\cos^2(\alpha)\;.$$

The first polarizer only polarizes the light, which we have already described above, so let us now examine the losses on the other four polarizers. There, we can use the identity $\cos(\alpha - \pi/2) = -\sin(\alpha)$ for the total measured intensity I to write

$$I = \frac{I_0}{2}\cos^2(\alpha)\cos^2\left(\alpha - \frac{\pi}{2}\right)\cos^2(\alpha)\cos^2\left(\alpha - \frac{\pi}{2}\right) = \frac{I_0}{2}\cos^4(\alpha)\sin^4(\alpha)\;.$$

It remains to find the maximum of this expression. By deriving and setting it equal to zero, we get

$$\cos^3(\alpha)\sin^3(\alpha)\left(\cos^2(\alpha) - \sin^2(\alpha)\right) = 0\,,$$

which for $0 < \alpha < \pi/2$ is equivalent to

$$\cos(2\alpha) = 0\,,$$

where $\alpha = \pi/4$. Now all we have to do is add and we get

$$\frac{I_{\max}}{I_0} = \frac{1}{32} = 0.03125\,.$$

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Problem 38 ... dog walking 7 points

Jarda is walking his dog. They are walking down a long straight street. Jarda's distance from the garden fences is 5.0 m *while his dog is tied to a* 4.0 m *long leash. Already from afar, the dog knows about his friend Žeryk, who is lying contently in a kennel by the fence of one of the gardens and is looking forward to a barking encounter. For this reason, he always tries to be as close to Žeryk as possible on his leash. What is the highest velocity Jarda's dog will move at? Jarda walks with a constant velocity* $v = 1.0\,\mathrm{m{\cdot}s^{-1}}$.

Jarda invents problem assignments even while walking his dogs.

Let us introduce a Cartesian coordinate system with its center at the point where Jarda stands when he is closest to Žeryk, i.e. perpendicular to the fence at a distance of 5 m. Let $t = 0$ correspond to the moment when Jarda passes through this point. Jarda's dog is thus exactly 1 m from Žeryk.

At the time t, Jarda's position is $x = vt$. Let us determine the position of Jarda's dog. The dog is always on the line between Jarda and the point $x = 0, y = D = 5\,\mathrm{m}$ where Žeryk is. For the angle between this line and the x axis, the following applies

$$\sin\varphi = \frac{D}{\sqrt{D^2 + v^2t^2}}\,,\ \cos\varphi = \frac{vt}{\sqrt{D^2 + v^2t^2}}\,.$$

Then the position of Jarda's dog is

$$x_\mathrm{p} = vt - r\frac{vt}{\sqrt{D^2 + v^2t^2}}\,,\ y_\mathrm{p} = r\frac{D}{\sqrt{D^2 + v^2t^2}}\,.$$

By deriving with respect to time, we find the components of its velocity

$$\dot{x}_\mathrm{p} = v\left(1 - \frac{r}{\sqrt{D^2 + v^2t^2}} + \frac{v^2t^2r}{\sqrt{D^2 + v^2t^2}^3}\right)\,,\ \dot{y}_\mathrm{p} = -\frac{rDv^2t}{\sqrt{D^2 + v^2t^2}^3}\,.$$

After multiplying and summing the two components, we obtain

$$v_\mathrm{p}^2 = \frac{v^2}{(D^2 + v^2t^2)^2}\left(\left(D^2 + v^2t^2\right)^2 + r^2D^2 - 2rD^2\sqrt{D^2 + v^2t^2}\right)\,.$$

We derive the expression and set it equal to zero. The solution to this equation is either $t = 0$ or it leads to a simple condition

$$3\sqrt{D^2 + v^2t^2} = 2r\,,$$

which cannot be satisfied for any real t. At the time $t = 0$ the dog's velocity is minimal, and no other extreme value is attained. Thus the maximal velocity of the dog will be $1.0\,\mathrm{m{\cdot}s^{-1}}$ at times $t = \pm\infty$.

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Problem 39 ... ordinary estimate of mean air height 6 points

Consider a hollow cylinder with base area $S = 327\,\mathrm{cm}^2$, which is closed at the bottom and infinitely tall at the top. The whole cylinder is placed in a homogeneous field with a gravitational acceleration g. Let us fill it with diatomic nitrogen molecules with temperature $T_0 = 25\,^\circ\mathrm{C}$ so that the pressure $p_0 = 101\,\mathrm{kPa}$ is at the bottom. What is the mean height of these molecules above the base of the cylinder? Assume that nitrogen behaves like an ideal gas.

Lego neglected as much as possible.

Classical solution

Of course, we will use the ideal gas equation of state $pV = NkT$. From this we can immediately see that in a small air layer of height $\mathrm{d}h$, located at height h, where the pressure is $p(h)$; there will be

$$\mathrm{d}N(h) = \frac{p(h)\,S}{kT_0}\,\mathrm{d}h$$

molecules. When we find out exactly how the pressure (and hence the "linear density" of the particles) evolves with height, we will be able to find their mean height as

$$\langle h \rangle = \frac{\int_0^\infty h\,\mathrm{d}N(h)}{\int_0^\infty \mathrm{d}N(h)} = \frac{\int_0^\infty hp(h)\,\mathrm{d}h}{\int_0^\infty p(h)\,\mathrm{d}h}\,.$$

We have a given pressure at the bottom, so we need to find some (obviously differential) relationship for the decreasing pressure with the upward direction. If we take a layer of molecules and divide its total gravity by the area S, we get the pressure it exerts on the gas below. The pressure below this layer will be greater than the pressure above it by this amount. Let's move from words to equations. Let's denote the mass of one molecule m. Then the described pressure difference will be

$$\mathrm{d}p = -\frac{mg}{S}\,\mathrm{d}N(h) = -\frac{mg}{kT_0}p(h)\,\mathrm{d}h\,.$$

This is a differential equation that can be solved simply by the separation of variables. However, you need to watch out for the correct sign – the pressure will decrease with increasing height! So we solve the equation

$$\frac{1}{p}\,\mathrm{d}p = -\frac{mg}{kT_0}\,\mathrm{d}h\,,$$

$$\int_{p_0}^{p(h)} \frac{1}{p}\,\mathrm{d}p = \int_0^h -\frac{mg}{kT_0}\,\mathrm{d}h\,,$$

$$\ln\left(\frac{p(h)}{p_0}\right) = -\frac{mgh}{kT_0}\,,$$

$$p(h) = p_0 \exp\left(-\frac{mgh}{kT_0}\right)\,.$$

We get that the pressure will decrease exponentially with height. We plug the obtained expression into the formula for the mean height that we provided above

$$\langle h \rangle = \frac{\int_0^\infty hp_0 \exp\left(-\frac{mgh}{kT_0}\right)\mathrm{d}h}{\int_0^\infty p_0 \exp\left(-\frac{mgh}{kT_0}\right)\mathrm{d}h} = \frac{\left(\frac{kT_0}{mg}\right)^2}{\frac{kT_0}{mg}} = \frac{kT_0}{mg} = 9020\,\mathrm{m}\,.$$

The mean height of the molecules above the bottom base of the cylinder is, therefore 9015 m. We may notice that it does not depend on p_0 at all. This makes sense since one of the fundamental properties of an ideal gas is that the molecules do not affect each other. Hence, no matter how many molecules we add, the mean height won't change... Now someone might ask if we couldn't obtain the result in a simpler way. The answer is that we could.

Statistical solution

Probably the best-known relationship from statistical physics is the Boltzmann distribution. This tells us that the probability that some degree of freedom (for example, the height of a single molecule h) will have a particular value is proportional to $\exp(-E/kT)$, where E is the energy associated with that value. In this case, it is the potential energy of the molecule, so $E(h) = mgh$. Then the average height of one molecule is obtained immediately as

$$\langle h \rangle = \frac{\int_0^\infty h \exp\left(-\frac{mgh}{kT_0}\right) \mathrm{d}h}{\int_0^\infty \exp\left(-\frac{mgh}{kT_0}\right) \mathrm{d}h} \,.$$

We see that we get the same integrals as at the end of the classical solution. The denominator (also called sum over states Z) is there again due to normalization.

Finally, we would add that the result in the order of higher thousands of meters makes quite a lot of sense if we realize that Mount Everest is usually climbed with oxygen equipment, but it is possible to climb it even without it. So the characteristic air height could be in the order of 8000 m.

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Problem 40 ... modern art 7 points

We have 2 rectangular cuboids, one with mass $M = 9.50\,\mathrm{kg}$ and the other with mass m. We lift the second one and press it against the wall by the first one, which is standing on the ground. What is the largest possible mass m to prevent the second cuboid from falling if the coefficient of friction between the cuboids, the cuboid and the wall, and between the cuboid and the ground is $f = 0,288$? Assume that the rectangular cuboid standing on the ground is long enough not to topple backward. *Lego really didn't know how to name this problem.*

A rectangular cuboid of mass m is pressed upwards only by friction forces – between the cuboid and the wall and between the cuboids themselves. The friction force can be calculated by multiplying the friction coefficient f (which is the same for all pairs of surfaces in this problem) with the normal force that presses the two surfaces together. Since the cuboids are not moving in the horizontal direction, the forces in this direction must be balanced. Thus, the normal force on one side of the lifted cuboid must be the same as on the other. If we denote this force by F_1, the friction force on each side will be $F_{t1} = fF_1$. The sum of the friction forces must compensate for the weight of the cuboid, so $mg = 2F_{t1}$. From here, we can express the force by which the cuboids push on each other as $F_1 = mg/(2f)$.

It comes from Newton's 3rd law of motion that cuboid M is pressed away by force F_1. To remain standing, a force of the same magnitude must act on it in the opposite direction. The only other force acting on it in the horizontal direction is the friction force from the ground. Its magnitude must therefore be equal to F_1, and again, we can express it as the product of f

and the normal force. However, here comes the trickiest part of the problem – the normal force will not only be Mg, but we have to consider Newton's third law. The friction force between the cuboids acts both on the cuboid held in the air and on the cuboid of mass M that holds it there (and the same is true for the wall).

So $F_2 = Mg + F_{t1} = Mg + mg/2$, and then the friction force between the ground and the cuboid is $F_{t2} = fg(M + m/2)$. We set this friction force equal to F_1 and get the equation for the maximum mass m, at which the lifted cuboid will not fall yet

$$\begin{aligned} F_1 &= F_{t2} \\ \frac{mg}{2f} &= fg\left(M + \frac{m}{2}\right) \\ \frac{m}{f} - fm &= 2fM\,, \end{aligned}$$

from that

$$m = 2M\frac{1}{\frac{1}{f^2} - 1} = 1.72\,\mathrm{kg}\,.$$

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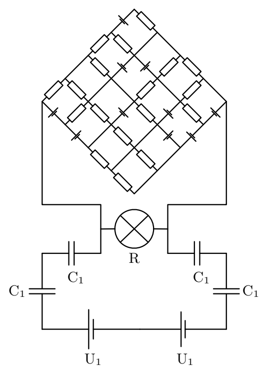
Problem 41 ... maze of resistance

6 points

All resistors (including the bulbs) in the circuit in the figure have a value of $R = 1.0\,\Omega$, capacitors have a capacitance of $C = 1.0\,\mathrm{F}$ and all batteries have a DC voltage of $U = 1.0\,\mathrm{V}$. Calculate the current through the bulb when the current is settled.

Marek J. encountered resistance while walking.



The circuit can be separated into upper and lower parts. The lower part consists of a voltage source and capacitors only. We know that DC current will not flow through the capacitors. Therefore, after initially charging the capacitors and rearranging the charge to compensate for the voltage, the situation settles down (zero current), and we can ignore the lower part of the circuit in our analysis of the problem.

The upper part, however, is really a maze of resistors and batteries. After wandering for a while, however, we can see that there is a path of least resistance for electric current (literally)! This is the part of the circuit where there are only batteries (and "empty" parts with just the wire). The batteries have differently oriented polarities, but since the value of their voltage is always the same and constant, the total voltage on that part of the circuit is ultimately the simple sum of the positive and negative voltages.

Looking at the figure, we get the total voltage value $U_c = 3\,\mathrm{V}$. The loop with the least resistance in the circuit is unambiguous and is completed by the bulb in parallel. Thus, from Kirchhoff's second law, we know the voltage across the bulb, which is equal to the aforementioned three volts. Finally, using Ohm's law, we calculate the current through the bulb as

$$I = \frac{U_c}{R} = 3.0\,\mathrm{A}\,.$$

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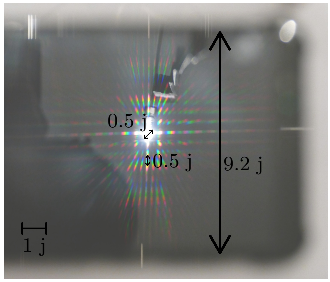
Problem 42 ... screen resolution

8 points

Jarda noticed a diffraction on his cell phone, so he tried to calculate the size of the individual pixels, which he thought were squares. He placed the cell phone 1.05 m *under a light source with a size of* 1.5 cm *and photographed the reflection on the screen with a camera whose lens was* 40 cm *above it. In the photograph, the width of the screen was* 9.2 j*, the diameter of the light was* 0.5 j*, and the distance of the yellow maxima was* 0.5 j*, where* 1 j *is the scale on the photograph. The screen of Jarda's phone has dimensions* 13 cm *and* 7.5 cm*. How many pixels are there?*



Jarda introduced an experimental problem to Physics Brawl Online.

First, let's note that the cell phone screen will act as a diffraction grating but also as a mirror that will show the source "behind the screen". From the grating equation

$$d \sin \alpha = k\lambda$$

we calculate the grating constant (i.e., the pixel size) d. Since we are working with small angles, we use the approximation $\alpha \approx \sin \alpha \approx \tan \alpha$ and measure the angle $\Delta\alpha$ between two adjacent maxima. Then we get

$$d = \frac{\lambda}{\Delta\alpha}\,,$$

where $\lambda = 580\,\mathrm{nm}$ is the wavelength of the yellow color.

The only task left is to determine $\Delta\alpha$ from the geometry of the apparatus and the photograph. To do this, we use the given lengths of the important objects in the photograph and the known width of the cell phone. The phone is a mirror, so we can imagine that the light source is behind it and the screen acts only as a diffraction grating. Consider that the light comes from parallel rays from behind the screen, where it is deflected due to diffraction and interference. Therefore, we are only interested in what happens in front of the display.

Consider the camera lens as the optical system that projects the incoming rays onto the sensor. For simplicity, we replace the camera lens in the diagram with a lens with focal length f. The sensor is in such a plane behind the focal point that the image is in focus. It is a known fact that the position of the image behind the optical system (i.e., the position in the photograph) is proportional to the angle at which the rays enter the system (at least in the geometric optics approximation). Since we don't need to have the center of the phone exactly on the optical axis, we'll just use angle differences and distance differences in the image. From the knowledge of the real width of the cell phone and its size in the picture, we find a proportionality constant, which we use to calculate $\Delta\alpha$.

The difference in the angles taken by the rays coming from the two edges of the phone is

$$\beta = \frac{s}{h} = \frac{7.5\,\mathrm{cm}}{40\,\mathrm{cm}}\,,$$

where s is the width of the screen and h is the distance of the lens from the screen. This angle is proportional to the width of the display in the photo $s_\mathrm{f} = 9.2\,\mathrm{j}$.

Now consider the rays from two adjacent yellow maxima. The difference in their angles (relative to the optical axis, for example) is exactly $\Delta\alpha$. This angle difference is, in turn, proportional to the distance in the photo $s_\mathrm{z} = 0.5\,\mathrm{j}$. Therefore, we can relate the angles β and $\Delta\alpha$ to the positions in the photograph and we get

$$\frac{s_\mathrm{f}}{\beta} = \frac{s_\mathrm{z}}{\Delta\alpha}\,.$$

We have already expressed β from the true dimensions of the objects. So now we are all set to calculate d as

$$d = \lambda\frac{s_\mathrm{f}}{s_\mathrm{z}\beta} = \lambda\frac{hs_\mathrm{f}}{s_\mathrm{z}s} = 57\,\mathrm{\mu m}\,,$$

where $\lambda = 580\,\mathrm{nm}$ is the wavelength of yellow light.

The number of these units on the whole display is thus given by the ratio of the area of the whole display, which is sv, where $v = 13\,\mathrm{cm}$, and the area of one pixel d^2. The numerical value is

$$N = \frac{sv}{d^2} = 3.0\cdot 10^6\,.$$

We adjusted the display dimensions in the problem statement from $12.8\,\mathrm{cm}$ and $6.4\,\mathrm{cm}$ compared to the experiment. With these values, we would get a result of $1.9\cdot 10^6$. The stated resolution of the mobile is 2160 by 1080, so there are $2.3\cdot 10^6$ pixels. The difference between these values is due to the error in the measurement of the height h or an approximation of the ray that passes through the lens.

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Problem 43 ... wired 7 points

Two adjacent electric transmission towers are distanced $s = 400\,\mathrm{m}$ in a flat landscape. One of the wires between them is fixed on both ends at a height of $H = 20\,\mathrm{m}$ above the ground. In winter, when the air temperature is $t_\mathrm{z} = -10\,\mathrm{°C}$, the lowest point of the wire is due to sagging at $h_\mathrm{z} = 19\,\mathrm{m}$ above ground. At what height will this point be in summer when the temperature is $t_\mathrm{l} = 20\,\mathrm{°C}$? The coefficient of thermal expansion of the steel from which the wire is made is $\alpha = 13\cdot 10^{-6}\,\mathrm{K^{-1}}$. For simplicity, neglect the elongation of the wire due to its weight.

Matej Rz was outside.

It is a well-known fact that the shape of the sagging wire corresponds to the so-called catenary, i.e. the curve described by the equation

$$r(x) = a\cosh\left(\frac{x}{a}\right)\,,$$

where a is a parameter specifying the "sag of the rope", the lowest point of the catenary is thus defined as the point $[0, a]$. We will need to be able to determine the length of the catenary – we can find this either by solving the simple integral[2] or we can state it as a known statement. Since the problem is symmetric, the length of our catenary will be

$$d = a\sinh\left(\frac{b}{2a}\right) - a\sinh\left(\frac{-b}{2a}\right) = 2a\sinh\left(\frac{b}{2a}\right)\,,$$

[2] $\int_\alpha^\beta \sqrt{1 + (a\cosh'(x/a))^2}\,\mathrm{d}x$

where b denotes the distance between the ends of the catenary. Next, we introduce a natural coordinate system with the x axis parallel to the ground and the y axis parallel to the transmission towers, placing its origin at a distance a "under" the top of the chain. This gives us a natural way to work with the catenary using the approach described above.

We determine the length of the catenary in winter. It holds $b = s$ and $r(b/2) = a + H - h_z$. From this, we can already determine the parameter a and then the length d. Unfortunately, we have to solve the equation for a numerically

$$a + H - h_z = a\cosh\left(\frac{s}{2a}\right) \quad \Rightarrow \quad a \doteq 20\,000.166\,7\,\mathrm{m}\,,$$

from where then

$$d = 2a\sinh\left(\frac{s}{2a}\right) \doteq 400.007\,\mathrm{m}\,.$$

The length of d' in summer is easily determined as

$$d' = (1 + \alpha(t_l - t_z))\,d = (1 + \alpha(t_l - t_z))\,2a\sinh\left(\frac{s}{2a}\right)$$

To determine the lowest point of the wire, we must now apply the same procedure as above backward. First, we introduce a coordinate system with its origin in $[0, a']$, determine the parameter a' numerically from the new length of the string, and finally calculate the height h_l as the difference $H - (r'(b/2) - a')$, where $r'(x)$ describes the shape of the catenary in summer. It applies

$$2a'\sinh\left(\frac{s}{2a'}\right) = (1 + \alpha(t_l - t_z))\,2a\sinh\left(\frac{s}{2a}\right)\,,$$

which we have to solve numerically again, but we get quite straightforwardly

$$a' \doteq 4\,049.097\,\mathrm{m}\,.$$

Finally, we can write

$$h_l = H - a'\cosh\left(\frac{s}{2a'}\right) + a' \doteq 15.1\,\mathrm{m}\,.$$

In our solution, we rounded the intermediate results, while for the final solution we substituted values with accuracy to multiple significant digits.

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Problem 44 ... electromagnet

8 points

Consider a long thin bar made from a material with high relative permeability $\mu_r = 7\ 000$. A wire is wrapped around the rod to obtain a solenoid with $N = 1\ 000$ turns. The solenoid is then bent into a circle so that the opposite ends of the bar almost touch, but a narrow gap of width $s = 1.00\,\mathrm{mm}$ is left between them. This gap is small compared to both the radius of the circle $r = 20.0\,\mathrm{cm}$ and the radius of the bar. If a current $I = 500\,\mathrm{mA}$ flows through the wire, determine the magnitude of the magnetic induction in the gap.

Jindra stole the problem from a textbook.

From Maxwell's equations it is possible to prove the continuity of the perpendicular component of the magnetic induction $\mathbf{B}$ and the parallel component of the magnetic field strength $\mathbf{H}$ at the interface of two materials (see the proof below).

In the gap, the magnetic induction is perpendicular to the surface of the bar, so the magnetic induction B there will be the same as everywhere else in the solenoid. The magnetic intensity in the core of the coil is H_{in} and the magnetic intensity in the gap is H_m. We know that $\mu_r\mu_0 H_\mathrm{in} = \mu_0 H_\mathrm{m} = B$.

The radius of the bar is negligible compared to the radius of the circle r. The line integral along the core of the coil along a circle of radius r gives

$$
\begin{aligned}
NI &= \oint \mathbf{H}\cdot \mathrm{d}\mathbf{l}\,,\\
NI &= H_\mathrm{in}(2\pi r - s) + H_\mathrm{m} s\,,\\
NI &= \frac{B}{\mu_r\mu_0}(2\pi r - s) + \frac{B}{\mu_0}s\,,\\
B &= \frac{\mu_r\mu_0 NI}{2\pi r + (\mu_r - 1)s} = 0.533\,\mathrm{T} \approx 0.53\,\mathrm{T}\,.
\end{aligned}
$$

We can notice that the magnetic field in the gap with this approximation is determined by the width of the gap and is greatly amplified compared to the situation where we would use a coil without a core (for a coil without a core we use the same relation, we substitute $\mu_r = 1$ and $s = 0\,\mathrm{mm}$, and get $B' = 5.00\cdot 10^{-4}\,\mathrm{T} \ll B = 0.533\,\mathrm{T}$).

Proof of the continuity of the perpendicular component $\mathbf{B}$ *and the tangential component* $\mathbf{H}$ *at the interface*

For the proof, we need two of Maxwell's equations for the magnetic field

$$
\oint \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0, \qquad \oint \mathbf{H}\cdot \mathrm{d}\mathbf{l} = I + \int \frac{\partial \mathbf{D}}{\partial t}\cdot \mathrm{d}\mathbf{S}\,.
$$

The first equation states that there are no magnetic monopoles. The second equation links the line integral of the magnetic field strength to the current flowing through that closed curve and the time variation of the electric induction flux $\mathbf{D}$.

Consider two different materials in contact with relative magnetic permeabilities μ_1 and μ_2. Let us take a closer look at a small section of their contact area so that we can neglect the curvature of the interface and treat it as a plane surface.

Around the planar interface, we form a Gaussian surface – a cube with one base lying in material 1 and the other base lying in material 2. The bases with surface A are parallel to the material interface. The cube has a negligible height h (see the figure on the left). From the equation for the absence of magnetic monopoles, we derive that the perpendicular component of the magnetic induction vector is continuous at the interface of any two materials

$$
\begin{aligned}
0 &= \oint \mathbf{B}\cdot \mathrm{d}\mathbf{S} \approx A\mathbf{B}_1\cdot \mathbf{n}_1 + A\mathbf{B}_2\cdot\mathbf{n}_2 = -AB_{1,\perp} + AB_{2,\perp}\,,\\
B_{1,\perp} &= B_{2,\perp}\,.
\end{aligned}
$$

When deriving, we neglected the flux of magnetic induction through the side walls of the cube, as we are working in the limit $h \to 0$. This way we proved a fact on which the solution to the problem relies. For interest, we shall also derive the continuity of the tangential component of the magnetic intensity vector.

To prove that the tangential component of the magnetic intensity vector is continuous at the interface, we draw a closed curve around the interface – a rectangle of length a and negligible height h. The sides of length a are parallel to the interface (see the figure on the right). For the proof, we use the equation

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I + \int \frac{\partial \mathbf{D}}{\partial t} \cdot \mathrm{d}\mathbf{S}$$

If we push the height h to zero, the current I flowing through the rectangle and the magnetic flux $\int (\partial \mathbf{D}/\partial t) \cdot \mathrm{d}\mathbf{S}$ will also go to zero. This follows from the fact that they are both proportional to the area of the rectangle, which goes to zero in the limit. However, the integral $\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l}$ on the left-hand side remains, as it depends on the perimeter of the rectangle, which remains non-zero

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} = 0 \approx \mathbf{H}_1 \cdot \mathbf{a}_1 + \mathbf{H}_2 \cdot \mathbf{a}_2 = aH_{1,\parallel} - aH_{2,\parallel} \,,$$

$$H_{1,\parallel} = H_{2,\parallel} \,.$$

The contribution from the vertical sides of the rectangle has been neglected since we have sent their height h to zero.

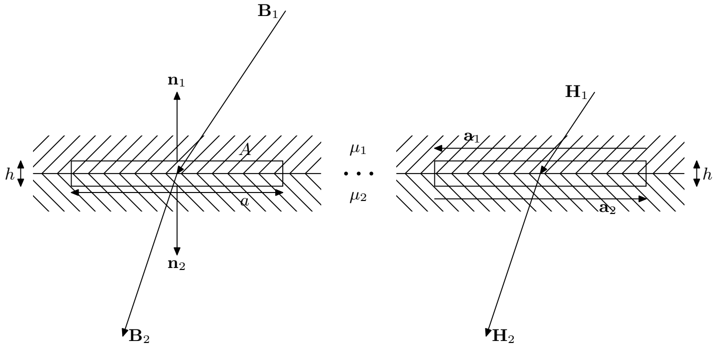

Fig. 2: For the proof of the continuity of the tangential and normal components.

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Problem 45 ... ladder 7 points

A ladder leaning against a wall has a length of $l = 5\,\mathrm{m}$ and a mass of $m = 10\,\mathrm{kg}$, with its center of gravity located exactly at its midpoint. When Jindra stands on the ladder, his center of gravity is at a distance of $h = 4\,\mathrm{m}$ along the ladder from the ground, and a distance of $x = 30\,\mathrm{cm}$ perpendicularly to it. Jindra's mass is $M = 70\,\mathrm{kg}$. The coefficient of kinetic friction between the ladder and the floor is $f_1 = 0.4$ and between the ladder and the wall is $f_2 = 0.6$. Determine the maximum angle between the ladder and the vertical wall, before Jindra starts falling. *Jindra painted the walls of a room.*

The amount of data complicates the problem. Let's analyze all the forces at work. At the contact with the ground, a normal force N_1 (size unknown) acts perpendicularly upwards on the ladder and a frictional force T_1 (size unknown) acts towards the wall. At the point of contact with the wall, a normal force N_2 (magnitude unknown) and a frictional force T_2 (magnitude unknown) act on the ladder perpendicular to the wall. The downward gravitational force Mg, where g is the acceleration due to gravity, acts on Jindra, and the downward gravitational force mg acts on the ladder.

If the linked system of Jindra and the ladder is supposed to be in equilibrium, the sum of all forces and the sum of all torques must be zero

$$
\begin{gathered}
(M+m)g = N_1 + T_2\,,\\
T_1 = N_2\,,\\
Mg(h\sin\alpha - x\cos\alpha) + mg\frac{l}{2}\sin\alpha = lT_2\sin\alpha + lN_2\cos\alpha\,,\\
Mg(l\sin\alpha - h\sin\alpha + x\cos\alpha) + mg\frac{l}{2}\sin\alpha + lT_1\cos\alpha = lN_1\sin\alpha.
\end{gathered}
$$

The first equation represents the balance of forces in the vertical direction, the second equation describes the balance of forces in the horizontal direction, the third equation represents the balance of moments of forces with respect to the point of contact of the ladder with the ground, and the fourth equation describes the balance of moments of forces with respect to the point of contact of the ladder with the wall. There are five unknowns in the system of four equations, the forces N_1, T_1, N_2, T_2, and the angle α. However, we are only interested in the specific maximal value of the angle $\alpha = \alpha_{max}$ at which the ladder starts to slip.

Let's think about what is happening at that point. The ladder starts sliding on the ground and along the wall at a critical moment. This means that none of the frictional forces T_1, T_2 can prevent the ladder from moving. The maximum magnitudes of the friction forces are $T_1 = f_1N_1$ and $T_2 = f_2N_2$. Thus, these two equations hold at the largest possible angle α_{max}. Our four equations reduce to two (taking advantage of the fact that $T_2 = f_2N_2 = f_1f_2N_1$)

$$
\begin{gathered}
(M+m)g = (1+f_1f_2)N_1\,,\\
Mg(h\sin\alpha_{\mathrm{max}} - x\cos\alpha_{\mathrm{max}}) + mg\frac{l}{2}\sin\alpha_{\mathrm{max}} = lf_1f_2N_1\sin\alpha_{\mathrm{max}} + lf_1N_1\cos\alpha_{\mathrm{max}}\,.
\end{gathered}
$$

Express N_1 from the first equation and plug it into the second equation

$$
Mg(h\sin\alpha_{\mathrm{max}} - x\cos\alpha_{\mathrm{max}}) + mg\frac{l}{2}\sin\alpha_{\mathrm{max}} = \frac{f_1l(M+m)g}{1+f_1f_2}\left(f_2\sin\alpha_{\mathrm{max}} + \cos\alpha_{\mathrm{max}}\right)\,,
$$

cancel g and convert the sine and cosine terms to each other

$$
\left(Mx + \frac{f_1l(M+m)}{1+f_1f_2}\right)\cos\alpha_{\mathrm{max}} = \left(Mh + m\frac{l}{2} - \frac{f_1f_2l(M+m)}{1+f_1f_2}\right)\sin\alpha_{\mathrm{max}}\,.
$$

We divide the equation by $\cos\alpha_{\mathrm{max}}$ (we're not afraid of dividing by zero) and we get

$$
\tan\alpha_{\mathrm{max}} = \frac{Mx + \frac{f_1l(M+m)}{1+f_1f_2}}{Mh + m\frac{l}{2} - \frac{f_1f_2l(M+m)}{1+f_1f_2}} \doteq 33.4^\circ\,.
$$

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Problem 46 ... mushroom hunting 7 points

Jarda went to an oak forest to collect parasol mushrooms. The wind was blowing quite strongly, so acorns from trees were falling with a frequency per unit area of $N = 1.0 \cdot 10^{-4}\,\mathrm{m^{-2}{\cdot}s^{-1}}$. The area that Jarda normally occupies is $A = 1100\,\mathrm{cm^2}$. What is the probability that an acorn will hit him if he plans on picking mushrooms in the forest for $t = 35\,\mathrm{min}$?

Jarda was hit by an acorn three times in a row!

From the problem statement, the mean value of the number of acorns that will fall on Jarda in 35 minutes can be directly determined as

$$\lambda = NAt\,.$$

Afterward, it is important to realize that the fall of an acorn is an independent event and that the number of independent events per unit of time is represented by the Poisson distribution. This distribution states that for a random variable X with a mean value of the number of occurrences of λ, the probability of x occurrences is

$$P(X = x) = \frac{\lambda^x}{x!}\mathrm{e}^{-\lambda}\,.$$

The probability that at least one acorn falls on Jarda is then given by

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \mathrm{e}^{-\lambda} \doteq 0.023\,.$$

One could argue that this problem has little to do with physics, but the mathematical model – studying independent phenomena – is an inseparable part of nuclear or statistical physics.

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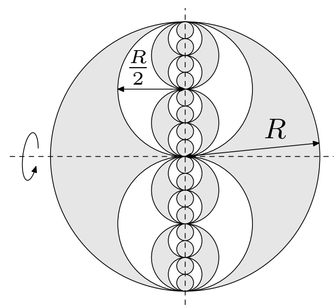
Problem 47 ... a ball for Jáchym reloaded 7 points



In successive steps, we will make a special ball for Jáchym. We start with a homogenous solid ball of radius R and material density ρ. We choose one major axis passing through the center of the ball. Next, we extract two balls from the large ball that have radii $R/2$, whose centers lie on the major axis and are at distances $R/2$ from the center, such that the resulting holes are closely adjacent to each other. Then we add 4 balls of radii $R/4$ inside. Their centers again lie on the main axis and the balls are placed next to each other. We repeat this process of making two spherical holes in each of these smaller balls and then putting the half balls into these again ad infinitum. What is the moment of inertia of Jáchym's ball with respect to the axis perpendicular to the main axis? Express it as a ratio to the moment of inertia of a full ball of radius R and equal density ρ.

Karel (even Vojta and the other Vojta) thinks that Jáchym deserves the ball not once, but twice.

The moment of inertia of a ball of mass m and radius r rotating about the axis through its centre corresponds to

$$J = \frac{2}{5}mr^2\,.$$

In order to solve the problem, in addition to knowing that the moments of inertia of bodies rotating with respect to the same axis can be added to obtain the total moment of inertia,[3] we will also need Steiner's theorem. The latter states that if we move the axis of rotation of a rigid body passing through a center of gravity of mass m to a distance r, the moment of inertia changes from J to

$$J' = J + mr^2\,.$$

To avoid having to laboriously add up the geometric series, we note that we can use the recurrence of the whole the situation. Specifically, Jáchym's ball is composed of one large ball of radius R, in which two spherical cavities are excavated containing four smaller Jáchym's balls (of quartered radii) whose centers are $R/4$ and $3R/4$ away from the center of the original ball.

Before we get to the actual calculation, we need to consider how the moment of inertia of a body scaled by a factor α (while maintaining a constant density) will change. The moment of inertia depends on the square of the distance from the axis of rotation and on the mass, which is proportional to the third power of the magnitude.[4] Therefore if we reduce all the dimensions of the body in proportion to the coefficient α, the moment of inertia drops by α^5. Therefore, Jáchym's ball of a quarter radius will have a rotational inertia of $1/4^5$ of the original rotational inertia.

Finally, we will also need the masses of the quarter-sized Jáchym's balls. From the problem statement of "a ball for Jáchym" we know that the mass of large Jáchym's ball is $4M/5$, so the mass of the reduced balls will be equal to $M/80$ (i.e. $1/4^3$ of the mass of the original ball, since the mass decreases with the third power of "size").

Now we can finally do the math. For the moment of inertia of Jáchym's ball J, following equation must hold

$$J = \frac{2}{5}MR^2 - 2\left(\frac{2}{5}\frac{M}{8}\left(\frac{R}{2}\right)^2 + \frac{M}{8}\left(\frac{R}{2}\right)^2\right) + 2\left(\frac{J}{4^5} + \frac{M}{80}\left(\frac{R}{4}\right)^2\right) + 2\left(\frac{J}{4^5} + \frac{M}{80}\left(\frac{3R}{4}\right)^2\right)\,,$$

since it must be equal to the moment of inertia of the full ball without the moments of inertia of the two smaller balls of halved radius, in turn increased by the moments of inertia of the four small Jáchym's balls – all modified by Steiner's theorem. Solving this equation will then give us

$$J = \frac{28}{85}MR^2 = \frac{14}{17}J_0\,,$$

where $J_0 = 2/5MR^2$ is the moment of inertia of the solid ball. Therefore, the answer to the question in the problem statement is $14/17 \doteq 0.82$.

We may incidentally realize that the same result is obtained if we use Steiner's theorem to "subtract" two Jáchym's balls of half the size from the full ball.

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Problem 48 ... up and down 8 points

Ping pong is played with balls of diameter $d = 4.0\,\mathrm{cm}$ and mass $m = 2.7\,\mathrm{g}$. After losing a match, Jarda took one of these balls and out of anger fired it vertically in the air. The ball reached

[3]This stems trivially from the fact that the moment of inertia is the integral of the square of the distance from the axis of rotation over the total mass of the body.

[4]More precisely, it is the integral of the square of the distance from the axis of rotation over the mass.

the height $H = 8.5\,\mathrm{m}$. *Determine the ratio of the time the ball was going up and the time it was falling down.* *Jarda tried to hit the window in his dorm with a ping-pong ball, but he failed.*

Right at the beginning of the problem, it is important to remember that the ping-pong ball is not a point mass, so the drag force of air must be considered. We will use the following formula for the whole problem

$$F_\mathrm{o} = \frac{1}{2} C S \rho v^2 \,,$$

where the designation of the individual quantities will be explained later. The drag force depends on the square of the velocity v because the motion of the ball is fast enough to cause turbulent flow.

It is obvious that the ball will be falling longer than it will be going up because it will travel the same distance, but its average speed will be lower due to the dissipation of energy due to the drag force.

We divide the trajectory into two parts - the flight upwards and the flight downwards; we have different equations for the motion of both. The process of solving the equations is left as an exercise for an interested reader; here, we give only the necessary results. The solution process is also possible to find on the Internet or by using one of the computing software.

The equation of motion for upward flight is

$$ma = -(mg + kv^2)\,,$$

where m is the mass of the ball, g is the gravitational acceleration, a is acceleration of the ball, v is its speed, and $k = CS\rho/2$ is a constant. Here $S = \pi d^2/4$, ρ is the density of air, and $C = 0.5$ for a ball. If we orient the velocity upwards, the acceleration upwards is negative.

The solution to this equation is

$$v = \sqrt{\frac{gm}{k}} \tan\left(\sqrt{\frac{gk}{m}}\,(T - t)\right)\,,$$

where T is the time when v is zero, so the ball is at the top of its path. By integration, we get

$$h = \frac{m}{k} \log\cos\left(\sqrt{\frac{gk}{m}}\,(T - t)\right) + K\,.$$

From the initial condition, where $h = 0$ in $t = 0$, we get the value of the integration constant as

$$K = -\frac{m}{k} \log\cos\left(T\sqrt{\frac{gk}{m}}\right)\,.$$

By substituting in time $t = T_1$, we get

$$H = -\frac{m}{k} \log\cos\left(T_1\sqrt{\frac{gk}{m}}\right)\,,$$

from where we express the time T_1 as

$$T_1 = \sqrt{\frac{m}{gk}} \arccos e^{\frac{-kH}{m}}\,.$$

For a downfall, the equation of motion has the form

$$ma = mg - kv^2\,,$$

where the acceleration points downwards. We get the velocity in the form

$$v = \sqrt{\frac{gm}{k}}\tanh\sqrt{\frac{gk}{m}}t\,,$$

where in time $t = 0$ the velocity is also zero. By integration, we get

$$h = H - \frac{m}{k}\log\cosh\sqrt{\frac{gk}{m}}t\,,$$

where we have determined the integration constant from the initial condition. We find the time T_2 for which the ball falls down as

$$\sqrt{\frac{m}{gk}}\operatorname{argcosh} e^{\frac{kH}{m}} = T_2\,.$$

The solution to the problem is thus

$$\frac{T_1}{T_2} = \frac{\arccos e^{\frac{-kH}{m}}}{\operatorname{argcosh} e^{\frac{kH}{m}}} = 0.68\,.$$

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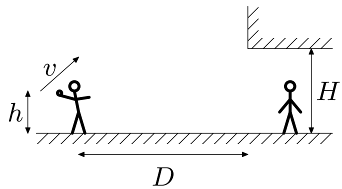
Problem 49 ... revenge for the machines 7 points

Jarda got really angry at Viktor for his noisy vacuum cleaners and started chasing him around the dormitory. Viktor is now hiding in a passage under the dorms. The ceiling is $H = 5.00\,\mathrm{m}$ *high there. Jarda is standing at a distance* $D = 12.0\,\mathrm{m}$ *from the beginning of the passage when he throws an object at Viktor with velocity* $v = 13.5\,\mathrm{m{\cdot}s^{-1}}$ *from a height of* $h = 2.10\,\mathrm{m}$*. How far from the beginning of the passage must Viktor stand to avoid being hit by Jarda?*



Viktor created an inspiring environment in his room...

The solution breaks down into several cases. If the velocity is small, it is not possible to hit the passage ceiling at any initial angle, so we do not have to reckon it. So we find the maximum distance to which an object can be hit from a height h at a given velocity.

For high velocities, it is worth throwing at a small angle so that the object does not rise too high and can get into the passage at all. For this case, it is important whether the vertex of the parabola along which the object is moving can be at a greater distance from Jarda than D. Let us explore this possibility. The equation of the parabola along which the object moves is

$$y = h + x\tan\alpha - \frac{g}{2}\frac{x^2}{v^2\cos^2\alpha}\,, \tag{5}$$

where g is the acceleration due to gravity, α is the initial angle of the trajectory with respect to the ground, y is the vertical coordinate upwards, and x is the horizontal coordinate towards the passage. At the apex of the parabola, y does not change, so the derivative of the previous function is zero, which allows us to find the position of the apex as

$$x_{\mathrm{v}} = \frac{v^2 \sin 2\alpha}{2g} \,,$$

where we used the goniometric identity $2 \sin \alpha \cos \alpha = \sin 2\alpha$. The maximum value of the sine function is 1, so the maximum possible value for x_{v} in our case is $9.3\,\mathrm{m}$, which is less than D. Thus, when entering the passage, the object will already be falling.

From equation 5 we find the distance of the point of impact from Jarda by setting $y = 0$ and solving the quadratic equation. We get

$$x_{\mathrm{d}} = \frac{v \cos \alpha}{g} \left(v \sin \alpha \pm \sqrt{v^2 \sin^2 \alpha + 2gh} \right) \,. \tag{6}$$

But at the same time, the condition $y(D) \leq H$ must be satisfied; otherwise, Jarda's object will end up hitting the roof of the passage. We find the extreme angles under which equality occurs; that is, the object's trajectory intersects the bottom corner of the passage roof. We plug $x = D$, $y = H$ into equation 5 and use the identity $1/\cos^2 \alpha = 1 + \tan^2 \alpha$. This gives us

$$0 = \frac{g}{2} \frac{D^2}{v^2} \tan^2 \alpha_{\mathrm{m}} - D \tan \alpha_{\mathrm{m}} + \frac{g}{2} \frac{D^2}{v^2} + H - h$$

$$\Rightarrow \quad \tan \alpha_{\mathrm{m}} = \frac{v}{gD} \left(v \pm \sqrt{v^2 - 2g \left(\frac{gD^2}{2v^2} - h + H \right)} \right) .$$

The limiting angles are therefore $67.0\,^{\circ}$ and $36.6\,^{\circ}$. Between these angles, the object will not enter the passage.

It is possible that the maximum of function 6 does not lie in this interval of angles so that the object might not touch the mentioned corner of the passage at all. The angle for which the object will travel the furthest is found as

$$\alpha_{\max} = \arctan \frac{v}{\sqrt{v^2 + 2gh}} \doteq 42.1\,^{\circ} \,.$$

Since this angle lies in the forbidden interval, we would hit the passage. Since function 6 is first increasing and then decreasing for $\alpha > 0$, the longest possible range occurs for either $67.0\,^{\circ}$ or $36.6\,^{\circ}$. Substituting in 6, we find that the maximum range is for $36.6\,^{\circ}$, namely $20.3\,\mathrm{m}$. From this distance, we still need to subtract the distance D, which gives us the value $8.3\,\mathrm{m}$. If Viktor had not hidden in the passage, Jarda would still have only reached $20.6\,\mathrm{m}$, which is only $30\,\mathrm{cm}$ farther.

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Problem 50 ... water in a bowl

7 points

We pour 650 ml *of water into a bowl shaped like a perfect homogenous hemisphere of radius* 12 cm *and mass* 610 g*. Then we start tilting the bowl. What work we must do to get the water to start pouring out? Assume that the water is an ideal liquid, and neglect the thickness of the walls.* *Jarda was rinsing a bowl which had had cucumber salad in it.*

Firstly, we note a few important observations. When we tilt the bowl, the water stays in place, so we do not change its potential energy. Furthermore, if we assume there is no friction between the water and the bowl, we are not doing any work at all because of the water. The only thing we are doing in the terms of the work is raising the center of gravity of the bowl itself.

Another interesting thing is the shape of the bowl. Since the thickness of the walls is small, we consider the bowl to be a hemisphere. Obviously, the bowl has some area density, which is a constant. Therefore, the mass is everywhere proportional to the surface area. Moreover, for a sphere, the area of the spherical segment is directly proportional to its height. From here, it simply follows that the center of gravity of the hemisphere is located at half of its height on the axis of symmetry, i.e. at $h_1 = R/2$, where $R = 12\,\mathrm{cm}$ denotes the radius of the bowl.

So far, we know the position of the center of gravity and that the work done is proportional to the change in its position. Now we need to calculate how much we need to tilt the bowl to get the water to start pouring out. That will occur when the water level is at the same height as the bottom of the bowl. Let us denote by α the angle that the normal to the water (and the ground) makes with the plane of the rim of the bowl.

At this moment, the height of the center of gravity above the ground

$$h_2 = R - \frac{R}{2}\sin\alpha\,.$$

It remains to obtain the value of the angle α from knowing the volume. These two quantities are bound by the equation[5]

$$V = \frac{\pi R^3}{3}\left(2+\cos\alpha\right)\left(1-\cos\alpha\right)^2\,.$$

That is a cubic equation for $\cos\alpha$. Its numerical solution is

$$\cos\alpha = 0.630\,47 \quad\Rightarrow\quad \alpha = 0.887\,28 \quad\Rightarrow\quad \sin\alpha = 0.776\,27\,.$$

The difference in potential energy of the bowl is thus

$$E_{\mathrm{p2}} - E_{\mathrm{p1}} = mg\left(h_2 - h_1\right) = mgR\frac{\left(1-\sin\alpha\right)}{2} = 80\,\mathrm{mJ}\,.$$

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[5] See Wikipedia `https://en.wikipedia.org/wiki/Spherical_cap`.

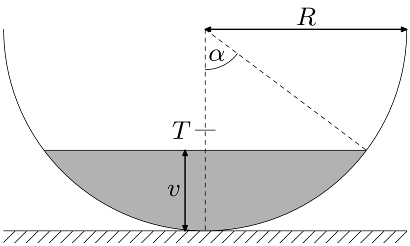

Fig. 3: Before tilting.

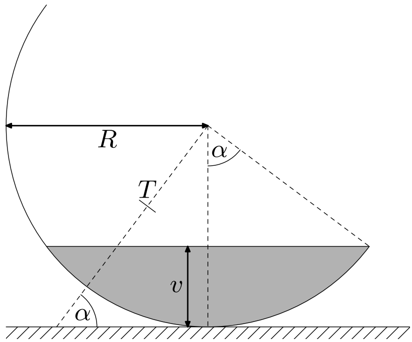

Fig. 4: When water pours out.

Problem 51 ... where to hide? 9 points

It is the year 2311 and aliens are about to attack Earth. They arrived in their spaceship and stopped at an altitude of $h = 12\,200\,\mathrm{km}$ above the South Pole. Fortunately, the Earthlings had already developed a defensive missile system and destroyed the attackers' ship. It fell apart into thousands of tiny fragments, which flew in all directions at speeds up to a maximum of $v = 300\,\mathrm{m{\cdot}s^{-1}}$. What percentage of the Earth's surface do the fragments hit? The ship was made of refractory material, so the debris does not burn in the atmosphere.

Jarda saw Anubis' fleet destroyed in Stargate.

The spaceship was shattered above Earth's pole, so we need not consider the Earth's rotation. It is clear that this problem is axisymmetric, so we will tackle the problem in two dimensions. Firstly we verify that the fragments have insufficient velocity to escape from the Earth's gravity by looking at the sign of the total mechanical energy per unit mass (denoted as E). We easily see that the value

$$E = \frac{v^2}{2} - \frac{\mu}{r_0} \doteq -21.4\,\mathrm{MJ{\cdot}kg^{-1}}$$

is negative, therefore all of the fragments have elliptical trajectories. The Earth's gravitational parameter, i.e. the product of Earth's density and gravitational constant, is denoted by $\mu \doteq 3.99 \cdot 10^{14}\,\mathrm{m^3{\cdot}s^{-2}}$, and $r_0 = R_\mathrm{Z} + h$ is the distance between the point of explosion and Earth's center.

Firstly we will find the region of space which can be hit by fragments. Then we will find the intersection of this set of points with the surface of the Earth. We consider only fragments with the maximal velocity (v). All of these fragments thus have the same total energy (sum of kinetic and potential energy) since their distance from Earth's center is the same. The total energy stays constant during motion and it corresponds with the major half-axis of ellipse a according to

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}\,.$$

Thus all trajectories of fragments have the same major half-axis

$$a = \frac{\mu r_0}{2\mu - r_0 v^2}\,.$$

We use the following reasoning to find the boundary of the space where fragments can sweep. We choose a ray p pointing from the Earth's center (i.e. focus F_1) in an arbitrary direction. We then try to find the farthest point on p, which still intersects a trajectory k of any fragment. Let us denote this point as B and its distance from Earth's center (first focus) as ρ. Let s be the line segment between this point and the second focus F_2 of a given trajectory k, while we still don't know the position of this focus. Certainly, it must hold $\rho + s = 2a$ since the point B lines on the ellipse k.

The second focus F_2 can be connected with the point of the explosion by a line segment with length t and it also holds $r_0 + t = 2a$. We already know the length of a, thus we know the length of the line segment t. The focus F_2 must lie somewhere on the circle k_1 with radius $2a - r_0$ and center in the point of the explosion.

The point B thus lies un the ray p and its distance from the focus F_1 is ρ. The focus F_2 must lie on the circle k_2 with radius $2a - \rho$ and center in the point B. The value of ρ should be the largest, so the radius k_2 should be the lowest. This focus also lies on the circle k_1. Both of these conditions are fulfilled iff the circles k_1 and k_2 have exactly one common point. This point is the focus F_2 and it lies on the line connecting the centers of both circles.

Distance of point B is then readily computed as a sum of both radii, thus $2a - r_0 + 2a - \rho$. The important fact is that the sum of distances from point B to the center of Earth and from point B to the point of the explosion (i.e $4a - r_0 - \rho$) is constant and equal to $4a - r_0 - \rho + \rho = 4a - r_0$ for each ray on which point B may lie. Therefore, the set of all such points is an ellipse with one focus in the center of Earth and the second focus in the center of the explosion. The major half-axis of the ellipse is

$$a_\mathrm{o} = \frac{4a - r_0}{2} = \frac{r_0}{2} \frac{2\mu + r_0 v^2}{2\mu - r_0 v^2} \doteq 9\,328\,\mathrm{km}\,.$$

The only remaining task is to find the intersection of this ellipse with Earth. Let's introduce the cartesian coordinate system with the origin at the center of the line segment between the point of the explosion and Earth's center, i.e. in the center of the envelope ellipse (the ellipse no fragment can pass through). We know the major half-axis of the envelope ellipse and we can calculate the minor half-axis

$$b_\mathrm{o} = \sqrt{a_\mathrm{o}^2 - e^2} = \sqrt{a_\mathrm{o}^2 - \frac{r_0^2}{4}} = \frac{r_0 \sqrt{2\mu r_0 v^2}}{(2\mu - r_0 v^2)}\,.$$

The equation of envelope ellipse is thus

$$\frac{x^2}{a_\mathrm{o}^2} + \frac{y^2}{b_\mathrm{o}^2} = 1\,,$$

while the equation of circle (Earth's surface) is

$$\left(x + \frac{r_0}{2}\right)^2 + y^2 = R_\mathrm{Z}^2\,.$$

We plug y^2 from the second equation to the first one and then we solve a quadratic equation for x. After a series of adjustments we obtain the result

$$x_{1,2} = \frac{2a_\mathrm{o}}{r_0}\left(-a_\mathrm{o} \pm R_\mathrm{Z}\right) \quad \Rightarrow \quad x = \frac{2a_\mathrm{o}}{r_0}\left(-a_\mathrm{o} + R_\mathrm{Z}\right) \doteq -2\,962\,\mathrm{km}\,.$$

We choose the positive sign (negative sign does not make sense).

We compute how far are the points of intersection of envelope ellipse from the South Pole

$$l = \left|\frac{r_0}{2} + x - R_\mathrm{Z}\right| \doteq 51.4\,\mathrm{km}\,.$$

The area which can be hit by the fragments of the spaceship is a spherical canopy with height l. Hence the fragments will hit Úlomky tak bude zasaženo

$$\frac{2\pi R_\mathrm{Z} l}{4\pi R_\mathrm{Z}^2} \doteq 0.004\,0 = 0.40\,\%$$

of Earth's surface.

The correct way of solving this problem would employ the ellipsoidal shape of the Earth, which would make the calculations even more difficult. The polar radius of Earth is $R_\mathrm{p} = 6\,357\,\mathrm{km}$, while the equatorial radius is $R_\oplus = 6\,378\,\mathrm{km}$. The value of l is relatively small so we can consider an osculating sphere with the same curvature at the South Pole $R = R_\oplus^2/R_\mathrm{p} \doteq 6\,399\,\mathrm{km}$. But we cannot just simply plug this value into the above expressions, because the center of the osculating sphere is not at the same point as the center of Earth. Using this more accurate approach correctly we obtain a more accurate value of $0.398\,\%$.

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Problem 52 ... reading against the light — 9 points

Imagine a large plane with an albedo (a measure of reflectivity) of 0.69*. At a height* $h = 1.0\,\mathrm{m}$ *above the plane, there is a point source of light illuminating it. Precisely below the source, we place a black disc with a radius* h *at a height* x*, parallel to the plane. Determine the value of* x *for which the center of the bottom face of the disc receives the greatest luminous flux. Assume that the reflected light is scattered uniformly in all directions and the plane has a Lambertian surface.* *Matěj wanted to read while lying down, but the only light source was the chandelier.*

First, realize that albedo is redundant information. Since we are only interested in the maximum, we don't have to worry about "how much light" reaches the center of the disk in absolute terms.

Before we get into the solution to the problem, we need to understand the properties of a point source. The essential characteristic of a point source is that it radiates power evenly in all directions. This can be interpreted as follows. If the total radiated power is equal to P, the measured intensity I at a distance r will be proportional to P/r^2 since all the power will be uniformly distributed over a spherical area $4\pi r^2$.

Furthermore, we will need Lambert's cosine law, which states that the luminous flux scattered by a perfectly opaque surface is directly proportional to the cosine of the scattering angle (measured from the perpendicular). We should also remember that the incident light flux (whether in the case of a plane or a black disk) is proportional to the cosine of the incident angle. Finally, since the problem is axially symmetric with respect to the line passing through the source and the center of the disk, it will be useful to divide the plane on which the light is scattered into concentric, infinitesimally thin rings centered at the point where the axis of symmetry intersects the plane. If we denote the radius ρ of one such ring of width $\mathrm{d}\rho$, its area will be equal to $2\pi\rho\,\mathrm{d}\rho$. Equipped with this knowledge, we can then say that the luminous flux delivered to the center of the bottom of the disk by the rays scattered by one infinitesimally thin ring of radius ρ will be

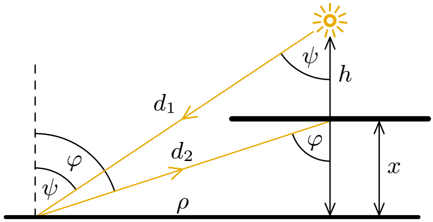

Fig. 5: Simple geometry.

- inversely proportional to the square of the distance between the source and the point of incidence on the plane (denoted by d_1),
- directly proportional to the cosine of the angle of incidence on this plane (let's denote ψ),
- directly proportional to the area of the ring $2\pi\rho \, d\rho$,
- directly proportional (according to Lambert's law) to the cosine of the scattering angle towards the center of the disk (let's denote φ),
- inversely proportional to the square of the distance between the scattering point and the center of the black disc (let's denote d_2), and finally
- directly proportional to the cosine of the incidence angle to the target location (also equal to φ, from the alternation of angles).

By simple geometry, we can then express these quantities as

$$d_1 = \sqrt{h^2 + \rho^2}\,, \quad \cos(\psi) = \frac{h}{d_1} = \frac{h}{\sqrt{h^2 + \rho^2}}\,,$$
$$d_2 = \sqrt{x^2 + \rho^2}\,, \quad \cos(\varphi) = \frac{x}{d_2} = \frac{x}{\sqrt{x^2 + \rho^2}}\,.$$

Now we determine the minimum value of $\rho = \rho_0$ and integrate over all from ρ_0 to ∞. The value of ρ_0 corresponds to the radius of the shadow from the disc, which we determine from the similarity of the triangles as

$$\frac{\rho_0}{h} = \frac{h}{h-x} \quad \Rightarrow \quad \rho_0 = \frac{h^2}{h-x}\,.$$

The luminous flux at the center of the disk will therefore be proportional to the value of

$$hx^2 \int_{\frac{h^2}{h-x}}^{\infty} \frac{\rho}{(h^2+\rho^2)^{3/2}(x^2+\rho^2)^2} \, d\rho\,.$$

We can solve this integral numerically. By plugging in Desmos, we can even plot the graph of this function, from which we can see that the maximum is $x \doteq 0.311\,\mathrm{m}$.

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Problem 53 ... rattlesnake 11

8 points

Semir and his wingman are chasing a criminal in a police car. The perpetrator drove off the road onto a horizontal forest path, thinking he could outrun the highway cops. However, he did not notice that the road ended with a cliff, $H = 20\,\mathrm{m}$ deep. The criminal is approaching the cliff with a velocity $v = 144\,\mathrm{km{\cdot}h^{-1}}$. At what angle does the front bumper hit the ground? The parameters of the perpetrator's car are: mass $m = 1.32\,\mathrm{t}$, wheelbase $D = 2.46\,\mathrm{m}$, distance from the front axle to bumper $d_\mathrm{n} = 0.71\,\mathrm{m}$, the height of the center of mass of the car above the ground $h_\mathrm{t} = 42\,\mathrm{cm}$, the horizontal distance of the center of mass from the front axle $d_\mathrm{t} = 80\,\mathrm{cm}$, the height of the front bumper above the ground $h_\mathrm{n} = 15\,\mathrm{cm}$, the moment of inertia of the car with respect to the center of mass $I = 1.67 \cdot 10^3\,\mathrm{kg{\cdot}m^2}$. The car uses front wheel drive. Assume that the rear wheels roll without resistance. *Jindra watches German crime series.*

At the moment when the front axle crosses the edge of the cliff, the car loses support at the front and gravity causes rotation around the axis passing through the points of contact between the rear wheels and the ground.

The normal force F exerted by the ground on the rear axle may change depending on the angle of rotation of the car. Therefore, we'll work with the assumption that crossing the edge of the cliff itself takes such a short time ($t_1 = D/v \doteq 6.15 \cdot 10^{-2}\,\mathrm{s}$) that the car remains almost horizontal during that time. Then we can consider the force F to be constant. This force F, which imparts rotation to the car, acts at the rear axle, so the length of the moment arm is $x_\mathrm{t} = D - d_\mathrm{t} \doteq 1.64\,\mathrm{m}$.

The motion of the car is described by two equations. The first one describes acceleration in the vertical direction (the axis y points upwards), the second equation describes rotational motion (we define angular acceleration to be positive)

$$\begin{aligned} m\ddot{y} &= -mg + F,\\ I\dot{\omega} &= F x_\mathrm{t}. \end{aligned}$$

The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. We can express F from the second equation and substitute into the first equation to obtain

$$\ddot{y} = -g + \frac{I}{m x_\mathrm{t}}\dot{\omega}.$$

We know that the drop of the center of mass due to vertical forces must correspond to drop of the center of mass due to rotation around the rear axle, so

$$\dot{\omega} = -\frac{\ddot{y}}{x_\mathrm{t}}.$$

Both equations derived above can be combined into one, which lets us express the angular acceleration $\dot{\omega}$ using known quantities as

$$\dot{\omega} = \frac{g}{x_\mathrm{t}\left(1 + \frac{I}{m x_\mathrm{t}^2}\right)}.$$

Alternative approach: We would find the same angular acceleration in the reference frame of the rear axle, where gravity instead imparts rotation to the car and the moment of inertia can be found using Steiner's theorem.

This angular acceleration is only imparted onto the car during the time $t_1 = D/v$, while the rear wheels stay in contact with the ground. During this time, the car gains an angular velocity

$$\omega = \dot{\omega} t_1 = \frac{gD}{x_\mathrm{t} v \left(1 + \frac{I}{m x_\mathrm{t}^2}\right)} \doteq 0.25\,\mathrm{rad{\cdot}s^{-1}}$$

and since its initial angular velocity was zero, it rotates by an angle

$$\theta = \frac{1}{2}\dot{\omega} t_1^2 = \frac{gD^2}{2x_\mathrm{t} v^2 \left(1 + \frac{I}{m x_\mathrm{t}^2}\right)} \doteq 7.7 \cdot 10^{-3}\,\mathrm{rad} \doteq 0.44^\circ .$$

Our assumption that the angle of rotation while crossing the cliff should be small turns out to be justified.

From the moment when the rear wheels also cross the edge of the cliff, the car is falling towards the ground along a parabolic trajectory and rotating with constant angular velocity $\omega = 0.25\,\mathrm{rad{\cdot}s^{-1}}$ around the rear axle. The center of mass would hit the ground after a time

$$t_\mathrm{t} = \sqrt{\frac{2H}{g}} \doteq 2.0\,\mathrm{s}\,,$$

where we neglected the depth of the front bumper under the rear axle at the moment of impact, since $D + d_\mathrm{n}$ is much smaller than H. At that moment, the angle of rotation of the car is approximately

$$\theta_\mathrm{n} \approx \omega t_\mathrm{t} \doteq 29^\circ\,,$$

where we can see that the contribution of the angle θ is also negligible.

Is this result sufficiently accurate? It's true that in reality, both the fall of the car and its rotation would be affected by air resistance, and therefore, more complex calculations still wouldn't correspond to the real world. However, if we didn't disregard anything except friction and air resistance, we could find the result 28.2° numerically.

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Problem 54 ... spring with friction — 8 points

Consider a massless spring with stiffness $k = 9.0\,\mathrm{N{\cdot}m^{-1}}$. *We attach a weight with mass* $m = 110\,\mathrm{g}$ *to its free end. The other end of the spring is anchored in a wall. The spring is horizontal and the weight slides on the floor with friction. The coefficient of friction is* $f = 0.35$, *and the rest length of the spring is* $l_0 = 30\,\mathrm{cm}$. *By pulling on the weight, the spring is stretched by* $x_0 = 20\,\mathrm{cm}$ *and then released with zero initial speed. When the weight comes to a complete stop, what will the length of the spring be?* *Jindra was playing with a slinky.*

The movement of the weight is described by the differential equation

$$m\ddot{x} = -kx - fmg\,\mathrm{sgn}(\dot{x})\,, \tag{7}$$

where g is the acceleration due to gravity and $\operatorname{sgn}(\dot{x})$ is the sign function

$$\operatorname{sgn}(\dot{x}) = \begin{cases} +1 & \dot{x} > 0\,, \\ 0 & \dot{x} = 0\,, \\ -1 & \dot{x} < 0\,. \end{cases}$$

The sign function ensures that the frictional force always acts against the direction of velocity.

Recall the basic property of the friction force, where at a non-zero velocity it has the magnitude fmg. In the static case, the friction force has a magnitude less than or equal to fmg, always so that the resultant of the forces acting on the body is zero. To set the weight in motion from rest, the applied force of the spring must be greater than fmg. The limiting case is

$$x_{\max} = \frac{fmg}{k} \doteq 4.2\,\mathrm{cm}\,.$$

If the weight stands still less than 4.2 cm from the equilibrium position, it will stay in place. The spring does not exert sufficient force to move the weight. For each swing, the direction of the friction force will change, so we will have to successively find a solution to the equation (7) with several different initial conditions. The equation describing the motion of the weight from release to reaching the extreme position is of the form

$$m\ddot{x} + kx = fmg$$

and the initial conditions are $x(0) = x_0 = 20\,\mathrm{cm}$ and $\dot{x}(0) = 0\,\mathrm{m{\cdot}s^{-1}}$. The general solution of this equation is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{fmg}{k}\,,$$

where A and B are the integration constants and $\omega = \sqrt{k/m}$ we denote the angular frequency of the oscillations. The values of the constants A and B are found from the initial conditions $A = x_0 - fmg/k$ and $B = 0\,\mathrm{m}$. The body oscillates to the extreme position $x_1 = -x_0 + 2fmg/k$ at time $t_1 = \pi/\omega$. We quantify $x_1 \doteq -11.6\,\mathrm{cm}$ and see $|x_1| > x_{\max}$, which means the body starts moving in the opposite direction.

At that point the direction of velocity changes, so the motion is now described by the equation

$$m\ddot{x} + kx = -fmg$$

with initial conditions $x(t_1) = x_1$ and $\dot{x}(t_1) = 0\,\mathrm{m{\cdot}s^{-1}}$. The general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) - \frac{fmg}{k}\,,$$

where the integration constants are $A = -x_1 - fmg/k = -x_0 - 3fmg/k$ and $B = 0\,\mathrm{m}$. The body will oscillate to the extreme position $x_2 = x_0 - 4fmg/k$ at time $t_2 = 2\pi/\omega$ and at that instant its velocity will be zero. We quantify $x_2 \doteq 3.2\,\mathrm{cm}$ and see $|x_2| < x_{\max}$, so the body stops at that position.

The length of the spring after the weight stops will be $l = l_0 + x_2 \doteq 33.2\,\mathrm{cm}$.

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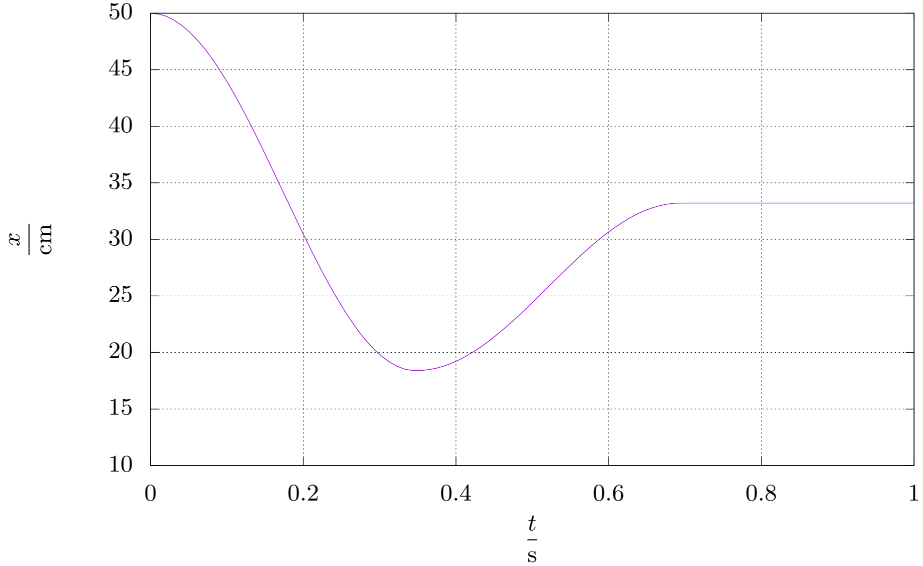

Fig. 6: Spring length vs. time.

Problem 55 ... central spring

9 points

Consider a spring attached at a point $r = 1\,321\,\mathrm{km}$ away from the center of a planet with mass $M = 4.11 \cdot 10^{22}\,\mathrm{kg}$ and radius $R = 1\,220\,\mathrm{km}$. The spring has a free length $x_0 = 89\,\mathrm{km}$ and a homogeneous linear density $\lambda = 8.7\,\mathrm{kg{\cdot}m^{-1}}$. What is the stiffness which the spring must have if we require it to exactly reach the surface of the planet? Do not consider any rotation of the system.

Jáchym was sorry that he had forgotten about a problem with a non-massless spring, so he decided to correct it.

Let us introduce a usual notation for heavy springs. A quantity x denotes the section of a length $\mathrm{d}x$ of the relaxed spring, whereas $y(x)$ is a coordinate of the given section on the stretched spring. We measure the distances from the point of attachment towards the center of the planet.

A gravitational force acting on the section $\mathrm{d}x$ is

$$\mathrm{d}F_\mathrm{g} = \frac{GM}{(r-y)^2}\lambda\,\mathrm{d}x\,.$$

Furthermore, the section $\mathrm{d}x$ is subject to an elastic (which we denote F) force emerging from adjacent sections. We can find this force by thinking of the section as a small individual spring with its stiffness $k_{\mathrm{d}x} = kx_0/\,\mathrm{d}x$, which extends to a length $\mathrm{d}y$. Therefore

$$F = k_{\mathrm{d}x}\left(\mathrm{d}y - \mathrm{d}x\right) = kx_0\frac{\mathrm{d}y - \mathrm{d}x}{\mathrm{d}x} = kx_0\left(y' - 1\right)\,,$$

where k is the sought stiffness of the original spring.

At the beginning of the selected section, this force has a magnitude $F(x)$, at the end it would be $F(x+\mathrm{d}x)$. The difference between these forces is compensated by the gravitational force

$$F(x+\mathrm{d}x) - F(x) = -\,\mathrm{d}F_\mathrm{g}\,.$$

This happens to look similar when compared with the derivative definiton, so we can write

$$\frac{F(x+\mathrm{d}x)-F(x)}{\mathrm{d}x} = F' = -\frac{\mathrm{d}F_\mathrm{g}}{\mathrm{d}x} = -\frac{GM}{(r-y)^2}\lambda\,.$$

We've already had written F explicitly, so we just need to calculate its derivative and obtain

$$(r-y)^2\,y'' = -\frac{GM\lambda}{kx_0} \quad\Rightarrow\quad z^2 z'' = -\frac{GM\lambda}{kx_0} = -K\,, \tag{8}$$

where we have introduced a new coordinate $z = y - r$ and a constant K to simplify it. We use a common trick

$$z'' = z'\frac{\mathrm{d}z'}{\mathrm{d}z} \quad\Rightarrow\quad z'\,\mathrm{d}z' = -Kz^{-2}\,\mathrm{d}z \quad\Rightarrow\quad \frac{z'^2}{2} = Kz^{-1} + C\,.$$

Now is the right time to think about boundary conditions. Apparently $y(0) = 0$ and $y_0 = y(x_0) = r - R$. If we define $z_0 = z(y_0) = y_0 - r$, $z_0 = -R$ holds true. Furthermore, at point y_0 the spring is no longer tensioned, so we can write $y'(x_0) = 0$. Then, $z' = y'$ and $z'(y_0) = y'(x_0) = 0$. We can determine the constant of integration from

$$0 = Kz_0^{-1} + C \quad\Rightarrow\quad C = -Kz_0^{-1} \quad\Rightarrow\quad z' = \sqrt{2K}\sqrt{z^{-1} - z_0^{-1}}\,.$$

Notice that z is in a range $z(0) = y(0) - r = -r$ to $z_0 = -R$. After reasoning, we can tell that $y' > 0$, so even $z' > 0$, therefore, z is an ever-increasing negative function. We verified that the square root is well defined and we can proceed further. We separate the equation again and rewrite in an integral form

$$\sqrt{2K}\int \mathrm{d}x = \int \left(z^{-1} - z_0^{-1}\right)^{-\frac{1}{2}}\mathrm{d}z = \int\sqrt{\frac{z}{1-\frac{z}{z_0}}}\,\mathrm{d}z = z_0\sqrt{-z_0}\int\sqrt{\frac{\zeta}{\zeta-1}}\,\mathrm{d}\zeta\,,$$

where we used a substitution $\zeta = z/z_0$ and factored the terms so that the argument of square root $\sqrt{-z_0}$ is positive. There is no nice antiderivative of this function; however, we do not need it to solve the problem. We figure out the limits of integration as we integrate from the beginning till the end of spring. Now, we can solve the integral numerically

$$I_0^{x_0} = \int_{\frac{r}{R}}^{1}\sqrt{\frac{\zeta}{\zeta-1}}\,\mathrm{d}\zeta \doteq -0.583\,.$$

To express K, we also need to integrate the other side of the equation

$$z_0\sqrt{-z_0}I_0^{x_0} = \sqrt{2K}\int_0^{x_0}\mathrm{d}x = \sqrt{2K}x_0\,,$$

$$K = \frac{1}{2}\left(\frac{z_0\sqrt{-z_0}I_0^{x_0}}{x_0}\right)^2 = \frac{1}{2}\left(\frac{-R^{\frac{3}{2}}I_0^{x_0}}{x_0}\right)^2 = \frac{R^3}{2}\left(\frac{I_0^{x_0}}{x_0}\right)^2.$$

Finally, we plug the sought stiffness into the expression for K (8) and the result is

$$k = \frac{GM\lambda}{Kx_0} = \frac{2GM\lambda x_0}{\left(I_0^{x_0}\right)^2 R^3} \doteq 6.9\,\mathrm{N{\cdot}m^{-1}}\,.$$

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Problem 56 ... foggy glass 9 points

Consider horizontal glass with thickness $d_\mathrm{g} = 16.3\,\mathrm{\mu m}$ and refractive index $n_\mathrm{g} = 1.68$. We spill water with refractive index $n_\mathrm{w} = 1.33$ on the glass in such a way that it forms a layer with uniform thickness $d_\mathrm{w} = 15.5\,\mathrm{\mu m}$. From the bottom, we shine a vertical beam of light with wavelength $\lambda = 0.590\,\mathrm{\mu m}$ on the glass. What is the ratio of the intensity of light which passes through in this situation to the intensity of light which would pass through if there was no water? Assume that light is only reflected or transmitted by the materials, not absorbed.

Jarda's window got foggy.

When a plane electromagnetic wave hits an interface, reflections and passage of waves occur. Since there are multiple interfaces in the problem, there will be several internal reflections and interference. The reflection and transmission coefficients depend on the angles of the wave direction relative to the interfaces, the mediums' refractive indexes, and the polarization of the incident light. In the case of perpendicular incidence, the Fresnel equations for the transmission coefficients $t_{s,p}$ for both polarizations have the same form. For the reflection coefficients, the equation $r_\mathrm{s} = -r_\mathrm{p}$ holds, thus, the two coefficients differ only in sign. These relations, which depend on the refractive indices of the two media, tell us what the intensity of the reflected and transmitted radiation will be.

Since we have three interfaces in the problem, it is better to work with the electric intensity E rather than directly with I. We first solve the problem for one of the polarizations (e.g., for s-type) and then assign a negative sign to all r-reflection coefficients to obtain an expression for the p-type polarization. Thus, for a perpendicular incidence, the reflection coefficient at the interface for the s-polarization is equal to

$$r = \frac{E_r}{E_\mathrm{i}} = \frac{n_1 - n_2}{n_1 + n_2}\,,$$

where E_i is the vector of the electric intensity of the incident wave and E_r of the reflected wave. The refractive index n_1 belongs to the medium where the incident wave travels, and n_2 is the refractive index behind the interface. In our case, we have three media whose refractive indexes are $n = 1$ for air, $n_g = 1.5$ for glass, and $n_w = 1.33$ for water. We can see that if the wave is reflected from a medium with a higher refractive index, there is a phase change due to the coefficient being negative.

The transmission coefficient t is established analogically and has the following form

$$t = \frac{E_\mathrm{t}}{E_\mathrm{i}} = \frac{2n_1}{n_1 + n_2}\,,$$

where E_t is the transmitted electric intensity.

Radiation passes from one medium to another, but part of it gets reflected. Some of it then reaches the other interface, where a certain part of the radiation passes through again and some is reflected. This reflected portion returns to the first interface, where, of course, another passage and reflection occurs. It is evident that a simple estimation of the amount of total radiation that passes through is impossible. Iterating these considerations, we could arrive at a summation of infinite series. This method can be used for plane-parallel plates (2 interfaces), but in our case, it is no longer wise. Let us look at it differently.

In the next section, we introduce the notation for each value of intensity. However, the phase of these intensities changes as they pass through the environment. For this, we will use complex labeling. We will subscript the intensities in each environment and put their phase at the point of origin equal to zero. The change in phase depends on the thickness of the glass and water as well as on their refractive indexes.

Let us denote by E_i an incident wave for which (contrary to the definitions in the previous paragraph) we set the phase at impact as zero. The wave with direction from the first medium to the second is E_{s1}, and the one in the opposite direction is E_{s2}. The wave going from the first interface to space is E_r. It is important to note that in our notation we will not be dealing only with a reflected wave, but also with parts of waves that have been reflected at other interfaces.

The wave going from the second interface to the third is E_{v1}, in the opposite direction E_{v2}. The wave passing through the entire double layer is E_t. Again, we must remember that these are the sum of all the partial waves that are formed by the infinite reflections at the interfaces.

Let us denote the coefficient of passage from air to glass t_{ag}, from glass to water t_{gw}, from water to air t_{wa}, from glass to air t_{ga}, and from water to glass t_{wg}. Then

$$t_\mathrm{ag} = \frac{2}{1+n_\mathrm{g}}, \quad t_\mathrm{gw} = \frac{2n_\mathrm{g}}{n_\mathrm{g}+n_\mathrm{w}}, \quad t_\mathrm{wa} = \frac{2n_\mathrm{w}}{n_\mathrm{w}+1}, \quad t_\mathrm{ga} = \frac{2n_\mathrm{g}}{n_\mathrm{g}+1}, \quad t_\mathrm{wg} = \frac{2n_\mathrm{w}}{n_\mathrm{g}+n_\mathrm{w}}\,.$$

Similarly, let us denote the reflection coefficient between air and glass r_ag, between glass and water r_gw, between water and air r_wa, between glass and air r_ga, and between water and glass r_wg (the first environment is the one in which the incident wave is at the interface). Then

$$r_\mathrm{ag} = \frac{1-n_g}{1+n_g} = -r_\mathrm{ga}, \quad r_\mathrm{gw} = \frac{n_g-n_w}{n_g+n_w} = -r_\mathrm{wg}, \quad r_\mathrm{wa} = \frac{n_w-1}{n_w+1}\,.$$

Let us now denote the phase shift between the air-glass and glass-water interface by δ_g which is equal to

$$\delta_\mathrm{g} = \frac{2\pi n_\mathrm{g} d_\mathrm{g}}{\lambda}\,,$$

where d_g is the thickness of the glass and *lambda* is the wavelength of light. By analogy, we introduce the phase shift δ_w in water.

We see that we must distinguish the order of the environments. Now we can start building relationships between the different electrical intensities. For E_g1

$$E_\mathrm{g1} = t_{ag}E_\mathrm{i} + r_{ga}E_\mathrm{g2}\mathrm{e}^{i\delta_\mathrm{g}}\,,$$

is the sum of the transmitted and reflected waves multiplied by their respective coefficients. The reflected wave E_g2 must be multiplied by the change in its phase. Then we have

$$E_\mathrm{g2} = r_\mathrm{gw}E_\mathrm{g1}\mathrm{e}^{i\delta_\mathrm{g}} + t_\mathrm{wg}E_\mathrm{w2}\mathrm{e}^{i\delta_\mathrm{w}}, \quad E_r = r_\mathrm{ag}E_\mathrm{i} + t_\mathrm{ga}E_\mathrm{g2}\mathrm{e}^{i\delta_\mathrm{g}}\,.$$

For waves in water

$$E_\mathrm{w1} = t_\mathrm{gw}E_\mathrm{g1}\mathrm{e}^{i\delta_\mathrm{g}} + r_\mathrm{wg}E_\mathrm{w2}\mathrm{e}^{i\delta_\mathrm{w}}, \quad E_\mathrm{w2} = r_\mathrm{wa}E_\mathrm{w1}\mathrm{e}^{i\delta_\mathrm{w}}, E_\mathrm{t} = t_\mathrm{wa}E_\mathrm{w1}\mathrm{e}^{i\delta_\mathrm{w}}\,.$$

If we consider that E_i is known, then we have just obtained a system of six equations with six unknowns. Our task is to express E_t in terms of E_i.

We have

$$E_\mathrm{t} = t_\mathrm{wa} E_\mathrm{w1} \mathrm{e}^{i\delta_\mathrm{w}} = t_\mathrm{wa} t_\mathrm{gw} \frac{E_\mathrm{g1} \mathrm{e}^{i\delta_\mathrm{w} + i\delta_\mathrm{g}}}{1 - r_\mathrm{wg} r_\mathrm{wa} \mathrm{e}^{i2\delta_\mathrm{w}}}$$

and

$$E_\mathrm{g1} - r_\mathrm{ga} \mathrm{e}^{i\delta_\mathrm{g}} \left(r_\mathrm{gw} \mathrm{e}^{i\delta_\mathrm{g}} + t_\mathrm{wg} r_\mathrm{wa} t_\mathrm{gw} \frac{\mathrm{e}^{i\delta_\mathrm{w} + i\delta_\mathrm{g}}}{1 - r_\mathrm{wg} r_\mathrm{wa} \mathrm{e}^{i2\delta_\mathrm{w}}} \right) E_\mathrm{g1} = t_\mathrm{ag} E_\mathrm{i} \,,$$

from where

$$E_\mathrm{t} = \frac{t_\mathrm{wa} t_\mathrm{gw} t_\mathrm{ag} \mathrm{e}^{i\delta_\mathrm{g} + i\delta_\mathrm{w}}}{\left(1 - r_\mathrm{wg} r_\mathrm{wa} \mathrm{e}^{i2\delta_\mathrm{w}}\right)\left(1 - r_\mathrm{ga} r_\mathrm{gw} \mathrm{e}^{i2\delta_\mathrm{g}}\right) - r_\mathrm{ga} t_\mathrm{wg} r_\mathrm{wa} t_\mathrm{gw} \mathrm{e}^{i2\delta_\mathrm{g} + i2\delta_\mathrm{w}}} E_\mathrm{i} \,.$$

We've got a symmetric expression, which is good news. In addition, the reflection coefficients r always occur as couples in products. Thus, if we replace all these numbers with negative ones (when we change the polarization from s to p), the result does not change because the negative signs are multiplied to positive ones. From this point on, we no longer need to distinguish between s and p polarizations. We now must calculate the square of the absolute value. The complex expression in the numerator will not play a role. However, we must multiply the denominator by its complex conjugate number.

In the denominator we substitute $a = r_\mathrm{wg} r_\mathrm{wa}$, $b = r_\mathrm{ga} r_\mathrm{gw}$ and $c = r_\mathrm{ga} t_\mathrm{wg} r_\mathrm{wa} t_\mathrm{gw}$. We then expand it and multiply the whole by its complex conjugated value. After the adjustment, we get

$$\begin{aligned} &1 + a^2 + b^2 + a^2 b^2 + c^2 \\ &-2\left(a\left(1 + b^2 - \frac{bc}{a}\right)\cos 2\delta_\mathrm{w} + b\left(1 + a^2 - \frac{ac}{b}\right)\cos 2\delta_\mathrm{g}\right) \\ &-2\left((c - ab)\cos(2\delta_\mathrm{g} + 2\delta_\mathrm{w}) - ab\cos(2\delta_\mathrm{w} - 2\delta_\mathrm{g}) + abc\right) \,. \end{aligned}$$

Now consider that $n_\mathrm{w} = n_\mathrm{a} = 1$, meaning that the air is straight behind the glass. Then, after plugging for a, b and c we have $a = 0$, $b = r_\mathrm{ga} r_\mathrm{ga} = \left(\frac{1-n_g}{1+n_g}\right)^2$ and $c = 0$. The square of the absolute value of the electric intensity is then

$$I_{tg} = \frac{(t_\mathrm{ga} t_\mathrm{ag})^2}{1 + (r_\mathrm{ga} r_\mathrm{ga})^2 - 2 r_\mathrm{ga} r_\mathrm{ga} \cos 2\delta_\mathrm{g}} I_i \,,$$

where in the numerator we no longer have $(t_\mathrm{wa})\, ani\, (t_\mathrm{gw})$, since there is no water.

By comparison we get the final result (in which we have neglected some small terms in the denominator)

$$\frac{I_\mathrm{tgw}}{I_\mathrm{tg}} = \frac{t_\mathrm{wa}^2 t_\mathrm{gw}^2 t_\mathrm{ga}^{-2}\left(1 + (r_\mathrm{ga} r_\mathrm{ga})^2 - 2 r_\mathrm{ga} r_\mathrm{ga} \cos 2\delta_\mathrm{g}\right)}{1 + a^2 + b^2 + c^2 - 2\left(a\left(1 - \frac{bc}{a}\right)\cos 2\delta_\mathrm{w} + b\left(1 - \frac{ac}{b}\right)\cos 2\delta_\mathrm{g} + (c - ab)\cos(2\delta_\mathrm{g} + 2\delta_\mathrm{w}) - ab\cos(2\delta_\mathrm{w} - 2\delta_\mathrm{g})\right)} \,,$$

$$\frac{I_\mathrm{tgw}}{I_{tg}} = 0.96 \,.$$

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Problem 57 ... Matěj's black warm box

7 points

Determine the number of photons in a sealed box of exact volume $1\,\ell$*, which is in thermal equilibrium in a room with a temperature of* $25.0\,\mathrm{K}$.

Karel was wondering what the head of Physics Brawl Online does when out of the public eye.

To determine the number of photons in the box, we need to find the number density. It could seem we only need to use the Stefan–Boltzmann law to determine the energy density, but it's not that simple because we don't know the mean energy of a single photon. We must, therefore, proceed from the first principles – Planck's law

$$B(T,\nu) = h\nu\frac{2\nu^2}{c^2}\frac{1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}}-1}\,,$$

where ν is the frequency of radiaton, T is the absolute temperature of black body, h is the Planck constant, c is the speed of light and k_B is the Boltzmann constant. The spectral radiance of a body B describes the spectral emissive power per unit area, per unit solid angle for particular radiation frequencies from $[\nu,\nu+\mathrm{d}\nu]$ frequency interval. Hence the energy density is

$$u = \frac{4\pi}{c}L = \frac{4\pi}{c}\int_0^\infty B(T,\nu)\,\mathrm{d}\nu\,.$$

The result of this integration is the Stefan–Boltzmann law. However, our goal is to determine the number density of photons, so we divide the integrand $B(T,\nu)$ by energy of a single photon $h\nu$

$$n = \frac{4\pi}{c}\int_0^\infty \frac{B(T,\nu)}{h\nu}\,\mathrm{d}\nu = \frac{4\pi}{c}\int_0^\infty \frac{2\nu^2}{c^2}\frac{1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}}-1}\,\mathrm{d}\nu\,.$$

The next step is to convert this integral containing physical quantities into a mathematical integral (to get a dimensionless integrand by a substitution – a frequently used method of converting a computational physical problem to a mathematical one). The most important is to have a simple variable in the argument of the exponential function, therefore, we use

$$\frac{h\nu}{k_\mathrm{B}T} = x \quad\Rightarrow\quad \nu = \frac{xk_\mathrm{B}T}{h}\,.$$

After this substitution we get

$$n = \frac{4\pi}{c}\int_0^\infty \frac{2\left(\frac{xk_\mathrm{B}T}{h}\right)^2}{c^2}\frac{1}{\mathrm{e}^x-1}\frac{k_\mathrm{B}T}{h}\,\mathrm{d}x = 8\pi\left(\frac{k_\mathrm{B}T}{ch}\right)^3\int_0^\infty \frac{x^2}{\mathrm{e}^x-1}\,\mathrm{d}x\,.$$

There would be x to the third power in the numerator in case of determining the energy. This integral can be solved numerically, or by integral definition of the Riemann zeta function

$$\int_0^\infty \frac{x^2}{\mathrm{e}^x-1}\,\mathrm{d}x = \zeta(3)\,\Gamma(3) = 2\zeta(3) \doteq 2.404\,.$$

The final equation we use to get the answer is

$$N = Vn = 2.404\cdot 8\pi\left(\frac{k_\mathrm{B}T}{ch}\right)^3\cdot V \doteq 3.170\cdot 10^8\,.$$

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Problem M.1 ... fast and even faster for the first time 3 points

Two cars travel the same distance s. Both are moving with constant acceleration. The first moves with acceleration a_1, and the second with acceleration $a_2 = 1.25\,a_1$. The initial velocity of both cars is zero. How fast will the second car travel the required distance? Enter the result into the system as a ratio of the second car's time to the first car's time.

Karel varied the problems.

We will start with the known equation for accelerated motion, which is very similar for both cars

$$s = \frac{1}{2}a_1 t_1^2 = \frac{1}{2}a_2 t_2^2\,,$$

where t_1 and t_2 is the time it takes the first and the second car, respectively to travel the distance s. This way we immediately obtained an equation from which we express t_2/t_1

$$\frac{t_2}{t_1} = \sqrt{\frac{a_1}{a_2}} = \sqrt{\frac{1}{1.25}} \doteq 0.894\,.$$

The correct answer is that the second car travels the same distance 0.894 times faster than the first one. We can also note that even if we change the acceleration of the second car by a quarter, the time will shorten just by 0.106 times the time it took the first car.

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Problem M.2 ... fast and even faster for the second time 3 points

We have two cars, and we want them to travel the distance s in the same time t. During their journey, they will both have constant acceleration in the direction of travel. The first car will travel with an acceleration of a_1, whereas the second car will have an acceleration of $a_2 = 1.250\,a_1$. The second car starts its movement from the rest. With what initial velocity v_0 must the first car start? Give the result as a multiple of $a_1 t$ (enter only the number by which you multiply these quantities). *Karel varied the problems for the second time.*

Both cars have to travel the same distance, so

$$s = \frac{1}{2}a_1 t^2 + v_0 t = \frac{1}{2}a_2 t^2\,.$$

From the equation, we can simply express the velocity

$$v_0 t = \frac{1}{2}a_2 t^2 - \frac{1}{2}a_1 t^2 = \frac{1}{2}\left(a_2 - a_1\right) t^2\,,$$
$$v_0 = \frac{1}{2}\left(a_2 - a_1\right) t = \frac{1}{2}\cdot 0.250\,a_1 t = 0.125\,a_1 t\,.$$

The initial velocity expressed using the acceleration of the first car and total time is $v_0 = 0.125\,a_1 t$.

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Problem M.3 ... fast and even faster for the third time 3 points

Two cars have to travel the same distance s. Both will be moving with the constant jerk, which is a change in acceleration over time (analogous to acceleration being a change in velocity). The first car will travel with jerk g_1 and the second with jerk $g_2 = 1.25\, g_1$. The initial velocities and accelerations of both cars are zero. How fast will the second car travel the required distance? Enter the result into the system as the ratio of the second car's time to the first.

Karel varied the problems for the third time.

This is a simple application of the jerk if we know the formula for position versus time (which can be found on the internet, but with a little more effort) or we derive it by integration easily. Knowing that the jerk is constant, the following holds for acceleration

$$a_i = \int_0^{t_i} g_i \,\mathrm{d}\tilde{t} = g_i t_i \,,$$

where i can be 1 or 2 – depending on whether we want the formula for the first or the second car, and a is acceleration. We proceed with the integration for the velocity v and the position s, remembering that we have initial conditions for velocity and acceleration equal to zero.

$$v_i = \int_0^{t_i} a_i \,\mathrm{d}\tilde{t} = \int_0^{t_i} g_i \tilde{t} \,\mathrm{d}\tilde{t} = \frac{1}{2} g_i t_i^2 \,,$$
$$s = \int_0^{t_i} v_i \,\mathrm{d}\tilde{t} = \int_0^{t_i} \frac{1}{2} g_i \tilde{t}^2 \,\mathrm{d}\tilde{t} = \frac{1}{6} g_i t_i^3 \,.$$

The last equation can be rewritten as an equation for both cars, and very quickly, we get to the result

$$\frac{1}{6} g_1 t_1^3 = \frac{1}{6} g_2 t_2^3 \qquad \Rightarrow \qquad \frac{t_2}{t_1} = \sqrt[3]{\frac{g_1}{g_2}} = \sqrt[3]{\frac{1}{1.25}} \doteq 0.928 \,.$$

The second car will finish in a time 0.928 times shorter than the first one.

We could reach the same result by logical reasoning about the analogy with accelerated motion. For motion with constant acceleration, the acceleration does not depend on time, the velocity depends on time linearly and the position quadratically. If we have a constant jerk, we will be adding powers of the time again. The acceleration will depend on time linearly, the velocity quadratically, and the position cubically. This simplified reasoning will not give us the factor 1/6, but we do not need to know this factor if we only want the ratio of the times of the two cars as the result.

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Problem M.4 ... fast and even faster for the fourth time 3 points

Two cars have to travel the same distance s. Both will move with constant power. The power of the first car will be P_1, and the power of the second car will be $P_2 = 1.25\, P_1$. The initial velocity of both cars is zero, and they have the same mass, including the load. How fast will the second car travel the required distance? Enter the result into the system as the ratio of the second car's time to the first.

Karel varied the problems for the fourth time.

We start with the formula for the kinetic energy $E_{k,i}$ of the motion of the ith car

$$E_{k,i} = \frac{1}{2} m v_i^2 = P_i t_i \,,$$

where m is the mass (without the index because it is the same for both cars), v_i is the velocity of ith car, and t_i is the travel time. If we express the velocity, we get its dependence on time and other parameters, which we consider to be constant with time

$$v_i = \sqrt{\frac{2P_i t_i}{m}} \,.$$

However, we need to know the time dependence of the distance traveled. Therefore, we integrate the formula for the velocity

$$s = \int_0^{t_i} v_i \, \mathrm{d}\tilde{t} = \int_0^{t_i} \sqrt{\frac{2P_i \tilde{t}_i}{m}} \, \mathrm{d}\tilde{t} = \frac{2}{3} \sqrt{\frac{2P_i t_i^3}{m}} \,.$$

The distance traveled is the same for both cars, so we set up the following equation

$$\frac{2}{3} \sqrt{\frac{2P_1 t_1^3}{m}} = \frac{2}{3} \sqrt{\frac{2P_2 t_2^3}{m}} \,,$$

$$P_1 t_1^3 = P_2 t_2^3 \,,$$

$$\frac{t_2}{t_1} = \sqrt[3]{\frac{P_1}{P_2}} \doteq 0.928 \,.$$

The second car will travel the distance in 0.928 times the first car's time when having 25 % more power. Coincidentally, this is the same ratio of times as in the "fast and even faster for the third time" problem, where you were comparing two cars with a constant jerk. This holds even though the time dependencies are different.

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Problem E.1 ... light circle

4 points

A thin convex lens is placed at a distance of 7.0 cm *from the screen. We then turn on a distant light source located on its optical axis. An illuminated circle appears on the screen behind the lens. After moving the lens to a distance of* 9.0 cm *from the screen, we notice that the circle on the screen now has the same radius as before. What is the focal length of the lens?*

Jarda saw the opposite of the eclipse.

While the source is far from the lens, all the rays passing through the lens converge at its focus. They form a cone after passing through the lens. If the screen is between the lens and its focus, the intersection of the cone and screen forms a circle. In this case, let the distance between the focus and the screen be x_1. However, a circle of the same radius will appear when the screen is behind the focus and the rays passing through the focus begin to diverge again. Now let there be a distance x_2 between the screen and the focus. Since the radii are equal, $x_1 = x_2$.

Moreover, in the first case $x_1 = f - d_1$, where $d_1 = 7\,\mathrm{cm}$, and in the second case $x_2 = d_2 - f$ (where $d_2 = 9\,\mathrm{cm}$). Since these distances are equal, we obtain

$$f = \frac{d_1 + d_2}{2} = 8\,\mathrm{cm}\,.$$

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Problem E.2 ... planparallel lens 3 points

A thin convex lens with radii of curvature $R_1 = 20\,\mathrm{cm}$ and $R_2 = 35\,\mathrm{cm}$ is cut perpendicular to its optical axis. The two halves are placed with the convex sides close to one another (the flat surfaces are facing away from each other). What is the ratio between the old and the new focal lengths of this arrangement of glasses? The refractive index of the glass is $n = 1.5$.

According to Jarda, shape does not matter.

Let's look at the passage of a beam through two lenses that are next to each other. Let a be the distance of an object from the first lens with focal length f_1. Just behind it is the second lens with a focal length f_2. Using the thin lens equation, we find the position of the image a' of the object as

$$a' = \frac{af_1}{a - f_1}\,.$$

This distance is positive if the object is imaged behind the first lens. We project the image through the second lens, with the object distance now equal to $-a'$. The second lens displays the object at

$$a'' = \frac{-a'f_2}{-a' - f_2} = \frac{-\left(\frac{af_1}{a-f_1}\right)f_2}{-\frac{af_1}{a-f_1} - f_2} = \frac{af_1f_2}{af_1 + af_2 - f_1f_2} = \frac{aF}{a - F}\,,$$

where $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$. Thus, we derived the interesting relation that the total optical power of two lenses that are close to each other is equal to the sum of the optical powers of the two lenses.

Now note that neither f_1 nor f_2 depends on the rotation of the lens. It is also important for this problem that we have derived that F does not depend on the order of the lenses. Thus, in both cases from the problem statement, F is the same and is equal to the focal length of the original lens. Thus, in both cases from the problem statement, F is the same and equal to the original lens's focal length. The ratio we were looking for is, therefore 1.

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Problem E.3 ... two magnifiers 4 points

Jarda found two magnifying glasses at home and started playing with them. When he put them close together, he discovered they had a focal length of $F = 7\,\mathrm{cm}$. He then placed the lenses at a suitable distance apart, stood far behind them, and looked at a distant tree. He saw it sharply and its magnitude was 1.5-times larger than without the lenses. What is the difference *in the focal lengths of the two lenses? Subtract the smaller value from the larger. Think of the magnifying glass as a thin lens.*

Jarda lives so high that he can't really see the ground from the window.

In the previous problem, we derived that for two lenses that are placed closely together, applies $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$, where f_1 and f_2 are the focal lengths of the two lenses and F is the focal length of the system we know from the problem.

Conversely, if Jarda looked at a distant tree through two lenses and saw it sharply, he built a Kepler telescope. For him, the foci of the two lenses are at the same point, so the lenses are $f_1 + f_2$ apart. The magnification of such a telescope is then $Z = \frac{f_1}{f_2}$ for $f_1 > f_2$.

From the two equations with two unknowns, we can calculate f_1 and f_2 as

$$f_1 = F\left(1 + Z\right) ,$$
$$f_2 = F\left(1 + \frac{1}{Z}\right) .$$

Since $Z > 1$, then $f_1 > f_2$, so we are interested in the result $f_1 - f_2$, which is

$$f_1 - f_2 = F\left(Z - \frac{1}{Z}\right) = 5.8\,\mathrm{cm} .$$

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Problem E.4 ... recording ants 8 points

We place two lenses and a screen on an optical axis. One lens, with focal length $f_1 = 3.6\,\mathrm{cm}$, is located $d = 8.3\,\mathrm{cm}$ in front of the screen. The second lens with focal length $f_2 = -1.6\,\mathrm{cm}$ is located between the first lens and the screen. An ant is initially located $a_0 = 9.7\,\mathrm{cm}$ in front of the first lens and is walking along the optical axis towards the lenses and the screen with velocity $v = 0.6\,\mathrm{cm{\cdot}s^{-1}}$. The image of the ant on the screen is initially in focus. What should be the velocity of the second lens if we want the image to stay in focus for the next moment during the walk? The sign of this velocity should be positive if the lens is moving in the same direction as the ant.

Jarda likes to take photos of insects.

Let's denote the position of the ant by $a = a_0 - vt$. The position of the focused image formed by the first lens is

$$a' = \frac{af_1}{a - f_1} .$$

The distance of this image from the second lens is

$$a'' = d - x - a' ,$$

where $x = x_0 + ut$ and x_0 is the distance of the second lens from the screen at the time $t = 0\,\mathrm{s}$.

The final focused image is formed by the second lens when we treat the image formed by the first lens as the object. The distance of the final image from the second lens is

$$a''' = \frac{a'' f_2}{a'' - f_2} = \frac{\left(d - x - a'\right) f_2}{d - x - a' - f_2} = x ,$$

where the last equality describes that we see the ant on the screen in focus. We solve this quadratic equation to find

$$x = \frac{-a' + d \pm \sqrt{a'-d}\sqrt{a'-d+4f_2}}{2}\,.$$

Then, we find

$$x_0 = \frac{1}{2}\left(-\frac{a_0 f_1}{a_0 - f_1} + d + \sqrt{\frac{a_0 f_1}{a_0 - f_1} - d}\sqrt{\frac{a_0 f_1}{a_0 - f_1} - d + 4f_2}\right) \doteq 3.69\,\mathrm{cm}\,,$$

where we chose the positive root according to the problem statement. Let's not fear the expressions under the square roots - they're both negative, so if we put them under a common square root, we end up taking the square root of a positive number.

Next, let's plug time-dependence into the formula for x. We're only dealing with the next moment of the walk, so we consider a very small value of t, for which $a_0 \gg vt$. Then

$$\begin{aligned} a' &= \frac{(a_0 - vt)\, f_1}{a_0 - vt - f_1} = \frac{(a_0 - vt)\, f_1}{\left(1 - \frac{vt}{a_0 - f_1}\right)(a_0 - f_1)} \\ &\approx \frac{(a_0 - vt)\, f_1}{(a_0 - f_1)}\left(1 + \frac{vt}{a_0 - f_1}\right) \approx \frac{a_0 f_1}{(a_0 - f_1)} + \frac{v f_1^2}{(a_0 - f_1)^2} t\,. \end{aligned}$$

For clarity, let's denote $b' = a'(t = 0\,\mathrm{s}) = \frac{a_0 f_1}{(a_0 - f_1)}$. Now we find the approximation

$$\sqrt{a'-d} = \sqrt{b'-d}\sqrt{1 + \frac{1}{b'-d}\frac{vb'^2}{a_0^2}t} \approx \sqrt{b'-d}\left(1 + \frac{1}{b'-d}\frac{vb'^2}{2a_0^2}t\right)\,.$$

Similarly, we find the approximation for the second square root

$$\begin{aligned} \sqrt{a'-d+4f_2} &= \sqrt{b'-d+4f_2}\sqrt{1 + \frac{1}{b'-d+4f_2}\frac{vb'^2}{a_0^2}t} \\ &\approx \sqrt{b'-d+4f_2}\left(1 + \frac{1}{b'-d+4f_2}\frac{vb'^2}{2a_0^2}t\right)\,. \end{aligned}$$

Again, for clarity, let's denote in the next steps $s_1 = \sqrt{b'-d}$ and $s_2 = \sqrt{b'-d+4f_2}$. Substituting into the formula for x and approximating, we get

$$\begin{aligned} x &= \frac{1}{2}\left[-b' - \frac{vb'^2}{a_0^2}t + d \pm s_1 s_2\left(1 + \frac{vb'^2}{2s_1^2 a_0^2}t\right)\left(1 + \frac{vb'^2}{2s_2^2 a_0^2}t\right)\right] \\ &\approx \frac{1}{2}\left\{-s_1^2 - \frac{vb'^2}{a_0^2}t \pm s_1 s_2\left[1 + \frac{vb'^2}{2a_0^2}\left(\frac{1}{s_1^2} + \frac{1}{s_2^2}\right)t\right]\right\}\,. \end{aligned}$$

Finally, we can express

$$x - x_0 = \frac{1}{2}\frac{vb'^2}{a_0^2}\left(\frac{s_1^2 + s_2^2}{2s_1 s_2} - 1\right)t\,.$$

The velocity of the lens is

$$u = \frac{x - x_0}{t} = \frac{1}{2} \frac{v f_1^2}{(a_0 - f_1)^2} \left(\frac{\frac{a_0 f_1}{a_0 - f_1} - d + 2 f_2}{\sqrt{\frac{a_0 f_1}{a_0 - f_1} - d}\sqrt{\frac{a_0 f_1}{a_0 - f_1} - d + 4 f_2}} - 1 \right) \doteq -0.23\,\mathrm{cm{\cdot}s^{-1}}\,.$$

Note that in this fraction, the numerator contains the arithmetic mean, and the denominator contains the geometric mean of the values $\frac{a_0 f_1}{a_0 - f_1} - d$ and $\frac{a_0 f_1}{a_0 - f_1} - d + 4 f_2$.

We still need to check in which direction the second lens moves, which determines the sign of the result. We defined its position x as its distance from the screen and the velocity u is then positive when moving away from the screen. Negative velocity u means that the second lens moves toward the screen. The ant is also moving toward the screen. The lens is moving in the same direction as the ant, so the result is $0.23\,\mathrm{cm{\cdot}s^{-1}}$.

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Problem X.1 ... we are cleaning up the garden 4 points

Autumn has come, and the leaves in our garden have started to fall. After raking them, we need to take them to a nearby pile, so we scoop them on a garden wheelbarrow and head towards the pile at a constant speed $v = 1.2\,\mathrm{m{\cdot}s^{-1}}$. We are pushing the wheelbarrow in front of us on a pavement made of interlocking paving stones, so there is always a small hole between the 15 cm *tiles, where the speed of the wheelbarrow is reduced by* 10 %. *Determine the constant horizontal component of the force with which we need to push the wheelbarrow forward to make its average speed also v. The wheelbarrow together with the leaves has a mass of* 19 kg.

Many leaves have fallen on Jarda's greenhouse.

The average speed of the wheelbarrow is the same as our walking speed, i.e. v. These speeds are constant, but the wheelbarrow periodically decreases its momentum. We have to compensate for this by applying a force to keep it at the average speed v.

We find the force F using the well-known relation

$$F = \frac{\Delta p}{\Delta t}\,,$$

where Δp is the change in momentum over time Δt. This time period corresponds to $\Delta t = \frac{l}{v}$, where l is the size of one tile.

On each tile the wheelbarrow loses momentum $\Delta p = m \Delta v$, where m is its mass and Δv is the change in speed. Let $v_{\max}$ denote the maximum speed of the wheelbarrow, and $v_{\min}$ its minimum speed. Then $\Delta v = v_{\max} - v_{\min}$ holds. At the same time the speed of the wheelbarrow is changing linearly in time (when it is not losing it on the edges of the tiles, so the average speed v has to be equal to the average

$$v = \frac{v_{\max} + v_{\min}}{2}\,.$$

We rewrite the last important piece of information in the problem statement as $v_{\min} = (1 - \eta)\, v_{\max} = 0.9\, v_{\max}$, where $\eta = 0.1$ is the ten-percent loss of the speed.

Then $v_{\max} = \frac{2v}{2-\eta}$ and $\Delta v = \eta v_{\max}$. By substituting into the original equation for the force, we get

$$F = \frac{mv\Delta v}{l} = \frac{2mv^2\eta}{l\,(2-\eta)} = 19.2\,\mathrm{N}\,.$$

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Problem X.2 ... watering the garden — 4 points

Jarda planted six fruit bushes next to a garden pond with a surface area of $A = 8\,\mathrm{m^2}$. These stand in a row, the first being $2.3\,\mathrm{m}$ from the pond and every other $1.0\,\mathrm{m}$ further than the previous one. Jarda always waters them using his ten-liter watering can that he refills at the pond. The total time it takes him to refill and pour one jug is $45\,\mathrm{s}$. Jarda walks at a speed of $v = 1.2\,\mathrm{m{\cdot}s^{-1}}$. In the meantime, the garden hose delivers water to the pond with a flow rate of $Q = 0.9\,\mathrm{l{\cdot}s^{-1}}$. After watering the last plant, he went to see how much the level of the pond had risen since he had started filling the first watering can. How much did he measure? If the level has dropped, submit a negative value. *Jarda found currants on sale.*

Let's determine the time it will take Jarda to water the garden. Let $N = 6$ be the number of bushes and $t_{\mathrm{m}} = 45\,\mathrm{s}$ be the time it takes to pour and refill the watering can, then the total time to handle the watering can is $t_{\mathrm{k}} = Nt_{\mathrm{m}}$.

The time it takes Jarda to walk back and forth is given by

$$t_{\mathrm{ch}} = 2\frac{1}{v}\left(\sum_{n=0}^{N-1} d_0 + nd\right) = \frac{2}{v}\left(Nd_0 + \frac{N\,(N-1)}{2}d\right)\,,$$

where $d_0 = 2.3\,\mathrm{m}$ is the distance of the first bush from the pond and $d = 1.0\,\mathrm{m}$ is the distance between the bushes.

The total watering time is

$$t = N\left(t_{\mathrm{m}} + 2\frac{d_0}{v} + d\frac{N-1}{v}\right)\,.$$

During this time, a volume of water Qt flows from the garden hose into the pond. However, Jarda has filled N gardening cans of volume $V_{\mathrm{k}} = 10\,\mathrm{l}$, so the increment in the volume of water in the pond is $\Delta V = Qt - NV_{\mathrm{k}}$.

The level has risen by

$$h = \frac{\Delta V}{A} = QN\frac{\left(t_{\mathrm{m}} + 2\frac{d_0}{v} + d\frac{N-1}{v} - \frac{V_{\mathrm{k}}}{Q}\right)}{A} = 2.83\,\mathrm{cm}\,.$$

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Problem X.3 ... winding the cable

6 points

Jarda mows his garden with an electric mower. Once he has finished, he has to wind the entire cable of $L = 20\,\mathrm{m}$ onto the cable reel. It is wound onto a cylinder with a diameter $D = 40\,\mathrm{cm}$ whose axis of symmetry is horizontal. The cable attachment on the reel is close to the ground. The coefficient of friction between the ground and the cable is $f = 0.7$, and the length density of the cable is $\lambda = 220\,\mathrm{g\cdot m^{-1}}$. Determine the amount of work that Jarda will do when winding.

Jarda was laying the cable.

The frictional force acting on the cable is proportional to its length x, which remains in contact with the ground throughout the winding. Then

$$F = fmg = fgx\lambda\,,$$

where $m = \lambda x$ is the mass of the unwound cable and g is the gravitational acceleration.

The work to overcome the frictional force is found by using the integral as

$$W_\mathrm{F} = \int_0^L F\,\mathrm{d}x = fg\lambda \int_0^L x\,\mathrm{d}x = fg\lambda \left[\frac{x^2}{2}\right]_0^L = \frac{fg\lambda L^2}{2}\,.$$

Next, we raised the center of gravity of the entire cable by coiling. It is wound on a drum of diameter D, so the number of turns of the cable around it is $\frac{L}{\pi D} \doteq 15.9$. The center of gravity of the cable can thus be placed in the center of the reel, because we wrap a lot of turns, and the displacement of the center of gravity due to the incompletion of the last turn can be neglected. So we've done the work

$$W_\mathrm{g} = mgh = \lambda L g h\,,$$

where $h = \frac{D}{2}$ is the height of the center of the reel (and therefore the center of gravity of the cable) above the ground. The total work done corresponds to

$$W = W_\mathrm{F} + W_\mathrm{g} = \lambda g L \left(\frac{D}{2} + \frac{fL}{2}\right) = 310\,\mathrm{J}\,.$$

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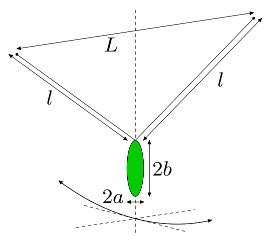
Problem X.4 ... swinging cucumber

6 points

A cucumber in Jarda's greenhouse has the shape of an elongated ellipsoid with semi-axes $a = 3\,\mathrm{cm}$ and $b = 10\,\mathrm{cm}$ and hangs in the middle of a stem of length $2l = 90\,\mathrm{cm}$. The ends of the stem are distanced $L = 51\,\mathrm{cm}$ from each other and the weight of the cucumber is $m = 450\,\mathrm{g}$. Determine the period of the small oscillations around the equilibrium position when the deflection is perpendicular to the plane of the suspension. *Jarda likes cucumbers.*



To find the period of the small oscillations we use the well-known relation for the physical pendulum

$$T = 2\pi\sqrt{\frac{J}{mgd}}\,,$$

where J is the moment of inertia of the oscillating body with respect to the axis of rotation, m is the mass of the body, g is the gravitational acceleration, and d is the distance of the center of gravity from the rotation axis.

The axis of rotation runs through the attachment points of the stems. The center of gravity of the cucumber is located at its center, so from the geometry of the problem we calculate the distance d as

$$d = b + \sqrt{l^2 - \frac{L^2}{4}} = 47\,\mathrm{cm}\,,$$

where $b = 10\,\mathrm{cm}$, $L = 51\,\mathrm{cm}$ and $l = 45\,\mathrm{cm}$ are the distances from the problem statement.

The moment of inertia of the cucumber with respect to this axis is found using Steiner's theorem as

$$J = J_T + md^2\,,$$

where J_T is the moment of inertia with respect to the axis passing through the object's center of gravity. For the ellipsoid in this particular geometry, we define it as $J_T = \frac{1}{5} m \left(a^2 + b^2\right)$.

After substituting into the original relation, we obtain

$$T = 2\pi\sqrt{\frac{J}{mgd}} = 2\pi\sqrt{\frac{\frac{a^2+b^2}{5} + \left(\sqrt{l^2 - \frac{L^2}{4}} + b\right)^2}{g\left(\sqrt{l^2 - \frac{L^2}{4}} + b\right)}} = 1.38\,\mathrm{s}\,.$$

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Solutions of Physics Brawl Online 2023

PhysicsBrawlOnline

Problem 1 ... rotating a force 3 points

There are two forces $F_1 = 2.0\,\mathrm{N}$ and $F_2 = 1.0\,\mathrm{N}$ acting on a point of mass. What is the angle between them if their resultant is the same magnitude as the larger of the forces, $F = F_1$?

May the Force be with you!

It is essential to map out the situation well.

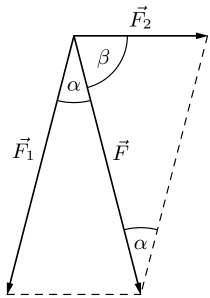

Figure 1: Force decomposition.

From the picture, we can see that we get two triangles with known side lengths. The angle we are looking for is $\alpha + \beta$. Since the triangles are isosceles, $\alpha + 2\beta = 180^\circ$, so we just need to calculate α because $\alpha + \beta = \alpha/2 + 90^\circ$.

We calculate the angle α using the law of cosines as

$$\cos\alpha = \frac{F_1^2 + F^2 - F_2^2}{2F_1F} = \frac{7}{8}\,.$$

From that, we get

$$\alpha/2 + 90^\circ = \arccos\left(\frac{7}{8}\right)/2 + 90^\circ = 104.5^\circ\,.$$

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Problem 2 ... five-second rule 3 points

You may have heard that if you drop your food on the floor but pick it up within five seconds, it will not be heavily contaminated with bacteria. Let us consider the following case. You drop a circular snack with a diameter of $4\,\mathrm{cm}$ on the ground. The bacteria from the floor will immediately stick to it. However, according to the common rule, this should not matter. Therefore, let us assume that most of the bacteria only arrive at the food from the vicinity of the snack during those five seconds. What would their velocity have to be for their numbers to multiply tenfold on a snack in five seconds? The surface density of bacteria on the ground is homogeneous. *Jarda always blows off the fallen food so that he can eat with a peace in mind.*

Let us assume that bacteria are a smart spieces, and as soon as the food falls on the ground, they instantly start moving towards it. Within five seconds, the bacteria can reach the snack from a distance $vt + r$, where v is their speed, $t = 5\,\mathrm{s}$ and $r = 2\,\mathrm{cm}$ is the radius of the snack.

In the beginning, $n_1 = \sigma\pi d^2/4$ of bacteria is stuck on the snack, where d is the diameter of the snack and σ is the surface density of bacteria on the floor, which is considered constant in the surroundings of the fallen food. To increase the number of bacteria tenfold, the area of the circle must increase by a factor of ten, which corresponds to the radius

$$R = \sqrt{10}r = vt + r\,,$$

from where the required bacteria speed is

$$v = \frac{R - r}{t} = \left(\sqrt{10} - 1\right)\frac{d}{2t} = 8.6\,\mathrm{mm\cdot s^{-1}}\,,$$

which is unrealistically high compared to their usual speed, which is at most in the order of tens of micrometers per second. Moreover, we must point out that the five-second rule has never been experimentally proven.

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Problem 3 ... game of tag 3 points

Two cars are driving on a road parallel to each other in the same direction. The trajectories of the cars are $\xi = 1.5\,\mathrm{m}$ apart. Nicolas is $d = 3\,\mathrm{m}$ away from the trajectory of the first car, which is moving at $v_1 = 55\,\mathrm{km\cdot h^{-1}}$. What is the velocity of the second car if it always stays hidden behind the first car from Nicolas's point of view? We approximate the cars as point masses. *Nicolas waited for far too long at the bus stop.*

We will solve this problem using the geometry in the figure.

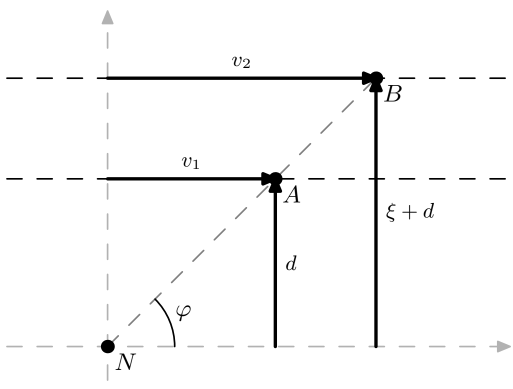

Figure 2: Sketch of the situation.

In the figure, instead of using the length of the sides, we have used vectors to represent the change in position of the point masses compared to the origin. We have set the origin at the point where Nicolas is standing, and we will use symmetry only for the case where both cars were directly in front of Nicolas at time t_0 and they have moved over the distance $s = v \cdot (t_1 - t_0)$, at t_1 to points A and B.

Consequently, we can notice that the triangles are similar based on the AA (Angle-Angle) theorem about the similarity of triangles. The similarity of the triangles is due to the angle φ and the right angle to the x-axis. The similarity of triangles means that the ratios of the hypotenuses are the same,[1] and thus we have

$$\frac{v_1 \cdot (t_1 - t_0)}{d} = \frac{v_2 \cdot (t_1 - t_0)}{d + \xi}$$

$$v_2 = \frac{d + \xi}{d} v_1 \,.$$

After substituting the values $v_1 = 55\,\mathrm{km{\cdot}h^{-1}}$, $\xi = 1.5\,\mathrm{m}$, $d = 3\,\mathrm{m}$, we got the value of $v_2 \doteq \doteq 83\,\mathrm{km{\cdot}h^{-1}}$.

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Problem 4 ... changing the transmission lines 3 points

We have a transmission line with high voltage $U_0 = 110\,\mathrm{kV}$, and we would like to increase it to extra-high voltage $U_1 = 400\,\mathrm{kV}$. Assuming the resistance of the line is constant, how much will the power losses on the line change? We are interested in the power loss ratio P_1/P_0.

Karel thought about changing the transmission lines.

We know that Ohm's law for a circuit or part of it says $U = RI$. Moreover, we can calculate the electric power P as $P = UI$. If we modify the formula by substituting the current from Ohm's law, we get

$$P = \frac{U^2}{R} \,.$$

So far, we have written the equation in general. Now, let us add the indices 0 and 1. Since the resistance remains constant, we will leave it without indices. Putting the indexed relations into the ratio, we easily get the result

$$\frac{P_1}{P_0} = \frac{\frac{U_1^2}{R}}{\frac{U_0^2}{R}} = \frac{U_1^2}{U_0^2} \doteq 13.2 \,.$$

Losses increase to 13.2 times the original value. In reality, the losses would probably increase more because with more power dissipation, the conductor temperature would stabilize at a higher temperature at which the conductor would have more resistance.

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[1] This stems from the fact that we can express the ratio of the hypotenuses in both triangles using the trigonometric function tangent.

Problem 5 ... swimming problem

3 points

Verča would like to go swimming, but she dislikes going into the water without eating. She has an average density of $945\,\mathrm{kg{\cdot}m^{-3}}$ *when she is hungry, and it is hard for her to dive. How many kilograms of food does Verča need to eat to have an average density of at least* $980\,\mathrm{kg{\cdot}m^{-3}}$*? Assume that she does not change her volume when she eats. The length of the pool is* $30\,\mathrm{m}$ *and its depth is* $3.1\,\mathrm{m}$*. Verča's weight before the meal is* $47\,\mathrm{kg}$*.*

Verča sometimes feels very empty inside.

First, we express the volume of Verča V. To do this, we use the information that before eating, her average density is ρ_0 and her mass is m_0

$$V = \frac{m_0}{\rho_0}\,.$$

Her volume does not change, while her average density must increase, hence

$$\frac{m_0}{\rho_0} = \frac{m_0 + \Delta m}{\rho_1}\,.$$

After simple algebraic manipulation, we get

$$\Delta m = m_0\left(\frac{\rho_1}{\rho_0} - 1\right) = 1.7\,\mathrm{kg}\,.$$

Thus, Verča has to eat $1.7\,\mathrm{kg}$ of food. The parameters of the pool were not needed to solve the problem.

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Problem 6 ... slide

3 points

The management of the Dormitories and Refectories decided to spend money meaningfully, so they built a slide from the roof of the "building A" of the 17th November dormitories directly to the door of the MFF's Impakt pavilion. The two buildings are $430\,\mathrm{m}$ *apart as the crow flies, and their height difference is* $59\,\mathrm{m}$*. What is the coefficient of friction of the slide if a student weighing* $60\,\mathrm{kg}$ *has thrown himself down it with a velocity* $v_0 = 8.5\,\mathrm{m{\cdot}s^{-1}}$ *and just stops at the end of it? Assume that the slide is an inclined plane.*

Eliška was late for a lecture.

For the resultant force acting on the student on the slide applies $F = F_{\parallel} - F_t$, where $F_{\parallel}$ is the force that accelerates the student down the slide and is parallel to the plane of the slide, and conversely F_t is the frictional force that acts against the force $F_{\parallel}$. Moreover, the resultant force is constant, and thus, the acceleration must also be constant.

The student is also subject to a component of the gravitational force $F_{\perp}$ in a direction perpendicular to the slide surface, which is compensated by the reaction of the slide R (also perpendicular to its plane). The forces $F_{\perp}$ and $F_{\parallel}$ are components of the gravitational force F_g. There is an angle α between F_g and $F_{\parallel}$, which also corresponds to the angle of inclination of the slide, and we calculate it as $\alpha = \arctan(59\,\mathrm{m}/(430\,\mathrm{m}))$.

Because the student started running, he began to slip at v_0. He stopped with $v_1 = 0\,\mathrm{ms^{-1}}$ at the end of the slide. For the acceleration, we have the relation

$$a = \frac{v_1 - v_0}{t} = \frac{-v_0}{t}\,.$$

The distance of uniformly accelerated motion is the same as that of uniformly decelerated motion, allowing us to calculate it as

$$s = \frac{1}{2}at^2 = \frac{1}{2}v_0 t\,,$$

and if we express the time, we get

$$t = \frac{2s}{v_0}\,.$$

The distance s can be found from the knowledge of the horizontal and vertical dimensions of the slide as $s = l/\cos\alpha$, where $l = 430\,\mathrm{m}$ is the distance from the dormitories to the CUNI MFF's Impakt building.

After substituting in the equation for acceleration, we get

$$a = \frac{-v_0}{\frac{2s}{v_0}} = -\frac{v_0^2}{2s}\,.$$

We obtain $F_\perp = F_g\cos\alpha$ for the normal force, and $F_t = fF_\perp = fF_g\cos\alpha$ for the friction force. Finally, we use $F_\| = F_g\sin\alpha$ and $F = ma$. We get

$$\begin{aligned} F &= F_\| - F_t\,,\\ ma &= mg\sin\alpha - fmg\cos\alpha\,. \end{aligned}$$

After some manipulation (we can notice that mass is irrelevant), we get

$$f = \frac{v^2}{2gl} + \tan\alpha\,,$$

and after inserting the numeric values, we obtain $f = 0.15$. Such a low coefficient is the result of the fact that the slope of the slide is very small. Next time, the Dormitories and Refectories management might choose a brachistochrone-shaped slide.

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Problem 7 ... Jarda's problems 3 points

When selecting problems for Physics Brawl Online, Jirka calculated that 20 % of the available problem assignments comes from Jarda. Yet, Jarda is the author of an incredible third of all the problems selected for the competition. Let us assume that FYKOS organizers do read the problem proposals when selecting them for the competition. How much more likely is it that the organizers will select a problem if the author is Jarda than if the proposal comes from another (average) organizer?

Jirka's original version of the problem did not pass the political censorship.

There are several ways of solving the problem. The first is by reasoning. Let's denote the total number of proposed problems by N and the number of tasks selected by n. We know that Jarda has proposed $0.2 \cdot N$ problems and $\frac{n}{3}$ were selected. We are interested in the quality of the tasks Jarda proposes, i.e., the ratio between the number of selected tasks and the number of tasks he proposed; which is equal to $n/(3 \cdot 0.2 \cdot N)$.

We now compare this ratio with the ratio for any other organizer. The other organizers proposed $0.8 \cdot N$ problems, from which a total of $\frac{2n}{3}$ were selected. Then the ratio is

$$\frac{\frac{n}{3}}{0.2 \cdot N} : \frac{\frac{2n}{3}}{0.8 \cdot N} = \frac{\frac{1}{3}}{\frac{2}{3}} \cdot \frac{0.8}{0.2} = 2$$

Thus, we discovered that Jarda's problems are approximately twice as successful as those of the other organizers.

Alternatively, we could solve the problem using conditional probability. Our goal is to calculate the probability that the problem is chosen given that it comes from Jarda (we will denote it by $P(\text{chosen} \,|\, \text{Jarda})$; the symbol $|$ denotes "under the condition"), and compare it with the probability of being selected, given that it comes from any other organizer.

For conditional probability

$$P(\text{selected,} \,|\, \text{Jarda}) = \frac{P(\text{selected} \cap \text{Jarda})}{P(\text{Jarda})}\,,$$

where the probability that the selected problem is Jarda's, i.e., $P(\text{selected} \cap \text{Jarda})$, is given again via conditional probability, this time using the probability that Jarda is the author of the selected task – so $P(\text{Jarda}|\text{selected})$. We know that this probability is equal to $1/3$.

In total, we have

$$P(\text{selected} \,|\, \text{Jarda}) = P(\text{Jarda} \,|\, \text{selected}) \cdot \frac{P(\text{selected})}{P(\text{Jarda})}\,,$$

while the probability that the task will be selected is unknown. Similarly, for any other organizer (denoted as "other") we have

$$P(\text{selected} \,|\, \text{other}) = P(\text{other} \,|\, \text{selected}) \cdot \frac{P(\text{selected})}{P(\text{other})}\,.$$

Now, we want to find out how much more likely Jarda's problems are to be selected, so we are interested in the proportion of conditional probabilities expressed. We get

$$\frac{P(\text{selected} \,|\, \text{Jarda})}{P(\text{selected} \,|\, \text{other})} = \frac{P(\text{Jarda,} \,|\, \text{selected})}{P(\text{other} \,|\, \text{selected})} \cdot \frac{P(\text{other})}{P(\text{Jarda})} = \frac{\frac{1}{3}}{\frac{2}{3}} \cdot \frac{0.8}{0.2} = 2\,.$$

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Problem 8 ... irresistibly attractive 4 points

Jindra can't find a girlfriend, so he orders a female-attracting device from a dubious online store. He received a pocket black hole. At a distance of $5\,\mathrm{m}$*, the black hole exerts gravitational acceleration of* $9.81\,\mathrm{m\cdot s^{-2}}$ *on all bodies (including girls). Calculate the Schwarzschild radius of this black hole.* *Jindra thought about complaining, but the black hole swallowed him.*

The well-known relation for the Schwarzschild radius of a black hole is

$$R_S = \frac{2GM}{c^2}\,, \tag{1}$$

where G is the gravitational constant, M is the mass of the black hole, and c is the speed of light. We assume the Schwarzschild radius of this black hole to be orders of magnitude smaller than 5 m. Therefore at a distance of $r = 5\,\mathrm{m}$, we can use the relation from classical physics

$$a = \frac{GM}{r^2}\,. \tag{2}$$

where $a = 9.81\,\mathrm{m{\cdot}s^{-2}}$ is the gravitational acceleration. We express the mass M from the equation (1) and plug it into the equation (2)

$$a = \frac{R_\mathrm{S} c^2}{2r^2}.$$

Now we express the Schwarzschild radius R_S and plug in the numbers

$$R_\mathrm{S} = \frac{2ar^2}{c^2} \doteq 5.5 \cdot 10^{-15}\,\mathrm{m}\,.$$

The pocket black hole has a Schwarzschild radius $R_\mathrm{S} \doteq 5.5 \cdot 10^{-15}\,\mathrm{m}$, so our initial assumption of a negligible black hole radius was confirmed. Our usage of the classical physics calculation for gravitational acceleration was justified.

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Problem 9 ... flying droplets 4 points

Rotating around a horizontal axis is a wheel with an outer diameter of $d = 65.8\,\mathrm{cm}$ upon which it is raining. Water droplets collide inelastically with the wheel's surface, but subsequently they can detach from it. What is the minimum angular rotation speed for water droplets to depart from the entire upper half of the wheel's perimeter?

Jindra rode his bike through the puddles.

Let's move into a system associated with the rotating wheel. Water droplets experience a gravitational acceleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ downward and a centrifugal acceleration $\omega^2 r$ outward from the axis of rotation, where $r = 32.9\,\mathrm{cm}$ is the radius of the wheel, and ω is the angular velocity of rotation.

Water will depart from the wheel if the centrifugal force overcomes the radial component of the gravitational acceleration. We will measure the angle α from the vertical. The radial component of the gravitational acceleration is

$$g_r = g\cos\alpha,$$

where a positive sign indicates the direction inward towards the axis. A droplet located at position α on the wheel will depart if

$$g\cos\alpha < \omega^2 r,$$
$$\omega^2 > \frac{g\cos\alpha}{r}.$$

For this inequality to hold for all angles α from 0 to 2π it must be the case that

$$\omega > \sqrt{\frac{g}{r}}.$$

After substituting the given values, the result is $\omega > 5.46\,\mathrm{rad\cdot s^{-1}}$. The minimum angular rotation speed of the wheel for droplets to depart from the entire circumference is $5.46\,\mathrm{rad\cdot s^{-1}}$.

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Problem 10 ... stealing with a spring 4 points

We attach a spring with a stiffness $k = 5.2\,\mathrm{N\cdot m^{-1}}$ and a relaxed length $l_0 = 15\,\mathrm{cm}$ to a body with mass $m = 120\,\mathrm{g}$. We then start pulling at its other end at a constant speed $v = 65\,\mathrm{cm\cdot s^{-1}}$. To what maximum length does the spring stretch? The motion takes place on a smooth horizontal plane. *Jarda would like to become a pickpocket.*

Let's analyze the situation in the reference system of the hand. In this system, the body initially attains speed v, resulting in kinetic energy $mv^2/2$. That is all transformed into elastic potential energy $kx^2/2$ when the spring is the most stretched. This provides us with its maximum length

$$l = l_0 + v\sqrt{\frac{m}{k}} \doteq 25\,\mathrm{cm}\,.$$

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Problem 11 ... radiocarbon dating 5 points

Zuzka went to Abusir for archaeological excavations. In the tomb of an ancient Egyptian dignitary, she found a sample of wood, which she took to analyze on a mass spectrometer. The ratio of carbon isotopes in the sample was measured as $p_{^{14}\mathrm{C}/^{12}\mathrm{C}} = 9.22\cdot 10^{-13}$. In what year was the dignitary buried? Using the Gregorian calendar, write the years BCE with a minus sign. The half-life of ^{14}C is $T = 5\,730\,\mathrm{yr}$, and the ratio of isotopes in the atmosphere is historically constant $p_0 = 1.25\cdot 10^{-12}$. Assume that the tree was cut down shortly before the dignitary's burial. *Jindra came up with the origin of the problem only after Terka pointed it out to him.*

In the upper atmosphere, radioactive carbon atoms are naturally formed from nitrogen $^{14}\mathrm{N}$ by exposure to cosmic rays. Through photosynthesis, the radioactive carbon isotope is incorporated into plant cells and through the food chain into the bodies of animals. This fact establishes the same ratio p_0 of carbon isotopes $^{14}\mathrm{C}/^{12}\mathrm{C}$ in living organisms and the atmosphere.

When a living organism dies (e.g., when a tree is cut down), its carbon exchange with the environment stops. Thus, there is no replenishment of the decaying isotope $^{14}\mathrm{C}$, and the ratio of $^{14}\mathrm{C}/^{12}\mathrm{C}$ decreases exponentially with time, with a half-life of $T = 5\,730\,\mathrm{yr}$. We calculate the age of the tomb t from the equation

$$\frac{p_{^{14}\mathrm{C}/^{12}\mathrm{C}}}{p_0} = 2^{-\frac{t}{T}}\,,$$

$$t = -T\log_2\left(\frac{p_{^{14}\mathrm{C}/^{12}\mathrm{C}}}{p_0}\right)\,,$$

$$t = 2\,516\,\mathrm{yr}\,.$$

Subtracting the age of the wood from the current year of 2023, we get a burial year of 493 BCE, which we round and write in the solution as −490.

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Problem 12 ... curious cyclist

4 points

Kuba bought a new bicycle and wanted to find out the size of the rolling resistance coefficient ξ of his wheels. He noticed that the bike goes downhill on its own if the plane of the slope makes an angle with the horizontal greater than $\alpha = 0.5\,^\circ$. The diameter of the bicycle wheels is $d = 67\,\mathrm{cm}$. Can you help Kuba? *Kuba likes cycling.*

If Kuba goes down a slope that has a deviation from the horizontal direction equal to α, the force in the direction of motion of the cyclist $\vec{F_1}$ and the rolling resistance force $\vec{F_v}$ will be in equilibrium. Thus, the bicycle will move in a uniform linear motion and

$$F_1 = F_v = F_n \frac{\xi}{d/2}\,.$$

The figure shows that $F_1 = F_G \sin(\alpha)$ and $F_n = F_G \cos(\alpha)$, so

$$F_G \sin(\alpha) = F_G \cos(\alpha) \frac{\xi}{d/2}\,.$$

After expressing ξ and substituting, we get the result

$$\xi = \frac{d}{2} \tan(\alpha) \doteq 2.9\,\mathrm{mm}.$$

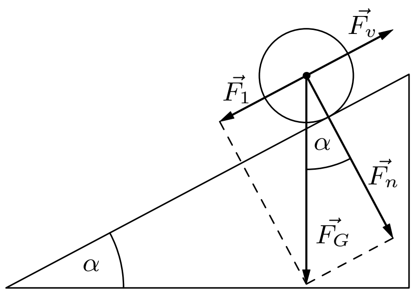

Figure 3: Decomposition of forces.

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Problem 13 ... carrying a box 5 points

As Lego carried the box, he was pondering about force he was applying to it. The box has a mass $m = 7.5\,\mathrm{kg}$ and is shaped like a rectangular cuboid. Lego holds it by pushing on two opposite (vertical) sides, and the coefficient of friction between the sides and Lego's hands is $f = 0.45$. What is the magnitude of the force exerted by one of Lego's hands on the box?

Lego was carrying a lot of things at the camp

The hands are pushing on the box from opposite sides, so they both have to push with the same normal force, let's call it F_N. Then the friction force between each of the hands and the box is $F_\mathrm{t} = fF_\mathrm{N}$. A gravity $F_g = mg$ acts on the box as well. In order for Lego to carry the box, the frictional forces between the box and the hands must compensate for this force. Thus

$$2F_\mathrm{t} = F_g$$
$$2fF_\mathrm{N} = mg$$
$$F_\mathrm{N} = \frac{mg}{2f}\,.$$

One might expect this to be the result, but it is not! The point is that the frictional force between the box and the hand is also a force that the hand exerts on the box. So each hand exerts forces F_N and F_t on the box, and these forces are perpendicular to each other, so if we want to know the magnitude of the total force exerted by the hand on the box, we get it as

$$|F| = \sqrt{F_\mathrm{N}^2 + F_\mathrm{t}^2} = F_\mathrm{N}\sqrt{1+f^2} = \frac{mg}{2f}\sqrt{1+f^2} = 90\,\mathrm{N}\,.$$

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Problem 14 ... divider No.1 4 points

Petr needed a $3\,\mathrm{V}$ voltage source, but he only had a $12\,\mathrm{V}$ source and resistors. So he decided to build a voltage divider from the schematic 4. What value of R did Peter have to choose to get $3\,\mathrm{V}$ at V_+?

I wanted to reminisce about electrical engineering.

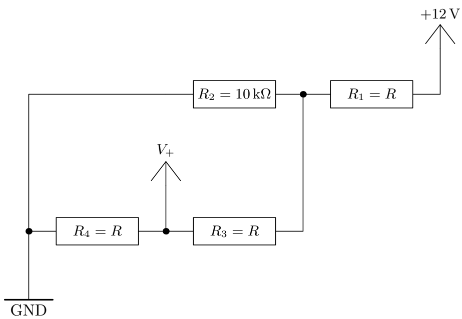

Figure 4: Circuit diagram

The problem's name reveals that we need a voltage divider (sometimes called a potential divider). It is a device consisting of two resistors connected in series, where we use the voltage on the second resistor (in the direction of the current) as a source. The formula for the voltage of an unloaded voltage divider on the second resistor in series is

$$U_{R_2} = U_{\text{in}} \frac{R_2}{R_1 + R_2} \,,$$

where U_{in} is the voltage across the series circuit and U_{out} is the voltage across the second resistor. For the loaded divider, we have

$$U_{R_2} = U_{\text{in}} \frac{R_2 R_{\text{L}}}{R_1 R_2 + R_1 R_{\text{L}} + R_2 R_{\text{L}}} \,,$$

where R_{L} is the load resistance connected in parallel to R_2. First, let us consider that there are two resistive dividers, where the second one uses the output voltage of the first divider as its input voltage. However, the first divider is loaded by the second, and the schematic shows that the load resistance is $R_3 + R_4 = 2R$. Let us note that both resistances are the same in the second divider, and the previous formula for the unloaded divider shows that its output voltage will always be half of its input voltage. Thus, we only need to solve the circuit for the output voltage of the first divider. Then we apply the formula for the voltage of the loaded resistive divider, and we get

$$U_{\text{out}} = U_{\text{in}} \frac{R_2 \cdot 2R}{R_2 \cdot R + 2R^2 + R_2 \cdot 2R} \frac{1}{2} \,,$$

from this, we construct an equation for R

$$R = \frac{R_2 U_{\text{in}} - 3 R_2 U_{\text{out}}}{2 U_{\text{out}}} \,.$$

After inserting the values from the assignment, we get

$$R = 5\,000\,\Omega \,.$$

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Problem 15 ... steel sphere floats — 4 points

Karel found a steel sheet with a thickness of $\Delta r = 0.84\,\mathrm{mm}$ and was thinking about what to do with it. Since Karel likes to experiment, he made a hollow sphere of such a radius that when placed in the water, one half was above the surface and one below. Consider that the thickness of the sphere is just Δr and that the density of the steel is $\rho' = 7\,840\,\mathrm{kg{\cdot}m^{-3}}$. Find the outer radius of this sphere. *Karel was thinking on a boat.*

Let m be the sphere's mass, and ρ_{w} the density of water. Half of the volume of the sphere is immersed in water. In equilibrium, the buoyant force must be equal to the gravitational force, so we get the equation

$$mg = \frac{V}{2} \rho_{\text{w}} g \,,$$

from which, after canceling g, we get

$$\frac{m}{V} = \frac{\rho_{\text{w}}}{2} \,.$$

We express the sphere's density as $\rho = \frac{m}{V}$, where V is the total volume of the sphere and m is the mass of the sheet from which we made it. We determine the mass of the sheet by its density and the volume formed by the space between two spheres. The volume of the sheet will therefore be $V' = \frac{4}{3}\pi\left(r^3 - (r-\Delta r)^3\right)$, where r is the radius of the sphere. Substituting into the equation and rearranging, we get

$$\rho = \frac{m}{V} = \frac{\rho'\left(r^3 - (r-\Delta r)^3\right)}{r^3}\,,$$

while the weight of the air inside the sphere is equal to the buoyant force of the air acting on the part of the sphere above the water.

Substituting into the equation for the equality of forces, we get the equation of the third degree:

$$0 = \rho'\left(r^3 - (r-\Delta r)^3\right) - \frac{\rho_\mathrm{w}}{2}r^3\,,$$

which we solve numerically. The solutions will give one real root and two complex. The real root is $r = 0.0387\,\mathrm{m} = 3.87\,\mathrm{cm}$, which is the sought radius.

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Problem 16 ... modified solar system — 5 points

Suppose the Sun had an effective temperature $T_2 = 8\,000\,\mathrm{K}$. By how many percent would the period of Jupiter's orbit have to be extended for the same amount of power to fall on it as it does now? *Danka was trying hard to come up with an interesting problem.*

According to the Stefan-Boltzmann law, the heat output of a star (an absolute black body) is

$$L = S_\odot \sigma T^4\,,$$

where $S_\odot$ is the surface area of the star, T is its temperature, and σ is the Stefan-Boltzmann constant.

The Sun radiates evenly throughout the space. The planet captures a fraction of this energy that is proportional to the cross-sectional area of the planet. So we want the following to hold

$$\pi r_\mathrm{J}^2 \frac{L_1}{4\pi d_1^2} = \pi r_\mathrm{J}^2 \frac{L_2}{4\pi d_2^2}\,,$$

where r_J is the radius of Jupiter and d_1 and d_2 are the distances from the Sun for surface temperatures T_1 and T_2, respectively. We substitute the heat output L from the first equation and get

$$d_2 = d_1 \left(\frac{T_2}{T_1}\right)^2\,.$$

Kepler's third law binds the orbital periods of the planets around the Sun

$$\frac{t_1^2}{d_1^3} = \frac{t_2^2}{d_2^3}\,,$$

from where we can already find the prolongation we are looking for as

$$p = \frac{t_2 - t_1}{t_1} = \left(\left(\frac{d_2}{d_1}\right)^{\frac{3}{2}} - 1\right) = \left(\left(\frac{T_2}{T_1}\right)^{3} - 1\right) = \left(\left(\left(\frac{4\pi R_\odot^2 \sigma}{L_1}\right)^{\frac{1}{4}} T_2\right)^{3} - 1\right) = 166\,\%\,.$$

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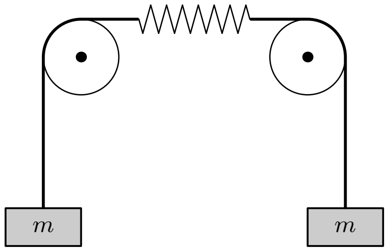
Problem 17 ... pulleys with elastic rope

4 points



Lego likes tasks with pulleys. But this time he wanted to come up with a problem that would model the fact that the rope has some elasticity. He decided to do this by splitting a perfectly stiff, weightless rope in the middle and then joining it with a weightless spring with stiffness $k = 78\,\mathrm{N{\cdot}m^{-1}}$. He then placed the modified rope on two fixed pulleys placed at the same height and hung a weight with mass $m = 9.0\,\mathrm{kg}$ on each end. He then held both the weights with just enough force so that the tension in the rope was $T = 12\,\mathrm{N}$. What is the acceleration with which the weights will move if they are released at the same time?

The problem came up to Lego's mind, when he had a lecture on pulleys at camp.

A quick way to solve this is to consider that a gravitational force mg will be applied to the block in the downward direction and a force from the rope in the upward direction. At the moment of letting go, the spring has not yet had time to stretch from the state it was in when we held the blocks. Thus, the force T will be applied to the rope, and hence the tension in the intangible rope must still be T. So the resulting force $mg - T$ is acting on both blocks and hence they will move with acceleration $a = g - T/m \doteq 8.5\,\mathrm{m{\cdot}s^{-2}}$.

But we will also describe a more complicated way (since it came to our mind earlier). The whole situation is symmetrical with respect to the centre of the spring, so it will not move. So we can imagine that this point is perfectly fixed to some wall. Thus we get a situation where a block of mass m oscillates on a half spring. That is effectively on a spring of stiffness $2k$. The angular frequency of the oscillation will be $\omega = \sqrt{2k/m}$.

The equilibrium position will be when this half of the spring is extended by $l_r = mg/2k$ compared to its rest length. We release the blocks when there is a tension T in the rope, then the spring half must be extended by $l_m = T/2k$. Since we are releasing the weight from rest, it will be at its maximum at the moment of release, and the magnitude of the oscillatory motion amplitude will therefore be $l_a = l_r - l_m = (mg - T)/2k$. At the same time, in addition to the position, there will be an acceleration in the amplitude, the magnitude of which will therefore be

$$a = l_a\omega^2 = \frac{mg - T}{2k}\frac{2k}{m} = g - \frac{T}{m} \doteq 8.5\,\mathrm{m{\cdot}s^{-2}}\,.$$

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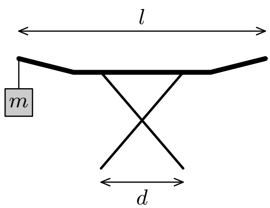
Problem 18 ... falling clothes horse

4 points



Matěj has a clothes horse with a width of $l = 180\,\mathrm{cm}$ on which he spreads his clothes evenly after washing. On the left edge of the dryer, he places a weight of $m = 3\,\mathrm{kg}$. How long after we hang up the clothes will the dryer tip over if its leg span is $d = 60\,\mathrm{cm}$? The mass of the wet clothes is $M_0 = 6\,\mathrm{kg}$; this will decrease to $M_1 = 2\,\mathrm{kg}$ when the clothes are completely dry. The weight of the dryer itself is $2\,\mathrm{kg}$. For simplicity, consider that the water evaporates at a constant rate and the clothes dry up in one day.

Matěj cannot dry his clothes.

Let us denote the mass of the dryer by M_s. We can calculate the position of the center of gravity of the entire system as the weighted average of the positions of the centers of the individual bodies. Thus, at the beginning of the drying process, the total center of gravity is located at

$$x_0 = \frac{m\frac{l}{2}}{m + M_0 + M_\mathrm{s}} = 24.5\,\mathrm{cm}$$

from the center of the dryer. When the laundry dries up, it moves to

$$x_1 = \frac{m\frac{l}{2}}{m + M_1 + M_\mathrm{s}} = 38.6\,\mathrm{cm}\,,$$

which is more than $x_\mathrm{max} = 30\,\mathrm{cm}$, and therefore the dryer must tip over at some point. From the relation above, we express the mass of the laundry and substitute x_max for the position of the center of gravity to obtain the minimum mass of the laundry that will not yet allow the clothes horse to tip over

$$M_\mathrm{min} = \frac{ml}{2x_\mathrm{max}} - m - M_\mathrm{s} = 4\,\mathrm{kg}\,.$$

If the laundry dries at a constant rate, it will dry to this critical mass in $\frac{M_\mathrm{min} - M_1}{M_0 - M_1} = 1/2$ of a day, or 12 hours.

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Problem 19 ... acoustic speedometer 5 points

When the car is stationary, 80 raindrops per second hit its windshield. What velocity is it traveling at if the current frequency of impacts is $230\,\mathrm{s^{-1}}$? The windshield has an area S and is inclined at an angle of $33\,^\circ$ with respect to the ground. Raindrops fall vertically to the ground at a velocity $4.5\,\mathrm{m{\cdot}s^{-1}}$. *Jarda was afraid of a speeding fine.*

Let's find the number of raindrops that fall on the glass in one second. Let us denote the velocity at which they fall as v and their volumetric density in the air as n. Then, f_1 raindrops fall on the stationary glass during one second

$$f_1 = nvS\cos\alpha\,.$$

However, the situation is a bit more complicated when the car is moving with velocity u. Now, the volume collected by the front windshield of the car per unit time is equal to

$$Q = uS\sin\alpha + vS\cos\alpha\,.$$

We see that an additional term has been introduced here. From the difference in frequencies, we obtain

$$f_2 - f_1 = nuS\sin\alpha\,,$$

From there, we can easily express the final velocity of the car

$$u = \frac{f_2 - f_1}{nS\sin\alpha} = v\frac{f_2 - f_1}{f_1\tan\alpha} = 47\,\mathrm{km{\cdot}h^{-1}}\,.$$

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Problem 20 ... the Synchrotron in Grenoble

5 points

At the ESRF synchrotron in Grenoble, electrons with an energy of 6.03 GeV *orbit along the track with a circumference of* 844.4 m, *generating a current of* 35.8 mA. *How many electrons are there in the entire synchrotron at one moment?*

Jarda participated in a diffraction experiment.

Since the rest mass of an electron is $E_0 = 511\,\mathrm{keV}$, it is negligible compared to their total energy of $E = 6.03\,\mathrm{GeV}$, so they move strongly relativistically. We can calculate their speed if we start from the equation:

$$E = \frac{E_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}\,,$$

where v is their speed and c is the speed of light. From this:

$$\frac{v}{c} = \sqrt{1 - \left(\frac{E_0}{E}\right)^2} = 0.999\,999\,996\,.$$

Furthermore, due to the precision of quantities in the task and the precision of the required result, we can solve the problem with approximation $v = c$.

The frequency of electron circulation in the synchrotron is $f = c/l$, where $l = 844.4\,\mathrm{m}$ is its circumference. The current produced by a single electron is $I_\mathrm{e} = ef$, where e is the elementary charge.

The total number of electrons is then:

$$N = \frac{I}{I_\mathrm{e}} = \frac{I}{ef} = \frac{Il}{ec} = 630 \cdot 10^9\,.$$

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Problem 21 ... maximal pétanque

6 points

Vojta played pétanque, but he was not good at it. Therefore, he got angry and threw the ball he held in his hand as far away as he could. At what angle must it be thrown with velocity $v = 11\,\mathrm{m\cdot s^{-1}}$ *to go as far as possible if the ball continues to roll after impact? When the ball hits the ground, only the vertical component of the velocity is absorbed, the ball does not slip, and the rolling resistance coefficient is* $c = 0.17$. *Assume that the ball falls on a nearby horizontal plateau that is at the same height as was the ball when it left Vojta's hand.*

Vojta misunderstood the rules of pétanque.

We can determine the range from the known relation

$$d_\mathrm{v} = \frac{v^2}{g}\sin 2\alpha\,,$$

where $0^\circ < \alpha < 90^\circ$ denotes the angle at which we launched the ball, which we are looking for. If all the vertical component of the velocity is absorbed, the kinetic energy of a sphere of mass m immediately after impact will be

$$E_\mathrm{k} = \frac{1}{2}m\left(v\cos\alpha\right)^2\,,$$

against which the rolling resistance will do the work. Thus, the condition for the ball to stop will be

$$E_\mathrm{k} = d_\mathrm{k} mgc \quad \Rightarrow \quad d_\mathrm{k} = \frac{1}{2cg}\left(v\cos\alpha\right)^2\,,$$

and in total, the ball travels a distance

$$d = d_\mathrm{k} + d_\mathrm{v} = \frac{v^2}{g}\left(\frac{1}{2c}\cos^2\alpha + \sin 2\alpha\right)\,.$$

For this distance to be maximal, the following expression must be maximal

$$\frac{1}{2c}\cos^2\alpha + \sin 2\alpha\,.$$

So let's find the derivative with respect to α and set the result equal to zero

$$-\frac{1}{2c}2\cos\alpha\sin\alpha + 2\cos 2\alpha = 0 \quad \Rightarrow \quad \tan 2\alpha = 4c\,,$$

from where we get the optimal angle as $17\,^\circ$.

Note that if we informally put $c = \infty$, we get the well-known result $45\,^\circ$ for the ball not rolling.

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Problem 22 ... taxidermy of a cylinder
5 points

Let's have a homogeneous cylinder. Around its axis, we cut a smaller cylinder out of it. The hollow and the smaller cylinder are then released down the inclined plane. What is the radius of the smaller cylinder if it starts with $20\,\%$ more acceleration than the rest of the larger cylinder? Provide the answer in multiples of the original radius.

Jarda wanted to state a problem without any number. It didn't work out.

When moving on an inclined plane, the law of conservation of energy applies to a rolling body of mass m in the form

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2\,, \tag{3}$$

where g is the gravitational acceleration, h is the height by which the body has descended, v is the velocity attained by its center, J is the moment of inertia with respect to the axis of symmetry, and ω is the angular velocity of rotation. For a cylinder, $J = \frac{1}{2}mr^2$ holds, where r is its radius. If the body is not circular, $\omega r = v$ must hold.

On an inclined plane, we release two bodies of different mass and radius. For each of them, we calculate its acceleration. We denote the mass of the original cylinder M, its radius R, the mass of the small cylinder as m, and its radius as r.

For a small cylinder

$$\frac{1}{2}mv^2 + \frac{1}{2}J\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\frac{1}{2}mr^2\omega^2 = \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2\,. \tag{4}$$

The law of conservation of energy is therefore in the form

$$mgh_1 = \frac{3}{4}mv_1^2 \quad \Rightarrow \quad a_1 = \frac{2}{3}g\sin\alpha\,,$$

where the index 1 denotes the change in height, velocity, and acceleration of the cylinder, and α is the angle of inclination of the plane with respect to the horizontal direction. We came to the acceleration by the complete time derivative of the law of conservation of energy, since $\dot{h}_1 = v_1 \sin\alpha$ and $\dot{v_1^2}/2 = a_1 v_1$, with v_1 then reduced on both sides of the equation.

The mass of the large cylinder is $M - m$ and its moment of inertia with respect to the axis of symmetry is

$$\frac{1}{2}MR^2 - \frac{1}{2}mr^2 \,.$$

The right hand side of the equation 3 is thus

$$\frac{1}{2}\left(M-m\right)v^2 + \frac{1}{2}\left(\frac{1}{2}MR^2 - \frac{1}{2}mr^2\right)\frac{v^2}{R^2} = \frac{1}{2}\left(M-m\right)v^2 + \frac{1}{4}\left(MR^2 - mr^2\right)\frac{v^2}{R^2} \,. \tag{5}$$

By the same process as above we find the acceleration of the rest of the cylinder as

$$\left(M-m\right)gh_2 = \frac{1}{2}\left(M-m\right)v_2^2 + \frac{1}{4}\left(MR^2 - mr^2\right)\frac{v_2^2}{R^2} \,,$$

from which we get

$$a_2 = \frac{M-m}{\left(M-m\right) + \frac{1}{2}\left(M - m\frac{r^2}{R^2}\right)} g\sin\alpha \,.$$

From the condition $\frac{a_1}{a_2} = K = 1,2$ we get the equation

$$3M - 2m - m\frac{r^2}{R^2} = 3KM - 3Km \,.$$

We express m and M in terms of r and R, the cylinder length h and the density ρ as $m = \pi r^2 h\rho$ and $M = \pi R^2 h\rho$. Substituting into the previous equation and after subtracting π, h and ρ we get the biquadratic equation

$$r^4 + \left(2 - 3K\right)R^2 r^2 + 3\left(K-1\right)R^4 = 0 \,,$$

where the variable is r^2. The solution to this equation is

$$r^2 = \frac{-\left(2-3K\right) \pm \sqrt{\left(2-3K\right)^2 - 12\left(K-1\right)}}{2}R^2 = \frac{-2 + 3K \pm \left(4 - 3K\right)}{2}R^2 \,.$$

If we choose the $+$ sign, we get $r = R$ and independence on K, which makes no sense. So we choose the $-$ sign, which leads to

$$r = \sqrt{3\left(K-1\right)}R \,.$$

We see that it can never happen that the rest of the cylinder goes faster than the small cylinder. But at the same time, the problem has no solution for $K > \frac{4}{3}$ either, because then $r > R$ comes out. If $r \to R$, the rest of the cylinder becomes a hoop and achieves an acceleration of $\frac{1}{2}g\sin\alpha$. After setting $K = 1.2$, we get the result we are looking for

$$\frac{r}{R} = \sqrt{\frac{3}{5}} \doteq 0.775 \,.$$

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Problem 23 ... rats on rats

4 points

Imagine a heap of rats, each of which has mass m. We arrange the rats in a 2D pyramid similar to Pascal's triangle. There will be one rat at the top, two rats below it, three in the next row, and so on. We have a lot of them, infinitely many. Each rat distributes its weight and the weight it carries to the rats below it. What is the total weight the legs of the rat on the far left must carry? Express the result as a multiple of m. If it would be infinite, enter 0. Consider the rats in a homogeneous gravitational field. *Karel was thinkig about Jára (da) Cimrman*

The main difficulty of the problem lies in the scope of the problem statement and in the understanding of the question itself. There are several possible approaches. A practical option is, for example, to make a spreadsheet in Excel and see what the values are close to; indeed, after a few rows, they start to settle around $2m$.

The alternative is to manually count the elements. The weight that the rat must bear in each successive row is obtained by always taking half of the previous rat's load and adding m, which represents the weight of the rat itself (we must not forget this, as the assignment asks for the total weight that the rat's legs must bear). For simplicity, let's consider only multiples of m – if the rat in the n-th row on the left carries a weight of m_n, we denote $a_n = m_n/m$. Let's write

$$a_1 = 1\,, \quad a_2 = \frac{a_1}{2} + 1 = \frac{3}{2}\,, \quad a_3 = \frac{7}{4}\,, \quad a_4 = \frac{15}{8}\,, \quad \dots$$

After a few more tries, it looks like we're still getting closer to 2 – in this way, this approach is similar to the previous one mentioned. For both, however, we don't know for sure if the value 2 is accurate, but you'll find that during the competition after entering it.

A better approach is to note what each member satisfies

$$a_n = 1 + \frac{a_{n-1}}{2} = 1 + \frac{1}{2} + \frac{a_{n-2}}{4} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{a_{n-3}}{8} = \dots = \sum_{i=0}^{n-1} \frac{1}{2^i}\,,$$

where in the notation on the right side we choose the summation index i from 0 to $n-1$ so that the sum corresponds to the actual situation. Now we can compute the limit as $n \to \infty$, respectively add up the infinite geometric series to get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2\,.$$

So we actually got the expected result of 2.

We will show one more, so far the most crafty method, that will also give us the correct result. Assuming we know the result, let's denote it x. If our sequence is indeed close to some real number, then it must be true that in the next row, the result is practically the same; so we construct the equation

$$x = 1 + \frac{x}{2} \qquad \frac{x}{2} = 1 \qquad x = 2\,.$$

Yet again we get the result that the multiple of the weight m carried by the bottom left (or bottom right) rat is 2. This approach is probably the fastest.

In all cases, the result is only valid for obedient rats that exert an even load on their mates below them, they all stand in a homogeneous gravitational field, and there is an infinite number of them. However poor rats in the middle have to carry an infinite load. *But the more rats, the more hatred towards them, as the classic, Jára Cimrman, wished in his pedagogical rules.*

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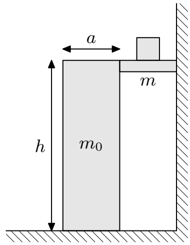
Problem 24 . . . postmodern art 5 points



Lego wanted to create a sculpture. He therefore took a rectangular cuboid with mass $m_0 = 18.5\,\mathrm{kg}$, a square base with side length $a = 60.0\,\mathrm{cm}$ and height $h = 185\,\mathrm{cm}$, which he placed next to a solid wall (the sides of the cuboid are parallel to it). He then inserted a slab between the block (cuboid) and the wall so that it was held horizontally at height h only by friction. The coefficients of friction between the slab and the block, and the slab and the wall are $f = 0.42$. The coefficient of friction between the cuboid and the ground is effectively infinite. Lego, nevertheless, wants to put a weight on the middle of the slab. What is the largest sum of masses of the weights and the slab that the block can hold? The slab and the cuboid are homogeneous.

Lego's problems are evolving.

We are investigating when the block will not hold the slab anymore. We know that the friction between the cuboid and the ground is effectively infinite so that the cuboid does not slip (we can imagine a stopper behind the block that will not let it go any further). Thus, the forces acting on the block can always be in equilibrium. What, however, can cause a block to fail to hold the slab? It may be that the torques are not in equilibrium. It might seem that due to the unlimited friction between the block and the ground, the slab can push on the block with an arbitrary force. However, this is not true because the cuboid would topple over for some magnitude of force. It cannot exert such forces on the slab, so for cases where such a force would be needed to hold the slab, the slab would fall.

Thus, we are interested in the total sum of the torques of the forces acting on the block. We will choose the back edge on the ground as the axis of rotation since this is the edge around which the cuboid would start to overturn if the force exerted on it by the slab were too great.

What are all the forces acting on the block? It is gravity, the normal and frictional force between it and the slab, and the normal and frictional force between it and the ground. We will discuss these forces, and especially, their torque, in that order.

The block has mass m_0, so the weight of the block is $m_0 g$. Since the block is homogeneous, this force will act at its center, and hence, at a horizontal distance $a/2$ from the back edge. The torque with which this force acts on the cuboid will, therefore, be

$$M_g = \frac{1}{2} a m_0 g\,.$$

Let's examine the forces between the block and the slab. Let us denote the sum of the mass of the slab and the weight as m. The slab is pushed upwards by frictional forces only (between the slab, the wall, and the block). We can calculate the friction force as the friction coefficient f (f is the same for both surfaces) multiplied by the normal force that pushes the two surfaces together. Since the slab is not moving in the horizontal direction, the forces in that direction are balanced. Thus, the normal force on one side of the slab must be as large as that on the other. If we denote this force by F_N, the frictional force on each side will be $F_\mathrm{t} = f F_\mathrm{N}$. The sum of the frictional forces must compensate for the gravity of the slab and the weight, that is, $mg = 2F_\mathrm{t}$. From here, we can express the force pushing the block and the slab together as $F_\mathrm{N} = mg/(2f)$.

At what torques will these two forces act on the block? The slab will be pushing the block down with the frictional force, and this torque will, therefore, act in the same direction as the gravitational torque of the block itself. The horizontal distance of the center of this force from the axis of rotation is a, so the torque from the frictional force is

$$M_\mathrm{t} = aF_\mathrm{t} = \frac{1}{2}amg\,.$$

The torque from the normal force between the slab and the block will rotate the block in the opposite direction as the torque from the gravity and friction forces, so we assign the opposite sign to it. This force is horizontal, so its torque vector will be the same as the vertical distance of the origin and the axis of rotation, or h. Together, this torque will be

$$M_\mathrm{N} = -hF_\mathrm{N} = -\frac{1}{2f}hmg\,.$$

It is important to consider the effect between the block and the ground. The friction acts on the plane of the ground, which means that it will not exert any torque in the axis of rotation. However, it is crucial to understand the impact of the normal force. Normally, this force is distributed continuously over the entire contact area. In the absence of external forces, it can be assumed that the normal force will act uniformly at the center of the base.

When a force is applied to a block, its weight distribution changes. To better understand this concept, you can try standing upright while having a friend push you from the front. Even if you do not fall over, you'll notice that your weight shifts towards your heels. The same thing happens to the normal force distribution when we apply a force to the block.

Just as the frictional force is enough to keep the object from sliding, this normal force distribution is enough to keep the object from tipping over, unless the applied torque is too great.

The torque from the normal force from the ground will rotate the block in the same direction as the torque from the normal force from the slab. Thus, in the limit situation, when the normal force between the block and the slab is the maximum possible, the normal force between the ground and the block will act entirely in the axis of rotation (because it has zero torque there). If it did not operate entirely there and therefore had a non-zero torque, this would mean that it is possible to increase the force between the block and the slab, because a small enough increase would only tilt the block more backward. That is, when we are interested in the maximum possible weight of the slab and the weight, the torques of the forces between the block and the ground must be taken as zero in our chosen axis of rotation.

So, we get an identity that will hold in the limit case

$$\begin{aligned} M_g + M_\mathrm{t} + M_\mathrm{N} &= 0 \\ \frac{1}{2}am_0g + \frac{1}{2}amg - \frac{1}{2f}hmg &= 0 \\ m &= \frac{m_0}{\frac{h}{fa} - 1} = 2.92\,\mathrm{kg}\,. \end{aligned}$$

This is the maximum possible mass m that the block can support.

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Problem 25 ... ice in a cube

5 points

We enclose ice with a mass of 15 g *and temperature* 0 °C *under normal conditions in a hermetically sealed cubic container with a volume* 0.10 l. *Afterwards, we start heating it. What will be the pressure inside the cube when the temperature there reaches* 120 °C*?*

Jarda wanted to trick the participants, but he tricked himself instead.

Firstly, we will attempt the calculation while considering water an ideal gas, even though we might conclude that it was not the best idea.

The volume of air in the cube at the beginning is $V_\mathrm{v} = V - m/\rho_\mathrm{L} = 83.6\,\mathrm{cm^3}$, where $m = 15\,\mathrm{g}$ is the mass and $\rho_\mathrm{L} = 916.2\,\mathrm{kg{\cdot}m^{-3}}$ the density of ice at 0 °C. Because it is air under normal conditions, we can use the state equation to determine the amount of substance present in the cube as

$$n_\mathrm{v} = \frac{p_\mathrm{n} V_\mathrm{v}}{R T_\mathrm{n}} = 3.48\,\mathrm{mmol}\,,$$

where p_n and T_n are pressure and temperature under normal conditions.

At a temperature $T = 120\,\mathrm{^\circ C}$, all the water will have evaporated. The chemical amount of water is

$$n_{\mathrm{H_2O}} = \frac{m}{M_{\mathrm{H_2O}}} = 833\,\mathrm{mmol}\,.$$

The chemical amount of water is thus much higher; therefore, we can neglect the partial pressure of air. The total pressure will ultimately be

$$p = \frac{n_{\mathrm{H_2O}}}{V} R T = 27\,\mathrm{MPa}\,.$$

That is a very high pressure. However, water boils at higher temperatures under increased pressure. Therefore, not all the water will have evaporated even at 120 °C. Only a portion will have. Water will exert a partial pressure, which is the pressure of saturated water vapor at 120 °C, as at this point, no more water will evaporate. Its value is approximately $p_\mathrm{w} = 198.9\,\mathrm{kPa}$. It is possible to find this data on the internet, e.g., `https://www.engineeringtoolbox.com/water-vapor-saturation-pressure-d_599.html`.

To this pressure must be added the partial pressure of the air inside the cube. The volume of air is

$$V_\mathrm{v2} = V - V_\mathrm{w} = V - \frac{m}{\rho_{120\,\mathrm{^\circ C}}} = 84.1\,\mathrm{cm^3}\,.$$

In this calculation, we assumed that the mass of water that evaporated and is in the cube in gaseous form is negligible compared to the mass still in the liquid state. We observed that when all the water evaporates, its pressure is at least two orders of magnitude higher than the pressure of saturated water vapor. Therefore, the amount of evaporated water will also be two orders of magnitude lower than the mass of the remaining water. For this calculation, we used the density of water $\rho_{120\,\mathrm{^\circ C}} = 943\,\mathrm{kg{\cdot}m^{-3}}$ from source `https://www.engineeringtoolbox.com/water-density-specific-weight-d_595.html`.

For partial pressure of air

$$p_{\mathrm{v}} = n_{\mathrm{v}} \frac{RT}{V_{\mathrm{v2}}} = p_{\mathrm{n}} \frac{V_{\mathrm{v}}}{V_{\mathrm{v2}}} \frac{T}{T_{\mathrm{n}}} \doteq 135\,\mathrm{kPa}\,.$$

The total pressure inside the cube is

$$p_{\mathrm{tot}} = p_{\mathrm{w}} + p_{\mathrm{v}} = 334\,\mathrm{kPa} \doteq 330\,\mathrm{kPa}\,.$$

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Problem 26 ... When will I be there? 7 points

Jarda rides in an elevator that ascends at a constant speed. Since he is getting impatient, he is throwing his keys into the air. They always fly to the height of 62 cm. However, between one ejection and the subsequent catching of the keys, the elevator begins to brake steadily, so that the keys fly to a height of 72 cm and spend 0.15 s more time in the air. Determine the acceleration with which the elevator slowed down.

Jarda is in the elevator and already looking forward to bed.

When the elevator is braking on the way up, the inertial force in the elevator is acting upwards, so the acceleration of the keys will be smaller (we denote it as a). Since the height to which the keys flew is greater than the original height, braking occurs as the keys ascend.

When the elevator was moving steadily, the keys always spent in the air

$$h = \frac{1}{2} g \frac{t^2}{4} \quad \Rightarrow \quad t = 2\sqrt{\frac{2h}{g}} = 0.711\,\mathrm{s}\,.$$

We divide the time of the throw of the keys in the air when the elevator brakes into two – t_1 is the time from the ejection when the elevator is not yet braking and t_2 is the time the keys spend in the air when the acceleration a is applied to them. Thus $T = 2\sqrt{2h/g} + \Delta t = t_1 + t_2 = 0.861\,\mathrm{s}$. Thus for the initial velocity, we get $v_0 = \sqrt{2gh} = 3.49\,\mathrm{m{\cdot}s^{-1}}$.

Let's denote the velocity that the keys had at the moment of change of acceleration as v_1. Then

$$v_1 = \sqrt{v_0^2 - 2gh_1}\,,$$

where h_1 is the height at which they were at that moment. For time t_1 we then have

$$t_1 = \frac{v_0 - v_1}{g}\,.$$

Time t_2 is then composed of two parts – during the first part the keys were still flying up, and during the second they were falling down. We can write it as

$$t_2 = \frac{v_1}{a} + \sqrt{\frac{2H}{a}}\,.$$

Substituting into the equation for total time, we get

$$t_1 + t_2 = \frac{v_0}{g} + v_1 \frac{g-a}{ag} + \sqrt{\frac{2H}{a}} = T\,.$$

The last unknowns are v_1 and the variable a, which we want to find. From the law of conservation of energy, we know the velocity of the keys at the moment of change of acceleration. This is then converted entirely into a change in potential energy with the new acceleration in the lift, so we have an equation

$$v_0^2 = 2gh_1 + 2a\left(H - h_1\right) = 2\left(g-a\right)h_1 + 2aH \quad \Rightarrow \quad \frac{v_0^2 - 2aH}{2\left(g-a\right)} = h_1\,.$$

From here we substitute in the equation for v_1, which is

$$v_1 = \sqrt{a\frac{2gH - v_0^2}{g-a}}\,.$$

Substituting this into the equation for the times, we have an equation in which the only unknown is the acceleration. We have

$$\sqrt{\frac{g-a}{a}2g\left(H-h\right)} + g\sqrt{\frac{2H}{a}} = gT - v_0\,.$$

We could further modify the equation for a, but we will compute the value of a numerically. We get $a = 7.74\,\mathrm{m\cdot s^{-2}}$. The elevator therefore decelerated with acceleration $a_v = g - a \doteq 2.1\,\mathrm{m\cdot s^{-2}}$.

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Problem 27 ... insidious horn 5 points

At the FYKOS camp, participants measured a car's speed using the frequency shift of the horn. However, they encountered a problem – the horn's pitch changed between individual repetitions. Lego, therefore, came up with the following modification of the experiment: we measured the horn's frequency when the car was approaching us as $f_1 = 437\,\mathrm{Hz}$. Right after that, the frequency $f_2 = 415\,\mathrm{Hz}$ was measured when the car passed us closely. Assuming that neither the car's speed nor the frequency emitted by the horn has changed, what speed was the car traveling at? *Lego really came up with the idea at the camp during the presentations.*

The Doppler shift when the source approaches us is expressed by the relation

$$f_1 = f_0 \frac{v_c}{v_c - v}\,,$$

where f_1 is the frequency which we measure, f_0 is the frequency emitted by the source (in our case, the horn), $v_c = 343\,\mathrm{m\cdot s^{-1}}$ is the speed of sound in the air, and v is the speed of the source (in our case, the speed of the car).

When the source moves away from us, only the sign of the velocity changes, so the following holds

$$f_2 = f_0 \frac{v_c}{v_c + v}\,.$$

We get a system of two equations for the unknowns f_0 a v, where the task is to express v. We can solve the system, for example, by dividing the second equation by the first

$$\frac{f_2}{f_1} = \frac{v_c - v}{v_c + v}\,.$$

We multiply both sides by the denominator on the right side, move all terms containing v to one side, isolate v, and obtain the result as

$$v = v_c \frac{1 - \frac{f_2}{f_1}}{1 + \frac{f_2}{f_1}} = 31.9\,\mathrm{km{\cdot}h^{-1}}\,.$$

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Problem 28 ... incorrect voltage 5 points

A testing water electrolyzer produces $1.43\,\mathrm{g}$ *of hydrogen per hour. The device is powered by direct current from the source connected by cables, each having a resistance of* $3.1\,\mathrm{m\Omega}$*. Although the source exhibits an output voltage of* $1.95\,\mathrm{V}$*, this value is not directly associated with electrolysis. What voltage would be measured if we connected a voltmeter directly to the electrolyzer?* *Jarda produces hydrogen.*

Excluding the voltage necessary for water electrolysis, the source must supply voltage to overcome the ohmic resistance of supply cables. When measuring the voltage using the four-probe method, a lower value is obtained

$$U_\mathrm{e} = U - 2RI\,,$$

where I is the current passing through the entire device. The coefficient 2 must be included because one cable goes from the source to the electrolyzer and the second goes the other way. The amount of released hydrogen, according to Faraday's law of electrolysis, is proportional to the passing current

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{M_{\mathrm{H}_2}}{N_A}\frac{I}{2e}\,,$$

where the coefficient 2 accounts for the fact that two electrons are needed for the formation of every molecule H_2. The molar mass of hydrogen is $2.016\,\mathrm{g{\cdot}mol^{-1}}$. After obtaining I and substituting it into the first equation, the voltage under which the electrolysis occurs is

$$U_\mathrm{e} = U - 2R\frac{2eN_A}{M_{\mathrm{H}_2}}\frac{\mathrm{d}m}{\mathrm{d}t} \doteq 1.71\,\mathrm{V}\,.$$

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Problem 29 ... banded archerfish 6 points

Banded archerfish is a fish species that has found an original way to hunt for food. It approaches the surface and spits out a stream of water to knock down unsuspecting insects nearby. The insect falls into the water and has little time to escape. If the archerfish sees the insect sitting at an angle 35° relative to surface normal, how far must the insect sit from it to be knocked down? Consider that the fish can knock down insects at a maximum height of $3.0\,\mathrm{m}$ above the surface. *The fish splashed Jarda.*

From the last condition in the statement, we get that the speed at which the archerfish can spit water from its mouth is

$$v = \sqrt{2gh}\,,$$

where $h = 3.0\,\mathrm{m}$.

Furthermore, we need to use a protective parabola, whose equation for spraying from zero height is

$$y = -\frac{1}{4h}x^2 + h\,.$$

At the point where this parabola intersects with the direction towards the insect is the farthest position where the food can still be hit. Therefore, we need to determine this specific direction. It might seem to be the given 35° from the task, but the archerfish's eyes are below the water surface, so it is necessary to account for the refraction of light at the water-air interface. Using Snell's law for water with a refractive index of $n = 1.333$, we obtain

$$\beta = \arcsin(n \sin\alpha) = 50^\circ\,.$$

The line $y = x\cot\beta$ intersects with the protective parabola at points

$$0 = \frac{1}{4h}x^2 + x\cot\beta - h \quad \Rightarrow \quad x_{1,2} = 2h\frac{-\cos\beta \pm 1}{\sin\beta}\,,$$

where we are interested in the positive root of our solution. The total distance from the archerfish can, therefore, be at most

$$d = \frac{x_1}{\sin\beta} = 2h\frac{1-\cos\beta}{\sin^2\beta} = 2h\frac{1}{1+\cos\beta} \doteq 3.6\,\mathrm{m}\,.$$

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Problem 30 ... digging up 6 points

Inside a hollow planet, a special life form has evolved. The inhabitants of this vacuum bubble with a radius of $r = 1\,000\,\mathrm{km}$ decided to dig their way up to the planet's surface. Their scientists measured the density of the rock in several places and found that it decreases linearly with distance from the center of the planet. At the surface of their bubble, they measured a density of $9\,000\,\mathrm{kg{\cdot}m^{-3}}$ and $100\,\mathrm{km}$ further from the centre they measured $8\,800\,\mathrm{kg{\cdot}m^{-3}}$. This gave them an estimate that their planet was unlikely to have a radius larger than $5\,000\,\mathrm{km}$.

While digging the tunnel to the planet's surface, they decided to take a lunch break at a distance $R = 3\,000\,\mathrm{km}$ from the center of the planet. But they encountered a strange difficulty

– gravity. There's zero gravitational acceleration inside their bubble, so they were surprised to find something pulling them back down. Calculate the gravitational acceleration at the point of their lunch break. Assume that all material at distance x from the center of the planet has the same density. *Kuba was reading The Wandering Earth.*

We use Newton's shell theorem, which states that the gravitational field strength inside a spherical shell is zero. This is also the reason why there is zero gravitational acceleration inside this hollow planet. At the same time, this means that we can ignore all the mass that is above our explorers.

So we just need to determine how the mass in the sphere below the explorers acts gravitationally on them. In the center of this sphere is a vacuum bubble. Here we will use Gauss's law. It says that a spherical surface acts gravitationally on external objects as if all its mass is concentrated at its center. So we just need to determine the gravitational effect of a spherical surface with radius x and integrate that from r to R.

We know that the density of the planet ρ is some linear function depending on x. So it has the form $\rho(x) = ax + b$. From the measurements of scientists, we know that firstly $9\,000\,\mathrm{kg{\cdot}m^{-3}} = a \cdot 1\,000 \cdot 10^3\,\mathrm{m} + b$, and secondly $8\,800\,\mathrm{kg{\cdot}m^{-3}} = a \cdot 1\,100 \cdot 10^3\,\mathrm{m} + b$. After solving this system of equations, we get the result $a = -2 \cdot 10^{-3}\,\mathrm{kg{\cdot}m^{-4}}$ a $b = 11\,000\,\mathrm{kg{\cdot}m^{-3}}$. Thus, we get

$$\mathrm{d}g = \frac{G}{R^2}\,\mathrm{d}M = \frac{G}{R^2}\rho(x)\,\mathrm{d}V = \frac{G}{R^2}(ax+b)4\pi x^2\,\mathrm{d}x = \frac{4\pi G}{R^2}\left(ax^3 + bx^2\right)\mathrm{d}x\,.$$

We integrate this result from r to R

$$g = \frac{4\pi G}{R^2}\int_r^R \left(ax^3 + bx^2\right)\mathrm{d}x = \frac{4\pi G}{R^2}\left[\frac{a}{4}x^4 + \frac{b}{3}x^3\right]_r^R = \frac{\pi G}{3R^2}\left[3ax^4 + 4bx^3\right]_r^R\,.$$

After some manipulations and substitution, we then get

$$g = \frac{\pi G}{3R^2}\left[3a\left(R^4 - r^4\right) + 4b\left(R^3 - r^3\right)\right] \doteq 5.15\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem 31 ... echooooooooooo — 6 points

In the middle of a long tunnel with radius $R = 15\,\mathrm{m}$ stands a point source of sound which emits a short beep. At a distance $D = 210\,\mathrm{m}$, also in the middle of the tunnel, we hear its intensity level as $60\,\mathrm{dB}$. How long after we initially hear the beep do we stop hearing its echo if each time it reflects off the wall its intensity decreases by $60\,\%$? The lowest sound intensity that can still be heard in the tunnel is $22\,\mathrm{dB}$.

In his younger days, Jarda often visited caves of the Moravian Karst.

The reduction of the sound intensity level takes place in two ways – by reflection, and by propagation in space, since the sound source is a point source. We express the sound intensity level as

$$L = 10\log\left(\frac{I}{I_0}\right)\,,$$

where I is the sound intensity at a given location and $I_0 = 10^{-12}\,\mathrm{W{\cdot}m^{-2}}$ is the intensity of the threshold of hearing.

A point source is characterized by the fact that we can neglect its dimensions with respect to the distance from it. Assume that the source isotropically radiates some power P. The intensity at distance r is thus

$$I = \frac{P}{4\pi r^2}\,.$$

Hence, the intensity decreases, not very surprisingly, with the square of the distance. We can easily calculate the acoustic power of the source from the problem statement and use it in further calculations.

We need to determine the number of reflections at which the sound intensity level will still be greater than 20 dB, which corresponds to an intensity of $100I_0$. Sound is reflected off the walls, so it travels a greater distance to the point where we hear it. Since the situation is rotationally symmetric, we can only consider a 2D cross section. We mirror reflect the point at which we listen around the wall, several times. We then connect each of these points with a sound source and measure the distance. For the n-th reflected point, this will be

$$r_n = \sqrt{(2nR)^2 + D^2}\,.$$

We also note how many times the sound passes through the mirror walls, this represents the number of reflections, which is n. We then multiply the sound intensity calculated from the distance r_n by a factor of 0.4^n. We record the results in the table 1.

Table 1: Dependence of the sound intensity level on the number of reflections from the source.

$\frac{n}{1}$	$\frac{r_n}{\mathrm{m}}$	$\frac{I}{\mathrm{W{\cdot}m^{-2}}}$	$\frac{L}{\mathrm{dB}}$
0	210	$1.0 \cdot 10^{-6}$	60.0
1	212	$3.9 \cdot 10^{-7}$	55.9
2	218	$1.5 \cdot 10^{-7}$	51.7
3	228	$5.4 \cdot 10^{-8}$	47.3
4	242	$1.9 \cdot 10^{-8}$	42.9
5	258	$6.8 \cdot 10^{-9}$	38.3
6	277	$2.4 \cdot 10^{-9}$	33.7
7	297	$8.2 \cdot 10^{-10}$	29.1
8	319	$2.8 \cdot 10^{-10}$	24.5
9	342	$9.9 \cdot 10^{-11}$	19.9

We can see that after nine reflections the sound intensity level dropped below 22 dB. This corresponds to a time delay

$$\Delta t = \frac{r_8 - r_0}{c} \doteq 0.32\,\mathrm{s}\,.$$

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Problem 32 ... maximal activity I

6 points

Jindra has $N_0 = 10^7$ atoms of the isotope ^{211}Bi. This isotope, with a half-life of $T_{\mathrm{Bi}} = 2.14\,\mathrm{min}$, transforms into the isotope ^{207}Tl, which then, with a half-life of $T_{\mathrm{Tl}} = 4.77\,\mathrm{min}$, transforms into the stable isotope ^{207}Pb. What is the maximal activity that the system can reach?

Jindra does not disdain any activity.

It might seem that to answer this question, we need to solve a system of differential equations describing the decay series, but this is not true. The maximal activity in the system will be at time $t = 0$, which we can also prove via simple reasoning.

The decay of a radioactive isotope can be described either by the half-life $T_{1/2}$ or by the decay constant λ. If there were N atoms of a given isotope in the system at time $t = 0$, then at time t there will only be

$$N(t) = N \cdot 2^{-\frac{t}{T_{1/2}}} = N\mathrm{e}^{-\lambda t} \tag{6}$$

atoms of that isotope, as some of them have already decayed. The relationship between the half-life and the decay constant is

$$\lambda = \frac{\ln 2}{T_{1/2}}\,.$$

The activity R of a radioactive isotope (the number of decays per second) depends on the number of atoms N in the system and the decay constant λ

$$R = \lambda N\,.$$

In our system, there are two radioactive isotopes ^{211}Bi and ^{207}Tl. Let's call the instantaneous number of bismuth atoms N_{Bi} and the instantaneous number of thallium atoms N_{Tl}. The activity depends on the instantaneous number of atoms of both isotopes as

$$R = \lambda_{\mathrm{Bi}} N_{\mathrm{Bi}} + \lambda_{\mathrm{Tl}} N_{\mathrm{Tl}}\,, \tag{7}$$

where λ_{Bi} and λ_{Tl} are the decay constants of the given isotopes. Given that $T_{\mathrm{Bi}} < T_{\mathrm{Tl}}$, it follows that $\lambda_{\mathrm{Bi}} > \lambda_{\mathrm{Tl}}$.

Now comes the key part of our reasoning. Because the decay constant of the bismuth isotope is greater than that of the thallium isotope, with the same number of atoms of both isotopes, the bismuth sample will exhibit higher activity. We start purely with bismuth atoms in the amount N_0 in our system. Over time, some of them decay into the thallium isotope. Further, some thallium atoms decay into the stable isotope of lead ^{207}Pb. If M atoms of bismuth have decayed, then there are $N_{\mathrm{Bi}} = N_0 - M$ atoms of bismuth and $N_{\mathrm{Tl}} \leq M$ atoms of thallium in the system. Looking again at equation (7), we see that some bismuth atoms have been replaced by thallium atoms. As mentioned earlier, the thallium isotope has a lower activity than the same amount of the bismuth isotope. But the number of bismuth atoms only decreases according to equation (6), and there can never be more thallium atoms in the system than the number of decayed bismuth atoms. Therefore, the maximal activity in the system occurred at time $t = 0$ and had a value of

$$R_{max} = \lambda_{\mathrm{Bi}} N_0 = 5.40 \cdot 10^4\,\mathrm{s}^{-1}\,.$$

The maximal activity in our system had a value of $54.0\,\mathrm{kBq}$.

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Problem 33 ... immersing

5 points

We have a negligibly thin hollow sphere and a hollow cube, both with volume $V = 5\,\mathrm{l}$ and full of air. What is the ratio of the work required to submerge them under water? The base of the cube is parallel to the surface (we don't rotate it during motion), both bodies are starting at the surface, and we want to get them to a state where they are completely submerged and touching the surface. As a result, sumbit a number greater than 1. *Matěj was in the bathtub.*

Solution via forces

To calculate the work, we will have to overcome the buoyant force – the gravitational force is in this case negligible and compensated by the buoyant force of the air (relative to $V \gg 0$). Its magnitude is always proportional to the volume V of the submerged part of the body according to Archimedes' law as $F = V\rho_\mathrm{v}g$, where ρ_v is the density of water and g is the gravitational acceleration.

The work required to submerge the cube is calculated straightforwardly as

$$W_\square = \int_0^a F\,\mathrm{d}h = \int_0^{\sqrt[3]{V}} g\rho_\mathrm{v}\sqrt[3]{V}^2 h\,\mathrm{d}h = \frac{1}{2}g\rho_\mathrm{v}\sqrt[3]{V}^4 .$$

For a sphere, the situation is a little more complicated – the submerged part is a spherical cap, and its volume is determined by the well-known relation

$$V_\mathrm{v} = \frac{\pi h^2}{3}(3r - h) ,$$

where h is the height of the cap and $r = \sqrt[3]{3V/(4\pi)}$ is the radius of the sphere. We can then write

$$W_\bigcirc = \int_0^{2r} F\,\mathrm{d}h = \int_0^{2r} g\rho_\mathrm{v}\frac{\pi h^2}{3}(3r - h)\,\mathrm{d}h = g\rho_\mathrm{v}\frac{4\pi}{3}r^4 = \sqrt[3]{\frac{3}{4\pi}}g\rho_\mathrm{v}\sqrt[3]{V}^4 .$$

The work for the immersion of the sphere is therefore greater, and the ratio we are looking for is obtained as

$$\frac{W_\bigcirc}{W_\square} = \sqrt[3]{\frac{6}{\pi}} \doteq 1.24 .$$

Solution without forces and without integrals

For simplicity, we will neglect the weight of the air, but we will return to this simplification over time. With this assumption, the work done in immersion is transfers only as the increase in the potential energy of the water. Thus, our question becomes what is the ratio of the increase in potential energy of water when we submerge a sphere versus when we submerge a cube.

The assignment does not say what the surface area is, so we will assume for simplicity that it is infinitely large. In that case, by submerging the object below the surface, we move the water that was previously in its place to the surface. The change in potential energy is $\Delta E_p = mg\Delta h$, where g is the gravitational constant and $m = V\rho$ is the mass of the displaced water, which is the same for both the cube and the sphere, so the ratio of the work done will be equal to Δh.

The Δh is equal to the change of height of the center of gravity of the displaced water. As we have already said, the water is lifted to the water surface, so the new position of the center of gravity will be at the surface. Since the water is homogeneous and both the sphere and the cube are symmetric objects, we can say that the original centers of gravity were located where the centers of the submerged objects are after submergence. That is, for the sphere r below the surface and for the cube $a/2$ below the surface. All this reasoning can be symbolically rewritten as

$$\frac{W_{\bigcirc}}{W_{\square}} = \frac{\Delta E_{p\bigcirc}}{\Delta E_{p\square}} = \frac{mg\Delta h_{\bigcirc}}{mg\Delta h_{\square}} = \frac{r}{a/2}\,,$$

where $r = \sqrt[3]{3V/(4\pi)}$ is the radius of the sphere and $a = \sqrt[3]{V}$ is the side of the cube. By utilizing these expressions and evaluating, we receive the result

$$\frac{W_{\bigcirc}}{W_{\square}} = \frac{\sqrt[3]{3V/(4\pi)}}{\sqrt[3]{V}/2} = \sqrt[3]{\frac{6}{\pi}} \doteq 1.24\,.$$

Finally, we return to the neglection of the weight of air, we can note that the decreases in the potential energy of air will be in the same ratio, so in fact, neglecting air does not affect the resulting ratio at all.

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Problem 34 ... frog on a water lily 6 points

Jarda spends so much time in the garden that he enjoys getting to know its inhabitants. However, a frog sitting on a water lily in the garden pond always gets scared and jumps away. If the water lily leaf has a mass of 43 g *and the mass of the frog is* 150 g*, how far can it jump if it jumps up to a distance of* 2.1 m *on solid ground? Assume that the water lily does not bend and moves only in a horizontal direction and that it moves freely.*

Jarda is woken up by frog croaking at home.

Consider that this frog is highly intelligent and jumps up from the ground at such an angle that it can jump as far as possible. It is a well-known fact that this angle is 45 ° and that the maximum distance that an object launched from the ground at speed v_0 reaches is

$$L = \frac{v_0^2}{g}\,,$$

from which we can find the speed of the frog right after the jump as $v_0 = \sqrt{gL}$.

Thus, with this speed, the frog is able to jump from a surface on which it is sitting. However, when the frog jumps from the water lily, by the law of conservation of momentum, its speed relative to the ground is smaller. Let the initial horizontal velocity of the frog in the reference frame with the water lily be $v_0 \cos\alpha$, where α is the angle at which the frog jumps. Then the law of conservation of momentum has the form

$$Mv_{\mathrm{h}} = mv_{\mathrm{l}}\,,$$

where v_h is the horizontal speed of the frog, M its mass and v_l is the speed of the water lily and m is its mass. In the water lily reference frame, the speed of the frog is $v_0 \cos\alpha = v_\mathrm{l} + v_\mathrm{h}$, from which

$$v_\mathrm{h} = \frac{v_0 \cos\alpha}{1 + \frac{M}{m}} \,.$$

So, at this speed, the frog is moving horizontally with respect to the water surface (with respect to the ground). The time the frog spends in the air is

$$t = 2\frac{v_0 \sin\alpha}{g} \,,$$

so that it can reach a distance of

$$l = v_\mathrm{h} t = \frac{v_0^2 2 \cos\alpha \sin\alpha}{g\left(1 + \frac{M}{m}\right)} \,.$$

We got an interesting result. The best angle for the frog to jump relative to the water lily in its rest frame is again $45\,^\circ$. The maximum distance the frog can reach is

$$l = \frac{v_0^2}{g\left(1 + \frac{M}{m}\right)} = \frac{L}{1 + \frac{M}{m}} \doteq 0.47\,\mathrm{m} \,.$$

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Problem 35 ... relativistic star — 6 points

What would be the radius of a star with the same mass as the Sun, but its redshift would be so large that the wavelength of the radiation coming from its surface would double for an observer at infinity? For simplicity, consider a non-rotating spherically symmetric star.

Karel was thinking about neutron stars and black holes.

The phenomenon described in the problem statement is called a gravitational redshift. For a spherically symmetric gravitational field, we describe it with the equation

$$\frac{\lambda_\infty}{\lambda_0} = \left(1 - \frac{r_\mathrm{S}}{R}\right)^{-1/2} \,,$$

where λ_∞/λ_0 is the ratio of the wavelengths of the radiation at infinity and the source (in this case equal to 2), R is the radius of the star, and r_S denotes its Schwarzschild radius. We determine the Schwarzschild radius from the equation

$$r_\mathrm{S} = \frac{2GM}{c^2} \,,$$

where M is the star's mass and G is the gravitational constant. By substituting and modifying the equation, we get

$$R = \frac{1}{1 - \left(\frac{\lambda_\infty}{\lambda_0}\right)^{-2}} \frac{2GM}{c^2} = \frac{4}{3} r_\mathrm{S} \doteq 3\,938\,\mathrm{m} \,.$$

This number implies that it will not yet be a black hole, but it is not far from it, and such a star would probably collapse into a black hole. Typical neutron stars with masses close to the mass of the Sun can have radii on the order of 10 km.

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Problem 36 ... billiard

6 points

The FYKOS-bird is at the pool table, which is 3.5 by 7 feet and has a hole for balls in each corner. There are white and green balls on the table, which are otherwise the same, fitting just right into the holes and bouncing off the walls and each other with perfect elasticity. The FYKOS-bird originally placed the green one on the longer axis of symmetry of the table. He nudged the white towards the green. The green crashed into one of the corner holes. However, after several impacts from the longer edges of the table, the white ball ended up in the same hole. How many positions on the table are there to place the green ball for this situation to play out? Ignore the rotation of the balls. *Jarda mostly hits the hole with white.*

First, we will show that the sphere flies off at right angles in an elastic impact. Let us denote the initial velocity vector of one of the spheres as $\mathbf{v}_0$, its velocity vector after the collision as $\mathbf{v}$, and the velocity vector of the other sphere as $\mathbf{u}$. Then, the law of conservation of momentum and the law of conservation of energy hold in the form

$$\mathbf{v}_0 = \mathbf{v} + \mathbf{u}, \qquad v_0^2 = v^2 + u^2 \,.$$

By squaring the first equation, we get

$$v_0^2 = v^2 + u^2 + 2\mathbf{v}\cdot\mathbf{u} \,,$$

which when compared with the law of conservation of energy, gives the condition $\mathbf{v}\cdot\mathbf{u} = 0$, or that indeed, the velocity vectors after the collision are perpendicular to each other and the spheres move along perpendicular trajectories.

The table is symmetrical according to two axes. In the beginning, we chose one hole to hit the balls into. Since we are placing the green ball on one of the axes of symmetry, two out of four solutions for the four holes will be the same. On the other hand, because of the symmetry, the solutions for two holes will be different, each of which will be on one side of the table along the longer edge. Thus, we must multiply the number of solutions found for one ball by two at the end.

In an elastic impact with a wall, the tangential component of the sphere's velocity is preserved because the force from the wall acts only perpendicularly. This force will change the perpendicular component of the velocity to the opposite since the sphere's energy is conserved. Therefore, we can think of the wall as a mirror and the sphere's trajectory as a ray. We can stretch this behind the wall and mirror the holes in the table and the other side of the wall. We can do this several times in a row, showing other such reflections on the same and opposite wall behind the first wall.

According to the previous paragraph, we used mirror reflections to project the hole into which the two balls are to fall (see figure). The trajectories of the balls form a right triangle with legs leading from the original location to the real and depicted hole. The point at which the collision occurs thus lies on Thales's circles, which always have centers at a distance of

$$d_n = \frac{2ns}{2}$$

from the hole hit along the shorter side of the pool table. Its length is $s = 3.5$ feet. The radius of the circles is then d_n. The distance of the point of collision from this wall is

$$y_n = \sqrt{d_n^2 - \left(d_n - \frac{s}{2}\right)^2} = s\sqrt{n - \frac{1}{4}}\,.$$

Since the ratio of the longer side $l = 7$ feet to the shorter side s is two, all possible values of y_n must be less than $2s$, which corresponds to

$$\sqrt{n - \frac{1}{4}} < 2\,,$$

and we find that the highest index that satisfies this is $n = 4$. For only one hole, just four such positions have the properties as in the statement. As we commented above, we must multiply this result by two for all remaining holes, quickly verifying that no position lies in the center of the table. The correct answer to the problem is therefore 8.

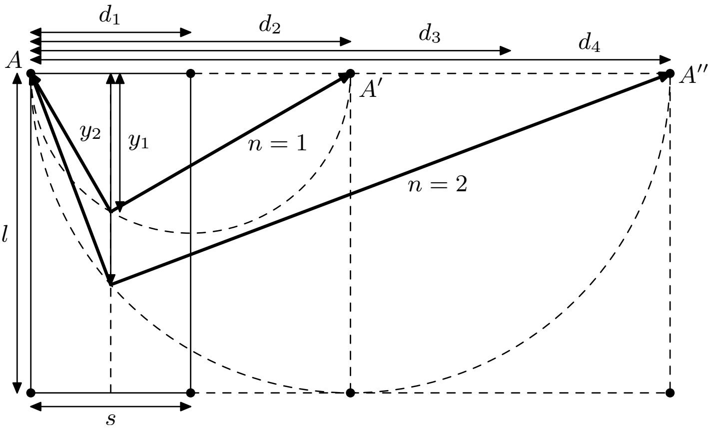

Figure 5: Zrcadlové zobrazení vybrané díry a situace pro $n = 1$ a $n = 2$.

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Problem 37 ... impedance spectroscopy 7 points

In electrochemical experiments, we often encounter a method of sending an AC signal to the device, we then change its frequency and monitor the evolution of the impedance. Consider an alternative circuit, which consists of a parallel connection of a capacitor with capacitance C and a resistor with R_2, which are connected in series with R_1. Determine the largest possible phase shift by which the voltage is delayed behind the current if $4R_1 = R_2$.

Jarda examined data from his bachelor's thesis on electrolyzers.

The impedances of the individual elements are R_1, R_2 and $-i/(C\omega)$. The total impedance can therefore be written as

$$Z = R_1 + \left(\frac{1}{R_2} + iC\omega\right)^{-1} = R_1 + \frac{R_2}{1 + iR_2C\omega} = R_1 + \frac{R_2}{1 + R_2^2C^2\omega^2} - i\frac{R_2^2C\omega}{1 + R_2^2C^2\omega^2}\,.$$

If we plot a graph showing the imaginary component on the vertical axis and the real component of impedance as a function of frequency on the horizontal axis, we find that all points lie on a semicircle centered at $R_1 + R_2/2$ and with radius $R_2/2$. The entire semicircle lies below the real axis. Indeed, if we denote $\sin\psi = 2R_2C\omega/\left(1 + R_2^2C^2\omega^2\right)$, we get

$$\begin{aligned}
Z &= R_1 + \frac{R_2}{2} + \frac{R_2}{2}\exp(-i\psi) = R_1 + \frac{R_2}{2} + \frac{R_2}{2}\left(\cos(\psi) - i\sin(\psi)\right) = \\
&= R_1 + \frac{R_2}{2} + \frac{R_2}{2}\left(\frac{1 - R_2^2C^2\omega^2}{1 + R_2^2C^2\omega^2} - i\frac{2R_2C\omega}{1 + R_2^2C^2\omega^2}\right) = \\
&= R_1 + \frac{R_2}{2}\left(\frac{1 - R_2^2C^2\omega^2}{1 + R_2^2C^2\omega^2} + 1\right) - i\frac{R_2^2C\omega}{1 + R_2^2C^2\omega^2} = R_1 + \frac{R_2}{1 + R_2^2C^2\omega^2} - i\frac{R_2^2C\omega}{1 + R_2^2C^2\omega^2}\,.
\end{aligned}$$

The largest phase shift will be achieved if the absolute value of the ratio of the imaginary to the real component is the largest. Consider a straight line in the graph of the imaginary and real component that is gradually tilted counterclockwise from the vertical axis. This reduces the ratio of the imaginary to the real component. At one point, this line intersects the semicircle, which shows all possible impedances as a function of frequency. Thus, at this point the absolute value of the phase shift is the largest. At the same time, this straight line is now tangent to the circle. We get a right triangle whose hypotenuse is $R_1 + R_2/2$ and one of its legs is $R_2/2$. So the phase shift between voltage and current is finally

$$\varphi = -\arcsin\left(\frac{\frac{R_2}{2}}{R_1 + \frac{R_2}{2}}\right) = -\arcsin\left(\frac{R_2}{2R_1 + R_2}\right) = -\arcsin\left(\frac{2}{3}\right) \doteq -41.8^\circ\,.$$

The minus sign represents that the voltage is delayed behind the current, so the answer is the absolute value of the result.

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Problem 38 ... small hole in the water balloon — 7 points

We have a hollow sphere with thin walls of radius $R = 10.5\,\mathrm{cm}$. This sphere lies on (or is attached to at its lowest point) a horizontal plane; it is completely filled with water and has a hole at its highest point. Where do we need to drill another hole so that the water will spray out as far as possible from the point of contact of the sphere with the plane? We make the hole infinitely small. Enter the result as the angle formed by the line joining this hole and the center of the sphere with the vertical direction (i.e., a vector perpendicular to the plane and pointing away from it forms an angle of 0). *Lego thought, that a nice problem on hydro...*

We could use Bernoulli's law to get the rate at which the water will spray out of the hole, but we will simply use the reasoning from the law of energy conservation (from which Bernoulli himself is derived). As the water sprays, the water level will fall. Thus, the kinetic energy of the splashing water is gained at the expense of a decrease in the potential energy of the water. We can imagine that instead of water splashing out of the newly formed hole, which is there now, an element of water of mass m always teleports in from the top of the sphere, and it also flies out. We can do this reasoning because nothing happens to the water in all the other places in the container (which in turn is a consequence of the fact that we are considering a limitingly small hole; if it were not negligible, currents with non-negligible kinetic energy would be generated in the container). We have also subtly exploited that, thanks to the hole at the top of the vessel, the air pressure at the surface and in the new hole is the same. The splash element will thus have a kinetic energy corresponding to the decrease in potential energy due to the difference in the surface heights and the new hole. If we denote this difference by h, then

$$\frac{1}{2}mv_0^2 = mgh \rightarrow v_0 = \sqrt{2gh}\,.$$

So, we have the magnitude of the velocity at which the water is spraying. We still need the direction. This will be perpendicular to the wall at that point. This is because force is pressure times area, whereas pressure (in fluids) is a scalar quantity, so the direction of the force is given purely by the "direction of the area." Thus, the force pushing the water in the hole pushes it perpendicular to the surface of the hole.

Let us denote the angle between the line of the hole with the center of the sphere and the vertical direction as φ. Then the horizontal component of the initial velocity will be $v_{0x} = v_0 \sin\varphi$, and the vertical component will be $v_{0y} = v_0 \cos\varphi$.

The position of the hole relative to the point where the sphere touches the plane on which it lies can be expressed using the angle φ as $x_0 = R\sin\varphi$ and $y_0 = R(1+\cos\varphi)$. The top of the sphere is, of course, at height $2R$, so we can express the height difference between the hole and the top of the sphere h as $h = R(1-\cos\varphi)$.

Thus, in the vertical direction, the height above the plane will evolve as

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2\,.$$

Putting $y(t_\mathrm{d}) = 0$ gives the time for the water to fall. In general

$$t_\mathrm{d} = \frac{v_{0y} \pm \sqrt{v_{0y}^2 + 2gy_0}}{g}\,,$$

where we are only interested in the positive root, we don't yet account for the initial velocity and position, although we could (depending on preference).

The velocity in the horizontal direction does not change, so the distance from the point where the sphere touches the plane to the point where the water hits will be the product of the horizontal component of the velocity v_{0x} and the time to impact t_d, plus the initial horizontal distance x_0. Together, then, we get

$$x_\mathrm{d} = v_{0x}t_\mathrm{d} + x_0 = v_{0x}\frac{v_{0y} + \sqrt{v_{0y}^2 + 2gy_0}}{g} + x_0\,.$$

Insert for the initial positions and times

$$x_\mathrm{d} = \sqrt{2gh}\sin\varphi\frac{\sqrt{2gh}\cos\varphi + \sqrt{2gh\cos^2\varphi + 2gR(1+\cos\varphi)}}{g} + R\sin\varphi\,.$$

We can see that g is completely removed. Add h, and we get a relation depending only on φ

$$x_\mathrm{d} = \sqrt{2R(1-\cos\varphi)}\sin\varphi\left(\sqrt{2R(1-\cos\varphi)}\cos\varphi + \sqrt{2R(1-\cos\varphi)\cos^2\varphi + 2R(1+\cos\varphi)}\right) + R\sin\varphi\,.$$

We can take out R and get

$$x_\mathrm{d} = R\left(2\sqrt{1-\cos\varphi}\sin\varphi\left(\sqrt{1-\cos\varphi}\cos\varphi + \sqrt{(1-\cos\varphi)\cos^2\varphi + 1 + \cos\varphi}\right) + \sin\varphi\right)\,.$$

where R is a given constant, so we need to find φ that maximizes that bracket. Determining the maximum of such an expression analytically is difficult but probably completely impossible. So we'll plot the bracket into the graph 6.

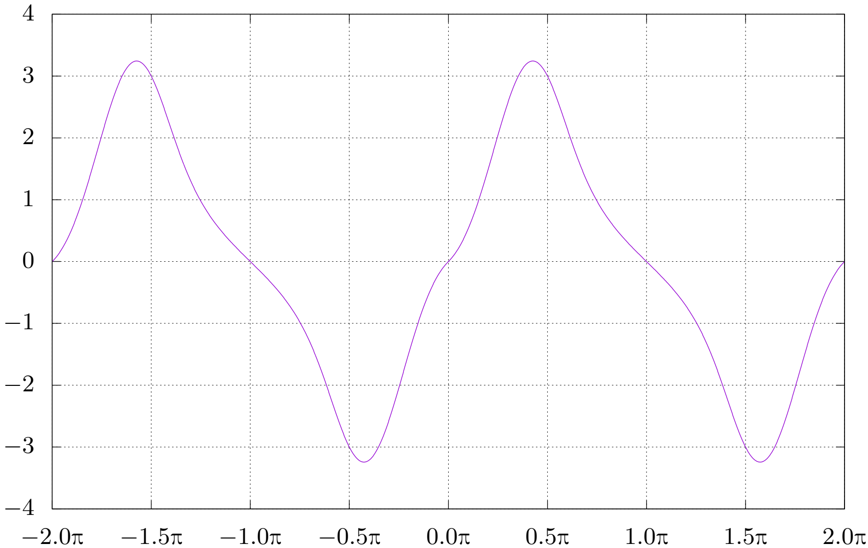

Figure 6: Distance to which the water x_d will penetrate, normalized to the radius of the sphere R depending on the angle of the hole φ.

The graph shows that the bracket takes a maximum for $\varphi = 1.34\,\mathrm{rad}$.

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Problem 39 ... changing of our star — 6 points

Consider a star similar to our sun that emitted most of its energy at a wavelength of $504.7\,\mathrm{nm}$ *and had a radius corresponding to* $0.990R_\odot$*, where* $R_\odot$ *is the radius of today's Sun. Over time, its composition has changed, increasing its radius to* $R_\odot$*, while shifting the wavelength* *of maximum emission by* 3.7 nm *towards the ultraviolet part of the spectrum. How much has the luminosity of this star increased during this time?*

Karel wondered about the history and future of the Sun.

For the luminosity of a black body

$$L = 4\pi R^2 \sigma T^4 \,,$$

where σ is the Stefan-Boltzmann constant. At the beginning, the star has radius R_0 and temperature T_0. After the transformation, its radius changes to $R_1 = R_\odot$ and its temperature to T. We calculate this temperature from the change in maximum wavelength. From Wien's displacement law we have

$$T_0 = \frac{b}{\lambda_0} \,, \qquad T = \frac{b}{\lambda_0 - \Delta\lambda} \,,$$

where $b = 2.898 \cdot 10^{-3}\,\mathrm{m \cdot K}$ is the *Wien constant* and $\Delta\lambda = 3.7\,\mathrm{nm}$ has a negative sign because its shift is to shorter wavelengths. For temperature, we have

$$T = \frac{\lambda_0}{\lambda_0 - \Delta\lambda} T_0 \,.$$

The ratio of luminosities is then equal to

$$\frac{L_1}{L_0} = \frac{R_1^2}{R_0^2} \cdot \left(\frac{\lambda_0}{\lambda_0 - \Delta\lambda}\right)^4 \,.$$

We are interested in how much the luminosity has changed, i.e., the number $L_1/L_0 - 1$ expressed as a percentage. We get

$$\frac{L_1}{L_0} - 1 = \frac{1}{0.99^2} \cdot \left(\frac{504.7\,\mathrm{nm}}{504.7\,\mathrm{nm} - 3.7\,\mathrm{nm}}\right)^4 - 1 = 0.0508 \,,$$

which is 5.08 %.

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Problem 40 ... top of the class 6 points

During one of the boring lessons at school, the students invented their own fun – they threw a small heavy object perpendicularly upwards so that it would be as close to the ceiling as possible, but at the same time, would not touch it. The ceiling is at a height $H = 2.7\,\mathrm{m}$ *from the point of the throw. However, when they did this fun in physics class, the teacher made them measure the initial velocity and its standard deviation as* $(7.0 \pm 0.5)\,\mathrm{m{\cdot}s^{-1}}$. *How likely are the students to hit the ceiling if the distribution of initial velocities is Gaussian?*

Jarda was listening to the Vašek's stories.

The velocity required to hit the ceiling is

$$u = \sqrt{2gH} = 7.28\,\mathrm{m{\cdot}s^{-1}} \,.$$

Let $v_0 = 7.0\,\mathrm{m{\cdot}s^{-1}}$ denote the mean initial velocity and $\Delta v = 0.5\,\mathrm{m{\cdot}s^{-1}}$ the standard deviation of the previous quantity. Then, the Gaussian curve for the given values has the form of

$$f(v) = \frac{1}{\sqrt{2\pi}\Delta v} \exp\left(-\frac{(v - v_0)^2}{2\,(\Delta v)^2}\right) .$$

We can find the probability that the velocity of the throw is greater than v by integrating the Gaussian curve in the limits from u to ∞ as

$$p = \frac{1}{\sqrt{2\pi}\Delta v} \int_u^\infty \exp\left(-\frac{(v - v_0)^2}{2\,(\Delta v)^2}\right) \mathrm{d}v \ = 28.8\,\% \doteq 29\,\% ,$$

We had to calculate the integral numerically (e.g., using WolframAlpha).

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Problem 41 ... pressure difference 6 points

Jindra has a solid cylindrical tube with a $H = 50.0\,\mathrm{cm}$ length and a radius of $r = 4.00\,\mathrm{cm}$. One end of the tube is airtight sealed with a lid. Then, he held the tube by the lid and began to submerge it perpendicularly beneath the sea level with the open end. What will be the maximum difference between the pressures acting on the top and bottom lids of the tube during submersion? Jindra submerged the tube to a depth allowed by the length of his arm, $l = 65.0\,\mathrm{cm}$. The density of water is $\rho = 1\,024\,\mathrm{kg{\cdot}m^{-3}}$. The air at sea level is under normal conditions. Both sea water and air have the same temperature.

Jindra didn't know how else to entertain himself on a holiday by the sea.

Let's denote the depth of the bottom edge of the tube as d. The water inside the tube will rise above the bottom edge and compress the air inside. Let's denote the water column height above the bottom of the tube as h. Equilibrium occurs when the water pressure $p_\mathrm{a} + \rho g(d - h)$ balances the pressure of the compressed air inside the cylinder. The gravitational acceleration is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$, and the atmospheric pressure is $p_\mathrm{a} = 101\,325\,\mathrm{Pa}$ (see constants). Due to the slow submersion of the tube beneath the surface, air compression occurs isothermally. The air inside has the same temperature as the atmosphere and the sea.

As long as the tube lid is above the surface, atmospheric pressure p_a is applied to the top of the lid. The air inside the tube, compressed at a pressure $p_\mathrm{a} + \rho g(d - h)$, pushes on its bottom side. The difference in pressure acting on the lid is $\rho g(d - h)$.

The quantity $d - h$ is the depth of the water surface in the tube below sea level. If we sink the tube deeper, we increase d, the height of the water column in the tube will also increase h, but less than the increase in d. If the water level of the column remained at the same depth $d - h$ below sea level as before, the water pressure would remain the same, but the air volume would be less – the air would push the water level of the column down. If the height of the water column h remained the same when submerged, the air would have the same pressure as before, but the depth $d - h$ of the column would be greater than before – the water pressure would push the air in the tube.

Therefore, the depth $d - h$ increases continuously as the tube is submerged. Thus, the pressure difference $\rho g(d - h)$ increases until the lid is submerged at sea level.

The pressure applied from above changes once the lid is submerged below sea level. Now, the pressure of the water column above the lid and the atmosphere $p_\mathrm{a} + \rho g(d - H)$ is acting on the lid from above. From below, the pressure of the compressed air $p_\mathrm{a} + \rho g(d - h)$ is applied. The pressure difference is $\rho g(H - h)$. As we sink the tube deeper, the air is compressed under increasing water pressure. As the water column height inside the h increases, the pressure difference across the lid $\rho g(H - h)$ decreases.

The maximum pressure difference, therefore, occurs when the tube lid is aligned with the sea surface. The isothermal compression of the air inside the tube is described by equation:

$$p_1 V_1 = p_2 V_2 \,,$$

where p_1, p_2 are the initial and final pressures and V_1, V_2 are the initial and final gas volumes. The initial air pressure in the tube before plunging is $p_1 = p_\mathrm{a}$ and the initial volume is $V_1 = \pi r^2 H$. The final volume of air is $V_2 = \pi r^2 (H - h)$ and the final pressure is

$$p_2 = \frac{V_1}{V_2} p_1 = \frac{\pi r^2 H}{\pi r^2 (H - h)} p_\mathrm{a} = \frac{H}{H - h} p_\mathrm{a} \,.$$

The pressure of the water in the tube when the lid is at sea level $d = H$ is

$$p_2 = \rho g(H - h) + p_\mathrm{a} \,.$$

These two pressures must be equal, so we get the equation for h

$$\frac{H}{H - h} p_\mathrm{a} = \rho g(H - h) + p_\mathrm{a} \qquad \Rightarrow \qquad h^2 - \left(\frac{p_\mathrm{a}}{\rho g} + 2H\right) h + H^2 = 0 \,,$$

thus

$$h_{1,2} = \frac{1}{2}\left[\frac{p_\mathrm{a}}{\rho g} + 2H \pm \sqrt{\left(\frac{p_\mathrm{a}}{\rho g} + 2H\right)^2 - 4H^2}\right] = \frac{p_\mathrm{a}}{2\rho g} + H \pm \sqrt{\left(\frac{p_\mathrm{a}}{2\rho g}\right)^2 + \frac{p_\mathrm{a} H}{\rho g}} \,.$$

The height of the water column h must be less than the height of the tube H, so we are only interested in the root of the quadratic equation with the minus sign

$$h = \frac{p_\mathrm{a}}{2\rho g} + H - \sqrt{\left(\frac{p_\mathrm{a}}{2\rho g}\right)^2 + \frac{p_\mathrm{a} H}{\rho g}} \,.$$

This is plugged into the equation for the pressure difference acting on the lid

$$\Delta p = \rho g(d - h) = \rho g(H - h) = \rho g \sqrt{\left(\frac{p_\mathrm{a}}{2\rho g}\right)^2 + \frac{p_\mathrm{a} H}{\rho g}} - \frac{p_\mathrm{a}}{2} \,,$$

which gives the highest pressure difference $\Delta p = 4.80\,\mathrm{kPa}$.

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Problem 42 ... fireworks 6 points

After the competition, the FYKOS organizers planned a fireworks display. They will launch their firework at $45\,\mathrm{m\cdot s^{-1}}$ perpendicularly upwards, which will disintegrate into many small pieces in $3.3\,\mathrm{s}$. These fly off in all directions from the point of disintegration at $15\,\mathrm{m\cdot s^{-1}}$ relative to the reference frame of the original firework and glow for $5.5\,\mathrm{s}$. Determine the volume of space into which the fragments have managed to spread when they are extinguished.

Jarda likes to toast on fireworks.

The velocity of the reference frame of the firework at the moment of disintegration is $u = v_0 - gT = 12.63\,\mathrm{m\cdot s^{-1}}$ in the upward direction, where $T = 3.3\,\mathrm{s}$ is the time of the disintegration of the firework since launch and $v_0 = 45\,\mathrm{m\cdot s^{-1}}$ is the initial velocity. The problem is rotationally symmetric, so let us consider a cut through only one plane in which we introduce coordinates y upward and x to one of the sides so that the explosion occurred just on the $x = 0$ axis. Next, we will investigate the individual fragments into which the firework breaks up. Let us denote by α the angle of the fragments with respect to the ground, $v = 15\,\mathrm{m\cdot s^{-1}}$ the velocity of the fragments after the disintegration of the firework, $\tau = 5.5\,\mathrm{s}$ the time of their glow. The fragment then moves in the horizontal direction with velocity $v_x = v\cos\alpha$, while in the vertical direction its velocity varies due to gravitational acceleration as $v_y = u + v\sin\alpha - gt$, where t is the time since the disintegration of the firework. Integrating the two velocities with respect to time gives the dependence of position on time as

$$x = v\cos\alpha t,\; y = H + ut + v\sin\alpha t - \frac{1}{2}gt^2\,,$$

using $H = v_0T - 1/2\cdot gT^2 = 95.1\,\mathrm{m}$ as the disintegration height of the original firework.

We can notice that these coordinates form a circle with centre at $H + ut - 1/2\cdot gt^2$ and radius vt depending on the angle α. Now we need to find out if all the fragments are still in the air when the light goes out, or if some have already hit the ground. Plugging τ into the equation for the y coordinate and putting $y = 0$ gives the condition

$$0 = H + u\tau + v\sin\alpha_\mathrm{d}\tau - \frac{1}{2}g\tau^2 \quad\Rightarrow\quad \sin\alpha_\mathrm{d} = \frac{\frac{1}{2}g\tau^2 - H - u\tau}{v\tau} = 0.196\,.$$

The fragments, whose original angle was thus less than α_d, hit the ground while the others are still in the air. The shape whose volume we are now seeking is thus a spherical canopy. We calculate the volume of the spherical canopy as $V = \pi h^2\left(3r - h\right)/3$, where r is the radius of the sphere and h is the height from the cut-off wall. In our case, h is the height above the ground of the fragments that flew perpendicularly upwards in time τ after the disintegration of the firework, i.e.

$$h = H + \left(u + v\right)\tau - \frac{1}{2}g\tau^2 = 98.7\,\mathrm{m}\,,$$

while the radius is $r = v\tau = 82.5\,\mathrm{m}$. The search volume is thus

$$V = \frac{\pi\left(H + \left(u+v\right)\tau - \frac{1}{2}g\tau^2\right)^2}{3}\left(2v\tau - H - u\tau + \frac{1}{2}g\tau^2\right)\,,$$

$$V = \frac{\pi\left(v_0\left(T+\tau\right) - \frac{1}{2}g\left(T+\tau\right)^2 + v\tau\right)^2}{3}\left(2v\tau - v_0\left(T+\tau\right) + \frac{1}{2}g\left(T+\tau\right)^2\right)\,,$$

which is approximately $V \doteq 1.52 \cdot 10^{6}\,\mathrm{m}^{3}$.

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Problem 43 ... electron collision 6 points

In one accelerator, colliding electrons are flying towards each other, each with an energy of $104.5\,\mathrm{GeV}$. *In a second accelerator, electrons with an energy of* $209\,\mathrm{GeV}$ *are flying towards a target made of stationary electrons. How many times more energy is available for the creation of matter in the first accelerator compared to the second?*

Jindra felt like colliding his electrons.

The electrons in both accelerators are highly relativistic $E \gg m_0 c^2$, where $m_0 = 9.109 \cdot 10^{-31}\,\mathrm{kg} \doteq 511.0\,\mathrm{keV/c^2}$ is the rest mass of an electron, which we can find in the "overview of constants". The sum of the energies of the colliding particles is the same in both accelerators, but this does not determine the energy of the collision. In the second accelerator, the center of mass of the system of both electrons is also moving. Due to the conservation of momentum during the collision, the center of mass will move after the collision as well – thus, part of the kinetic energy is associated with the motion of the center of mass and cannot be used to create new particles. To find the energy available for the creation of new particles in the second accelerator, we must also study it in the coordinate system associated with the center of mass of the colliding electrons. In the first accelerator, the situation is simple – we are already in the system associated with the center of mass of the colliding electrons. Both electrons are identical, and both fly towards each other with the same kinetic energy, thus also with the same momentum and velocity. The energy available for the creation of new particles is the total energy of both electrons

$$E_1 = 2E_A = 209\,\mathrm{GeV}\,,$$

where we denoted E_A the energy of one electron in the first accelerator.

In the second accelerator, the moving electron has energy

$$E_B = \gamma m_0 c^2\,,$$

where $E_B = 209\,\mathrm{GeV}$ and γ is the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,.$$

The Lorentz factor and the velocity of the electron are

$$\gamma = \frac{E_B}{m_0 c^2} = 4.090 \cdot 10^{5}\,,$$

$$v = c\sqrt{1 - \frac{1}{\gamma^2}} = c\sqrt{1 - \left(\frac{m_0 c^2}{E_B}\right)^2}\,.$$

The center of mass of the system moves with velocity

$$u = \frac{\gamma m_0 v}{\gamma m_0 + m_0} = \frac{\gamma}{\gamma + 1} v$$

towards the stationary electron. Now, we must relativistically transform the velocities of both electrons and determine their velocity relative to the center of mass. The velocity of the flying electron relative to the center of mass is

$$v_1 = \frac{v - u}{1 - \frac{uv}{c^2}} = \frac{v - \frac{\gamma}{\gamma+1} v}{1 - \frac{\gamma}{\gamma+1} \frac{v^2}{c^2}} = \frac{\frac{1}{\gamma+1}}{1 - \frac{\gamma}{\gamma+1} \frac{v^2}{c^2}} v = \frac{1}{\gamma + 1 - \gamma \frac{v^2}{c^2}} v \,.$$

Now we will use the definition of γ and substitute it into the equation

$$v_1 = \frac{1}{1 + \sqrt{1 - \frac{v^2}{c^2}}} v = \frac{1}{1 + \frac{1}{\gamma}} v = \frac{\gamma}{\gamma + 1} v = u \,.$$

The stationary electron moves with the speed

$$v_2 = u = \frac{\gamma}{\gamma + 1} v \,,$$

towards the center of mass. Therefore, the total energy available during the collision is

$$E_2 = 2 \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} m_0 c^2 \,.$$

We simplify the fraction

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\gamma^2}{(\gamma+1)^2} \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\gamma^2}{(\gamma+1)^2} \left(1 - \frac{1}{\gamma^2}\right)}} = \frac{1}{\sqrt{1 - \frac{\gamma-1}{\gamma+1}}} = \sqrt{\frac{\gamma + 1}{2}} \,.$$

The total energy available during the collision is

$$E_2 = 2 \sqrt{\frac{\gamma + 1}{2}} m_0 c^2 \doteq 462.2\,\mathrm{MeV} \,.$$

The ratio of the energies available for creating new particles in the first accelerator relative to the second is

$$\eta = \frac{E_1}{E_2} = 452.2 \doteq 452 \,.$$

Although the sum of the kinetic energies of the electrons is the same in both accelerators, and thus the same work had to be done to accelerate them, the first accelerator offers 640 times more energy available for the creation of new particles. That's why modern large accelerators like the LHC are built to collide particles flying towards each other.

We would reach the same conclusion even if we summed the energies and momenta of all particles, subtracted their squares, and took the square root

$$E^2 = \left(\sum_i E_i \right)^2 - c^2 \left(\left| \sum_i \mathbf{p}_i \right| \right)^2 \,.$$

For the first accelerator, we get

$$E_1 = \sqrt{(2E_A)^2 - c^2 \left(|0|\right)^2} = 2E_A = 209\,\mathrm{GeV} \,.$$

For the second accelerator, it turns out

$$E_2 = \sqrt{(E_B + m_0c^2)^2 - (\gamma m_0 v c)^2} = \sqrt{((\gamma+1)m_0c^2)^2 - \left(\gamma m_0 c^2 \sqrt{1 - \frac{1}{\gamma^2}}\right)^2} =$$

$$= m_0c^2 \sqrt{(\gamma+1)^2 - (\gamma^2 - 1)} = m_0c^2 \sqrt{2\gamma + 2} = 2\sqrt{\frac{\gamma+1}{2}} m_0c^2 \doteq 462.2\,\mathrm{MeV}\,.$$

We arrived at the same result as with the more laborious method of calculating velocity relative to the center of mass. The ratio of available energies for creating new particles between our two accelerators again turns out the same

$$\eta = \frac{E_1}{E_2} = 452.2 \doteq 452\,.$$

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Problem 44 ... we want to breathe on Everest 7 points

What would be the air pressure at sea level if the Earth had an atmosphere with the same temperature lapse rate as it does today (i.e., a linear decrease of temperature by $0.65\,\mathrm{^\circ C}$ *per* $100\,\mathrm{m}$ *altitude), but at an altitude* $H = 8\,850\,\mathrm{m}$ *above sea level (on Mount Everest), the pressure* p_a *would be the same as it is today at sea level? Consider the sea level temperature* $T_0 = 15\,\mathrm{^\circ C}$.

Karel was thinking about the atmosphere again.

The atmosphere holds together due to the Earth's gravitational force; therefore, we perceive air pressure as hydrostatic pressure. The change in pressure with altitude is

$$\mathrm{d}p = -\rho g \,\mathrm{d}h\,,$$

where ρ is the unknown density of air at the given altitude and the gravitational acceleration g changes very little with altitude, so we consider it to be constant. Furthermore, we know that for air, the state equation holds, from which we express the dependence of density on pressure and temperature as

$$pV = nRT \quad \Rightarrow \quad \rho = \frac{p}{T}\frac{M_\mathrm{m}}{R}\,,$$

where $M_\mathrm{m} = 28.9\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of air. We can find it in the tables, on the internet, or express it from the values of normal pressure, density, and temperature in a list of constants. We are also familiar with the dependence of temperature on altitude, which is equal to $T = T_0 - \tau h$, where $\tau = 0.65\,\mathrm{^\circ C/100\,m}$.

By combining the equations, we get a differential equation for pressure as a function of height

$$\frac{\mathrm{d}p}{p} = -\frac{M_\mathrm{m} g}{RT_0} \cdot \frac{\mathrm{d}h}{1 - \frac{\tau}{T_0}h}\,.$$

We integrate the equation and substitute the boundary conditions: the pressure at altitude H is p_a, and the pressure at sea level $0\,\mathrm{m}$ is p_0

$$[\ln p]_{p_0}^{p_\mathrm{a}} = -\frac{M_\mathrm{m} g}{RT_0}\left[\ln\left(1 - \frac{\tau}{T_0}h\right) \cdot \left(-\frac{T_0}{\tau}\right)\right]_0^H$$

from where

$$p_0 = p_a \left(1 - \frac{\tau}{T_0} H\right)^{-\frac{M_m g}{R\tau}} \doteq 327\,\mathrm{kPa}\,.$$

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Problem 45 ... a lake on a mirror reloaded 7 points

On a table, Jindra laid a hollow spherical mirror with a radius of curvature of $r = 2.00\,\mathrm{m}$ and a diameter $D = 12.0\,\mathrm{cm}$. The optical axis of the mirror points upwards. On the mirror, he poured water with a refractive index $n = 1.33$ so that the water formed a surface with a diameter $d = 6.00\,\mathrm{cm}$. Jindra illuminated the whole surface of the mirror with rays of light parallel to the optical axis. At what height above the table would an image with the smallest outer diameter possible be created on a hypothetical screen?

Jindra misses summer swimming.

The focal point of a concave mirror is the point where reflected rays parallel to the optical axis intersect. The focal length is the distance from the focal point to the vertex of the mirror (the point on the mirror's surface lying on the optical axis). In the paraxial approximation, we assume that the angles of all incoming and reflected light rays with the optical axis are small, i.e., $\alpha \ll 1$. Parallel rays to the optical axis hitting the mirror's surface at a perpendicular distance h from the optical axis will reflect at an angle $\alpha \approx 2h/r$. Therefore, in the paraxial approximation, all reflected rays intersect the optical axis at the same distance

$$f_0 = \frac{h}{\alpha} = \frac{r}{2}$$

independent of h. The focal length of a hollow spherical mirror is equal to half the radius of curvature. Therefore, the part of the mirror not covered by water has a focal length of

$$f_0 = \frac{r}{2} = 1.00\,\mathrm{m}.$$

The part of the mirror covered with water has a shorter focal length. The perpendicular water surface does not affect the direction of the rays coming parallel to the optical axis. However, the rays reflected from the mirror strike the water-air interface at an angle to the perpendicular $\alpha \approx \approx 2h/r$, where h is the perpendicular distance from the optical axis. According to Snell's law, these rays break at an angle $\alpha' \approx n\alpha$ and intersect the optical axis closer than the original focal length f_0, at a distance of

$$f = \frac{h}{\alpha'} = \frac{r}{2n} = 0.752\,\mathrm{m}.$$

The light reflected by the outer part of the mirror without water forms a ring of light with an outer diameter of

$$\delta_0 = D\frac{|f_0 - x|}{f_0},$$

where x is the distance of the hypothetical diaphragm from the top of the mirror. The central part of the mirror covered with water then forms a circle of light with a diameter of

$$\delta = d\frac{|f - x|}{f}.$$

The smallest outer diameter of the image occurs at a distance x where $\delta_0 = \delta$. For this position, $f < x < f_0$ necessarily holds. We, therefore, solve the equation

$$D\frac{f_0 - x}{f_0} = d\frac{x - f}{f}$$

for x. Substitute $f = f_0/n$, and we have the equation

$$D\frac{f_0 - x}{f_0} = nd\frac{x - \frac{f_0}{n}}{f_0},$$

which we further manipulate to

$$x(D + nd) = Df_0 + df_0.$$

The solution to the equation is

$$x = \frac{f_0(D + d)}{D + nd},$$

which gives $x = 0.901\,\mathrm{m}$.

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Problem 46 ... coin funnel 7 points

Consider a heavy hemispherical bowl of 19 cm *in diameter and take a point mass that can move in the bowl completely frictionless. We release it from the bowl's edge with velocity* $0.8\,\mathrm{m\cdot s^{-1}}$ *in the horizontal direction. What is the minimum height above the bottom of the bowl that the point mass can reach as it travels through the bowl?* *Jarda's wallet spilled out on the street.*

It is always possible to determine the position of a mass point using two parameters. Firstly, the angle θ represents the angle between the position vector of the mass point and the horizontal plane, originating from the center of the upper circle. Secondly, the angle φ describes the translation around the hemisphere's rotational axis from the beginning of the motion. From the initial condition we have $\theta(t = 0) = 0\,^\circ$ and we can put $\varphi(t = 0) = 0\,^\circ$. Moreover, let us denote $R = \frac{19\,\mathrm{cm}}{2} = 9.5\,\mathrm{cm}$.

Two forces act on the mass point during its motion in the hemisphere -– the weight and the bowl's reaction. Looking from above, we can see that the weight always acts downwards, and the bowl's reaction always acts in the direction of the center of the hemisphere. Thus, there is no force that, when viewed from above, acts tangentially on the motion of the point. That implies the law of conservation of the vertical component of the angular momentum of the mass point in relation to the rotational axis of symmetry of the whole hemisphere. We can write it as

$$(\mathbf{L})_z = m\,(\mathbf{r} \times \mathbf{v})_z = mr_xv_y - mr_yv_x\,.$$

Let's consider a rotation of the axes such that $r_y = 0$. Then $r_x = R\cos\theta$. The velocity component v_y is tangent to the horizontal circle on which the mass point currently lies, and the component's magnitude is $v_y = R\cos\theta\dot{\varphi}$, where $\dot{\varphi}$ is the angular velocity about the axis of symmetry of the whole hemisphere. We get the law of conservation of the vertical component of angular momentum as $L_z = mR^2\cos^2\theta\dot{\varphi}$.

Since the mass point is moving without friction, the law of conservation of mechanical energy also holds throughout the problem. The law of conservation of mechanical energy is

$$E = \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\cos^2\theta\dot{\varphi}^2\right) - gR\sin\theta\,.$$

Even though we can already calculate the rest of the problem from these two conserved quantities, we still have to determine their values from the initial conditions. For L_z we have $L_z = mR^2\frac{v}{R} = mvR$, while for the energy we have $E = \frac{1}{2}mv^2$. From the conservation law of L_z we substitute $\dot{\varphi}$ into the equation for energy and we get

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \frac{v^2}{\cos^2\theta}\right) - gR\sin\theta\,,$$

where we express $\dot{\theta}^2$ as

$$\dot{\theta}^2 = \frac{1}{R^2}\left(2gR\sin\theta - v^2\tan^2\theta\right)\,.$$

Now comes a crucial consideration. For an arbitrarily small initial velocity v, increasing θ will lead to a situation where the right-hand side of the equation is negative due to the properties of the function $\tan x$. However, the left side is the square of a real number, and this side must always be non-negative. The critical angle θ occurs when the angular velocity $\dot{\theta}$ is zero. At this point, its sign changes, and the mass point rises again. Thus, we get the equation

$$2gR\sin\theta_c = v^2\tan^2\theta_c \quad \Rightarrow \quad \sin^2\theta_c + \frac{v^2}{2gR}\sin\theta_c - 1 = 0\,.$$

From this quadratic equation, we find $\sin\theta_c$ as

$$\sin\theta_c = -\frac{v^2}{4gR} + \sqrt{\frac{v^4}{16g^2R^2} + 1}\,,$$

where we chose a positive sign because we expect a positive value of the result. This expression is always less than 1, so we can always find the angle θ_c. This result corresponds to the minimum height

$$h = R\left(1 - \sin\theta_c\right) = R\left(1 + \frac{v^2}{4gR} - \sqrt{\frac{v^4}{16g^2R^2} + 1}\right) = 1.5\,\mathrm{cm}\,,$$

which the mass point can reach.

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Problem 47 ... a point jumps over a cuboid 7 points

Once upon a time in the land of geometric shapes, a point mass was sitting still on a horizontal plane when it suddenly spotted a cuboid $h = 20.0\,\mathrm{cm}$ tall and $l = 280\,\mathrm{cm}$ long approaching it with a speed $v_k = 6.80\,\mathrm{m{\cdot}s^{-1}}$. Our poor point mass realizes the time to jump away ran out, so it just jumps over it. What is the minimum speed that the point mass needs to jump in order to be able to jump over the block? The point mass does not only need to jump vertically upwards. *This idea has been in Lego´s ideapad for such a long time*

If the point jumps along the optimal (the one that has the minimum speed) trajectory, it surely touches the upper front edge, stops rising exactly above the center of the cuboid, and touches the upper back edge as it is falling. There is an unlimited amount of possible trajectories satisfying the problem, so we look just for the one that needs the least speed.

When we denote the speed with which the point jumped off the plane v_0 and the speed it will have when touching the front edge of the cuboid v_1, then the law of conservation of energy tells us

$$\frac{1}{2}mv_0^2 = mgh + \frac{1}{2}mv_1^2$$
$$v_0 = \sqrt{2gh + v_1^2}\,,$$

so it is enough to minimize the speed v_1 and plug it all in.

If we start watching the motion of the point mass at the moment it touches the front edge, the conditions that it stops rising exactly above the center of the cuboid and that it touches the back edge falling are equivalent (the reader can prove this mathematically). We will use the first one.

The time it takes for a point to reach the same horizontal position as the center of the cuboid is obtained by dividing their distance (which is at first equal to half the length of the cuboid, or $l/2$) by their relative speed, which is $v_{1x} + v_k$. We denoted the horizontal component of the velocity vector v_{1x}. This is positive if the point jumps toward the block and negative if it jumps backward. Intuitively, to minimize the required speed, one would rather jump so that the mutual speed adds up, but it is not necessary to assume this.

We might obtain the time for the point to stop rising by dividing its vertical velocity component (let's denote it by v_{1y}) by the gravitational acceleration g.

Then the condition that it stops rising above the center of the cuboid can be expressed as the equality of these two times

$$\frac{l/2}{v_{1x} + v_k} = \frac{v_{1y}}{g}\,.$$

This condition therefore limits for which combinations of v_{1x} and v_{1y} the point stops rising exactly above the center of the cuboid. However, we are interested in just one of these combinations which also minimizes the speed $|v_1| = \sqrt{v_{1x}^2 + v_{1y}^2}$. Minimizing the speed is the same as minimizing the speed squared, so we raise the power and get rid of the square root. We plug v_{1y} obtained from the stopping condition and get

$$v_1^2 = v_{1x}^2 + v_{1y}^2 = v_{1x}^2 + \left(\frac{lg}{2}\frac{1}{v_{1x} + v_k}\right)^2\,,$$

where we expressed the quantity to minimize (v_1^2) as a single-variable function of (v_{1x}). Therefore we find the extrema by setting the derivative with respect to the only variable to zero

$$\frac{\mathrm{d}v_1^2}{\mathrm{d}v_{1x}} = 2v_{1x} + \frac{l^2g^2}{4}\frac{-2}{(v_{1x} + v_k)^3} = 0 \qquad \Rightarrow \qquad v_{1x} = \frac{l^2g^2}{4}\frac{1}{(v_{1x} + v_k)^3}\,,$$

which is a quartic equation we really don't want to solve (not even reasoning which one of the 4 solutions truly represents the global minimum), instead, we plot v_1^2 against v_{1x} in a graph 7.

We can clearly see the expression reaches the minimum for $v_1^2 \doteq 3.789\,\mathrm{m^2{\cdot}s^{-2}}$ for $v_{1x} \doteq$ $\doteq 0.487\,\mathrm{m{\cdot}s^{-1}}$ (which matches the intuition that one rather jumps toward). If one were to be

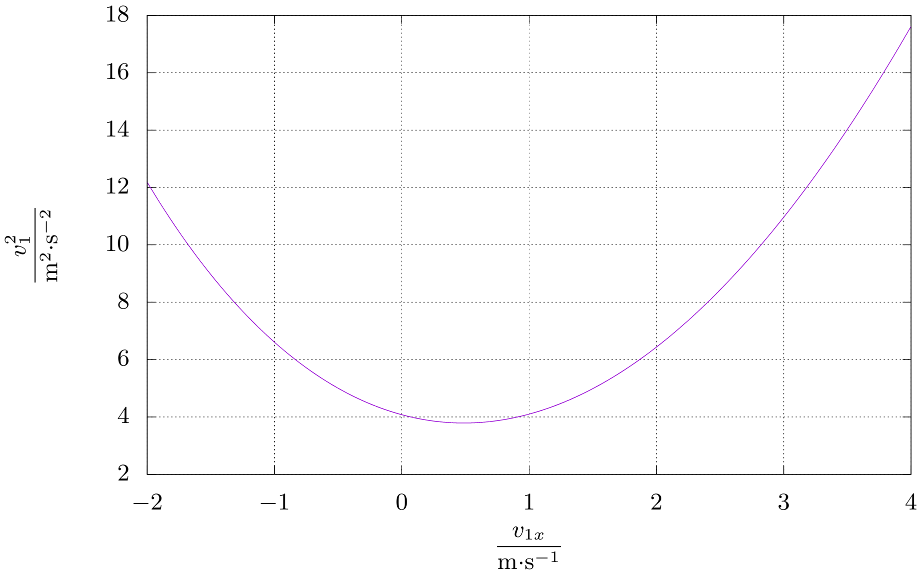

Figure 7: The graph of v_1^2 against v_{1x}

concerned about whether there might not be some even smaller minimum, just remember that $v_1^2 > v_{1x}^2$ must hold, so $|v_{1x}| > \sqrt{3.789\,\mathrm{m^2{\cdot}s^{-2}}} \doteq 1.95\,\mathrm{m{\cdot}s^{-1}}$ is truly the smallest.

Then we just have to remember to plug that back into v_1^2 and into v_0, which gives us that the point must jump off the ground with a speed equal to at least

$$v_0 = \sqrt{2gh + v_1^2} \doteq 2.78\,\mathrm{m{\cdot}s^{-1}}\,.$$

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Problem 48 . . . oscillating lens 8 points

Let's place an isotropic light source on the optical axis of a lens with a focal length of $f = 8.5\,\mathrm{cm}$ at a distance $A = 11\,\mathrm{cm}$ from its center. Then, we attach a spring to the lens, allowing it to perform torsional oscillations. The axis of oscillation is perpendicular to the optical axis of the lens. The moment of inertia of the lens with respect to this axis is $J = 63\,\mathrm{kg{\cdot}mm^2}$, and the torque acting on the lens is proportional to the angle of rotation from the equilibrium position as $M = -c\varphi$, where $c = 3.7\,\mathrm{mJ}$. What is the maximum speed at which the image of a point of light moves if the lens is deflected $5\,^\circ$ from its equilibrium position and let go?

Jarda wanted to combine optics and oscillation.

Let us first consider what movement the lens makes. From the problem statement, we know the momentum is $M = -c\varphi$, and from the rotational analogy of Newton's second law, we know that the time change of angular momentum is equal to the net torque. We therefore get

$$J\ddot{\varphi} = -c\varphi \,,$$

which is incidentally the equation of a harmonic oscillator. Its solution is

$$\varphi = \varphi_0 \cos(\omega t) \ ,$$

where $\omega = \sqrt{\frac{c}{J}}$ is the angular velocity of the lens' oscillation about its axis.

Using simple geometry, the position of an object relative to time on the lens' optical axis can be determined by the equation $a = A\cos\varphi$. Similarly, the object's distance from the axis can be calculated using the equation $y = A\sin\varphi$. Then, we can use the thin lens formula to display the image of a point located at a distance of $a' = \frac{af}{a-f}$ on the lens axis. Because of the rule that a ray passing through the center of the lens is not refractive, we know that the image will be at the junction of the object and the center of the lens. This line is deflected by an angle φ from the optical axis, and the position of the image is thus at a distance

$$A' = \frac{a'}{\cos\varphi} = \frac{Af}{A\cos\varphi - f} \,.$$

Substituting for the angle φ gives the time dependence of the image's position as

$$A' = \frac{a'}{\cos\varphi} = \frac{Af}{A\cos\left(\varphi_0\cos\left(\sqrt{\frac{c}{J}}t\right)\right) - f} \,.$$

By deriving with respect to time, we find the speed of the image as a function of time

$$V' = \frac{\mathrm{d}A'}{\mathrm{d}t} = \frac{-Af}{\left(A\cos\left(\varphi_0\cos\left(\sqrt{\frac{c}{J}}t\right)\right) - f\right)^2}\left(-A\sin\left(\varphi_0\cos\left(\sqrt{\frac{c}{J}}t\right)\right)\right)\left(-\varphi_0\sqrt{\frac{c}{J}}\sin\left(\sqrt{\frac{c}{J}}t\right)\right) \,.$$

We plot this function in some graphical editor and find that the magnitude of the velocity is maximal at $4.9\,\mathrm{cm{\cdot}s^{-1}}$.

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Problem 49 ... epic fail 8 points

Radek and Radka are merrily enjoying the ride on the carousel, with Radka sitting a quarter circle in front of Radek (in the direction of rotation). In a moment of mischief, Radka throws a rotten tomato directly at Radek, but after less than half of the carousel's rotation period, the tomato appears back in Radka's face. Determine the magnitude of the velocity (as the ratio of $\kappa > 0$ to the circumferential velocity of the carousel) with which Radka threw the tomato. This grotesque took place in weightlessness. *Radka didn't want to hit... Didn't want to hit herself.*

Since the velocity is to be expressed in units of circumferential velocity, let's set the circumferential velocity equal to 1. Then, κ will be the magnitude of the velocity at which Radka threw the tomato (relative to herself). Since she was throwing directly at Radek from her point of view, the angle that the tomato's velocity vector made with the position vector in the *rotating* system at the moment of the throw must have been equal to $\pi/4$. This position vector marks the boundary of the two half-planes.

If the tomato is to come back to Radka in less than half a period, the velocity of the tomato in the *nonrotating* system at the moment of ejection must be in the opposite half-plane to that of Radek.[2] In this half-plane, the tomato then knocks off Radka.

From now on, we solve the problem in a non-rotating (inertial) frame. At the instant of the ejection, the radial component of the tomato's velocity was equal to $\kappa/\sqrt{2}$, while the tangential component was $1-\kappa/\sqrt{2}$ (in the direction of rotation). Thus, the angle α that the velocity in the non-rotating system makes at the moment of the tomato's ejection with the position vector satisfies

$$\cos\alpha = \frac{\kappa/\sqrt{2}}{\sqrt{1+\kappa^2-\sqrt{2}\kappa}}\,,$$
$$\sin\alpha = \frac{1-\kappa/\sqrt{2}}{\sqrt{1+\kappa^2-\sqrt{2}\kappa}}\,.$$

Because the magnitude of the velocity of a tomato in a non-rotating system can be calculated from the law of cosine as

$$v = \sqrt{1+\kappa^2-\sqrt{2}\kappa}\,.$$

Let the radius of the carousel also be unitary because the result will certainly not depend on it. The distance the tomato travels before returning to the circumference of the carousel can be found from an isosceles triangle (with vertices Radka - the center of the carousel - the point where the tomato hits Radka, the angle at the first and last vertex being α) as

$$s = 2\cos\alpha\,.$$

So the tomato will return to the perimeter of the carousel in time

$$\tau = \frac{s}{v} = \frac{2\cos\alpha}{\sqrt{1+\kappa^2-\sqrt{2}\kappa}} = \frac{\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa}\,.$$

We will now try to manipulate this expression using the formulas for the sine and cosine of a double angle. We find that

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{\sqrt{2}\kappa-\kappa^2}{1+\kappa^2-\sqrt{2}\kappa}$$

and

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{\sqrt{2}\kappa-1}{1+\kappa^2-\sqrt{2}\kappa}\,.$$

[2] This can undoubtedly be achieved by choosing $\kappa < \sqrt{2}$.

Thus, we can rewrite τ as

$$\begin{aligned}\tau &= \frac{\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa} = \\ &= \frac{1+\kappa^2-\sqrt{2}\kappa+\sqrt{2}\kappa-1-\kappa^2+\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa} = \\ &= 1+\frac{\sqrt{2}\kappa-1-\kappa^2+\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa} = \\ &= 1+\sin 2\alpha+\cos 2\alpha\,.\end{aligned}$$

If the tomato is to smash against Radka's face, we must also have

$$\tau = \pi - 2\alpha\,.$$

Since the time τ takes Radka to reach the point where her face meets the tomato is due to the unit rotation speed of the carousel corresponding directly to the angle of its rotation. Therefore, after substituting $x = 2\alpha$, we solve the equation

$$1 + x + \sin x + \cos x = \pi\,.$$

The obvious root is $x = \pi$, which gives $\kappa = 0$, i.e., zero speed of the tomato (but this way, the tomato returns to Radka somewhat trivially). The second (and only other) root is found iteratively as

$$x \doteq 0.729\,581\,5\,.$$

From here, after expressing $\tan\alpha$ as a fraction of sin and cosine, we get the speed of a tomato

$$\kappa = \frac{\sqrt{2}}{1+\tan\frac{x}{2}} \doteq 1.023\,397\,.$$

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Problem 50 ... intermittent voltage 7 points

In the circuit, a capacitor with capacitance $C = 47\,\mathrm{nF}$ and a resistor with resistance $R = 220\,\mathrm{k\Omega}$ are connected in series to an intermittent voltage source with period $2T = 20\,\mathrm{ms}$. The voltage waveform over one period is

$$V(t) = \begin{cases} 0\,\mathrm{V} & -T < t < 0; \\ 9.0\,\mathrm{V} & 0 \le t < T. \end{cases}$$

Determine the difference between the highest and lowest voltage on the capacitor at steady state. *Jindra was playing with a switch in a DC circuit.*

We can describe the relationship between the charge Q and the voltage V_c across the capacitor as

$$Q = CV_\mathrm{c}\,,$$

where C is the capacitance of the capacitor.

At the stage of period $0 \leq t < T$, the voltage $V_0 = 9.0\,\mathrm{V}$ is constant at the source. During this phase, the voltage across the capacitor will increase and it will be the highest when the source switches to the $V = 0\,\mathrm{V}$ stage. At that point the voltage will start to decrease, and it will be the lowest at the moment of switching back to the $V = V_0 = 9.0\,\mathrm{V}$ stage. Let us call the lowest voltage on the capacitor V_- and the highest one V_+.

In the stage when a voltage of the source is $V = V_0$, the charge on the capacitor Q is governed by the differential equation

$$V_0 - \frac{Q}{C} - R\frac{\mathrm{d}Q}{\mathrm{d}t} = 0\,.$$

Using the relationship $Q = CV_\mathrm{c}$ we can rewrite the differential equation for the voltage V_c across the capacitor

$$V_0 - V_\mathrm{c} - RC\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} = 0\,.$$

The initial condition is $V_\mathrm{c}(0) = V_-$. The solution to this differential equation is a function

$$V_\mathrm{c}(t) = V_0 - (V_0 - V_-)\mathrm{e}^{-\frac{t}{RC}}\,.$$

The equation must hold at time $t = T$, therefore

$$V_+ = V_0 - (V_0 - V_-)\mathrm{e}^{-\frac{T}{RC}}\,.$$

During the zero voltage stage of the source, the charge Q on the capacitor is governed by the differential equation

$$\frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t} = 0\,,$$

which can again be rewritten using the voltage V_c across the capacitor

$$V_\mathrm{c} + RC\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} = 0\,.$$

The initial condition is $V_\mathrm{c}(0) = V_+$. The solution to this differential equation is a function

$$V_\mathrm{c}(t) = V_+\mathrm{e}^{-\frac{t}{RC}}$$

at time $t = T$, the following must hold

$$V_- = V_+\mathrm{e}^{-\frac{T}{RC}}\,.$$

We get a system of two equations with two unknowns V_-, V_+

$$V_- = V_+\mathrm{e}^{-\frac{T}{RC}}\,,$$
$$V_+ = V_0 - (V_0 - V_-)\mathrm{e}^{-\frac{T}{RC}}\,.$$

From the first equation, we substitute V_- into the second equation and get

$$V_+ = V_0\frac{1}{1 + \mathrm{e}^{-\frac{T}{RC}}}\,.$$

After putting this back into the first equation of V_+, we get

$$V_- = V_0 \frac{\mathrm{e}^{-\frac{T}{RC}}}{1 + \mathrm{e}^{-\frac{T}{RC}}} \,.$$

The difference between the highest and lowest voltage on the capacitor is

$$V_+ - V_- = V_0 \frac{1 - \mathrm{e}^{-\frac{T}{RC}}}{1 + \mathrm{e}^{-\frac{T}{RC}}} .$$

After substituting the numbers from the problem statement, we get the difference $4.04\,\mathrm{V} \doteq \doteq 4.0\,\mathrm{V}$.

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Problem 51 ... Van Allenovi is just bright 8 points

In the equatorial plane, at a distance of two Earth radii from the center of Earth, there is a proton with energy $E = 1\,\mathrm{keV}$, which is pointing at an angle $\alpha = 45^\circ$ from the force line towards the North Pole. As it approaches Earth, at one moment it is reflected by a magnetic mirror and returns towards the South Pole. Determine the (magnetic) latitude at which this happens. Treat the Earth's magnetic field as a dipole.

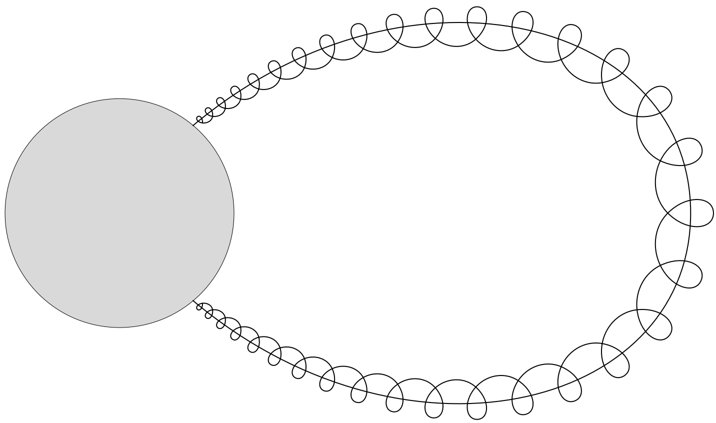

Figure 8: Representation of proton behavior.

Kačka got the result, but wanted to check it.

A proton in the Earth's magnetic field performs several motions, the simplest is the cyclotron motion around the magnetic field line, that is, the part of the motion perpendicular to the field line. In the direction parallel to the field line, the particle moves uniformly. However, as the proton approaches the pole along the magnetic field line, the amplitude of the magnetic induction increases because, in this case, it must maintain its magnetic moment. Due to the conservation of the magnetic moment $\mu = mv_{\perp}^2/(2B)$ the velocity parallel to the magnetic field changes to a velocity perpendicular to the magnetic field. Earth's magnetic field is considered a dipole, the expression of the magnetic field in polar coordinates is $\mathbf{B}(r,\theta,\varphi) = (B_0 R_E^3/r^3)\,(2\cos\theta, \sin\theta, 0)$ and the shape of the magnetic field line, which is in the equatorial plane in a distance L from the center of Earth, is $r(\theta) = L\sin^2(\theta)$. This gives us the condition for reflection, from the conservation of the magnetic moment we write:

$$\mu = \frac{mv_\perp^2}{2B(2R_Z, 0)} = \frac{mv^2}{2B(2R_z\sin^2\theta_r, \theta_r)}.$$

We substitute for the magnetic field and express the angle θ_r:

$$\begin{aligned}
\frac{mv^2\sin^2\alpha}{2B_0\frac{R_E^3}{(2R_E)^3}} &= \frac{mv^2}{2B_0\frac{R_E^3}{(2R_E\sin^2\theta_r)^3}\sqrt{4\cos^2\theta_r + \sin^2\theta_r}}\,,\\
\sin^2\alpha &= \frac{\sin^6\theta_r}{\sqrt{4\cos^2\theta_r + \sin^2\theta_r}}\,,\\
\sin^2\alpha\sqrt{4 - 3\sin^2\theta_r} &= \sin^6\theta_r\,,\\
\sin^4\alpha\left(4 - 3\sin^2\theta_r\right) &= \sin^{12}\theta_r\,.
\end{aligned}$$

Using the substitution $x = \sin^2\theta_r$ we get the equation $\sin^4\alpha\,(4 - 3x) = x^6$, which is analytically unsolvable. However, when we substitute the known angle $\alpha = 45\,^\circ$ then $\sin^2\alpha = 0.5$. The problem can be solved numerically, yielding a positive result of $\sin^2\theta_r = 0.846 \rightarrow \theta_r = 66.87°$. The magnetic width is then the complementary angle, $23.13\,^\circ$.

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Problem 52 ... flying mud 8 points

A tire with an outer diameter $d = 63.2\,\mathrm{cm}$ *is rolling on a flat surface at a constant speed* $v = 15.0\,\mathrm{m\cdot s^{-1}}$. *Suddenly, a piece of mud gets ejected from its rotating circumference. The soil flies through the air and eventually lands on the ground. Subsequently, the tire passes over the fallen piece of mud with the same part from which it was ejected. How long was the soil on the ground before the tire ran over it? Disregard air resistance during the flight of the mud. The mud did not fall farther than* $4.00\,\mathrm{m}$ *along the horizontal axis from the release point.*

While driving, a fallen tractor wheel whizzed past Jindra.

We will measure the angle α on the tire from the vertical direction. In the reference frame connected to the ground, the mud has a velocity vector

$$\mathbf{u} = v\,(1 + \cos\alpha, -\sin\alpha)\,.$$

When the piece of mud separates from the wheel, the mud with this initial velocity vector will continue in free fall toward the ground. However, we currently do not know the angle α that determines the location of the mud separation. Therefore, we will derive an equation to calculate the angle α. The piece of mud begins its fall to the ground from a height

$$h = r(1 + \cos\alpha)\,,$$

where $r = 31.6\,\mathrm{cm}$ is the radius of the tire. Its vertical position y and horizontal position x relative to the release point are

$$y = -vt\sin\alpha - \frac{1}{2}gt^2\,,$$
$$x = v\,(1+\cos\alpha)t\,.$$

We will express the time $t = x/(v(1+\cos\alpha))$ from the second equation and substitute it into the first equation, resulting in a dependence $y = y(x)$

$$y = -\frac{\sin\alpha}{1+\cos\alpha}x - \frac{g}{2v^2(1+\cos\alpha)^2}x^2.$$

The mud hits the ground when $y = -h = -r(1+\cos\alpha)$, thus obtaining a relationship between the impact location x and the separation angle α

$$\frac{g}{2v^2(1+\cos\alpha)^2}x^2 + \frac{\sin\alpha}{1+\cos\alpha}x - r(1+\cos\alpha) = 0\,,$$
$$\frac{1}{2}x^2 + \frac{v^2\sin\alpha(1+\cos\alpha)}{g}x - \frac{rv^2(1+\cos\alpha)^3}{g} = 0\,.$$

The roots of this equation are

$$x_{1,2} = -\frac{v^2\sin\alpha(1+\cos\alpha)}{g} \pm \sqrt{\left(\frac{v^2\sin\alpha(1+\cos\alpha)}{g}\right)^2 + \frac{2rv^2(1+\cos\alpha)^3}{g}}\,.$$

We are interested only in the positive root

$$x_1 = (1+\cos\alpha)\left(-\frac{v^2\sin\alpha}{g} + \sqrt{\left(\frac{v^2\sin\alpha}{g}\right)^2 + \frac{2v^2r(1+\cos\alpha)}{g}}\right)\,.$$

It is important to note that the current x-position is relative to the point of mud separation. However, we need to find the x-position relative to the point where the wheel contacts the ground at the time of mud separation. To achieve this, we must calculate the separation point's x-coordinate in relation to the point of contact, which can be expressed as $r\sin\alpha$.

For the wheel to run over the piece of mud at the same spot from which it previously separated, it must rotate by an angle $\pi - \alpha$ and complete k full revolutions. The mud must fall at a distance $r(2k\,\pi + \pi - \alpha)$ from the wheel's point of contact with the ground. Therefore, we obtain an equation for the separation angle

$$r(2k\,\pi + \pi - \alpha) = r\sin\alpha + (1+\cos\alpha)\left(-\frac{v^2\sin\alpha}{g} + \sqrt{\left(\frac{v^2\sin\alpha}{g}\right)^2 + \frac{2v^2r(1+\cos\alpha)}{g}}\right)\,,$$

which we have to solve numerically. If we divide both sides of the equation by r, we can introduce a dimensionless parameter $A = v^2/(gr)$, which will simplify the equation

$$(2k+1)\pi = \alpha + \sin\alpha + (1+\cos\alpha)\left(-A\sin\alpha + \sqrt{(A\,\sin\alpha)^2 + 2A\,(1+\cos\alpha)}\right)\,.$$

After substituting $A = 72.58$ according to the given numbers for various k, we find (for example, using the function `scipy.optimize.fsolve()` in Python) numerical solution

$$\begin{aligned} k &= 0, & \alpha &= \pi\,\mathrm{rad}, & x_1 &= 0\,\mathrm{m},\\ k &= 1, & \alpha &= 0.4070\,\mathrm{rad}, & x_1 &= 2.72\,\mathrm{m},\\ k &= 2, & \alpha &= 0.2042\,\mathrm{rad}, & x_1 &= 4.84\,\mathrm{m}.\\ &\vdots & &\vdots & &\vdots \end{aligned}$$

The case $k \geq 2$ does not satisfy the condition from the task that the mud landed closer than 4.00 m horizontally from the release point. The case $k = 0$ is again unsatisfactory because the soil did not fly through the air. It is a trivial case where the mud remained on the tire. The only possible solution is for $k = 1$.

The wheel traveled a distance $D = r(2\pi + \pi - \alpha)$ in time

$$t_k = \frac{D}{v} = 0.1900\,\mathrm{s}\,.$$

The mud was flying through the air for a duration before hitting the ground

$$t = \frac{x_1}{v(1 + \cos\alpha)} = 0.0947\,\mathrm{s}\,.$$

The time period when the mud lay on the ground before being run over again was $T = t_k - t = 0.0953\,\mathrm{s}$

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Problem 53 ... maximal activity II — 9 points

Jindra has $N_0 = 10^9$ atoms of the radioactive isotope ^{212}Bi. It decays to the isotope ^{212}Po with probability $P_\beta = 64.06\%$ by beta decay and to the isotope ^{208}Tl with probability $P_\alpha = 35.94\%$ by alpha decay. The half-life of bismuth is $T_{\mathrm{Bi}} = 60.6\,\mathrm{min}$. The polonium isotope decays further by alpha decay with a half-life of $T_{\mathrm{Po}} = 299\,\mathrm{ns}$ to the stable isotope ^{208}Pb. The thallium isotope decays by beta decay with a half-life of $T_{\mathrm{Tl}} = 3.05\,\mathrm{min}$ also to lead ^{208}Pb.

How long does it take Jindra to measure the maximum activity in the system?

Jindra measured the half-life of ^{212}Po.

For the purpose of solving the problem, it will be easier to work with decay constants

$$\lambda = \frac{\ln 2}{T_{1/2}}$$

than with half-lives $T_{1/2}$. The decay constants for each isotope are

$$\lambda_{\mathrm{Bi}} = 1.91 \cdot 10^{-4}\,\mathrm{s}^{-1}, \quad \lambda_{\mathrm{Po}} = 2.32 \cdot 10^{6}\,\mathrm{s}^{-1}, \quad \lambda_{\mathrm{Tl}} = 3.79 \cdot 10^{-3}\,\mathrm{s}^{-1}.$$

For clarity, we use the variable A for the number of bismuth atoms, B for the polonium atoms, C for the thallium atoms, and D for the lead atoms. The total activity R in the system depends on the actual amounts of the isotopes of bismuth A, polonium B, and thallium C.

$$R = \lambda_{\mathrm{Bi}} A + \lambda_{\mathrm{Po}} B + \lambda_{\mathrm{Tl}} C. \tag{8}$$

Therefore, we must solve a system of differential equations describing the decay series to find how A, B, and C evolve in time. The system of differential equations describing the amount of each isotope over time is

$$\begin{aligned}\dot{A} &= -\lambda_{\mathrm{Bi}} A,\\ \dot{B} &= P_\beta \lambda_{\mathrm{Bi}} A \ - \lambda_{\mathrm{Po}} B,\\ \dot{C} &= P_\alpha \lambda_{\mathrm{Bi}} A \ - \lambda_{\mathrm{Tl}} C,\\ \dot{D} &= \lambda_{\mathrm{Po}} B + \lambda_{\mathrm{Tl}} C,\end{aligned}$$

with initial conditions $A(0) = N_0, B(0) = 0, C(0) = 0, D(0) = 0$.

This system of differential equations can be solved from the top line by line. The first line of the equation has the solution

$$A(t) = N_0 \mathrm{e}^{-\lambda_{\mathrm{Bi}} t}.$$

Add this to the second line and solve the differential equation

$$\dot{B} = P_\beta \lambda_{\mathrm{Bi}} N_0 \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \lambda_{\mathrm{Po}} B$$

with initial condition $B(0) = 0$. The solution is the function

$$B(t) = \frac{P_\beta \lambda_{\mathrm{Bi}} N_0}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Po}} t}\right)$$

The differential equation on the third line has the same structure as the equation on the second line and the same initial condition, so its solution is

$$C(t) = \frac{P_\alpha \lambda_{\mathrm{Bi}} N_0}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Tl}} t}\right)$$

We plug the derived functions into the equation (8) for the activity

$$\begin{aligned}R(t) = \lambda_{\mathrm{Bi}} N_0 \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} + \frac{P_\beta \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Po}} N_0}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Po}} t}\right) +\\ + \frac{P_\alpha \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Tl}} N_0}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Tl}} t}\right)\end{aligned} \tag{9}$$

We verify that in the limit $t \to \infty$, the activity goes to zero $R \to 0$ due to decreasing exponentials. This is consistent with our expectation, since in a long time, most of the radioactive atoms decay and only stable ^{208}Pb remain.

To find the maximum of the activity, we have to derive the function (9)

$$\begin{aligned}\dot{R} = \lambda_{\mathrm{Bi}} N_0 \Bigg(- \lambda_{\mathrm{Bi}} \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \frac{P_\beta \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Po}}}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} + \frac{P_\beta \lambda_{\mathrm{Po}}^2}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Po}} t} -\\ - \frac{P_\alpha \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} + \frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Tl}} t} \Bigg).\end{aligned} \tag{10}$$

Let's see that at time $t = 0$ the change in activity over time is positive

$$\dot{R}(0) = \lambda_{\mathrm{Bi}} N_0 \left(-\lambda_{\mathrm{Bi}} + P_\beta \lambda_{\mathrm{Po}} + P_\alpha \lambda_{\mathrm{Tl}}\right) > 0,$$

therefore, initially, the decay activity in the system increases. At some point, it reaches a maximum, and then in the limit $t \to \infty$, the activity drops to zero.

We determine the time of maximum activity by setting the derivative of the activity with respect to time (10) equal to zero and solving for time t. Due to the fact $\lambda_{Po} \gg \lambda_{Tl}$ and $\lambda_{Po} \gg \lambda_{Bi}$, we can neglect the term with the exponential $\mathrm{e}^{-\lambda_{Po} t}$ in the equation.

$$-\lambda_{\mathrm{Bi}}\mathrm{e}^{-\lambda_{\mathrm{Bi}}t} - \frac{P_\beta \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Po}}}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}}\mathrm{e}^{-\lambda_{\mathrm{Bi}}t} - \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}\mathrm{e}^{-\lambda_{\mathrm{Bi}}t} + \frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}\mathrm{e}^{-\lambda_{\mathrm{Tl}}t} = 0$$

We gradually isolate the time t

$$\mathrm{e}^{(\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}})t} = \frac{\frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}{\lambda_{\mathrm{Bi}} + \frac{P_\beta \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Po}}}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} + \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}} \approx \frac{\frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}{\lambda_{\mathrm{Bi}} + P_\beta \lambda_{\mathrm{Bi}} + \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}$$

$$t = \frac{1}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \ln\left(\frac{\frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}{\lambda_{\mathrm{Bi}} + P_\beta \lambda_{\mathrm{Bi}} + \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}\right).$$

After plugging in the numbers, the time of maximum activity in the system comes out $t = 366\,\mathrm{s} = 6.09\,\mathrm{min}$.

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Problem 54 ... Mišo is shooting 9 points

Mišo likes to shoot from a laser. He needs to reduce its energy with reflective gray (neutral density, ND) filters. He would like to achieve a beam with an energy 37 J. *Mišo has 5 filter holders and 7 different filters, namely 2-stop ND, 3-stop ND, 5-stop ND, 7-stop ND, 11-stop ND, 13-stop ND, and 17-stop ND. Assume that all energy striking the filter is either reflected or transmitted. The laser has an energy of* 77 377 J. *With what accuracy can he achieve the desired* 37 J*? Give the result in* mJ. *Mišo was calculating the filtering at the PALS*

There will be an infinite number of reflections in the spaces between the filters. One possibility would be to compute infinite series. However, this is not necessary. We will be interested in the total amount of energy flowing in the spaces in between the filters. We denote the input energy 77 377 J by E and we assume that the laser is shining from the left. Firstly, let's suppose that we have used all five holders. We denote all the filters starting from the left by the indices 1 to 5. We then denote by the same indices the energies E flowing from the given filters towards the right and the returning energies R flowing into the given filters from the right. The resulting energy coming out of the system of filters will be E_5.

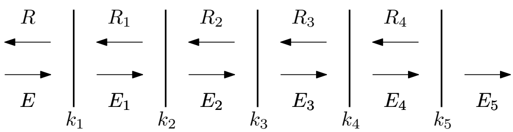

Figure 9: Energy in the spaces between the filters

The number n in the ND filter label indicates how much light passes through the filter

$$E_{\text{passes}} = E_{\text{enters}} \cdot 2^{-n} \,.$$

We substitute $k = 2^{-n}$ and then for the coefficient k we can get

$$k \;= \left\{ \frac{1}{4}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \frac{1}{2\,048}, \frac{1}{8\,192}, \frac{1}{131\,072} \right\}, .$$

Energy is not lost in the filters, so

$$E_{\text{reflect}} = E_{\text{enters}} \cdot (1 - k_i) \quad , \quad E_{\text{reflect}} + E_{\text{passes}} = E_{\text{enters}} \,.$$

We place five filters with coefficients k_1, k_2, k_3, k_4, k_5 whose values can be any combination from the set k in the holders. The following equations will hold on the interfaces

$$\begin{aligned}
E_5 &= k_5 E_4 \,, \\
E_4 &= k_4 E_3 + (1 - k_4) R_4 \,, \\
E_3 &= k_3 E_2 + (1 - k_3) R_3 \,, \\
E_2 &= k_2 E_1 + (1 - k_2) R_2 \,, \\
E_1 &= k_1 E + (1 - k_1) R_1 \,, \\
R_4 &= (1 - k_5) E_4 \,, \\
R_3 &= (1 - k_4) E_3 + k_4 R_4 \,, \\
R_2 &= (1 - k_3) E_2 + k_3 R_3 \,, \\
R_1 &= (1 - k_2) E_1 + k_2 R_2 \,,
\end{aligned}$$

where E_i are the energies flowing to the right and R_i are the energies flowing to the left.

We got nine equations with nine unknowns (5 times E_i and 4 times R_i), which we will solve using the matrix notation and Cramer's rule. The matrix notation of the equations above is as follows

$$\underbrace{\begin{pmatrix}
1 & -k_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -k_4 & 0 & 0 & k_4 - 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -k_3 & 0 & 0 & k_3 - 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -k_2 & 0 & 0 & k_2 - 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & k_1 - 1 \\
0 & k_5 - 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & k_4 - 1 & 0 & 0 & -k_4 & 1 & 0 & 0 \\
0 & 0 & 0 & k_3 - 1 & 0 & 0 & -k_3 & 1 & 0 \\
0 & 0 & 0 & 0 & k_2 - 1 & 0 & 0 & -k_2 & 1
\end{pmatrix}}_{\text{Túto maticu označme } A}
\begin{pmatrix} E_5 \\ E_4 \\ E_3 \\ E_2 \\ E_1 \\ R_4 \\ R_3 \\ R_2 \\ R_1 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_1 E \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

For Cramer's rule, we need the determinant of the matrix A and the determinant of the matrix A_1 where we replace the first column with the vector on the right-hand side of the equation. We replace the first one because we only want to find E_5, which is the first unknown in our ordering. We get

$$E_5 = \frac{\det A_1}{\det A} = E \cdot \frac{k_1 k_2 k_3 k_4 k_5}{k_1 k_2 k_3 k_4 + k_1 k_2 k_3 k_5 + k_1 k_2 k_4 k_5 + k_1 k_3 k_4 k_5 + k_2 k_3 k_4 k_5 - 4 k_1 k_2 k_3 k_4 k_5} \,,$$

which can be simplified as

$$E_5 = E \cdot \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} + \frac{1}{k_5} - 4} = \frac{1}{2^{n_1} + 2^{n_2} + 2^{n_3} + 2^{n_4} + 2^{n_5} - 4}\,,$$

Where n_i are the stop numbers of the ND filters.

By an analogous process for $N = \{1, 2, 3, 4, 5\}$ filters, we obtain in general

$$E_N = E \cdot \left(1 - N + \sum_{i=1}^{N} 2^{n_i}\right)^{-1}\,, \tag{11}$$

where E_N is the resulting energy leaving the system of N filters. This relation likely holds for any N. We can see that the resulting energy is not dependent on the order of the filters in the holders. All permutations for a given combination are the same. Our goal now will be to find a combination whose energy is as close to 37 J as possible, according to the relation (11).

From the relation (11), we can express and calculate the sum of the terms 2^{n_i}, putting $E_N = 37\,\mathrm{J}$, which is the desired output energy to which we want to get as close as possible.

$$\sum_{i=1}^{N} 2^{n_i} = \frac{E}{E_N} + N - 1 \doteq 2090 + N\,. \tag{12}$$

Now we can go through all the combinations using a script and find the best one. However, we can also obtain the result by simple reasoning.

We have the terms $2^{n_i} = 1/k_i$, so 4, 8, 32, 128, 2 048, 8 192, 131 072, to "make up" the number $2\,090 + N$. So we need to use the number $2\,048 = 2^{11}$, which is an 11-stop filter. Higher ND filters are too strong. That leaves $42 + N$ left from the sum, so 128 is too much, and we use a 5-stop filter, which corresponds to $2^5 = 32$. That leaves $10 + N$. The best we can do is to use the remaining 2 filters that we haven't eliminated yet. That is the 2-stop ND and the 3-stop ND, which together give $2^2 + 2^3 = 4 + 8 = 12$. We have used 4 filters, which means that $N = 4$. This gives us 2094 on the right-hand side of the equation (12), which is almost exactly equal to the sum on the left-hand side, which we determined to be $2\,048 + 32 + 8 + 4 = 2\,092$.

Finally, we calculate the deviation of ΔE energy E_N from 37 J using the relation (11) for the best set of filters, namely 2-stop ND, 3-stop ND, 5-stop ND and 11-stop ND,

$$\Delta E = 77\,377\,\mathrm{J} \cdot \left(1 - 4 + 2^2 + 2^3 + 2^5 + 2^{11}\right)^{-1} - 37\,\mathrm{J} \doteq 40\,\mathrm{mJ}$$

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Problem 55 ... Red Wednesday 9 points

Lego is situated aboard a space station in deep space. To avoid going crazy, he keeps watching what is the day today on Earth. He knows that last Sunday he launched a rocket (at rest) with a rest mass $m_0 = 31.0\,\mathrm{kg}$ *propulsed by an ion thruster which does not reduce the rest mass; however, it exerts a constant force* F*. The rocket glows with a golden light* $\lambda_0 = 600\,\mathrm{nm}$ *in such a way that when Lego points his telescope at it today at midday (specifically* $t_\mathrm{s} = 72.0\,\mathrm{h}$ *since the launch), he sees a red light* $\lambda_r = 670\,\mathrm{nm}$*. How big is the force* F*?*

Lego spotted a pattern in the date of Physics Brawl Online...

Redshift is a well-known case of a Doppler effect. To be specific, there is a relation between the emitted light's wavelength λ_0 and the received wavelength λ_r

$$\lambda_r = \lambda_0 \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}\,,$$

where v is the speed at which the source and the observer move mutually away and c is the speed of light. The fraction v/c is commonly denoted as β.

When we take the square and isolate β, we obtain

$$\beta = \frac{(\lambda_r/\lambda_0)^2 - 1}{(\lambda_r/\lambda_0)^2 + 1}\,,$$

which gives us the speed at which the rocket is receding from Lego's space station $v = 0.1c$.

Let's take a look at how the rocket will accelerate. Considering the acceleration up to $v = 0.1c$, we have to take relativistic equations into account for acceleration. To be precise, the spatial component of the motion equation looks the same as the classical one $\vec{F} = \mathrm{d}\vec{p}/\,\mathrm{d}t$, except $\vec{p} = m\vec{v}$, where $\vec{v}$ is a classical velocity relative to a certain inertial frame of reference. However, the mass m is relative to this reference frame by $m = m_0\gamma$, where m_0 is the (already given in the problem) rest mass and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

We choose the frame of reference associated with the station at which Lego is aboard to solve this problem. All distances, times, and speeds will be given in terms of this frame.

Now we can construct a differential equation for v and solve, but it is easier to realize that the momentum at time t will simply be $p(t) = Ft$, which gives us the relation between t and β

$$\begin{aligned}
Ft &= m_0 \frac{1}{\sqrt{1-\beta^2}} v \\
\frac{Ft}{m_0 c} &= \sqrt{\frac{\beta^2}{1-\beta^2}} \\
\left(\frac{Ft}{m_0 c}\right)^2 (1-\beta^2) &= \beta^2 \\
\beta &= \frac{Ft}{\sqrt{(m_0 c)^2 + (Ft)^2}}\,.
\end{aligned}$$

When we substitute β calculated from the redshift, we get

$$\frac{Ft}{\sqrt{(m_0 c)^2 + (Ft)^2}} = \frac{(\lambda_r/\lambda_0)^2 - 1}{(\lambda_r/\lambda_0)^2 + 1}\,.$$

It is crucial to realize that it is not enough to plug in the time t_s here since the light that Lego sees on Wednesday departed the rocket earlier. Light moves at the speed c, so if the rocket is at some point (from the perspective of the frame associated with Lego's station) x away from Lego, it will reach Lego in the time $t_\mathrm{c} = x/c$. That means Lego launched the rocket at the time $t = 0$. After some time $t = t_\mathrm{r}$, it will gain such a speed that it will be seen as red by Lego. But Lego will see it with a time delay t_c. So for Lego to be able to see this given wavelength at the time t_s, it must hold that $t_\mathrm{s} = t_\mathrm{r} + t_\mathrm{c}$.

Firstly, we will express t_r from the equation obtained by comparing β

$$\frac{1}{\sqrt{\left(\frac{m_0c}{Ft_\mathrm{r}}\right)^2+1}}=\frac{(\lambda_r/\lambda_0)^2-1}{(\lambda_r/\lambda_0)^2+1}$$

$$\frac{1}{\left(\frac{m_0c}{Ft_\mathrm{r}}\right)^2+1}=\left(\frac{(\lambda_r/\lambda_0)^2-1}{(\lambda_r/\lambda_0)^2+1}\right)^2$$

$$\left(\frac{(\lambda_r/\lambda_0)^2+1}{(\lambda_r/\lambda_0)^2-1}\right)^2-1=\left(\frac{m_0c}{Ft_\mathrm{r}}\right)^2$$

$$t_\mathrm{r}=\frac{m_0c}{F}\frac{(\lambda_r/\lambda_0)^2-1}{2\lambda_r/\lambda_0}\,.$$

Secondly, we calculate the distance traveled by the rocket at that time. We know its speed at the time t, so we just need to integrate from 0 to t_r

$$x_r=\int_0^{t_\mathrm{r}}\frac{c}{\sqrt{\left(\frac{m_0c}{Ft}\right)^2+1}}\,\mathrm{d}t=c\left[\sqrt{t^2+\left(\frac{m_0c}{F}\right)^2}\,\right]_0^{t_\mathrm{r}}=c\left(\sqrt{t_\mathrm{r}^2+\left(\frac{m_0c}{F}\right)^2}-\frac{m_0c}{F}\right)\,.$$

Thus, the time for the light to return will be

$$t_\mathrm{c}=\frac{x_r}{c}=\sqrt{t_\mathrm{r}^2+\left(\frac{m_0c}{F}\right)^2}-\frac{m_0c}{F}=\frac{m_0c}{F}\frac{(\lambda_r/\lambda_0)^2+1}{2\lambda_r/\lambda_0}-\frac{m_0c}{F}=\frac{m_0c}{F}\frac{(\lambda_r/\lambda_0-1)^2}{2\lambda_r/\lambda_0}\,.$$

And as we have said, $t_\mathrm{s}=t_\mathrm{r}+t_\mathrm{c}$ must hold, which finally gives us the equation for F

$$t_\mathrm{s}=\frac{m_0c}{F}\frac{(\lambda_r/\lambda_0)^2-1}{2\lambda_r/\lambda_0}+\frac{m_0c}{F}\frac{(\lambda_r/\lambda_0-1)^2}{2\lambda_r/\lambda_0}\,,$$

$$F=\frac{m_0c}{t_\mathrm{s}}\left(\frac{(\lambda_r/\lambda_0)^2-1}{2\lambda_r/\lambda_0}+\frac{(\lambda_r/\lambda_0)^2-2\lambda_r/\lambda_0+1}{2\lambda_r/\lambda_0}\right)\,,$$

$$F=\frac{m_0c}{t_\mathrm{s}}\left(\lambda_r/\lambda_0-1\right)=4\,183\,\mathrm{N}\,.$$

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Problem 56 ... pumping water using water — 8 points

In the 19th century, people began to make extensive use of steam engines and to think about their efficiency. One of the problems they could solve back then was pumping water out of the mine. However, we are not interested in the specific design of a process/engine that pumps water from a depth of $h=50\,\mathrm{m}$ to the surface. Consider that we have stumbled upon geothermally heated ideal water (it is incompressible, has a constant density ρ and specific heat capacity c) with volume $V_1=200\,\mathrm{m}^3$ and temperature $T_1=90\,\mathrm{°C}$. Nearby is also a lake (thermal bath) with constant temperature $T_\mathrm{J}=10\,\mathrm{°C}$. What is the maximum amount of water we can ideally pump to the surface? For the whole amount of water, consider a constant elevation h and a homogeneous gravitational field. *Marek made an excursion to the 19th century.*

From the problem statement, we can see that it is a problem where we have a certain amount of energy (geothermal water) available, which we want to convert into a certain amount of work (pumping water from a mine). Whatever we do, we consider that the total energy is conserved.

We are interested in the maximum amount of water we can pump out and, therefore, the maximum amount of work we can do. Let's do things efficiently and consider the ideal case where the geothermal energy is converted only into work and "waste" energy, which is received by the lake (a thermal bath of constant temperature T_J). At this point, let us note that the lake does have to accept a certain amount of heat/energy in this process, and thus, we cannot convert geothermal energy purely into work (this would violate the second law of thermodynamics). So, let's write the law of conservation of energy

$$\Delta Q_\mathrm{geo} + \Delta Q_\mathrm{J} + W = 0\,, \tag{13}$$

where ΔQ_geo denotes the heat/energy change of the geothermal water, ΔQ_J denotes the heat the lake receives, and W is the work we are looking for. We see that the problem actually reduces to finding and minimizing just the heat ΔQ_J, since ΔQ_geo is given by

$$\Delta Q_\mathrm{geo} = cV_1\rho\Delta T = cV_1\rho(T_\mathrm{J} - T_1)\,,$$

where the temperature at the beginning is set, and the temperature at the end is the temperature of the lake itself. This is because we want maximum work, so we will cool the geothermal water as long as we can efficiently (until the temperatures equilibrate, it would cost us work).

In the relation (13), we are left with two unknowns, so we need one more relation. And that is the condition that the whole process must be reversible! We know from Carnot that it is processes operating between two temperatures that are the most efficient – they can extract the most work. The law of energy conservation and the condition of reversibility of the whole process are at the heart of the so-called "Maximum work" theorem. The second law of thermodynamics in the form

$$\Delta S \ \geq 0\,,$$

respectively, for our process,

$$\Delta S_\mathrm{geo} + \Delta S_\mathrm{J} + \Delta S_W \geq 0\,. \tag{14}$$

And since a bath, by definition (being much larger than a geothermally heated water source) receives heat at a constant temperature, the entropy change is

$$\Delta S_\mathrm{J} = \frac{\Delta Q_\mathrm{J}}{T_\mathrm{J}}\,,$$

and we already know that we want to minimize ΔQ_J. So we can see that we want to consider equality in the equation (14) (which is easier to realize if we move the other terms without ΔQ_J to the right-hand side) if we are interested in maximizing the amount of work. Next, note that $\Delta S_W = 0$ helps us minimize ΔQ_J, meaning that we do the work reversibly/adiabatically. Finally, we indeed get a condition on the reversible process when $\Delta S = 0$

$$\Delta S_\mathrm{J} = \frac{\Delta Q_\mathrm{J}}{T_\mathrm{J}} = -\Delta S_\mathrm{geo} = -\int \frac{\mathrm{d}Q}{T} = -\int_{T_1}^{T_\mathrm{J}} c\rho V_1 \frac{\mathrm{d}T}{T}\,,$$

from where we have

$$\Delta Q_\mathrm{J} = -c\rho V_1 T_\mathrm{J} \ln\left(\frac{T_\mathrm{J}}{T_1}\right)\,.$$

For the work from (13), it holds

$$W = c\rho V_1 \left[T_1 - T_\mathrm{J} + T_\mathrm{J} \ln\left(\frac{T_\mathrm{J}}{T_1}\right)\right] ,$$

knowing that when pumping water, we are doing work against the gravitational field $W = \rho \Delta V gh$ and for the maximum amount of water V_max we have

$$V_\mathrm{max} = \frac{cV_1 \left[T_1 - T_\mathrm{J} + T_\mathrm{J} \ln\left(\frac{T_\mathrm{J}}{T_1}\right)\right]}{gh} .$$

This is for the values from the input $V_\mathrm{max} = 16\,277\,\mathrm{m^3}$. No matter what we do, no machine pumps more water using geothermal water energy. However, this is a remarkably large amount of water. Consider that to pump $16\,277\,\mathrm{m^3}$ from a depth of 50 meters; we only need $200\,\mathrm{m^3}$ of geothermally heated water and a bath. The reason for this is, of course, the high heat capacity of the water. Finally, let us note that while this is a lot of water, it is still much less than we might naively expect from $V = cV_1/gh = 136\,481\,\mathrm{m^3}$.

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Problem 57 ... a beer problem 9 points

Drinking beer is not as simple as it seems. You must pick up your half-pint and tilt it so much that it starts pouring into your mouth. For this, however, work must be done. Consider the half-liter weight of $m = 360\,\mathrm{g}$, its cylindrical shape of radius $r = 3.5\,\mathrm{cm}$ and height $h = 15\,\mathrm{cm}$ and density of beer $\rho = 1\,030\,\mathrm{kg{\cdot}m^{-3}}$. Determine the work done if you lift a half-liter filled with 350 ml of beer to drink. Your mouth is 40 cm above the table.

Jarda was thinking about how much he has to pay Viktor again...

To drink, we must do the work necessary to lift the tankard in a gravitational field, e.g. to increase its potential energy. At first, we will calculate the angle of the tilt of a half-liter so that a beer will start pouring out of it. The fluid will take shape consisting of a cylinder of height x and a shape that gets created by an oblique cut from one end of the first base to the other end of the other base. We will designate the height of this shape as y. Then, the overall volume of these two bodies is equal to the volume of the beer in the glass V

$$V = \pi r^2 x + \frac{1}{2}\pi r^2 y .$$

For the beer to pour out, a condition of $x + y = h$ must be fulfilled. The tilt angle can also be expressed as $\tan\alpha = y/2r$, where α is the angle by which the height of the half-liter is diverted from the vertical. From these three equations, we get the wanted angle as

$$\tan\alpha = \frac{\pi r^2 h - V}{\pi r^3} \quad \Rightarrow \quad \alpha = 59.3\,^\circ .$$

At this angle arises a situation where there is enough beer in the tankard that its whole base is still completely covered with beer. The situation would change at an angle of $\arctan(h/2r) = 65.0\,^\circ$.

From the relations before we can express numerically $y = 2r\tan\alpha = 11.81\,\mathrm{cm}$ and $x = 3.19\,\mathrm{cm}$.

Next, we will find at what height above the base of an empty half-liter is the center of mass. Since it is made out of a homogeneous material with a consistent width of walls, we can designate the areal density of the walls and the base as σ. The center of mass of the base is at a height of zero above the bottom, the center of mass of the walls is at a height of $h/2$, the overall height of the center of mass of the half-liter is therefore

$$y_\mathrm{k} = \frac{\frac{h}{2}2\pi r h\sigma}{2\pi r h\sigma + \pi r^2\sigma} = \frac{h^2}{2h+r} = 6.72\,\mathrm{cm}\,.$$

Calculating the center of mass of the space, which is taken by the beer, will be more difficult. We get it from the knowledge of the location of the center of mass of the cylinder, which lies $x/2$ above the base, and from the location of the center of mass of the second part of the shape the beer makes. We can calculate it using integration as

$$y_y = \frac{1}{\rho\left(\frac{1}{2}\pi r^2 y\right)}\int_{-r}^{r} 2\sqrt{r^2-u^2}\,(u+r)\,\frac{y}{2r}\rho\,(u+r)\,\frac{y}{4r}\,\mathrm{d}u \;=\frac{1}{(\pi r^3)}\frac{y}{2r}\frac{5\pi r^4}{8} = \frac{5y}{16} = 3.69\,\mathrm{cm}\,.$$

We will get the height of the center of mass of the beer above the base simply as

$$y_\mathrm{p} = \frac{1}{V}\left(\frac{x}{2}\pi r^2 x + \left(\frac{5y}{16}+x\right)\frac{1}{2}\pi r^2 y\right) = 5.03\,\mathrm{cm}\,.$$

The center of mass of the beer, however, will not be on the axis of symmetry of the cylinder but will be shifted in the direction closer to the ground. Analogically, we will calculate its distance from the axis of symmetry. The center of mass of the cylindrical part is on the axis of symmetry, and the center of mass of the other part is at a distance of

$$x_y = \frac{1}{\rho\left(\frac{1}{2}\pi r^2 y\right)}\int_{-r}^{r} 2\sqrt{r^2-u^2}\,(u+r)\,\frac{y}{2r}\rho u\;\mathrm{d}u \;=\frac{2}{\pi r^3}\frac{1}{8}\pi r^4 = \frac{1}{4}r = 0.875\,\mathrm{cm}\,.$$

The center of mass of the beer is, therefore, shifted by

$$x_\mathrm{p} = \frac{1}{V}\frac{1}{2}\pi r^2 y\frac{1}{4}r = 0.568\,\mathrm{cm}$$

from the axis of symmetry of the half-liter.

Let us designate $H = 40\,\mathrm{cm}$ as the height of the mouth above the table. The center of the base will be located at a height of

$$v_\mathrm{s} = H - h\cos\alpha + r\sin\alpha = 35.36\,\mathrm{cm}\,.$$

The center of mass of the half-liter itself is then higher by $y_\mathrm{k}\cos\alpha$. The center of mass of the beer is higher by $y_\mathrm{p}\cos\alpha - x_\mathrm{p}\sin\alpha$. The potential energy of the tilted beer and the half-liter is, relative to the ground,

$$E = g\left(m\left(v_\mathrm{s} + y_\mathrm{k}\cos\alpha\right) + V\;\rho\left(v_\mathrm{s} + y_\mathrm{p}\cos\alpha - x_\mathrm{p}\sin\alpha\right)\right) = 2.694\,\mathrm{J}\,.$$

From this value, it is necessary to subtract the potential energy of the center of mass of the half-liter when it is sitting on the table. That is equal to

$$E_0 = gm\frac{h^2}{2h+r} + gV\rho\frac{V}{2\pi r^2} = 0.398\,\mathrm{J}\,.$$

Therefore, the result is a work of 2 296 mJ.

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Problem K.1 ... the pendulum swings back... 3 points

In certain buildings, one might encounter Foucault's pendulum, which has a very long suspension. Jarda once encountered one, wanting to estimate the height of the ceiling from which the pendulum was suspended. He measured the period of oscillation as 14.2 s*, and when the pendulum was in its lowest position, it was* 70 cm *above the floor. Determine the height of the ceiling.* *Jarda has been in the Panthéon in Paris.*

In this solution, we consider the pendulum to be a mathematical one. The well-known relation that connects the length of the suspension L, gravity of Earth g and the period of oscillation T is given by the formula:

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \Rightarrow \quad L = \frac{T^2}{4\pi^2} g = 50.1\,\mathrm{m}\,.$$

In order to find the height of the point of suspension above the ground, it is neccessary to add the lowest height of the pendulum above the floor h to the length of the suspension. This yields the final result:

$$H = L + h = \frac{T^2}{4\pi^2} g + h = 50.8\,\mathrm{m}\,.$$

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Problem K.2 ... ...and forth... 3 points

To be able to use the approximation of a mathematical pendulum, we need a point of mass on an massless suspension. Let's consider a string made of steel with a diameter of 1.4 mm *on which a weight is suspended. For good accuracy, we require it to be* 80 *times more massive than the string on which it is suspended. If the tensile strength of the used steel is* $520\,\mathrm{N{\cdot}mm^{-2}}$*, what can be the maximum length of the suspension at rest? The density of the used steel is* $7\,900\,\mathrm{kg{\cdot}m^{-3}}$*.* *Such a thin and long rope, and yet it would still hold Jarda.*

The highest tension in the suspension will be at the point of suspension because it bears the mass of the weight and also of the rest of the suspension. The mass of the suspension will be

$$m_\mathrm{c} = \frac{\pi d^2}{4} \rho L\,,$$

where d is its diameter, ρ is the density of steel and L is its length. For the mass of the weight, it applies $m_\mathrm{w} = k m_\mathrm{c}$, where $k = 80$.

The force acting on the suspension at the suspension point is

$$F = (m_\mathrm{c} + m_\mathrm{w})\, g = m_\mathrm{c}\,(1 + k)\, g = \frac{\pi d^2}{4} \rho L\,(1 + k)\, g\,.$$

If we divide this force by the cross-sectional area of the suspension, we get the stress in the material. The suspension must not break, so the following holds

$$\sigma = \frac{F}{S} = \rho L \left(1 + k\right) g \quad \Rightarrow \quad L = \frac{\sigma}{\rho g \left(1 + k\right)} \doteq 83\,\mathrm{m}\,,$$

where σ is the tensile strength.

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Problem K.3 ... ...and back... 4 points

The horizontal displacement of a pendulum has lowered from $1.0\,\mathrm{m}$ *to* $0.9\,\mathrm{m}$ *in ten minutes. Determine the average work the resistance forces do during a single pendulum period. Again, assume a mathematical pendulum with a* $47\,\mathrm{kg}$ *weight and a period of oscillation of* $14.2\,\mathrm{s}$.

Jirka thought the statement of this problem was confusing.

It is important that the period of the oscillation is not dependent on the displacement of the pendulum. In time $t = 10\,\mathrm{min}$ it has made

$$N = \frac{t}{T} = 42.3\,.$$

Oscillations. Further, we know that the length of the pendulum can be calculated from the period of the oscillation as

$$L = \frac{T^2 g}{4\pi^2} = 50.2\,\mathrm{m}\,.$$

The pendulum has lost some energy, which we will calculate from the difference of potential energies. From the knowledge of the displacement $x = L \sin\varphi$, where φ is the angle of deviation from the vertical, we can calculate the decrease of potential energy of the pendulum as

$$\Delta E = mgL \left(\cos\varphi_\mathrm{f} - \cos\varphi_\mathrm{i}\right)\,.$$

We will designate index i as the initial angular displacement and index f as the final. Because the initial angle holds the equation $\varphi_\mathrm{i} = \arctan\left(\frac{x_\mathrm{i}}{L}\right) \doteq 0.020\,\mathrm{rad} \ll 1$, we can with a good precision use approximations $\tan\varphi \approx \sin\varphi \approx \varphi$ a $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$. Therefore, the difference of the potential energies can be written as

$$\Delta E = \frac{mgL}{2}\left(\varphi_\mathrm{i}^2 - \varphi_\mathrm{f}^2\right) = \frac{mg}{2L}\left(x_\mathrm{i}^2 - x_\mathrm{f}^2\right)\,.$$

This energy is equal to the work made by the resistance forces. On average, the pendulum loses during one period an energy

$$P = \frac{\Delta E}{N} = \frac{mg\left(x_\mathrm{i}^2 - x_\mathrm{f}^2\right)T}{2Lt}\,.$$

From the knowledge of the period of the oscillation, we will substitute for the length of the suspension $L = \frac{T^2 g}{4\pi^2}$, and we get

$$P = \frac{\Delta E}{N} = 2\pi^2 \frac{m\left(x_\mathrm{i}^2 - x_\mathrm{f}^2\right)}{Tt} \doteq 21\,\mathrm{mJ}\,.$$

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Problem K.4 ... ...and forth, over and over again 5 points

The estimate did not seem accurate enough to Jarda, so he decided not to consider the pendulum as a simple mathematical pendulum but went on to include the moments of inertia of its parts in his calculations. He found that the suspension consisted of $50.0\,\mathrm{m}$ *long string made from steel with a diameter of* $1.40\,\mathrm{mm}$ *and that at the end of the suspension, there was a ball of radius* $10.0\,\mathrm{cm}$ *with mass* $17.0\,\mathrm{kg}$*. What is the ratio of the period of such a pendulum to the period of a mathematical pendulum of length* $50.1\,\mathrm{m}$*? The density of steel is* $7.90\,\mathrm{g{\cdot}cm^{-3}}$*.*

Jarda needed a physical pendulum problem also for this year's Hurry-up.

We calculate the period using the formula for the physical pendulum, which is

$$T = 2\pi\sqrt{\frac{J}{mgx}}\,,$$

where J is the moment of inertia of the body with respect to the rotational axis, m is its mass, g is the gravitational acceleration, and x is the distance of the center of gravity of the body from the rotational axis.

The mass of the string is $m_d = \rho V = \rho\frac{\pi d^2}{4}l$, where the density of the steel is equal to $\rho = 7\,900\,\mathrm{kg{\cdot}m^{-3}}$. We denote the volume of the string as V and then express it in terms of diameter d and length l. The distance of the center of gravity from the rotational axis is thus

$$x = \frac{m_\mathrm{d}\frac{l}{2} + M\left(l+R\right)}{m_\mathrm{d} + M} = \frac{\rho\frac{\pi d^2}{8}l^2 + M\left(l+R\right)}{m_\mathrm{d} + M}\,,$$

where M is the mass of the ball at the end and R is its radius.

The moment of inertia of the whole body can be obtained by summing the moments of inertia of its individual parts. The suspension can be considered as a thin, rigid rod that rotates around one of its ends, which corresponds to the moment of inertia

$$J_\mathrm{d} = \frac{1}{3}m_\mathrm{d}l^3 = \frac{1}{12}\rho\pi d^2 l^3\,.$$

The moment of inertia of the ball with respect to the axis passing through its center of gravity is $\frac{2}{5}MR^2$, but in this situation, we have to shift the axis by $M\left(l+R\right)^2$ according to parallel axis theorem, and we get the total moment of inertia of the whole pendulum as

$$J = \frac{1}{12}\rho\pi d^2 l^3 + \frac{2}{5}MR^2 + M\left(l+R\right)^2\,.$$

By plugging in the initial relation and comparing it with the period of a mathematical pendulum of length $(l+R)$ we get

$$\frac{T}{T_\mathrm{mat}} = \sqrt{\frac{J}{\left(m_\mathrm{d}+M\right)\left(l+R\right)x}} = \sqrt{\frac{1 + \frac{\frac{1}{12}\rho\pi d^2 l^3 + \frac{2}{5}MR^2}{M(l+R)^2}}{1 + \frac{\rho\frac{\pi d^2}{8}l^2}{M(l+R)}}} \doteq 0.997\,.$$

The difference is so small that even for such a long period of oscillation, it would not be easy to measure using a stopwatch.

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Problem T.1 ... inflating a balloon 4 points

When we are inflating a balloon with a large enough radius, we can accurately assume the balloon's elastic potential energy to be proportional to its surface area. Jirka found such a balloon with a radius of 5 cm and measured the air pressure inside it to be equal to 107 kPa. However, the balloon was not very large, so he decided to inflate it to a radius of 15 cm. How many moles of air did he need to blow into the balloon? Assume a spherical balloon surrounded by room air with a temperature $T = 20\,^{\circ}\mathrm{C}$. *Optics tutorial was too boring for Jirka.*

We want to use the ideal gas law to compute the amount of air we have to blow:

$$pV = nRT\,.$$

In this equation, we know the temperature $T = 20\,^{\circ}\mathrm{C}$ and the volume $V_2 = \frac{4}{3}\pi r_2^3$, where $r_2 = 15\,\mathrm{cm}$ (spherical balloon). We have to relate the pressure inside the balloon to its size.

We already know that to increase the area of the balloon by $\mathrm{d}S$ (consider $\mathrm{d}S$ to be an infinitesimally small area, although the relation would hold even for a finite ΔS), we must do work $\mathrm{d}W$ against the balloon's forces such that

$$\mathrm{d}W = \frac{A}{2}\cdot \mathrm{d}S\,.$$

where we denoted the proportionality constant $\frac{A}{2}$ on behalf of consistency with other problems in this Hurry up series.

The balloon has a spherical shape, so the resultant force acts toward the center (the balloon is trying to shrink). During an enlargement by some small radius $\mathrm{d}r$, the work is done

$$\mathrm{d}W = F\cdot \mathrm{d}r = p\cdot S\cdot \mathrm{d}r = p\cdot 4\pi r^2\,\mathrm{d}r\,,$$

where p is the pressure exerted by the balloon at the radius r. We took advantage of the fact that $\mathrm{d}r$ is small, then during the increase of r by Δr we can consider the force F and the area S constant (i.e. we omit the terms $O(\mathrm{d}r^2)$). Similarly, we could proceed by reasoning that the pressure in the balloon does the same work in inflating it as the pressure of the gas does in expanding it. When the gas expands by a volume $\mathrm{d}V$, it does work $\mathrm{d}W = p\,\mathrm{d}V$, where we also have $\mathrm{d}W = p\cdot 4\pi r^2\,\mathrm{d}r$.

We do know as well that the work is proportional to the area $\mathrm{d}S = 8\pi r^2\,\mathrm{d}r$ and to sum up, we can write

$$p\cdot 4\pi r^2\,\mathrm{d}r = \frac{A}{2}\cdot 8\pi r^2\,\mathrm{d}r\,,$$

where we find the dependance of the pressure on the radius

$$p = \frac{A}{r}\,.$$

Please note that this is not yet the total pressure in the balloon. The air is also subjected to the atmospheric pressure $p_\mathrm{a} = 101\,325\,\mathrm{Pa}$. Finally, the total pressure is $p + p_\mathrm{a}$.

Given the initial pressure p_1 and the radius r_1, we determine the constant A as $A = (p_1 - p_\mathrm{a})r_1$. Then we obtain the number of moles of the air that Jirka must blow into the balloon

$$\Delta n = \frac{p_2 V_2}{RT} - \frac{p_1 V_1}{RT} \qquad \Rightarrow \qquad \Delta n = \frac{4\pi}{3RT}\left[\left(p_\mathrm{a} + (p_1 - p_\mathrm{a})\frac{r_1}{r_2}\right) r_2^3 - p_1 r_1^3\right]\,.$$

Substituting the numbers, we get $\Delta n = 0.576\,\mathrm{mol}$, which translates to around 13 liters of air at standard pressure. Humans are able to inhale approximately 3 liters of air, so Jirka needs about 5 breaths to inflate the balloon.

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Problem T.2 ... we are cooling down the balloon 5 points

Jirka was already satisfied with the size of his balloon and took it for a walk. He left the heated room at $20\,^\circ\mathrm{C}$ *wearing only a T-shirt and found out that he was really cold because the outside temperature was only* $3\,^\circ\mathrm{C}$*. What is the new radius of the balloon after it shrank during the walk if it initially had a radius of* $r_1 = 15\,\mathrm{cm}$*? Do not forget that in the previous problem, you derived a relationship between the overpressure in the balloon and its radius (assuming the balloon has a spherical shape)*

$$\Delta p = \frac{A}{r}\,,$$

where $A = 300\,\mathrm{Pa{\cdot}m}$*.* *Jirka attends secret night meetings.*

We start from the equation of state for an ideal gas. Assuming that the air inside the balloon does not escape during the walk, we have

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}\,,$$

where $T_1 = 20\,^\circ\mathrm{C}$, $T_2 = 3\,^\circ\mathrm{C}$. The balloon is spherical, so its volume is

$$V = \frac{4}{3}\pi r^3\,.$$

The pressure of the air in the balloon is equal to the sum of the atmospheric pressure p_a and the overpressure in the balloon. Therefore, we have

$$p = p_\mathrm{a} + \frac{A}{r}\,.$$

Now, we substitute all the information into the equation of state, and after some adjustments, we get

$$\frac{T_2}{T_1}\left(p_\mathrm{a} + \frac{A}{r_1}\right) r_1^3 = A r_2^2 + p_\mathrm{a} r_2^3\,.$$

Since it is a cubic equation, we will settle for a numerical solution. The only real root is $r_2 \doteq 14.7\,\mathrm{cm}$. Notice that in this case, the change in pressure in the balloon influences the result very little. If we considered a constant pressure inside the balloon, we would obtain a result that differs on the order of tenths of a percent. Therefore, we get

$$r_2 = r_1 \sqrt[3]{\frac{T_2}{T_1}} \doteq 14.7\,\mathrm{cm}\,.$$

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Problem T.3 ... our balloon has flown away 6 points

Viktor had gifted Jirka a little present for his drive during proofreading – a helium balloon with radius $r_0 = 11\,\mathrm{cm}$. Jirka weighted down the balloon in a way that the total weight of the material was $M = 5.2\,\mathrm{g}$ and hoped that it was enough for the balloon not to fly away. Unfortunately, he was mistaken, and the balloon started to rise. If we consider the same dependency between the radius and the pressure inside the balloon as in the problems before, i.e.

$$\Delta p = \frac{A}{r}\,,$$

where $A = 300\,\mathrm{Pa{\cdot}m}$, determine to what height the balloon will rise in an isothermic atmosphere with a temperature $T = 20\,{}^\circ\mathrm{C}$, provided that it won't pop. The molar mass of helium is $4.003\,\mathrm{g{\cdot}mol^{-1}}$.

Hint: *In an isothermic atmosphere both the pressure and the density decrease exponentially.*

Actually, Jarda invited Jirka to have a beer.

An isothermic atmosphere has a constant temperature everywhere and for its pressure p_a and density ρ_a the following holds

$$p_\mathrm{a} = p_\mathrm{a0}\exp\left(-\frac{h}{h_0}\right)\,, \rho_\mathrm{a} = \rho_\mathrm{a0}\exp\left(-\frac{h}{h_0}\right)\,,$$

where p_a0, ρ_a0 are the pressure and the density at the ground level and $h_0 = \frac{RT}{gM_\mathrm{a}} = 8600\,\mathrm{m}$ is the height ($M_\mathrm{a} = 28.96\,\mathrm{g \cdot mol^{-1}}$) where both pressure and density decrease to 1/e of the values at the ground level. Furthermore, the pressure and the density are in a relation

$$\frac{p_\mathrm{a}}{\rho_\mathrm{a}} = \frac{RT}{M_\mathrm{m}}\,,$$

where T is the temperature of the gas and M_m is its molar mass.

Another relation that will accompany us for the rest of the problem is the relation for pressure p_i of the helium inside the balloon

$$p_\mathrm{i} - p_\mathrm{a} = \frac{A}{r}\,.$$

Under normal circumstances, we can determine the density of the helium simply as

$$\rho_\mathrm{He0} = \frac{p_\mathrm{a0}M_\mathrm{He}}{RT} = 0.1664\,\mathrm{kg{\cdot}m^{-3}}\,,$$

where $M_\mathrm{He} = 4.003\,\mathrm{g \cdot mol^{-1}}$ is the molar mass of helium. The density of the helium inside the balloon will be somewhat greater because there is a pressure larger by $\frac{A}{r_0}$; therefore

$$\rho_\mathrm{He} = \rho_\mathrm{He0}\frac{p_\mathrm{a0} + A/r_0}{p_\mathrm{a0}} = 0.1709\,\mathrm{kg{\cdot}m^{-3}}\,.$$

For the balloon to float at a certain height, its gravitational force has to be equal to its upthrust force

$$mg = V\rho_\mathrm{a}g \quad \Rightarrow \quad m = \frac{4}{3}\pi r^3\rho_\mathrm{a}\,.$$

We know that at the ground level, the upthrust force was greater than the gravitational force, which is why we have weighted the balloon down.

We can find the weight of the balloon and the gas inside as

$$m = \frac{4}{3}\pi r_0^3 \rho_{\mathrm{He}} + M = 6.153\,\mathrm{g}\,.$$

The last important equation is the equation of state of an ideal gas in the balloon, according to which

$$p_{\mathrm{i}}\frac{4}{3}\pi r^3 = nRT\,.$$

From this equation, we will simply substitute in the equation for pressure and express the atmospherical pressure depending on the radius as

$$\frac{3nRT}{4\pi r^3} - \frac{A}{r} = p_{\mathrm{a0}} \exp\left(-\frac{h}{h_0}\right)\,.$$

We plug the exponential into the balance of forces equation

$$\exp\left(-\frac{h}{h_0}\right) = \frac{\rho_{\mathrm{a}}}{\rho_{\mathrm{a0}}} = \frac{m}{\frac{4}{3}\pi r^3 \rho_{\mathrm{a0}}}\,,$$

and we get the equation for r in the form

$$m = \frac{4}{3}\pi r^3 \frac{\rho_{\mathrm{a0}}}{p_{\mathrm{a0}}}\left(\frac{3nRT}{4\pi r^3} - \frac{A}{r}\right) \quad \Rightarrow \quad r = \sqrt{\frac{3}{4\pi A}\left(nRT - \frac{mp_{\mathrm{a0}}}{\rho_{\mathrm{a0}}}\right)}\,.$$

Because the temperature and the molar amount of helium in the ballon are constant, for the product nRT, we can write

$$nRT = p_{\mathrm{i0}}\frac{4}{3}\pi r_0^3 = \left(\frac{A}{r_0} + p_{\mathrm{a0}}\right)\frac{4}{3}\pi r_0^3$$

and substitute it into the preceding equation.

Finally, we substitute the radius into the forces equation, and after some additional adjustments and substitutions, we can express the height h as

$$h = h_0 \ln\left(\frac{\rho_{\mathrm{a0}}}{\rho_{\mathrm{a}}}\right) = h_0 \ln\left(\frac{4\pi r^3 \rho_{a0}}{3m}\right) = h_0 \ln\left(\frac{4\pi\left(\sqrt{r_0^2 + \frac{r_0^3 p_{\mathrm{a0}}}{A} - \frac{3mp_{\mathrm{a0}}}{4\pi A\rho_{\mathrm{a0}}}}\right)^3 \rho_{a0}}{3m}\right) \doteq 19\,\mathrm{km}\,.$$

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Problem T.4 ... connecting balloons

6 points

Because Jirka's helium balloon had flown away, he sadly had to ask Viktor for another one. He got two, but he had to inflate them himself. Jirka inflated one to a radius of $r_i = 15.0\,\mathrm{cm}$, then took a short, narrow tube and connected it to the other so that no air escaped. To his great surprise, once the balance was established, one balloon had a larger radius than the other. What was the ratio of the radius of the larger balloon to the radius of the smaller one? Now, suppose that the pressure difference between the balloon and the surrounding area depends on its radius as

$$\Delta p = \frac{A}{r}\left[1 - \left(\frac{r_0}{r}\right)^6\right] ,$$

where $A = 300\,\mathrm{Pa{\cdot}m}$ and $r_0 = 3.00\,\mathrm{cm}$. The experiment was conducted at a $20\,\mathrm{^\circ C}$. Assume the second balloon has radius r_0 before attaching it to the tube.

Jarda saw an interestig experiment at the lecture.

The situation stabilizes once the pressures between the balloons are equal while the total amount of substance of gas in the system is preserved. Knowing the relationship for pressure as a function of radius, we can use the equation of state to determine the amount of substance of gas in the system as

$$\begin{aligned} n_1 &= \frac{1}{RT} p \frac{4}{3}\pi r_i^3 = \frac{1}{RT}\left\{\frac{A}{r_i}\left[1 - \left(\frac{r_0}{r_i}\right)^6\right] + p_\mathrm{a}\right\}\frac{4}{3}\pi r_i^3 \doteq 0.599\,\mathrm{mol} ,\\ n_2 &= \frac{1}{RT} p_\mathrm{a}\frac{4}{3}\pi r_0^3 \doteq 0.005\,\mathrm{mol} . \end{aligned}$$

Let r_1 denote the radius of the original of the connected balloons and r_2 the radius of the second. With equal pressures, the following holds

$$\frac{1}{r_1}\left[1 - \left(\frac{r_0}{r_1}\right)^6\right] = \frac{1}{r_2}\left[1 - \left(\frac{r_0}{r_2}\right)^6\right] .$$

We can always use one radius from this equation to calculate the second radius.

At the same time, the equation of state for the whole system must be of the form

$$p\frac{4\pi}{3}\left(r_1^3 + r_2^3\right) = \left\{\frac{A}{r_1}\left[1 - \left(\frac{r_0}{r_1}\right)^6\right] + p_\mathrm{a}\right\}\frac{4\pi}{3}\left(r_1^3 + r_2^3\right) = (n_1 + n_2)RT \doteq 1472.18\,\mathrm{J} ,$$

We know the value of the right-hand side of the equation; the only variable on the left-hand side is r_1. So we will vary r_1 until we get the equality between the two sides to the desired precision. For example, we can use software like Wolfram Mathematica or a suitable graphing calculator like GeoGebra. We plot the dependence of the overpressure on the radius, place one point on it, and find another point that has the same overpressure and is, therefore, at the intersection of the overpressure curve with the parallel x axis, so we know r_1 and r_2. We find that for the right side to equal nRT, we need $r_1 = 14.994\,\mathrm{cm}$ and $r_2 = 3.119\,\mathrm{cm}$. Their ratio and, therefore, the solution to our problem is 4.81.

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Problem M.1 ... tram 3 points

What is the maximum angle at which a tram can travel downhill and still be able to stop? The coefficient of shear friction of the wheels and rails is $f = 0.15$.

David rode to the lecture suspiciously downhill.

Suppose the tram is on an inclined plane at an angle α with the ground. In that case, three forces act on it: the downward gravitational force F, the normal force $N = F\cos\alpha$, which is perpendicular to the inclined plane, and the frictional force F_f acting against the direction of motion. The maximum possible value of the friction force is

$$F_{\mathrm{f,\,max}} = fN = fF\cos\alpha\,,$$

where $f = 0.15$ is the coefficient of shear friction between the tram wheels and the rails. This is also the maximum braking force the tram can exert.

In the downhill direction, the tangential component of the gravitational force

$$F_{\|} = F\sin\alpha\,,$$

is trying to get the tram moving. The tram can only stop if its braking force is greater than the accelerating tangential component of the gravitational force. We get an inequality between these two forces

$$\begin{aligned} F_{\mathrm{f,\,max}} &> F_{\|} \\ fF\cos\alpha &> F\sin\alpha \\ \tan\alpha &< f \\ \alpha &< \arctan f \\ \alpha &< 8.53^\circ \doteq 8.5^\circ\,. \end{aligned}$$

The tram can only stop safely on slopes with an incline less than 8.5°.

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Problem M.2 ... the tram reloaded 3 points

The tram travels at $v = 27\,\mathrm{km{\cdot}h^{-1}}$ and its maximum deceleration is $a = 2.1\,\mathrm{m{\cdot}s^{-2}}$. What is the minimum distance required for the tram to come to a complete stop upon Davis's signaling at the stop? The reaction time of the driver to the David's wave is $t' = 0.3\,\mathrm{s}$.

David studied the T3 tram manual for far too long

First, we will calculate the distance the tram travels before the driver reacts to David as

$$s_1 = vt' = 2.25\,\mathrm{m}\,.$$

We will then calculate how long it takes for the tram to stop

$$v = at \quad \Rightarrow \quad t = \frac{v}{a} \doteq 3.57\,\mathrm{s}\,.$$

After that, we will determine the stopping distance using the well-known formula

$$s_2 = vt - \frac{1}{2}at^2 = \frac{1}{2}at^2 \doteq 13.39\,\mathrm{m}.$$

Finally, we will add the two distances together

$$s = s_1 + s_2 \doteq 16\,\mathrm{m}\,.$$

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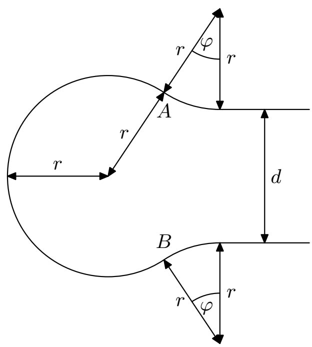
Problem M.3 ... parallel tracks

4 points

Consider two parallel tram tracks at a distance of $d = 11\,\mathrm{m}$. We want to build a system of arcs between them with a radius r, as shown in the right image. Find r such that a tram, capable of withstanding a centripetal acceleration of $a = 0.85\,\mathrm{m{\cdot}s^{-2}}$ in turn, passes through it in the shortest possible time. The velocity of the tram remains constant.

Adam would like to ride the Snowpiercer.



Obviously, only $r \geq d/2$ makes sense. Let's choose any r that satisfies this condition and calculate the total length of the arcs $s(r)$ and the maximum possible speed of the locomotive $v(r)$.

Speed is much simpler, so let's start with it. Because $a \leq v^2/r$ it holds

$$v = \sqrt{ar}\,.$$

To calculate the function $s(r)$, let's first determine the central angle corresponding to the middle arc. For that, we have $\alpha = \pi + 2\varphi$. Subsequently, we express distances A and B in two ways. Using the large arc as

$$2r\sin\left(\frac{\pi}{2} + \varphi\right) = 2r\cos\varphi$$

and using the small arcs as

$$d + 2r(1 - \cos\varphi).$$

We will compare these two expressions and obtain the relation $\varphi = \arccos\left[(d + 2r)/4r\right]$. Finally, it is sufficient to add up the individual arcs and obtain

$$s(t) = \pi r + 4r\arccos\left(\frac{d + 2r}{4r}\right).$$

From the known functions $s(r)$ and $v(r)$, we will express time as a function r

$$t(r) = \frac{s(r)}{v(r)} = \pi\sqrt{\frac{r}{a}} + 4\sqrt{\frac{r}{a}}\arccos\left(\frac{d + 2r}{4r}\right) \leq \pi\sqrt{\frac{r}{a}}\,.$$

Function $f(r) = \pi\sqrt{r/a}$ is increasing, and the tram will pass through the arc fastest when $r = d/2 = 5.5\,\mathrm{m}$.
Note: In fact, it was not necessary to express the angle φ (it is sufficient that it is non-negative), but it is an interesting geometric problem.

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Problem M.4 ... turboflies' troubles 5 points

Two trams at a distance of $s = 400\,\mathrm{m}$ are moving against each other, first with a velocity of $v_1 = 30\,\mathrm{km{\cdot}h^{-1}}$, second with a velocity of $v_2 = 35\,\mathrm{km{\cdot}h^{-1}}$. From the first one, a fly takes off with a velocity of $v_3 = 80\,\mathrm{km{\cdot}h^{-1}}$ and flies to the other tram, where it bounces off the windshield, flies back, and so on, as long as it manages to outfly the trams. The windshields are sticky, so with each bounce, its speed gets lowered to a q-factor of the velocity before the bounce, where $q = 0.97$. Assuming the bounce is instantaneous, what distance does the fly cover before the trams collide? *Marek has seen a fly in a tram.*

After time t since the moment the fly takes off, the trams are at a distance of $s - (v_1 + v_2)t$. We will define the time of the i-th collision with the trams as t_i (and assign $t_0 = 0\,\mathrm{s}$). Next, we will designate $v_1 = 30\,\mathrm{km{\cdot}h^{-1}}$ as the velocity of the first tram and v_2 as the velocity of the second one, while the initial velocity of the fly will be designated as v_3. Then, we can write for even and odd collisions:

$$s - (v_1 + v_2)t_{2n} = \left(v_2 + v_3 \cdot q^{2n}\right)(t_{2n+1} - t_{2n})\,,$$
$$s - (v_1 + v_2)t_{2n+1} = \left(v_1 + v_3 \cdot q^{2n+1}\right)(t_{2n+2} - t_{2n+1})\,,$$

because the fly and the oncoming tram have to cover the distance between the trams in the same time. Upon rearranging, we get iterative relations

$$t_{2n+1} = \frac{s - \left(v_1 - v_3 \cdot q^{2n}\right) t_{2n}}{v_2 + v_3 \cdot q^{2n}}\,,$$
$$t_{2n+2} = \frac{s - \left(v_2 - v_3 \cdot q^{2n+1}\right) t_{2n+1}}{v_1 + v_3 \cdot q^{2n+1}}\,.$$

We can notice that if the fly did not slow down, a formally infinite number of collisions with the trams would happen. However, the fly is slowing down, thus its velocity will be smaller at one point than the velocity of one of the trams, and it will only ride the windshield. This happens for least such k that one of the following inequalities will hold

$$v_3 q^{2k} \leq v_1\,, \quad v_3 q^{2k+1} \leq v_2\,.$$

This allows us to determine the number of collisions with trams N as $2k$ if the first inequality holds or as $2k + 1$ if the second inequality is met. k can be determined using, for example, any spreadsheet, such as Excel. Once we determine it, we will calculate the total path of the fly as

$$s = \sum_{n=1}^{N} v_3 q^{n-1} (t_n - t_{n-1})\,,$$

We could have observed that after about the tenth bounce, the trams are so close that the covered distances don't change significantly, so performing a summation until index $N = 10$ would be sufficient.

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